Lesson 1-5, 1-6

Inifinite Limits/Limits at Infinity

Objective

Students will…

- Be able to define and determine infinite limits.
- Be able to determine (finite) limits at infinity.
- Be able to find limits of rational functions at infinity by finding its horizontal asymptotes.

Infinite Limits

$\lim_{x \to c} f(x) = \infty$, means…

"the limit of $f(x)$ as x approaches c is ∞."

Or, as x approaches c , y or $f(x)$ grows positively without bound.

On the other hand, $\lim_{x \to c} f(x) = -\infty$

"the limit of $f(x)$ as x approaches c is $-\infty$."

Or, as x approaches c, y or $f(x)$ grows negatively without bound.

proves

Disclaimer: This actually shows that the limit **DOES NOT EXIST**. Infinity is **not** a number.

Vertical Asymptotes

We learned in our last study that vertical asymptotes are a type of a **nonremovable discontinuity**, i.e. the limit fails to exist. Better yet, the limit fails to exist because the limit is either ∞ or $-\infty$. Here is a quick way to find vertical asymptotes of a rational function.

Vertical asymptotes- For a rational function $h(x) = \frac{f(x)}{g(x)}$, and for some real number c, if $f(c) \neq 0$ and $g(c) = 0$, then $h(x)$ has a vertical asymptote at $x = c$.

In other words, by default $h(x)$ has a nonremovable discontinuity at $x = c$

Examples

a. $\lim_{x\to 1} \frac{x^2-3x}{x-1}$

$1^2-3(1) \neq 0$

V.A. @ $x=1$ Non rem.

DNE

b. $\lim_{x\to 0} \left(\cot x = \frac{\cos x}{\sin x}\right)$

$\cos(0) = 1 \neq 0$

non remov. V.A. @ $x=0$

DNE

Properties of Infinite Limits

THEOREM 1.15 PROPERTIES OF INFINITE LIMITS

Let c and L be real numbers and let f and g be functions such that

$$\lim_{x \to c} f(x) = \infty \quad \text{and} \quad \lim_{x \to c} g(x) = L.$$

DNE   Exists $\infty \pm L$ $\lim f(x) \pm \lim g(x) = \infty$

1. Sum or difference: $\lim_{x \to c} [f(x) \pm g(x)] = \infty$ DNE ± Exists = DNE

2. Product: $\lim_{x \to c} [f(x)g(x)] = \infty, \quad L > 0$ $\infty \cdot L = \infty$

$\lim_{x \to c} [f(x)g(x)] = -\infty, \quad L < 0$ $\infty \cdot -L = -\infty$

DNE · Exists = DNE

3. Quotient: $\frac{\infty}{L} = \infty$ $\lim_{x \to c} \frac{g(x)}{f(x)} = 0$ $\frac{L}{\infty} = \boxed{0}$ $\frac{L}{\infty} = 0$ $\frac{\text{Exists}}{\text{DNE}} =$ Exists

Similar properties hold for one-sided limits and for functions for which the limit of $f(x)$ as x approaches c is $-\infty$.

$\frac{L}{\to 0} = \infty$

Examples

a. $\lim_{x \to 0}(1 + \frac{1}{x^2})$

@ $x = 0$ V.A (non remov)

$= \lim_{x \to 0} 1 + \lim_{x \to 0} \frac{1}{x^2}$

$= 1 + \infty = \infty$

DNE DNE ✓

b. $\lim_{x \to 1^-} \frac{(x^2+1)}{\cot \pi x} = \frac{\lim_{x \to 1^-} (x^2+1) = 2}{\lim_{x \to 1^-} \cot \pi x}$

$= \frac{2}{\lim_{x \to 1^-} \frac{\cos \pi x}{\sin \pi x}} = \frac{-1}{\to 0} = -\infty \Rightarrow \frac{2}{-\infty} = \boxed{0}$

c. $\lim_{x \to 0^+} 3 \ln x$

$= 3 \lim_{x \to 0^+} \ln x$

$= -\infty$

$\lim_{x \to 0^-} 3 \ln x$

Limits at Infinity

$\lim_{x\to\infty} f(x) = L$, means…

"the limit of $f(x)$ as x grows positively without bound is L."

Or, as x approaches infinity, y or $f(x)$ approaches L.

On the other hand, $\lim_{x\to-\infty} f(x) = L$

"the limit of $f(x)$ as x grows negatively without bound is L."

Or, as x approaches negative infinity, y or $f(x)$ approaches L.

Horizontal Asymptote

The most useful way to evaluate limits at infinity is to find the **horizontal asymptote**. Recall from Pre-Calculus or Algebra 2….

1. If the degree of the numerator is less than the degree of the denominator, then the HA ~~limit of the rational function~~ is 0.
2. If the degree of the numerator is equal to the degree of the denominator, then the HA ~~limit of the rational function~~ is the ratio of the leading coefficients.
3. If the degree of the numerator is greater than the degree of the denominator, then the HA ~~limit of the rational function~~ does not exist, or the limit is $\pm\infty$.
   $x \to \pm\infty$

Remember: The bigger the denominator gets the closer it gets to zero.

Examples

a. $\lim_{x\to\infty} \frac{2x+5}{3x^2+1}$

$= 0$

b. $\lim_{x\to\infty} \frac{2x^2+5}{3x^2+1}$

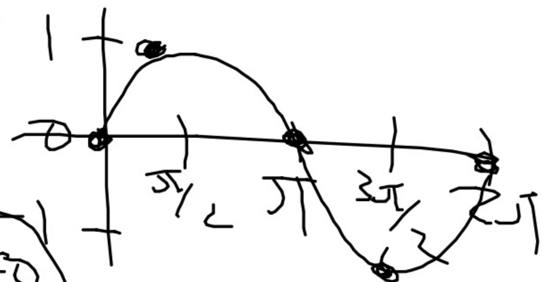
$= 2/3$

c. $\lim_{x\to\infty} \frac{2x^3+5}{3x^2+1} = \infty$

ONE

Limits at Infinity with Trig Functions

a. $\lim_{x \to \infty} \sin x$



$\boxed{= \text{DNE}.}$

b. $\boxed{\lim_{x \to \infty} \frac{\sin x}{x} = 0}$

$$-\frac{1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}$$

$$\lim_{x \to \infty} -\frac{1}{x} = 0 \qquad \lim_{x \to \infty} \frac{1}{x} = 0$$

By squeeze theorem

$$\lim_{x \to \infty} \frac{\sin x}{x} = 0.$$

Examples

a. $\lim_{x\to\infty} \frac{2x^2-4x}{x+1}$

$= \infty$

DNE

b. $\lim_{x\to-\infty} \frac{2x^2-4x}{x+1} = \frac{\lim_{x\to-\infty} 2x^2-4x}{\lim_{x\to-\infty} x+1}$ $\frac{\infty}{-\infty}$

$= -\infty =$ DNE

c. $\lim_{x\to-\infty} \frac{2x^6-3}{x^2+1}$

$\boxed{\infty}$

Homework 9/12

1.5 exercises #1-4, 5-8, 13-31 (odd)

1.6 exercises #1-6, 13-37 (odd), 49-50