Lesson 7-4

Arc Length and Surface Area

Objective

Students will…

- Be able to find the arc length of a smooth curve.
- Be able to find the surface area of revolution.

Arc-Length

Another application of integration is finding the arc length of a smooth curve. Of course finding the length of any curvature is no easy task. Using integration, however, this is achievable. For proof of this result, refer to page 476 of your textbook.

<u>Arc Length</u>- Let the function given by $y = f(x)$ represent a smooth curve on the interval $[a, b]$. The **arc length** of f between a and b is

$$s = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

$$y' = \frac{1}{2}x^2 - \frac{1}{2}x^{-2}$$

Examples

$$\frac{1}{6}x^3 + \frac{1}{2}x^{-1}$$

Find the arc length of the graph of $y = \frac{x^3}{6} + \frac{1}{2x}$ on the interval $\left[\frac{1}{2}, 2\right]$.

$$S = \int_{1/2}^{2} \sqrt{1 + \left(\frac{x^2}{2} - \frac{1}{2x^2}\right)^2}\, dx = \int_{1/2}^{2} \sqrt{1 + \left(\frac{x^4 - 1}{2x^2}\right)^2}\, dx$$

$$S = \int_{1/2}^{2} \sqrt{1 + \frac{x^{16} - 2x^4 + 1}{4x^4}}\, dx = \int_{1/2}^{2} \sqrt{\frac{x^{16} + 2x^4 + 1}{4x^4}}\, dx = \int_{1/2}^{2} \sqrt{\frac{(x^4+1)^2}{4x^4}}\, dx$$

$$= \int_{1/2}^{2} \frac{x^4+1}{2x^2}\, dx = \int_{1/2}^{2} \frac{\cancel{x^4}^{\,2}}{2\cancel{x^2}} + \frac{1}{2x^2}\, dx = \int_{1/2}^{2} \frac{1}{2}x^2 + \frac{1}{2}x^{-2}\, dx = \left(\frac{1}{6}x^3 - \frac{1}{2}x^{-1}\right)\Bigg|_{1/2}^{2}$$

$$= \frac{8}{6} - \frac{1}{4} - \left(\frac{1}{48} - 1\right) = \boxed{\frac{33}{16}}$$

$x' = \frac{3}{2}(y-1)^{1/2}$

Examples

$x = \pm(y-1)^{3/2}$ $\quad 8^2 = (y-1)^3$ $\quad 64 = (y-1)^3$ $\quad y = 5$

$0^2 = (y-1)^3 \Rightarrow y = 1$

Find the arc length of the graph of $(y-1)^3 = x^2$ on the interval $[0, 8]$.

$x_1 \quad x_2$

$$S = \int_1^5 \sqrt{1 + \left(\frac{3}{2}(y-1)^{1/2}\right)^2}\, dy = \int_1^5 \sqrt{1 + \frac{9}{4}(y-1)}\, dy = \int_1^5 \sqrt{\frac{9y-5}{4}}\, dy$$

$$= \int_1^5 \frac{\sqrt{9y-5}}{2}\, dy = \frac{1}{2}\int_1^5 (9y-5)^{1/2}\, dy$$

$u = 9y-5$

$du = 9\, dy$

$\frac{1}{9} du = dy$

$$\Rightarrow \frac{1}{18}\int_4^{40} u^{1/2}\, du = \frac{1}{18}\left(\frac{2}{3}u^{3/2}\right)\Big|_4^{40}$$

$$= \frac{1}{27}\left(40^{3/2} - 4^{3/2}\right) \approx \boxed{9.073}$$

Examples

$y' = \frac{1}{\cos x} \cdot -\sin x = \frac{-\sin x}{\cos x} = -\tan x$

Find the arc length of the graph of $y = \ln(\cos x)$ from $x = 0$ to $x = \frac{\pi}{4}$.

$$S = \int_0^{\pi/4} \sqrt{1+\tan^2 x}\, dx = \int_0^{\pi/4} \sqrt{\sec^2 x}\, dx = \int_0^{\pi/4} \sec x\, dx$$

$$= \ln\left|\sec x + \tan x\right| \Big|_0^{\pi/4} = \ln\left|\frac{2}{\sqrt{2}} + 1\right| - \ln\left|1 + 0\right| \approx \boxed{0.881}$$

Surface Area of Revolution

In our previous section we learned how to find the volume of the shape formed by revolution of a cross-section. Here, we learn how to find its surface area.

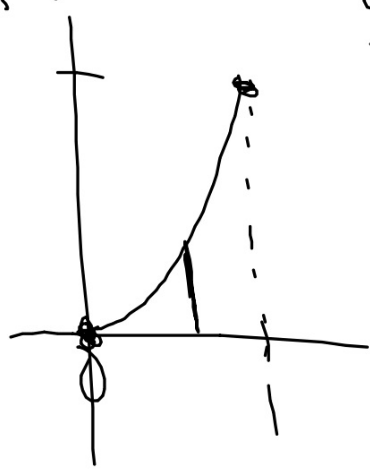
Surface Area of Revolution- Let $y = f(x)$ have a continuous derivative on the interval $[a, b]$. The area S of the surface of revolution formed by revolving the graph of f about a horizontal or vertical axis is

$S = 2\pi \int_a^b r(x)\sqrt{1 + [f'x]^2}\, dx$, where $r(x)$ represents the radius.

Examples

Find the area of the surface area formed by revolving the graph of $f(x) = x^3$ on the interval $[0, 1]$ about the x-axis.

$f'(x) = 3x^2$



$$S = 2\pi \int_0^1 x^3 \sqrt{1 + 9x^4}\, dx$$

$u = 1 + 9x^4$

$du = 36x^3\, dx$

$\frac{1}{36} du = x^3\, dx$

$$= \frac{\pi}{18} \int_1^{10} u^{1/2}\, du = \frac{\pi}{18} \left(\frac{2}{3} u^{3/2} \Big|_1^{10} \right)$$

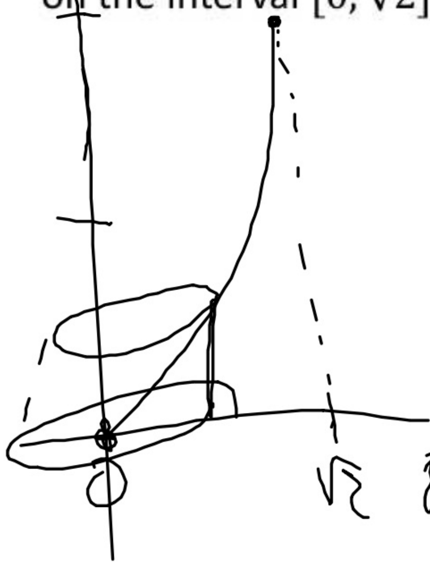
$$= \frac{\pi}{27} \left(10^{3/2} - 1^{3/2} \right)$$

$$\frac{(\sqrt{1000} - 1)\pi}{27}$$

Examples

$$f'(x) = 2x$$

Find the area of the surface formed by revolving the graph of $f(x) = x^2$ on the interval $[0, \sqrt{2}]$ about the y-axis.



$$S = 2\pi \int_0^{\sqrt{2}} x\sqrt{1+(2x)^2}\,dx = 2\pi \int_0^{\sqrt{2}} x(1+4x^2)^{1/2}\,dx$$

$u = 1+4x^2$

$du = 8x\,dx$

$\frac{1}{8}du = x\,dx$

$$\Rightarrow \frac{1}{4}\pi \int_1^9 u^{1/2}\,du = \frac{1}{4}\pi\left(\frac{2}{3}u^{3/2}\Big|_1^9\right)$$

$$= \frac{1}{6}\pi(27-1) = \frac{26}{6}\pi = \boxed{\frac{13}{3}\pi}$$

Homework 3/5

7.4 #3-13 (odd), 15, 19, 23, 39-44