Lesson 9-3, 9-4, 9-5, 9-6

Tests For Convergence

Objective

Students will…

- Be able to use the different tests for convergence to determine whether a series is convergent or divergent.

Integral Test

If f is positive, continuous, and decreasing for $x \geq 1$ and $a_n = f(n)$, then $\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} f(x)\,dx$ either both converge or both diverge.

Note: If the function is not positive, continuous, nor decreasing, then this test is not applicable! It is a very **strict and limited** test.

Example

Test for convergence: $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ + , cont. , dec.

$u = n^2+1$ $\quad \ln a = \quad e^? = \infty$ $\quad f'(n) = \frac{n^2+1-2n^2}{(n^2+1)^2} = \frac{-n^2+1}{(n^2+1)^2}$

$du = 2n\,dn$

$\frac{1}{2}du = n\,dn$ $\quad f(n)$

$$\int_1^{\infty} \frac{n}{n^2+1}\,dn = \lim_{b\to\infty} \int_1^b \frac{n}{n^2+1}\,dn = \frac{1}{2}\lim_{b\to\infty} \int_1^b \frac{1}{u}\,du$$

$$= \frac{1}{2}\lim_{b\to\infty} \left(\ln u \Big|_1^b \right)$$

$$= \frac{1}{2}\lim_{b\to\infty} \left(\ln(n^2+1) \Big|_1^b \right) = \ln(b^2+1) - \cancel{\ln(1)}$$

$$\frac{1}{2}\ln(\infty^2+1)$$

∞

D inverse by IT

Examples

$u = \ln n$
$du = \frac{1}{n}\,dn$

+, cont, dec.

$\frac{1}{n} \cdot \frac{1}{\ln n} = n^{-1} \cdot (\ln n)^{-1}$

$-n^{-2}(\ln n)^{-1} + -(\ln n) \cdot \frac{1}{n}$

$\frac{-1 - \ln n}{n^4 \ln n}$

Test for Convergence: $\sum_2^\infty \frac{1}{n \ln n}$

$$\int_2^\infty \frac{1}{n \ln n}\,dn = \lim_{b\to\infty} \int_2^b \frac{1}{n \ln n}\,dn$$

$$= \lim_{b\to\infty} \int_2^b \frac{1}{u}\,du = \lim_{b\to\infty}\left(\ln u \Big|_2^b\right) = \lim_{b\to\infty}\left(\ln(\ln n)\Big|_2^b\right)$$

$$= \lim_{b\to\infty}\left(\ln(\ln b) - \ln(\ln 2)\right)$$

$\infty - \ln(\ln 2)$

Divergent

By IT, series diverges.

P-Series

← power.

$\frac{1}{ar^n}$ ex. $\frac{1}{2^n}$ vs $\frac{1}{n^2}$

Geo. P-series

The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$

a. Converges if $p > 1$
b. Diverges if $0 < p \leq 1$

This can be used to show that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Direct Comparison Test

Note: 1. $\sum_{n=0}^{\infty} \frac{1}{2^n}$ is geometric, but $\sum_{n=0}^{\infty} \frac{n}{2^n}$ is not.

2. $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is a p-series, but $\sum_{n=1}^{\infty} \frac{1}{n^3+1}$ is not.

For this reason, there is another useful test for positive functions.

Direct Comparison Test- Let $0 < a_n \leq b_n$ for all n.

1. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
2. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

Suggestion: "Big C, small C. Small D, big D."

Examples

b_n

$r<1$ Conv.

Test for convergence: $\sum_{n=1}^{\infty} \frac{1}{2+3^n} \leq \frac{1}{3^n} = \left(\frac{1}{3}\right)^n$ Geo.

$\sum_{n=1}^{\infty} \frac{1}{3^n} = \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$ converges. Geometric series $r<1$.

By DCT, our series also converges.

$2+\sqrt{n}$		n
3	$>$	1
3…	$>$	2
4	$=$	4
5	$<$	9

$\frac{1}{n}$

Example

Test for convergence: $\sum_{n=1}^{\infty} \frac{1}{2+\sqrt{n}} = \frac{1}{2+n^{1/2}} \leq \frac{1}{n^{1/2}}$

$\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ diverges, p-series $p<1$. (Inconclusive)

after $n \geq 4$, $\frac{1}{n} < \frac{1}{2+\sqrt{n}}$

After $n \geq 4$, series diverges b/c harmonic series diverges.

Limit Comparison Test (l'hopitale)

Suppose that $a_n > 0$, $b_n > 0$, and $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = L$, where L is finite and **positive**. Then the two series $\sum a_n$ and $\sum b_n$ either **both** converge or diverge.

$a_n \div \quad b_n = \frac{1}{n^2}$

Ex. Test for convergence: $\sum_{n=1}^{\infty} \frac{1}{3n^2-4n+5}$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

p-series w/ $p > 1$.

$\lim_{n\to\infty}\left(\frac{n^2}{3n^2-4n+5}\right) \frac{\infty}{\infty} \underset{=}{\overset{lh}{}} \lim_{n\to\infty}\left(\frac{2n}{6n-4}\right)$

$\underset{=}{\overset{lh}{}} \lim_{n\to\infty}\left(\frac{2}{6}\right) = \frac{1}{3}$ ✓

∴ Series converges by LCT.

Therefore

Examples

Test for convergence: $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$

$a_n = \frac{n^{1/2}}{n^2+1}$ $\qquad b_n = \frac{1}{n}$

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverge. Harmonic series

$$\lim_{n\to\infty}\left(\frac{n\sqrt{n}}{n^2+1}\right) = \lim_{n\to\infty}\left(\frac{n^{3/2}}{n^2+1}\right)$$

Series must also diverge by LCT

$$\stackrel{LH}{=} \lim_{n\to\infty}\left(\frac{\frac{3}{2}n^{1/2}}{2n}\right) \stackrel{LH}{=} \lim_{n\to\infty}\left(\frac{\frac{3}{4}n^{-1/2}}{2}\right) = \lim_{n\to\infty}\left(\frac{\frac{3}{4}}{2n^{1/2}}\right) = \frac{\frac{3}{4}}{\infty} = \boxed{0}$$

$\frac{1}{n}$ $\therefore$

Examples

$\frac{1}{n} = a_n$

Test for convergence: $\sum_{n=1}^{\infty} \frac{1}{\sqrt{3n-2}}$

$b_n = \frac{1}{(3n-2)^{1/2}}$

$\sum_{n=1}^{\infty} \frac{1}{n}$ = diverge.

harmonic series

$$\lim_{n\to\infty} \left(\frac{(3n-2)^{1/2}}{n} \right) \underline{\underline{lh}} \lim_{n\to\infty} \left(\frac{\frac{1}{2}(3n-2)^{-1/2} \cdot 3}{1} \right)$$

$$= \lim_{n\to\infty} \left(\frac{\frac{3}{2}}{(3n-2)^{1/2}} \right) = \boxed{0}$$

$\therefore$ By LCT, series must diverge.

Alternating Series

alternating factor.

Let $a_n > 0$. The alternating series $\sum_{n=1}^{\infty}(-1)^n a_n$ and $\sum_{n=1}^{\infty}(-1)^{n+1} a_n$ converge if the following two conditions are met:

1. $\lim\limits_{n\to\infty} a_n = 0$ and

2. $a_{n+1} \le a_n$ for all n. (Following term is less ~~greater~~ than or equal to its previous). Decreasing..

Examples

Test for convergence: $\sum_{n=1}^{\infty} \boxed{(-1)^{n+1}} \frac{1}{n}$

① $\lim_{n\to\infty} \frac{1}{n} = 0$ ✓

② $\frac{1}{n+1} \leq \frac{1}{n}$ ✓

a_{n+1} a_n

By AST, series converges.

Examples AST,

Test for convergence: $\sum_{n=1}^{\infty} \frac{n}{(-2)^{n-1}} = \frac{\frac{n}{(-2)^n}}{(-2)^1} = \frac{\frac{n}{-2^n}}{-2} = \frac{-2n}{(-2)^n}$

$= \sum_{n=1}^{\infty} \frac{-2n}{(-1)^n (2)^n}$

1. $\lim_{n \to \infty} \frac{-2n}{2^n} = \frac{\infty}{\infty} \overset{lh}{=} \frac{-2}{\ln 2 (2^n)} = 0$ ✓

2. $\frac{-2(n+1)}{2^{n+1}} \leq \frac{-2n}{2^n}$ ✓

$\frac{-2(n+1)}{2^{n+1}} = \frac{-2n-2}{2^{n+1}} = \frac{-2n}{2^{n+1}} - \frac{2}{2^{n+1}}$

Examples

Test for convergence: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+1)}{n}$

① $\lim_{n \to \infty} \frac{n+1}{n} = \frac{\infty}{\infty} \overset{lh}{=} \frac{1}{1} = 1 \neq 0$

Divergent.

Absolute Convergence

1. $\sum a_n$ is absolutely convergent if $\sum |a_n|$ converges.
2. $\sum a_n$ is conditionally convergent if $\sum a_n$ converges but $\sum |a_n|$ diverges.

Ratio Test

Let $\sum a_n$ be a series with nonzero terms.

1. $\sum a_n$ converges absolutely if $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$.
2. $\sum a_n$ diverges absolutely if $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$ or $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$.
3. The Ratio Test is inconclusive if $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$.

Note: Inconclusive means cannot determine using the test. Thus, other tests must be used.

$2^{n+1} = 2^n 2^1$

$a_{n+1} = \dfrac{2^{n+1}}{(n+1)!} \div \dfrac{2^n}{n!}$

Examples

a_n

Test for convergence: $\sum_{n=0}^{\infty} \frac{2^n}{n!}$

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}\, n!}{2^n (n+1)!} = \frac{\cancel{2^n}\, 2^1\, \cancel{n!}}{\cancel{2^n}(n+1)\cancel{n!}} = \frac{2}{n+1}$$

$3! = 3 \cdot 2 \cdot 1$

$4! = 4 \cdot \underbrace{3 \cdot 2 \cdot 1}$

$(3+1)! =$

$(3+1)! = 4 \cdot 3!$

$= (3+1)\, 3!$

$(n+1)! = (n+1)\, n!$

Rat. Test

$$\lim_{n\to\infty}\left|\frac{2}{n+1}\right| = \frac{2}{\infty+1} = \frac{2}{\infty} = 0 < 1$$

Therefore, by Rat. Test series converges absolutely.

$$3^{n+1} = 3^n 3^1 \qquad (n+1)(n+1) \qquad a_{n+1} = \frac{(n+1)^2 2^{(n+1+1)}}{3^{n+1}} \div a_n$$

$$2^{n+1+1} = 2^{(n+1)+1} = 2^{n+1} 2^1$$

Examples

$\leftarrow a_n$

Test for convergence: $\sum_{n=0}^{\infty} \frac{n^2 2^{n+1}}{3^n}$

$$\frac{a_{n+1}}{a_n} = \frac{3^n (n+1)^2 2^{(n+1+1)}}{3^{n+1} n^2 2^{n+1}} = \frac{\cancel{3^n}(n+1)^2 \cancel{2^{n+1}} 2^1}{\cancel{3^n} 3^1 n^2 \cancel{2^{n+1}}} = \frac{2 (n+1)^2}{3 n^2} = \frac{2}{3} \frac{n^2+2n+1}{n^2}$$

Rat. Test

$$\frac{2}{3} \lim_{n\to\infty} \left| \frac{n^2+2n+1}{n^2} \right| \stackrel{eh}{=} \frac{2n+2}{2n} = \frac{2}{3} \lim_{n\to\infty} \frac{2}{2} = \frac{2}{3}(1) = \frac{2}{3} < 1$$

Therefore, by Rat. Test, the series converges absolutely by Rat. Test.

$(n+1)^{n+1} = (n+1)^n (n+1)^1$

$(n+1)! = (n+1)(n!)$ $\quad a_n \longrightarrow$

Examples

$a_{n+1} = \dfrac{(n+1)^{n+1}}{(n+1)!} \div \dfrac{n^n}{n!}$

Test for convergence: $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

$$\frac{a_{n+1}}{a_n} = \frac{n!\,(n+1)^{n+1}}{n^n (n+1)!} = \frac{\cancel{n!}(n+1)^n \cancel{(n+1)}}{n^n \cancel{n!} \cancel{(n+1)}} = \frac{(n+1)^n}{n^n}$$

$(1+0)^{\infty} = 1^{\infty} \neq 1$

Rat. Test

$$\lim_{n\to\infty}\left|\frac{(n+1)^n}{n^n}\right| = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty}\left(\frac{n}{n}+\frac{1}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

Therefore, by Rat. Test the series diverges absolutely.

$e > 1$

Examples

$$a_{n+1} = \frac{\sqrt{n+1}}{\underbrace{n+1+1}_{+2}} \quad \frac{\sqrt{n}}{n+1}$$

Test for convergence: $\sum_{n=1}^{\infty}(-1)^n \frac{\sqrt{n}}{n+1}$

Ratio test

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)\sqrt{n+1}}{(n+2)\sqrt{n}} \Rightarrow \lim_{n\to\infty} \frac{(n+1)\sqrt{n+1}}{(n+2)\sqrt{2}} = \lim_{n\to\infty}\left(\frac{n+1}{n+2}\right)\left(\frac{\sqrt{n+1}}{\sqrt{n}}\right)$$

$$= \lim_{n\to\infty}\left(\frac{n+1}{n+2}\right)\left(\sqrt{\frac{n+1}{n}}\right) = \lim_{n\to\infty}\left(\frac{n+1}{n+2}\right)\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^{1/2}$$

$$= \lim_{n\to\infty}\left(\frac{1}{1}\right)\lim_{n\to\infty}(1)^{1/2}$$

$$= 1\cdot\sqrt{1} = 1$$

Ratio Test is inconclusive

Root Test

Let $\sum a_n$ be a series.

1. $\sum a_n$ converges absolutely if $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$.

2. $\sum a_n$ diverges absolutely if $\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$

3. The Root Test is inconclusive if $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$.

Examples

Test for convergence: $\sum_{n=1}^{\infty} \frac{e^{2n}}{n^n}$

Root Test

$$\lim_{n\to\infty} \left| \sqrt[n]{\frac{e^{2n}}{n^n}} \right| = \lim_{n\to\infty} \left(\sqrt[n]{\frac{(e^2)^n}{n^n}} \right) = \lim_{n\to\infty} \left(\sqrt[n]{\left(\frac{e^2}{n}\right)^n} \right)$$

$$= \lim_{n\to\infty} \frac{e^2}{n} = \frac{e^2}{\infty} = 0 < 1$$

$\boxed{\text{By the Root Test, the series converges absolutely.}}$

Guidelines for Testing for Convergence

1. Does the nth term approach 0? If not, the series diverges.
2. Is the series one of the special types- geometric, p-series, telescoping, or alternating?
3. Can the Integral Test, the Root Test, or the Ratio Test be applied?
4. Can the series be compared favorably to one of the special types?

Note: In some instances, more than one test is applicable. However, your objective should be to learn to choose the most **efficient** test.

(Refer to pg. 644 for reference)

Homework 5/1

Pg. 643 Example 5 (a-g)

9.6 #51-67 (odd) (suggested to do all)