Lesson 2-6

Related Rates

Objective

Students will…

- Be able to find a related rate.
- Be able to solve related rate word problems.

Continuum of Time

One of the most useful things about Calculus is its allowance for solving real-life problems. Recall that the power of Calculus lies in instantaneous rate of change (derivatives). In reality, time never stops. Thus, when it comes to instantaneous rate of change in real-life situations, the involvement of time can be quite useful. For example,

$$V = \pi r^2 h \text{ (Volume of a Cylinder)}$$

In differentiating with respect to t, as in time,

$\frac{dV}{dt} =$ Volume ROC over time $\qquad \frac{dr}{dt} =$ Radius ROC over time

$\frac{dh}{dt} =$ Height ROC over time.

Implicit Differentiation 2.0

Recall that the technique of implicit differentiation called for the extra "y'," or $\frac{dy}{dx}$ after each y term was differentiated, since the equations were differentiated with respect to x. Thus, if we were to differentiate an equation with respect to t (time), we would need a $\frac{d\blacksquare}{dt}$ after each differentiated term.

$$V = \pi r^2 h$$

$$\frac{d}{dt}(V) = \frac{d}{dt}(\underbrace{\pi r^2}_{f}\underbrace{h}_{g})$$

$$1\frac{dV}{dt} = 2\pi r\frac{dr}{dt}h + 1\frac{dh}{dt}\pi r^2$$

$$\boxed{\frac{dV}{dt} = 2\pi rh\frac{dr}{dt} + \pi r^2\frac{dh}{dt}}$$

Example

For the equation, $y = x^2 + 3$, find $\frac{dy}{dt}$ when $x = 1$, given that $\frac{dx}{dt} = 2$.

$$\frac{d}{dt}(y) = \frac{d}{dt}(x^2+3)$$

$$1\frac{dy}{dt} = 2x\frac{dx}{dt} + 0$$

$$\frac{dy}{dt} = 2(1)(2) = \boxed{4}$$

Example

Example: For the equation, $V = \frac{4}{3}\pi r^3$, find $\frac{dV}{dt}$ when $r = 3$, given that $\frac{dr}{dt} = \cancel{1.5}$ $\frac{3}{2}$

$$\frac{d}{dt} V = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right)$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$= \overset{2}{\cancel{4}}\pi(9)\left(\frac{3}{\cancel{2}}\right) = \boxed{54\pi}$$

Implicit Differentiation

GUIDELINES FOR SOLVING RELATED-RATE PROBLEMS

1. Identify all *given* quantities and quantities *to be determined*. Make a sketch and label the quantities.
2. Write an equation involving the variables whose rates of change either are given or are to be determined.
3. Using the Chain Rule, implicitly differentiate both sides of the equation *with respect to time t.*
4. *After* completing Step 3, substitute into the resulting equation all known values for the variables and their rates of change. Then solve for the required rate of change.

Example

A pebble is dropped into a calm pond, causing ripples in the form of concentric circles. The radius r of the outer ripple is increasing at a constant rate of 1 foot per second. When the radius is 4 feet, at what rate is the total area A of the disturbed water changing?

$\frac{dr}{dt} = 1$ $\quad r = 4$ $\quad \frac{d}{dt}(A) = \frac{d}{dt}(\pi r^2)$ $\quad \frac{dA}{dt} = ?$

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

$$= 2\pi(4)(1) = \boxed{8\pi \text{ ft}^2/\text{sec.}}$$

Example

Volume

Air is being pumped into a spherical balloon at a rate of 4.5 cubic feet per minute. Find the rate of change of the radius when the radius is 2 ft.

$\frac{dV}{dt} = 4.5 \quad \frac{dr}{dt} = ? \quad r = 2$

$\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right)$

$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

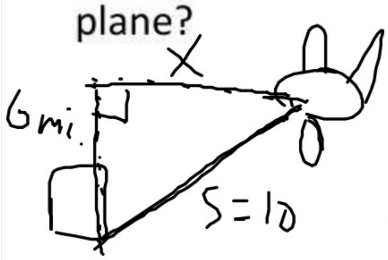
$4.5 = 4\pi(4)\frac{dr}{dt} \rightarrow$

$\frac{dr}{dt} = \frac{4.5}{16\pi}$ ft/min

Example

6 miles above ground

An airplane is flying on a flight path that will take it directly over a radar tracking station. If the distance between the station and the plane is decreasing at 400mph when $s = 10$ miles, what is the speed of the plane?



$\frac{ds}{dt} = -400 \qquad \frac{dx}{dt} = ?$

$6^2 + x^2 = s^2$

$0 + 2x\frac{dx}{dt} = 2s\frac{ds}{dt}$

$\frac{\cancel{2(8)}\frac{dx}{dt}}{\cancel{16}} = \frac{\cancel{2}(10)(\cancel{-400})}{\cancel{8}\,\cancel{16}} \Rightarrow \boxed{\frac{dx}{dt} = -500 \text{ mph}}$

$6^2 + x^2 = 10^2$

$x^2 = 64$

$x = 8$

Example

$\tan\theta = \frac{5000}{2000}$

$\tan^{-1}\frac{5}{2} \approx 1.1902 \approx \theta$

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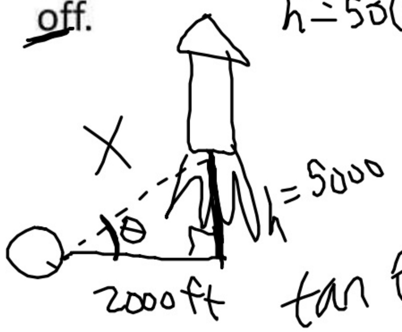
$\frac{d\theta}{dt} = ?$

Find the rate of change in the angle of elevation of the ground camera that is following a rocket lift off from 2000 ft away at 10 seconds after lift off.

$h = 50(100) = 5000$ $\boxed{h = 50t^2}$ $t =$

$\frac{dh}{dt} = 100t$

$\frac{dh}{dt} = 1000$



$\tan\theta = \frac{h}{2000} = \frac{1}{2000}h$

$\Rightarrow \sec^2\theta \frac{d\theta}{dt} = \frac{1}{2000}\frac{dh}{dt}$

$\Rightarrow \frac{\cancel{\sec^2(1.1902)}\frac{d\theta}{dt}}{\cancel{\sec^2(1.1902)}} = \frac{\frac{1000}{2000}}{\sec^2(1.1902)} \approx \boxed{0.069 \text{ rad/sec.}}$

Homework 10/12

2.6 Ex (from website) #1-7(odd), 11, 13, 15, 16, 18, 23, 24, 25, 29, 32, 43, 47