Lesson 2-3

Derivative III

Objective

Students will…

- Be able to know and use the product rule.
- Be able to know and use the quotient rule.
- Be able to find higher-order derivatives.

Product Rule

Recall the laws of derivatives:

$$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}[f(x)] \pm \frac{d}{dx}[g(x)] = f'(x) \pm g'(x)$$

But what about products or quotients?

Ex. $f(x) = (2x^4)(3x^9) = \frac{d}{dx}(2x^4)\frac{d}{dx}(3x^9) = (8x^3)(27x^8) = \boxed{216x^{11}}$

$f'(x)\ \frac{d}{dx}(6x^{13}) = 78x^{12}$

*<u>**NOTE**</u>: $\frac{d}{dx}[f(x) \times\div g(x)] \neq f'(x) \times\div g'(x)$

Product Rule

For products of functions, we must apply the **product rule**.

Product Rule- The product of two differentiable functions f and g is itself differentiable. Moreover, the derivative of fg is the derivative of the first function times the second, plus the derivative of the second function times the first. Or,….

$f'g + g'f$

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + g'(x)f(x)$$

Though tedious, the proof of this rule is not difficult to understand. Refer to pg. 119 in your textbook.

Product Rule (Extension)

Note that the product rule can be extended to more than two products. For example…

$$\frac{d}{dx}[f(x)g(x)h(x)] = f'(x)g(x)h(x) + g'(x)f(x)h(x) + h'(x)f(x)g(x)$$

Ex. For $y = \overset{f}{(x^2)}\overset{g}{(\sin x)}\overset{h}{(\cos x)}$ $= 2x\sin x\cos x + x^2\cos x\cos x + x^2\sin x(-\sin x)$

$= \boxed{2x\sin x\cos x + x^2(\cos^2 x - \sin^2 x)}$

Example

Find the derivative.

a. $y = \underbrace{(3x^2)}_{f}\underbrace{(\sin x)}_{g}$

$y' = f'g + g'f$

$= \boxed{6x\sin x + 3x^2\cos x}$

Example

Find the derivative.

b. $h(x) = (3x - 2x^2)(5 + 4x)$

$f \qquad g$

$h'(x) = f'g + g'f = (3 - 4x)(5+4x) + 4(3x-2x^2)$

$= 15 + 12x - 20x - 16x^2 + 12x - 8x^2$

$= \boxed{15 + 4x - 24x^2}$

$-24x^2 + 4x + 15$

Example

Find the derivative.

c. $y = \underbrace{2x}_{f}\underbrace{\cos x}_{g} - 2\sin x$

$y' = \cancel{2\cos x} + -2x\sin x - \cancel{2\cos x}$

Quotient Rule

For quotient of functions we must apply the **quotient rule**.

Quotient Rule- The quotient $\frac{f}{g}$ of two differentiable functions f and g is itself differentiable at all values of x for which $g(x) \neq 0$. Moreover, the derivative of $\frac{f}{g}$ is given by…

$$\frac{f'g - g'f}{g^2}$$

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - g'(x)f(x)}{\left(g(x)\right)^2}$$

Again, the proof isn't too difficult to understand. Refer to pg. 121.

Example

Find the derivative of $y = \frac{5x-2}{x^2+1}$ f g

$$y' = \frac{f'g - g'f}{g^2} = \frac{5(x^2+1) - 2x(5x-2)}{(x^2+1)^2} = \frac{5x^2+5-10x^2+4x}{(x^2+1)^2}$$

$$= \boxed{\frac{-5x^2+4x+5}{(x^2+1)^2}}$$

Example

Find an equation of the tangent line to the graph of $f(x) = \frac{3-\frac{1}{x}}{x+5}$ at $(-1,1)$

$(-1)^2 (-1)x^{-2}$

$3-x^{-1}$ $(-1)^{-2}$

$f = \frac{1}{(-1)^2}$

$$f'(x) = \frac{-(-1x^{-2})(x+5) - (3-x^{-1})}{(x+5)^2} = \frac{x^{-2}(x+5) - 3 + x^{-1}}{(x+5)^2}$$

$$m = \frac{(-1)^{-2}(-1+5) - 3 + (-1)^{-1}}{(-1+5)^2} = \frac{1(4) - 3 - 1}{4} 0$$

$y = 1$

Example

Note: Not every fraction requires the quotient rule!

Find the derivative of $y = \frac{x^2+3x}{6}$ $\frac{f}{g} = \frac{1}{6}(x^2+3x)$

$\boxed{y' = \frac{1}{6}(2x+3)}$

Find the derivative of $y = \frac{5x^4}{8}$ $\frac{f}{g} = \frac{5}{8}x^4 \Rightarrow y' = 4\left(\frac{5}{8}\right)x^3$

$= \boxed{\frac{5}{2}x^3}$

Trig Derivatives

Another big application of quotient rule is regarding trig derivatives.

1. $\frac{d}{dx}[\tan x] = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$ ($f = \sin x$, $g = \cos x$)

$$\Rightarrow \frac{f'g - g'f}{g^2} = \frac{\cos x(\cos x) + \sin x(\sin x)}{\cos^2 x}$$

$$= \frac{\overbrace{\cancel{\cos^2 x + \sin^2 x}}^{1}}{\cos^2 x} = \frac{1}{\cos^2 x} = \boxed{\sec^2 x}$$

2. $\frac{d}{dx}[\cot x] = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right)$ ($f = \cos x$, $g = \sin x$)

$$\frac{f'g - g'f}{g^2} = \frac{(-\sin x)(\sin x) - \cos x \cos x}{\sin^2 x}$$

$$= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = \frac{-\overbrace{\cancel{(\sin^2 x + \cos^2 x)}}^{1}}{\sin^2 x} = -\frac{1}{\sin^2 x} = \boxed{-\csc^2 x}$$

Trig Derivatives

3. $\frac{d}{dx}[\sec x] = \frac{d}{dx}\left(\frac{1}{\cos x}\right)$ f g

$$= \frac{0 + \sin x}{\cos^2 x} = \frac{\sin x}{\cos^2 x}$$

$$= \frac{1}{\cos x}\,\frac{\sin x}{\cos x}$$

$$= \boxed{\sec x \tan x}$$

4. $\frac{d}{dx}[\csc x] = \frac{d}{dx}\,\frac{1}{\sin x}$ f g

$$\frac{0 - \cos x}{\sin^2 x} = \frac{-\cos x}{\sin^2 x}$$

$$= -\frac{1}{\sin x}\,\frac{\cos x}{\sin x}$$

$$= \boxed{-\csc x \cot x}$$

Higher-Order Derivatives

In many instances, you can take the derivative of a derivative.

Ex. $f(x) = 5x^3 - 6x^2 + 11$

$f'(x) = 15x^2 - 12x$

$f''(x) = 30x - 12$

$f'''(x) = 30$

$f^{IV}(x) = f^{4}(x) = 0$

<u>Note</u>: In motion problems, the second derivative is **<u>acceleration</u>**.

Higher-Order Derivatives

First derivative:	y',	$f'(x)$,	$\frac{dy}{dx}$,	$\frac{d}{dx}[f(x)]$,	$D_x[y]$
Second derivative:	y'',	$f''(x)$,	$\frac{d^2y}{dx^2}$,	$\frac{d^2}{dx^2}[f(x)]$,	$D_x^2[y]$
Third derivative:	y''',	$f'''(x)$,	$\frac{d^3y}{dx^3}$,	$\frac{d^3}{dx^3}[f(x)]$,	$D_x^3[y]$
Fourth derivative:	$y^{(4)}$,	$f^{(4)}(x)$,	$\frac{d^4y}{dx^4}$,	$\frac{d^4}{dx^4}[f(x)]$,	$D_x^4[y]$
$\vdots$					
nth derivative:	$y^{(n)}$,	$f^{(n)}(x)$,	$\frac{d^ny}{dx^n}$,	$\frac{d^n}{dx^n}[f(x)]$,	$D_x^n[y]$

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2.3 Exercises #1-11 (odd), 39-53 (odd), 73, 75, 99, 102