Lesson 1-3

Limits III

Objective

Students will…

- Be able to evaluate limits of composite functions.
- Be able to define and use the Squeeze Theorem.
- Learn three "special limits"

Composite Functions

Recall that a composition of functions, or the composite function $f \circ g$ (also called a composition of f and g) is defined by

$$(f \circ g)(x) = f(g(x))$$

When it comes to limits, the following is true:
If f and g are functions such that

$(f \circ g)(x) = f(g(x))$ i.e. $\lim_{x \to c} g(x) = L$ and $\lim_{x \to c} f(x) = f(L)$,

then,

$\lim_{x \to c} f(g(x)) = f\left(\lim_{x \to c} g(x)\right) = f(L)$

(see proof in the textbook)

Example

Let $f(x) = \sqrt{x}$ and $g(x) = x^2 + 4$. Find $\lim_{x \to 0} f(g(x))$

$$\lim_{x \to 0} f(g(x)) = f\left(\lim_{x \to 0} g(x)\right) = f\left(\lim_{x \to 0} x^2 + 4\right) = f(0^2 + 4) = f(4) = \sqrt{4} = \boxed{2}$$

$$\lim_{x \to 0} f(g(x)) = \lim_{x \to 0} \left(\sqrt{x^2 + 4}\right) = \sqrt{0^2 + 4} = \sqrt{4} = \boxed{2}$$

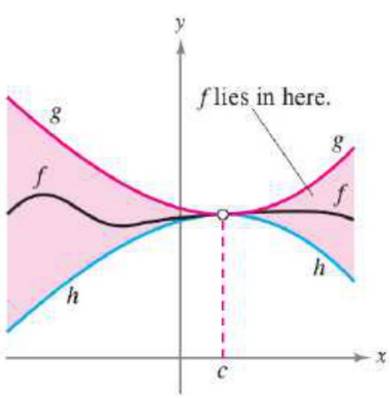
The Squeeze Theorem

If $h(x) \leq f(x) \leq g(x)$ for all x in an open interval containing c, except possibly at c itself, and if $\lim_{x \to c} h(x) = L = \lim_{x \to c} g(x)$, then…

$\lim_{x \to c} f(x)$ exists, and $\lim_{x \to c} f(x) = L$

(barring any jump discontinuity, i.e. piecewise functions)

$h(x) \leq f(x) \leq g(x)$



The Squeeze Theorem
Figure 1.21

Application of the Squeeze Theorem

The main application and the usefulness of the Squeeze Theorem is how it can be used to show a few "special" limits. Now, to actually use the theorem to show the work requires some rigor and finesse that won't be necessary for this particular course. Again, see the textbook if you want to see the proofs of them.

Theorem 1.9- Three Special Limits

1. $\lim_{x \to 0} \frac{\sin x}{x} = 1$ 2. $\lim_{x \to 0} \frac{1-\cos x}{x} = 0$ 3. $\lim_{x \to 0}(1+x)^{\frac{1}{x}} = e$

4. $\lim_{x \to 0} \frac{\tan x}{x} = 1$ 5. $\lim_{x \to 0} \frac{\sin nx}{nx} = n$

Examples

a. Find $\lim_{x\to 0}\left(\frac{\tan x}{x} = \frac{\frac{\sin x}{\cos x}}{x} = \frac{\sin x}{x\cos x} = \frac{\sin x}{x}\cdot\frac{1}{\cos x}\right)$ $\left(\lim_{x\to 0}\frac{\sin x}{x} = \right)$

$$\Rightarrow \lim_{x\to 0}\left(\frac{\sin x}{x}\cdot\frac{1}{\cos x}\right) = \lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\frac{1}{\cos x}$$

$$= 1\cdot\frac{1}{1} = \boxed{1}.$$

$$\lim_{x\to 0}\frac{\sin x}{x} = 1$$

Example

b. Find $\lim_{x\to 0}\frac{\sin 4x}{x}$

$$\lim_{x\to 0}\left(\frac{\sin 4x}{x}\cdot\frac{4}{4}\right) = \frac{4\sin 4x}{4x} = 4\left(\frac{\sin 4x}{4x}\right)$$

$x \to 0$

$4x \to 4(0)$

$4x \to 0$

Let $y = 4x$

$$\Rightarrow \lim_{x\to 0} 4\left(\frac{\sin 4x}{4x}\right) = 4\lim_{x\to 0}\left(\frac{\sin 4x}{4x}\right)$$

$$= 4\lim_{y\to 0}\left(\frac{\sin y}{y}\right) = 4(1) = \boxed{4}.$$

Homework 9/7

1.3 exercises #5-35 (e.o.o), 41-44, 45, 47, 49-61 (e.o.o), 66-74 (e.o.e)