Lesson 2-5

Implicit Differentiation

Objective

Students will…

- Be able to distinguish between implicit and explicit form.
- Be able to use implicit differentiation technique to find derivatives.

Implicit vs Explicit Form

Up to this point, we have been dealing with finding derivatives of functions that were written in **<u>explicit form</u>**, i.e. solved for a variable (dependent variable). However, functions may be written in **<u>implicit forms</u>**, where it is not clearly solved for a variable. For example,

Explicit Form: $y = 3x^2 - 5$

Implicit Form: $5 = 3x^2 - y$

$y = 3x^2 - 5$

Explicit Form: $y = \frac{1}{x}$

Implicit Form: $xy = 1$

In many cases, it would be easier to simply rewrite the equation in the explicit form before taking the derivative. But this may not always be easy to do!

Implicit Differentiation

When the function cannot easily be written in the explicit form, it's best to use the technique of **<u>implicit differentiation</u>**. Best way to interpret this technique is to treat any "y" term as a composition, thus requiring the use of the **<u>chain rule</u>**.

$$5 = 3x^2 - y^2 = 3x^2 - f(y)$$

Thus, in finding the derivative…

$$\frac{d}{dx}5 = \frac{d}{dx}3x^2 - \frac{d}{dx}\big(f(y)\big) \Rightarrow 0 = 6x - 2y \cdot y' \text{ (chain rule)}$$

Then, finally solving for y'…

$$y' = \frac{6x}{2y}$$

Implicit Differentiation

GUIDELINES FOR IMPLICIT DIFFERENTIATION

1. Differentiate both sides of the equation *with respect to x*.
2. Collect all terms involving dy/dx on the left side of the equation and move all other terms to the right side of the equation.
3. Factor dy/dx out of the left side of the equation.
4. Solve for dy/dx.

$y' = \frac{6x}{2y}$

$5 = 3x^2 - y^2 \Rightarrow \sqrt{y^2} = \sqrt{3x^2 - 5}$

$y = (3x^2 - 5)^{1/2}$

$= \frac{1}{2}(3x^2 - 5)^{-1/2} \cdot 6x$

$y' = \frac{6x}{2\sqrt{3x^2-5}} = \frac{3x}{\sqrt{3x^2-5}}$

$y' = \ldots$ $\left(\frac{dy}{dx}\right)$ or (y') Example

Find the derivative.

$$y^3 + y^2 - 5y - x^2 = -4$$

$$\frac{d}{dx} = 3y^2y' + 2yy' - 5y' \cancel{-2x} = 0$$

$$\cancel{+2x} \qquad +2x$$

$$3y^2y' + 2yy' - 5y' = 2x$$

$$y'(3y^2 + 2y - 5) = 2x$$

$$\frac{\cancel{y'(3y^2+2y-5)}}{\cancel{3y^2+2y-5}} = \boxed{\frac{2x}{3y^2+2y-5}}$$

Example

Find the slope of the tangent line to the graph: $x^2 + 4y^2 = 4$ at the point $\left(\sqrt{2}, -\frac{1}{\sqrt{2}}\right)$.

$x \quad y$

$$\frac{d}{dx} = \cancel{2x} + 8yy' = 0$$

$$\cancel{-2x} \qquad -2x$$

$$\frac{\cancel{8y}y'}{\cancel{8y}} = \boxed{\frac{-2x}{8y}}$$

$$\frac{-\cancel{2}(\sqrt{2})}{4\,\cancel{8}\left(-\frac{1}{\sqrt{2}}\right)}$$

$$= \frac{+\sqrt{2}}{1} \cdot \frac{\sqrt{2}}{4} = \frac{2}{4} = \boxed{\frac{1}{2}}$$

Example

Find the slope of the tangent line to the graph: $3(x^2 + y^2)^2 = 100\overset{f\,g}{xy}$ at the point $(3, 1)$.

$$\frac{d}{dx} = 6(x^2+y^2)(2x+2yy') = 100(y+xy')$$

$$= \cancel{12x^3} + 12x^2yy' + \cancel{12xy^2} + 12y^3y' = 100y + \cancel{100xy'}$$

$$-12x^3 \qquad -12xy^2 \qquad -100xy'$$

$$12x^2yy' + 12y^3y' - 100xy' = 100y - 12xy^2 - 12x^3$$

$$\boxed{y' = \frac{100y - 12xy^2 - 12x^3}{12x^2y + 12y^3 - 100x}} \qquad \boxed{m = \frac{100 - 36 - 324}{108 + 12 - 300}}$$

Example

Find the derivative.

$4\sin x \cos y = 1$

$\quad\; f \quad\;\; g$

$$\frac{d}{dx} \Rightarrow \cancel{4}\left(\cos x\cos y - y'\sin y \sin x\right) = 0$$

$$\frac{\cancel{4}y'\cancel{\sin y}\cancel{\sin x}}{\cancel{\sin y}\cancel{\sin x}} = \frac{\cancel{4}\cos x\cos y}{\sin y \sin x} \Rightarrow y' = \frac{\cos x\cos y}{\sin x \sin y}$$

$$= \boxed{\cot x \cot y}$$

Example

Find the second derivative.

$$x^2 + y^2 = 25$$

$$\frac{d}{dx} = 2x + 2yy' = 0$$

$$2yy' = -2x$$

$$y' = \frac{-2x}{2y}$$

$$y'' = \frac{-2(2y) + 2y'(2x)}{4y^2}$$

$$= \frac{-4y + 4xy'}{4y^2} = \frac{-4y + 4x\left(\frac{-2x}{2y}\right)}{4y^2}$$

$$= \boxed{\frac{-4y - \frac{8x^2}{2y}}{4y^2}}$$

$$\frac{\frac{-8y^2 - 8x^2}{2y}}{4y^2}$$

$$\boxed{\frac{-8y^2 - 8x^2}{8y^3}}$$

Homework 10/9

2.5 Exercises #1-13 (e.o.o), 15, 21-27 (e.o.o), 28, 29, 45, 47, 49