Warm Up 4/11

If $\mathbf{u} = \langle 4, -1 \rangle$ and $\mathbf{v} = \langle 1, 8 \rangle$, find $\mathbf{u} + \mathbf{v}$, $\mathbf{u} - \mathbf{v}$, $|\mathbf{u} + \mathbf{v}|$, and $|\mathbf{u} - \mathbf{v}|$

$$u+v = \langle 4+1, -1+8 \rangle$$
$$= \langle 5, 7 \rangle$$

$$|u+v| = \sqrt{a^2+b^2}$$
$$= \sqrt{5^2+7^2}$$
$$= \sqrt{25+49} = \boxed{\sqrt{74}}$$

$$u-v = \langle 4-1, -1-8 \rangle$$
$$= \langle 3, -9 \rangle$$

$$|u-v| = \sqrt{3^2+(-9)^2}$$
$$= \sqrt{9+81}$$
$$= \sqrt{90}$$
$$= \sqrt{9} \cdot \sqrt{10}$$
$$= \boxed{3\sqrt{10}}$$

Lesson 8-5

The Dot Product

Objective

Students will…

- Be able to compute the dot product of two vectors.
- Be able to define and identify orthogonal vectors.

Vectors

Vector- A line segment in a plane with an assigned direction.

Ex.

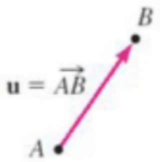


Figure 1

If a vector **v** is represented in the plane with initial point $P(x_1, y_1)$ and terminal point $Q(x_2, y_2)$, then

$$\boldsymbol{v} = \langle x_2 - x_1 \,,\; y_2 - y_1 \rangle$$

This is known as the **component form**.

Vector Operations

Another usefulness of writing vectors in component form is the ease of doing basic operations. Here is how to perform the three basic mathematical operations using vectors:

If $\mathbf{u} = \langle a_1, b_1 \rangle$ and $\mathbf{v} = \langle a_2, b_2 \rangle$, then

$\mathbf{u} + \mathbf{v} = \langle a_1 + a_2 , b_1 + b_2 \rangle$ **(Addition)**

$\mathbf{u} - \mathbf{v} = \langle a_1 - a_2 , b_1 - b_2 \rangle$ **(Subtraction)**

$c\mathbf{u} = \langle ca_1, cb_1 \rangle$, where $c \in \mathbb{R}$ **(Scalar Multiplication)**

The Dot Product

In the realm of vectors, multiplication of two vectors is known as the **Dot Product**.

Let $\mathbf{u} = \langle a_1, b_1 \rangle$ and $\mathbf{v} = \langle a_2, b_2 \rangle$. Then, their **dot product**, denoted by $\mathbf{u} \bullet \mathbf{v}$, is defined by

$$\mathbf{u} \bullet \mathbf{v} = a_1 a_2 + b_1 b_2$$

Ex. For $\mathbf{u} = \langle 3, -2 \rangle$ and $\mathbf{v} = \langle 4, 5 \rangle$, find their dot product.

$$u \cdot v = (3 \cdot 4) + (-2 \cdot 5)$$

$$= 12 + -10 = \boxed{2}$$

Properties of the Dot Product

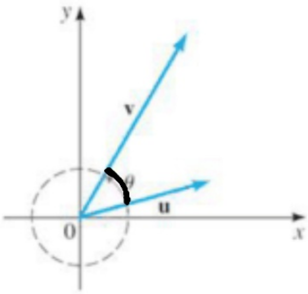
The properties of dot product are similar to other properties of vectors.

Properties of the Dot Product

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $(a\mathbf{u}) \cdot \mathbf{v} = a(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (a\mathbf{v})$
3. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$
4. $\boxed{|\mathbf{u}|^2 = \mathbf{u} \cdot \mathbf{u}}$

Dot Product Theorem

Let us consider the two vectors **v** and **u**.



As we can see, there is an angle that is formed between the two vectors. There is a very special relationship between this angle and the dot product of the two vectors.

<u>The Dot Product Theorem</u>- If θ is the angle between two nonzero vectors **u** and **v**, then

$$\mathbf{u} \bullet \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos \theta$$

Angle Between Two Vectors

We can now use this theorem to find the angle between two vectors, because solving for θ, we get the following:

$$\frac{u \cdot v}{|u||v|} = \frac{\cancel{|u||v|} \cos\theta}{\cancel{|u||v|}}$$

$$\cos^{-1} \frac{u \cdot v}{|u||v|} = \cancel{\cos}\,\theta$$

$$\Rightarrow \boxed{\theta = \cos^{-1}\left(\frac{u \cdot v}{|u||v|}\right)}$$

Example

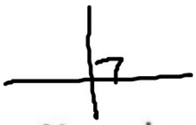
$16+9$

If **u** = ⟨2, 5⟩ and **v** = ⟨4, −3⟩, find the angle between the two vectors.

$$\boxed{\theta = \cos^{-1}\left(\frac{u \cdot v}{|u||v|}\right)}$$

$$\theta = \cos^{-1}\left(\frac{(2 \cdot 4)+(5 \cdot -3)}{(\sqrt{29})(\sqrt{25})}\right) = \cos^{-1}\left(\frac{8+-15}{(\sqrt{29})(5)}\right)$$

$$\theta = \boxed{\cos^{-1}\left(\frac{-7}{5\sqrt{29}}\right)}$$



Orthogonal Vectors

Now that we are able to find the angle measure between two vectors, we can use it to compare them.

We have learned in the past that perpendicular lines create $90°$ angles. Similarly, in vectors, when vectors are perpendicular to each other they are called **<u>orthogonal vectors</u>**. The next theorem tells us how to determine whether two vectors are orthogonal.

<u>Orthogonal Vectors Theorem</u>- Two nonzero vectors **u** and **v** are perpendicular if and only if $\boxed{\mathbf{u} \bullet \mathbf{v} = 0.}$

Though the proof of this theorem is very easy, we will leave it out. (Refer to TB pg. 620).

Example

Determine whether the vectors in each pair are orthogonal.

a. $\mathbf{u} = \langle 3\ 5 \rangle, \mathbf{v} = \langle 2, -8 \rangle$

$u \cdot v = (3 \cdot 2) + (5 \cdot -8)$

$0 \neq 6 - 40$

NO

b. $\mathbf{u} = \langle 2, 1 \rangle, \mathbf{v} = \langle -1, 2 \rangle$

$u \cdot v = (2 \cdot -1) + (1 \cdot 2)$

$= -2 + 2 = 0$

yes

c. $\mathbf{u} = \langle 1, 6 \rangle, \mathbf{v} = \langle 12, -2 \rangle$

$1 \cdot 12 + -12$

$= 0$

yes

Homework Problem

1–8 ■ Find **(a)** $\mathbf{u} \cdot \mathbf{v}$ and **(b)** the angle between $\mathbf{u}$ and $\mathbf{v}$ to the nearest degree.

1. $\mathbf{u} = \langle 2, 0 \rangle, \quad \mathbf{v} = \langle 1, 1 \rangle$

Homework 4/11

TB pg. 624 #1, 3, 5, 9-11