Lesson 3-4

Concavity and the Second Derivative Test

Objective

Students will…

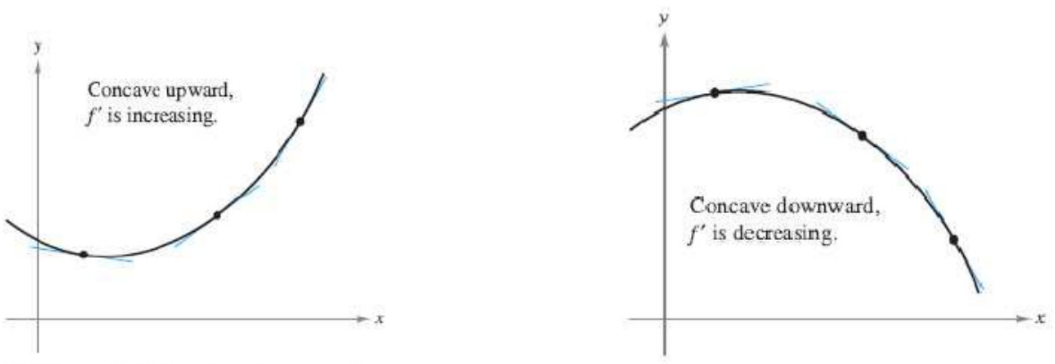
- Be able to define concavity.
- Be able to determine the different intervals of concavity.
- Be able to define and find points of inflections.
- Be able to know and apply the Second Derivative Test.

Concavity

Another useful information regarding graphs is concavity.

Concavity- Let f be differentiable on an open interval I. The graph of f is **concave upward** on I if f' is increasing on the interval and **concave downward** on I if f' is decreasing on the interval.

Graphically speaking…



(a) The graph of f lies above its tangent lines. (b) The graph of f lies below its tangent lines.
Figure 3.24

Concavity and the Second Derivative

That being said, if we used the first derivative f' to determine the intervals in which f increases or decreases, we would naturally use the second derivative f'' to do the same for the graph of f'.

Test for Concavity- Let f be a function whose second derivative exists on an open interval I.

1. If $f''(x) > 0$ for all x in I, then f' is increasing on I. Therefore, f is concave upward in I.
2. If $f''(x) < 0$ for all x in I, then f' is decreasing on I. Therefore, f is concave downward in I.
3. If $f''(x) = 0$ for all x in I, then f' is constant on I. Therefore, f is neither concave upward nor downward in I.

Note: Concavity is **not** defined for a linear line.

Example

$f(x) = 6(x^2+3)^{-1}$

Determine the open intervals on which the graph of $f(x) = \frac{6}{x^2+3}$ is concave upward or downward.

$f'(x) = -6(x^2+3)^{-2} \cdot 2x$

$= (-12x)(x^2+3)^{-2}$

$f''(x) = -12(x^2+3)^{-2} + -2(x^2+3)^{-3} \cdot 2x \cdot -12x$

$= -12(x^2+3)^{-2} + 48x^2(x^2+3)^{-3}$

$0 = \frac{-12}{(x^2+3)^2} + \frac{48x^2}{(x^2+3)^3} = \frac{-12(x^2+3)+48x^2}{(x^2+3)^3} = \frac{-12x^2-36+48x^2}{(x^2+3)^3} = \frac{36(x^2-1)}{(x^2+3)^3}$

$36(x^2-1) = 0$

$x = \pm 1$

$+$	-1	$-$	1	$+$

CU: $(-\infty, -1) \cup (1, \infty)$

CD: $(-1, 1)$.

Example

Determine the open intervals on which the graph of $f(x) = \frac{x^2+1}{x^2-4}$ is concave upward or downward.

$10(3x^4 - 8x^2 - 16) = 0$

$f'(x) = \frac{2x(x^2-4) - 2x(x^2+1)}{(x^2-4)^2} = \frac{2x(x^2-4-(x^2+1))}{(x^2-4)^2} = \frac{2x(x^2-4-x^2-1)}{(x^2-4)^2}$

$f''(x) = \frac{-10(x^2-4)^2 + 40x^2(x^2-4)}{(x^2-4)^4} = \frac{-10(x^4-8x^2+16) + 40x^4 - 160x^2}{(x^2-4)^4} = \frac{30x^4 - 80x^2}{(x^2-4)^4}$

$x = \pm 2$

-2		2	
+	−	+	

CU: $(-\infty, -2) \cup (2, \infty)$

CD: $(-2, 2)$

Points of Inflection

Recall from the First Derivative Test that **relative extrema** exist whenever f' switched signs (+ to –, or – to +). With regards to the second derivative and concavity, such occurrence gives us the **points of inflection**.

Points of Inflection- Let f be a function that is continuous on an open interval and let c be a point in the interval. If the graph of f has a tangent line at this point $(c, f(c))$, then this point is a point of inflection of the graph of f if the concavity of f changes from upward to downward (or downward to upward) at the point.

Theorem 3.8- If $(c, f(c))$ is a point of inflection of the graph f, then either $f''(c) = 0$ or f'' does not exist at $x = c$. (i.e. critical values of f'')

Example

Determine the points of inflection and discuss the concavity of the graph of $f(x) = x^4 - 4x^3$

$f'(x) = 4x^3 - 12x^2$

$f''(x) = 12x^2 - 24x$

$0 = 12x^2 - 24x$

$0 = 12x(x-2)$

$x = 0, 2$

	0		2	
+		−		+

$f(0) = 0$

$f(2) = 16 - 32$
$= -16$

pts. of inf. @ $(0,0)$, $(2,-16)$

Second Derivative Test

The second derivative can also be used to find the relative extrema of f.

c = CV of f'

Let f be a function such that $f'(c) = 0$ and the second derivative of f exists on an open interval containing c.

1. If $f''(c) > 0$, then f has a relative minimum at $(c, f(c))$.
2. If $f''(c) < 0$, then f has a relative maximum at $(c, f(c))$.
3. If $f''(c) = 0$, then the test is inconclusive. (Need to use the **First Derivative Test**).

Warning: Second derivative test cannot be used for critical values that does not exist in f'.

Example

Find the relative extrema for $f(x) = -3x^5 + 5x^3$

$f'(x) = -15x^4 + 15x^2$

$0 = -15x^4 + 15x^2$

$0 = -15x^2(x^2 - 1)$

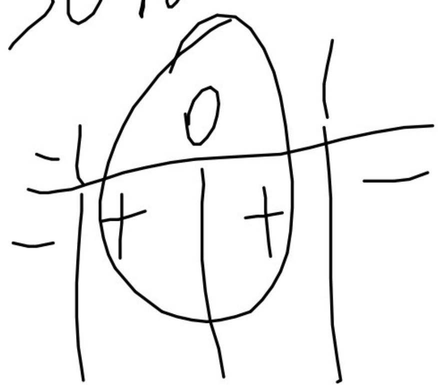
$x = 0, \pm 1$

$f''(x) = -60x^3 + 30x$

$f''(0) = 0$ inconclusive. use 1st deriv. test

$f''(1) < 0$ rel. max.

$f''(-1) > 0$ rel. min.



Example

Find the relative extrema $f(x) = \sqrt{x^2 + 1}$

Example

Find the relative extrema of the function $f(x) = \frac{x}{x-1}$

Homework 10/31

3.4 #1-6, 7-25 (odd), 27-39 (e.o.o)