Warm Up 3/5

Write the addition and subtraction formulas for sine, cosine, and tangent:

$\sin(x+y) = \sin x \cos y + \cos x \sin y$

$\sin(x-y) = \sin x \cos y - \cos x \sin y.$

$\cos(x+y) = \cos x \cos y + \sin x \sin y$

$\cos(x-y) = \cos x \cos y - \sin x \sin y.$

$\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$ $\tan(x-y) = \dfrac{\tan x - \tan y}{1 + \tan x \tan y}.$

Lesson 7-3

Double-Angle, and Half-Angle Formulas

Objective

Students will…

- Be able to know and derive the Double-Angle and Half-Angle formulas of sine, cosine, and tangent.
- Be able to apply the Double-Angle and Half-Angle formulas.

Double-Angle Formulas

The following formulas are direct results of addition and subtraction formulas. **Double-Angle** formulas allows us to find the values of the trigonometric functions at $2x$ from their values at x.

Double-Angle Formulas:

For Sine: $\sin(2x) = 2\sin x \cos x$

For Cosine: $\cos(2x) = \cos^2 x - \sin^2 x$

$$= 1 - 2\sin^2 x$$

$$= 2\cos^2 x - 1$$

For Tangent: $\tan(2x) = \frac{2\tan x}{1-\tan^2 x}$

Proof of Double-Angle Formulas

$2x = x + x.$

Prove the formula: $\cos(2x) = \cos^2 x - \sin^2 x$

$\cos(x+x) = \cos x \cos x - \sin x \sin x$

$= \boxed{\cos^2 x - \sin^2 x}$

Show that $\cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

$\cos^2 x - \sin^2 x = \cos^2 x - (1 - \cos^2 x) = \cos^2 x - 1 + \cos^2 x$

$= \boxed{2\cos^2 x - 1}$

$\cos^2 x - \sin^2 x = (1 - \sin^2 x) - \sin^2 x$

$= \boxed{1 - 2\sin^2 x}$

Proof of Double-Angle Formulas

Prove the formula: $\sin(2x) = 2\sin x \cos x$

$\sin(x+x)$

Prove the formula: $\tan(2x) = \frac{2\tan x}{1-\tan^2 x}$

$\tan(x+x)$

Using the Double-Angle Formulas

$\frac{4-\sqrt{5}}{\cancel{2}} \cdot \frac{\cancel{2}}{4+\sqrt{5}}$

If $\cos x = -\frac{2}{3}$ and x is in quadrant II, find $\cos 2x$, $\sin 2x$, and $\tan 2x$.

$$\cos(2x) = 2\cos^2 x - 1$$
$$= \frac{2}{1}\left(\frac{4}{9}\right) - 1$$
$$= \frac{8}{9} - \frac{1}{1}\cdot\frac{9}{9} = \boxed{-\frac{1}{9}}$$

$$\sin 2x = 2\sin x\cos x$$
$$= \frac{2}{1}\left(\frac{\sqrt{5}}{3}\right)\left(\frac{-2}{3}\right)$$
$$= \boxed{\frac{-4\sqrt{5}}{9}}$$

$$\sin^2 x = 1 - \cos^2 x$$
$$\sqrt{\sin^2 x} = 1 - \frac{4}{9} = \frac{9\cdot 1}{9\cdot 1} - \frac{4}{9} = \sqrt{\frac{5}{9}}$$

$$\sin x = \frac{\sqrt{5}}{3}$$

$$\tan x = \frac{\frac{\sqrt{5}}{3}}{-\frac{2}{3}} = \frac{\sqrt{5}}{\cancel{3}}\cdot\frac{\cancel{3}}{2} = -\frac{\sqrt{5}}{2}$$

$$\tan 2x = \frac{2\tan x}{1-\tan x}$$

$$\frac{\frac{2}{2}\,2\left(-\frac{\sqrt{5}}{2}\right)}{\frac{2}{2}\,1-\left(-\frac{\sqrt{5}}{2}\right)}$$

$$\frac{\frac{4-\sqrt{5}}{2}}{\frac{2+\sqrt{5}}{2}}$$

Homework Problems

Find $\cos 2x$, $\sin 2x$, and $\tan 2x$ from the given information.

5. $\sin x = -\frac{3}{5}$, x is in quadrant III.

Half-Angle Formulas

The next set of formulas relate the values of trig functions at $\frac{1}{2}x$ to their values at x. They are known as the **Half-Angle Formulas**.

Half-Angle Formulas:

$$\sin\frac{u}{2} = \pm\sqrt{\frac{1-\cos u}{2}} \qquad \cos\frac{u}{2} = \pm\sqrt{\frac{1+\cos u}{2}}$$

$$\tan\frac{u}{2} = \frac{1-\cos u}{\sin u} = \frac{\sin u}{1+\cos u}$$

*The choice of $+$ or $-$ depends on which quadrant $\frac{u}{2}$ lies in.

Using Half-Angle Formulas

Find the exact value of sin 22.5°

$$\sin\left(\frac{45^\circ}{2}\right) = \sqrt{\frac{1-\cos 45^\circ}{2}} = \sqrt{\frac{\frac{2}{2}-\frac{\sqrt{2}}{2}}{2}} = \sqrt{\frac{\frac{2-\sqrt{2}}{2}}{2}}$$

$$= \sqrt{\frac{2-\sqrt{2}}{2}\cdot\frac{1}{2}} = \sqrt{\frac{2-\sqrt{2}}{4}} = \frac{\sqrt{2-\sqrt{2}}}{\sqrt{4}} = \boxed{\frac{\sqrt{2-\sqrt{2}}}{2}}$$

Homework Problems

21. Find the exact value of $\tan\frac{\pi}{8}$

$$= \tan\left(\frac{\frac{\pi}{4}}{2}\right) = \frac{\sin\frac{\pi}{4}}{1+\cos\frac{\pi}{4}} = \frac{\frac{\sqrt{2}}{2}}{1+\frac{\sqrt{2}}{2}}$$

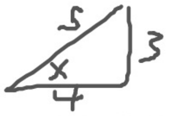
$$= \frac{1-\cos\frac{\pi}{4}}{\sin\frac{\pi}{4}} = \frac{\frac{2}{2}\cdot\frac{1}{1} - \frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = \frac{\frac{2-\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = \frac{2-\sqrt{2}}{\cancel{2}} \cdot \frac{\cancel{2}}{\sqrt{2}}$$

$$= \frac{(2-\sqrt{2})}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{2\sqrt{2}-2}{2} = \frac{\cancel{2}(\sqrt{2}-1)}{\cancel{2}}$$

$$= \boxed{\sqrt{2}-1}$$

Using Half-Angle Formulas

Find $\tan\frac{u}{2}$ if $\sin u = \frac{2}{5}$ and u is in quadrant II.



Homework Problems

$\sin^2 x + \cos^2 x = 1.$

$\cos x = \pm\sqrt{1-\sin^2 x}.$

$\sin x = \pm\sqrt{1-\cos^2 x}.$

Find $\sin\frac{x}{2}$, $\cos\frac{x}{2}$, and $\tan\frac{x}{2}$ from the given information.

$\cos x = \pm\sqrt{1-\left(\frac{3}{5}\right)^2} = \pm\sqrt{\frac{1}{1}\frac{25}{25} - \frac{9}{25}}$

$= \pm\sqrt{\frac{16}{25}} = \frac{4}{5}$

35. $\sin x = \frac{3}{5}$ $\quad 0° < x < 90°$

$\cos x = 4/5$

$\sin\frac{x}{2} = \sqrt{\frac{1-\cos x}{2}} = \sqrt{\frac{\frac{5}{5}1 - 4/5}{2}} = \sqrt{\frac{\frac{1}{5}}{2}} = \sqrt{\frac{1}{5}\cdot\frac{1}{2}} = \sqrt{\frac{1}{10}} = \frac{\sqrt{1}}{\sqrt{10}} = \frac{1}{\sqrt{10}} = \boxed{\frac{\sqrt{10}}{10}}$

$\cos\frac{x}{2} = \sqrt{\frac{1+\cos x}{2}} = \sqrt{\frac{\frac{5}{5}1 + 4/5}{2}} = \sqrt{\frac{\frac{9}{5}}{2}} = \sqrt{\frac{9}{10}} = \frac{\sqrt{9}}{\sqrt{10}}$

$= \frac{3}{\sqrt{10}} = \boxed{\frac{3\sqrt{10}}{10}}$

$\tan x = \frac{\sin x}{1+\cos x} =$

$\tan\frac{x}{2} = \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} = \frac{\frac{\sqrt{10}}{10}}{\frac{3\sqrt{10}}{10}} = \frac{\frac{\sqrt{10}}{10}}{\frac{3\sqrt{10}}{10}} = \boxed{\frac{1}{3}}$

$\frac{\cancel{5}}{\cancel{15}3} = \boxed{\frac{1}{3}}$

Homework 3/5

TB pg. 548 #1-7 (odd), 15, 17, 21, 23, 35, 36, 37

1. $\sin x = \frac{5}{13}$, x in Quad I. Find $\sin 2x$, $\cancel{\cos 2x}$, $\tan 2x$

$\cos x = \pm\sqrt{1-\sin^2 x}$

$= \pm\sqrt{\frac{1}{1} - \frac{25}{169}}$

$= \pm\sqrt{\frac{144}{169}}$

$= \boxed{\frac{12}{13}}$

$\sin 2x = 2\sin x\cos x$

$= 2\left(\frac{5}{13}\right)\left(\frac{12}{13}\right)$

$= \boxed{\frac{120}{169}}$

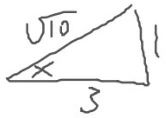
$\cos 2x = 1 - 2\sin^2 x$

$= 1 - 2\left(\frac{5}{13}\right)^2 = 1 - 2\left(\frac{25}{169}\right)$

$= \frac{169}{169} - \frac{50}{169} = \boxed{\frac{119}{169}}$

$\tan 2x = \frac{\sin 2x}{\cos 2x} = \frac{\frac{120}{169}}{\frac{119}{169}} = \frac{120}{\cancel{169}} \cdot \frac{\cancel{169}}{119} = \boxed{\frac{120}{119}}$

7. $\tan x = -\frac{1}{3}$, $\cos x > 0$. find $\sin 2x$, $\cos 2x$, $\tan 2x$.



$1^2 + 3^2 = c^2$

$1 + 9 = c^2$

$\sqrt{10} = c$

$\sin x = \frac{1}{\sqrt{10}} \cdot \frac{\sqrt{10}}{\sqrt{10}} = \frac{\sqrt{10}}{10}$;

$\cos x = \frac{3}{\sqrt{10}} \cdot \frac{\sqrt{10}}{\sqrt{10}} = \frac{3\sqrt{10}}{10}$

$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x} = \frac{2\left(-\frac{1}{3}\right)}{1 - \left(-\frac{1}{3}\right)^2}$$

$$= \frac{-\frac{2}{3}}{\frac{9}{9} - \frac{1}{9}} = \frac{-\cancel{2}^{1}}{3} \cdot \frac{9}{\cancel{8}\,4} = -\frac{9}{12} = \boxed{-\frac{3}{4}}$$

$$\sin 2x = 2\sin x\cos x = 2\left(\frac{\sqrt{10}}{10}\right)\left(\frac{3\sqrt{10}}{10}\right) = \frac{6\cancel{0}}{10\cancel{0}} = \frac{6}{10} = \boxed{\frac{3}{5}}$$

$$\cos 2x = \cos^2 x - \sin^2 x = \left(\frac{3\sqrt{10}}{10}\right)^2 - \left(\frac{\sqrt{10}}{10}\right)^2 = \frac{90}{100} - \frac{10}{100} = \frac{8\cancel{0}}{10\cancel{0}}$$

$$= \boxed{\frac{4}{5}}$$

53. $2 \sin 52.5 \quad \sin 97.5^\circ$

$= 2 \sin\left(\frac{105}{2}\right) \sin\left(\frac{195}{2}\right)$

$= 2 \sqrt{\frac{1-\cos 105}{2}} \sqrt{\frac{1-\cos 195}{2}}$

$2 \sqrt{\frac{1-\cos(60+45)}{2}} \sqrt{\frac{1-\cos(150+45)}{2}}$