Lesson 3-4

Concavity
and the
Second Derivative Test

Objective

Students will…

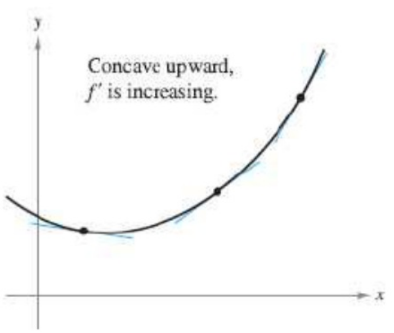
- Be able to define concavity.
- Be able to determine the different intervals of concavity.
- Be able to define and find points of inflections.
- Be able to know and apply the Second Derivative Test.

Concavity

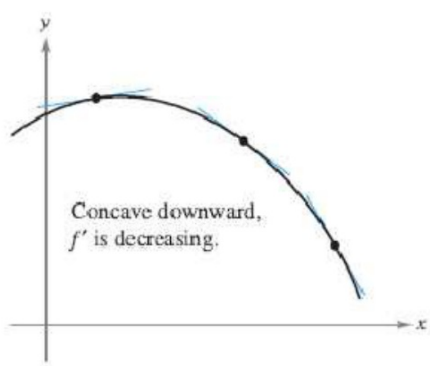
Another useful information regarding graphs is concavity.

<u>Concavity</u>- Let f be differentiable on an open interval I. The graph of f is **concave upward** on I if f' is increasing on the interval and **concave downward** on I if f' is decreasing on the interval.

Graphically speaking…



(a) The graph of f lies above its tangent lines.
Figure 3.24



(b) The graph of f lies below its tangent lines.

Concavity and the Second Derivative

That being said, if we used the first derivative f' to determine the intervals in which f increases or decreases, we would naturally use the second derivative f'' to do the same for the graph of f'.

Test for Concavity- Let f be a function whose second derivative exists on an open interval I.

1. If $f''(x) > 0$ for all x in I, then f' is increasing on I. Therefore, f is concave upward in I.
2. If $f''(x) < 0$ for all x in I, then f' is decreasing on I. Therefore, f is concave downward in I.
3. If $f''(x) = 0$ for all x in I, then f' is constant on I. Therefore, f is neither concave upward nor downward in I.

Note: Concavity is **not** defined for a linear line.

Example

Determine the open intervals on which the graph of $f(x) = \frac{6}{x^2+3}$ is $6(x^2+3)^{-1}$ concave upward or downward.

$f'(x) = -6(x^2+3)^{-2} \cdot 2x$

$= (-12x)(x^2+3)^{-2}$

$f''(x) = -12(x^2+3)^{-2} + -2(x^2+3)^{-3} \cdot 2x(-12x)$

$= -12(x^2+3)^{-2} + 48x^2(x^2+3)^{-3}$

CV: $0 = -12(x^2+3)^{-2}\left(1 - 4x^2(x^2+3)^{-1}\right)$

$-12(x^2+3)^{-2} = 0$

$\frac{-12}{(x^2+3)^2} = 0$ No sol.

$-4x^2(x^2+3)^{-1} + 1 = 0$

$\frac{4x^2}{x^2+3} = +1$

$4x^2 = x^2 + 3$

$3x^2 = 3$

$x = \pm 1$: CV

-2	-1	0	1	2
+		−		+

f is:

CU: $(-\infty, -1) \cup (1, \infty)$

CD: $(-1, 1)$

Example

Determine the open intervals on which the graph of $f(x) = \frac{x^2+1}{x^2-4}$ is concave upward or downward.

$\cancel{2x^3} - 8x - \cancel{2x^3} - 2x = -10x$

$f'(x) = \frac{2x(x^2-4) - 2x(x^2+1)}{(x^2-4)^2} = \frac{-10x}{(x^2-4)^2}$

$f''(x) = \frac{-10(x^2-4)^2 - 2(x^2-4) \cdot 2x(-10x)}{(x^2-4)^4}$

$= \frac{\boxed{-10(x^2-4)^2 + 40x^2(x^2-4)}}{(x^2-4)^4}$

$= \frac{-10\cancel{(x^2-4)}\left((x^2-4) - 4x^2\right)}{(x^2-4)^{\cancel{4}\,3}}$

CV:

$\cancel{-10}(-3x^2-4) = 0$

$3x^2 = -4$

$x^2 = -\frac{4}{3}$

No sol

VA: $x^2-4=0$

$x = \pm 2$

$-3 \quad -2 \quad 0 \quad 2 \quad 3$

$\frac{+}{+} = +$ | $\frac{+}{-} = -$ | $\frac{+}{+} = +$

f is CU: $(-\infty, -2) \cup (2, \infty)$

CD: $(-2, 2)$

Points of Inflection

Recall from the First Derivative Test that **relative extrema** exist whenever f' switched signs (+ to –, or – to +). With regards to the second derivative and concavity, such occurrence gives us the **points of inflection**.

Points of Inflection- Let f be a function that is continuous on an open interval and let c be a point in the interval. If the graph of f has a tangent line at this point $(c, f(c))$, then this point is a point of inflection of the graph of f if the concavity of f changes from upward to downward (or downward to upward) at the point.

Theorem 3.8- If $(c, f(c))$ is a point of inflection of the graph f, then either $f''(c) = 0$ or f'' does not exist at $x = c$. (i.e. critical values of f'')

Example

Determine the points of inflection and discuss the concavity of the graph of $f(x) = x^4 - 4x^3$ $\Rightarrow f'(x) = 4x^3 - 12x^2 \Rightarrow f''(x) = 12x^2 - 24x$

CU: $0 = 12x^2 - 24x$

$0 = 12x(x-2)$

$x = 0, 2$

$-1 \quad 0 \quad 1 \quad 2 \quad 3$

$+ \mid - \mid +$

$f(0) = 0$

$f(2) = 16 - 32$

$= -16$

$\boxed{\text{Pts. of inf. } (0,0), (2,-16)}$

CU: $(-\infty, 0) \cup (2, \infty)$.

CD: $(0, 2)$

Second Derivative Test

The second derivative can also be used to find the relative extrema of f.

Let f be a function such that $f'(c) = 0$ and the second derivative of f exists on an open interval containing c.

1. If $f''(c) > 0$, then f has a relative minimum at $(c, f(c))$.
2. If $f''(c) < 0$, then f has a relative maximum at $(c, f(c))$.
3. If $f''(c) = 0$, then the test is inconclusive. (Need to use the **First Derivative Test**).

Note: "c" is the critical value of f!

Warning: Second derivative test cannot be used for critical values that does not exist in f'.

Example

Find the relative extrema for $f(x) = -3x^5 + 5x^3$ $\Rightarrow f'(x) = -15x^4 + 15x^2$

$f''(x) = -60x^3 + 30x$

$f''(-1) = +$ $\boxed{\text{rel min} \quad (-1, -2)}$

$f''(0) = 0$ inc.

$f''(1) = -$ $\boxed{\text{rel max} \quad (1, 2)}$

CV: $0 = -15x^4 + 15x^2$

$0 = -15x^2(x^2 - 1)$

$x = 0, \pm 1$

	-1	0	1	
−	+	+	−	

Example

Find the relative extrema $f(x) = \sqrt{x^2 + 1}$

$(x^2+1)^{1/2} \Rightarrow f'(x) = \frac{1}{2} 2x (x^2+1)^{-1/2}$

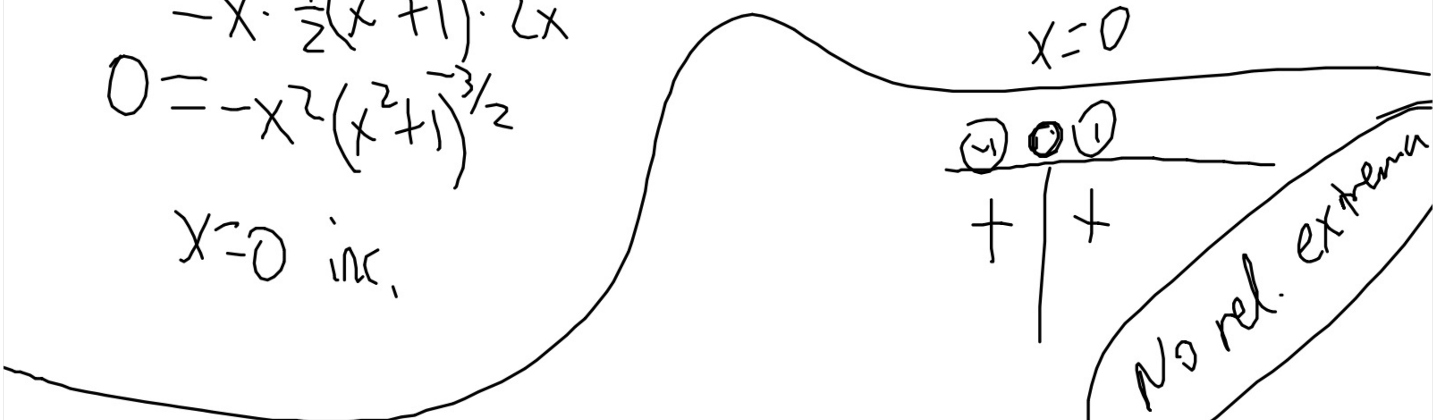
cv: $0 = x(x^2+1)^{-1/2}$

$f''(x) = (x^2+1)^{-1/2}$

$= x \cdot \frac{-1}{2}(x^2+1)^{-3/2} \cdot 2x$

$0 = -x^2(x^2+1)^{-3/2}$

$x=0$ inc.



$x = 0$

-1 0 1

+ | +

No rel. extrema

Example

Find the relative extrema of the function $f(x) = \frac{x}{x-1}$

Since $x=1$ does not make numerator zero, VA @ $x=1 \Rightarrow$ no rel. extremum

$$f'(x) = \frac{(x-1)-x}{(x-1)^2} = \frac{-1}{(x-1)^2}$$

CV: $1 \rightarrow$ VA: $x=1$.

disc. @ $x=1$

$-$ | $-$

no rel. extrema.

Homework 10/31

3.4 #1-6, 7-25 (odd), 27-39 (e.o.o)