Warm Up 4/24

Write the following matrix into row-echelon form:

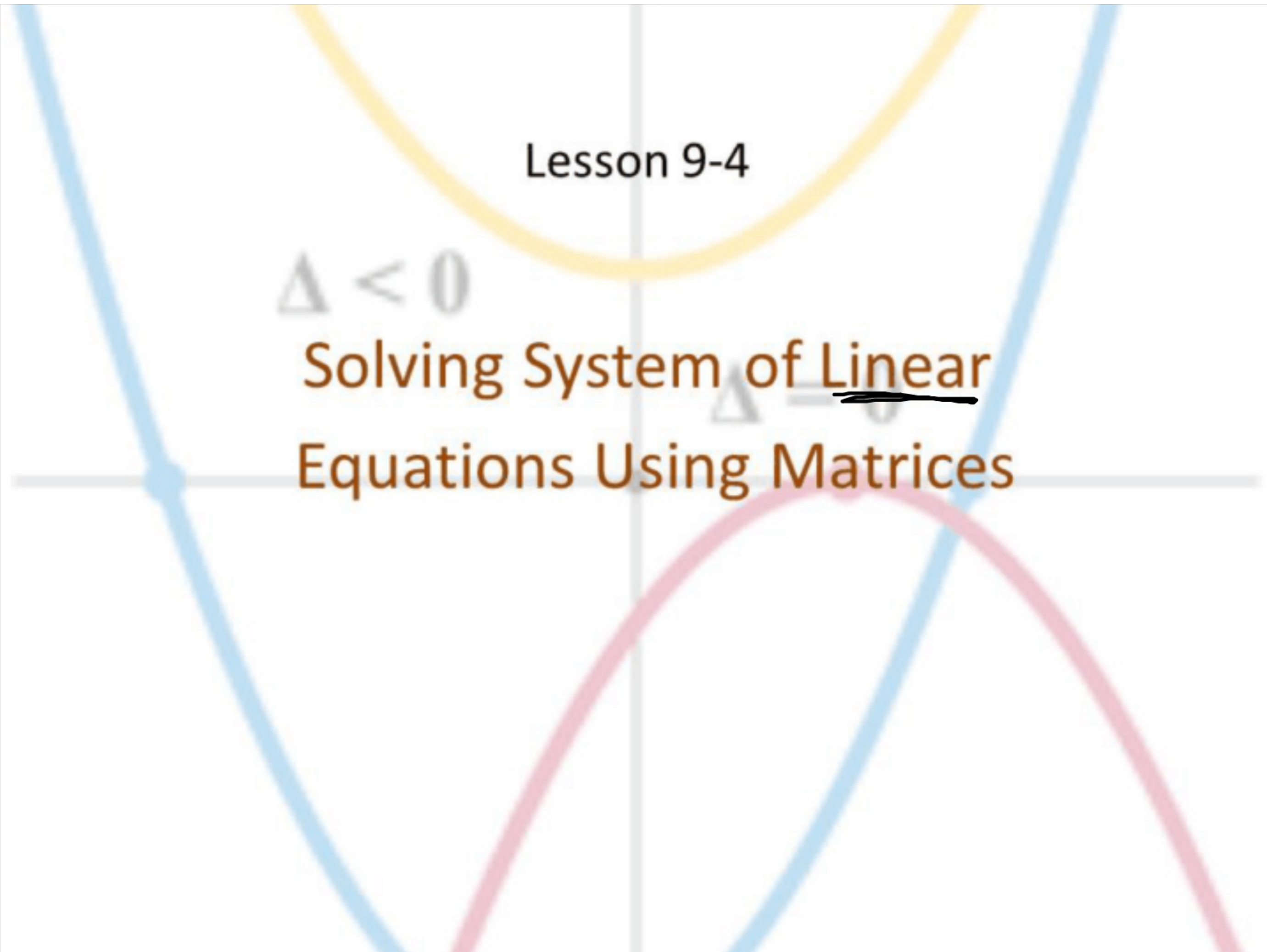
$$\begin{bmatrix} -2 & -3 & -2 \\ 3 & -2 & -2 \\ 3 & -2 & -1 \\ -1 & -1 & -2 \end{bmatrix} \xrightarrow{R_4 \leftrightarrow R_1} \begin{bmatrix} -1 & -1 & -2 \\ 3 & -2 & -2 \\ 3 & -2 & -1 \\ -2 & -3 & -2 \end{bmatrix} \xrightarrow{-1R_1} \begin{bmatrix} 1 & 1 & 2 \\ 3 & -2 & -2 \\ 3 & -2 & -1 \\ -2 & -3 & -2 \end{bmatrix}$$

$$\begin{array}{l} -3R_1+R_2 \\ -3R_1+R_3 \\ 2R_1+R_4 \end{array} \begin{bmatrix} 1 & 1 & 2 \\ 0 & -5 & -8 \\ 0 & -5 & -7 \\ 0 & -1 & 2 \end{bmatrix} \xrightarrow{-6R_4+R_2} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -20 \\ 0 & -5 & -7 \\ 0 & -1 & 2 \end{bmatrix} \xrightarrow[R_2+R_4]{5R_2+R_3} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -20 \\ 0 & 0 & -10 \\ 0 & 0 & -18 \end{bmatrix}$$

$$\xrightarrow{-\frac{1}{107}R_3} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -20 \\ 0 & 0 & 1 \\ 0 & 0 & -18 \end{bmatrix} \xrightarrow{18R_3+R_4} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -20 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Lesson 9-4

Solving System of Linear Equations Using Matrices



Objective

Students will…

- Be able to write augmented matrix of a linear system.
- Be able to solve system of linear equations using matrices.

Elementary Row Operations

There are certain ways a matrix can be "safely" modified. These are known as **<u>elementary row operations</u>**. They are given as follows.

<u>Elementary Row Operations</u>

1. Add a multiple of one row to another

   <u>Symbol:</u> $R_i + kR_j \to R_i$
2. Multiply a row by a nonzero constant

   <u>Symbol:</u> kR_i
3. Interchange two rows.

   <u>Symbol:</u> $R_i \leftrightarrow R_j$

These three operations can **<u>always be done</u>** in **<u>any matrix</u>**. The usefulness of these operations will be explored in the next section. For now, our goal is to get used to doing these operations on matrices.

Row-Echelon Form

The main reason for these operations is so that we can rewrite the matrix in what's known as, **Row-Echelon Form**. A matrix is in Row-Echelon Form if it satisfies the following conditions:

1. The first nonzero number in each row (reading from left to right) is 1. This is called the **leading entry**.
2. The leading entry in each row is to the right of the leading entry in the row immediately above it.
3. All rows consisting entirely of zeroes are at the bottom of the matrix.

Ex.
$$\begin{bmatrix} 1 & 3 & -6 & 10 & 0 \\ 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Augmented Matrices of a Linear System

The main reason why we study matrices in this course, is to solve **linear systems** using matrices. The first step to this process is to write a linear system (**system of linear equations**) into an **augmented matrix**, which is made up of the coefficient and the constants of the linear system.

Ex.

Linear system

$$\begin{cases} 3x - 2y + z = 5 \\ x + 3y - z = 0 \\ -x + 0y + 4z = 11 \end{cases}$$

Augmented matrix

$$\begin{array}{c} \begin{array}{cccc} x & y & z & = \square \end{array} \\ \begin{bmatrix} 3 & -2 & 1 & 5 \\ 1 & 3 & -1 & 0 \\ -1 & 0 & 4 & 11 \end{bmatrix} \end{array}$$

As you can see the **lone** constants appear on the **right-most** column.

Note: It's **imperative** that you line up the variables carefully.

Example

Write the following system of equation into an augmented matrix.

$$\begin{cases} 6x - 2y - z = 4 \\ x + 3z = 1 \\ 7y + z = 5 \end{cases} = \begin{bmatrix} 6 & -2 & -1 & 4 \\ 1 & 0 & 3 & 1 \\ 0 & 7 & 1 & 5 \end{bmatrix}.$$

Using Row Echelon Form

We can now solve system of linear equations using **row-echelon form**. The reason simply is that row-echelon form allows us to use substitution right away solve for the variables. Consider the following system.

$$\begin{cases} x - y + 3z = 4 \\ x + 2y - 2z = 10 \\ 3x - y + 5z = 14 \end{cases} = \begin{bmatrix} 1 & -1 & 3 & 4 \\ 1 & 2 & -2 & 10 \\ 3 & -1 & 5 & 14 \end{bmatrix} \xrightarrow[-3R_1+R_3]{-1R_1+R_2} \begin{bmatrix} 1 & -1 & 3 & 4 \\ 0 & 3 & -5 & 6 \\ 0 & 2 & -4 & 2 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{2}R_3 \leftrightarrow R_2} \begin{bmatrix} 1 & -1 & 3 & 4 \\ 0 & 1 & -2 & 1 \\ 0 & 3 & -5 & 6 \end{bmatrix} \xrightarrow{-3R_2+R_3} \begin{bmatrix} 1 & -1 & 3 & 4 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$x - y + 3z = 4$

$y - 2z = 1$

$\boxed{z = 3}$

$y - 2(3) = 1$

$\boxed{y = 7}$

$x - (7) + 3(3) = 4$

$x - 7 + 9 = 4$

$x + 2 = 4$

$\boxed{x = 2}$

Example

Solve the following linear system using row-echelon form.

$$\begin{cases} 4x + 8y - 4z = 4 \\ 3x + 8y + 5z = -11 \\ -2x + y + 12z = -17 \end{cases} = \begin{bmatrix} 4 & 8 & -4 & 4 \\ 3 & 8 & 5 & -11 \\ -2 & 1 & 12 & -17 \end{bmatrix} \xrightarrow{\frac{1}{4}R_1} \begin{bmatrix} 1 & 2 & -1 & 1 \\ 3 & 8 & 5 & -11 \\ -2 & 1 & 12 & -17 \end{bmatrix}$$

$$\xrightarrow[2R_1+R_3]{-3R_1+R_2} \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 2 & 8 & -14 \\ 0 & 5 & 10 & -15 \end{bmatrix} \xrightarrow{\frac{1}{2}R_2} \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & 4 & -7 \\ 0 & 5 & 10 & -15 \end{bmatrix}$$

$$\xrightarrow{-5R_2+R_3} \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & 4 & -7 \\ 0 & 0 & -10 & 20 \end{bmatrix} \xrightarrow{-\frac{1}{10}R_3} \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & 4 & -7 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$x + 2y - z = 1$

$y + 4z = -7$

$\boxed{z = -2}$

$y + 4(-2) = -7$

$\boxed{y = 1}$

$x + 2(1) - (-2) = 1$

$x + 2 + 2 = 1$

$\boxed{x = -3}$

$0 \neq 2$
No sol.

No or Infinitely Many Solutions

$0 = 0$
inf. sol.

Remember that when solving for a linear system, we may end up with a set of equations that are unsolvable (**no real solutions**) or equations that have **infinitely many** solutions. This can be seen in matrices as well.

No solution

$$\begin{bmatrix} 1 & 2 & 5 & 7 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Last equation says $0 = 1$.

One solution

$$\begin{bmatrix} 1 & 6 & -1 & 3 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 1 & 8 \end{bmatrix}$$

Each variable is a leading variable.

Infinitely many solutions

$$\begin{bmatrix} 1 & 2 & -3 & 1 \\ 0 & 1 & 5 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

z is not a leading variable.

Homework 4/24

TB pg. 674 #25, 31, 35, 37, 39