Lesson 2-5

Implicit Differentiation

Objective

Students will…

- Be able to distinguish between implicit and explicit form.
- Be able to use implicit differentiation technique to find derivatives.

Implicit vs Explicit Form

Up to this point, we have been dealing with finding derivatives of functions that were written in **explicit form**, i.e. solved for a variable (dependent variable). However, functions may be written in **implicit forms**, where it is not clearly solved for a variable. For example,

Explicit Form: $y = 3x^2 - 5$ VS Explicit Form: $y = \frac{1}{x}$

Implicit Form: $5 = 3x^2 - y$ Implicit Form: $xy = 1$

In many cases, it would be easier to simply rewrite the equation in the explicit form before taking the derivative. But this may not always be easy to do!

$f(g(x)) = f'(g(x)) \cdot g'$

Implicit Differentiation

When the function cannot easily be written in the explicit form, it's best to use the technique of **implicit differentiation**. Best way to interpret this technique is to treat any "y" term as a composition, thus requiring the use of the **chain rule**.

$$5 = 3x^2 - y^2 = 3x^2 - f(y)$$

Thus, in finding the derivative…

$$\frac{d}{dx}5 = \frac{d}{dx}3x^2 - \frac{d}{dx}(f(y)) \Rightarrow 0 = 6x - 2y \cdot y' \text{ (chain rule)}$$

Then, finally solving for y'…

$$y' = \frac{6x}{2y}$$

Implicit Differentiation

GUIDELINES FOR IMPLICIT DIFFERENTIATION

1. Differentiate both sides of the equation *with respect to x*.
2. Collect all terms involving ~~dy/dx~~ y' on the left side of the equation and move all other terms to the right side of the equation.
3. Factor ~~dy/dx~~ y' out of the left side of the equation.
4. Solve for ~~dy/dx~~ y'.

"solving for y'"

Example

Find the derivative.

$$\frac{d}{dx}\left(y^3 + y^2 - 5y - x^2\right) = \frac{d}{dx}(-4)$$

$$\Rightarrow 3y^2y' + 2yy' - 5y' \cancel{- 2x} = 0 \cancel{+2x}$$

$$\Rightarrow 3y^2y' + 2yy' - 5y' = 2x$$

$$\Rightarrow \frac{y'\left(3y^2 + 2y - 5\right)}{\cancel{3y^2 + 2y - 5}} = \boxed{\frac{2x}{3y^2 + 2y - 5}}$$

$y - y_1 = m(x - x_1)$

Example

Find the slope of the tangent line to the graph: $\frac{d}{dx}\, x^2 + 4y^2 = 4 \,\frac{d}{dx}$ at the point $\left(\sqrt{2}, -\frac{1}{\sqrt{2}}\right)$.

$\Rightarrow 2x + 8yy' = 0$

$\quad -2x \qquad\qquad -2x$

$\Rightarrow \frac{8yy'}{8y} = \frac{-2x}{8y}$

$\Rightarrow y' = \frac{-2x}{8y} = \boxed{\frac{-x}{4y}}$

$m = \frac{-\sqrt{2}}{4\left(-\frac{1}{\sqrt{2}}\right)} = \frac{-\sqrt{2}}{-\frac{4}{\sqrt{2}}} = \frac{-\sqrt{2}}{1} \cdot -\frac{\sqrt{2}}{4} = \frac{2}{4} = \boxed{1/2}$

$\boxed{y + \frac{1}{\sqrt{2}} = \frac{1}{2}\left(x - \sqrt{2}\right)}$

Example

Find the slope of the tangent line to the graph: $3(x^2+y^2)^2 = 100xy$ at the point $(3, 1)$.

$$6(x^2+y^2)\cdot(2x+2yy') = 100y + 100xy'$$

$$\Rightarrow 6(3^2+1^2)(2(3)+2(1)y') = 100(1) + 100(3)y'$$

$$\Rightarrow 60(6+2y') = 100 + 300y'$$

$$\Rightarrow 360 + 120y' = 100 + 300y'$$

$$-100 \quad -120y' \qquad -100 \quad -120y'$$

$$\Rightarrow \frac{260}{180} = \frac{180y'}{180} \Rightarrow \boxed{\frac{13}{9}} = m$$

Example

Find the derivative.

$4\sin x \cos y = 1$

$\underbrace{4\sin x}_{f}\,\underbrace{\cos y}_{g}$

$4\cos x\cos y - 4y'\sin y\sin x = 0$

$+4y'\sin y\sin x \qquad +4y'\sin y\sin x$

$\Rightarrow \dfrac{\cancel{4}\cos x\cos y}{\cancel{4}\sin y\sin x} = \dfrac{\cancel{4}y'\cancel{\sin y\sin x}}{\cancel{4}\cancel{\sin y\sin x}}$

$\boxed{y' = \cot x\cot y}$

Example

Find the second derivative. $\frac{d}{dx}\left(x^2 + y^2\right) = \frac{d}{dx}25$

$\Rightarrow 2x + 2yy' = 0$
$\quad -2x \qquad -2x$

$\Rightarrow y' = \frac{-2x}{2y} = \frac{-x}{y}$

$\Rightarrow y'' = \frac{-y + xy'}{y^2} = \frac{-y + x\left(\frac{-x}{y}\right)}{y^2} = \frac{-y - \frac{x^2}{y}}{y^2} = \frac{\frac{-y^2 - x^2}{y}}{y^2} = \boxed{\frac{-y^2 - x^2}{y^3}}$

Homework 10/9

2.5 Exercises #1-13 (e.o.o), 15, 21-27 (e.o.o), 28, 29, 45, 47, 49