Lesson 8-5

Partial Fractions

Objective

Students will…

- Be able to understand partial fraction decomposition.
- Be able to use partial fraction decomposition to integrate rational functions.

Partial Fractions

Consider the following problem: $\ln(\ldots$

$$\int \frac{1}{x^2 - 5x + 6} dx$$

This problem can be done using Trig Substitution (Section 8.4), but it becomes extremely tedious. (See example on pg. 552). Using Trig Substitution we see that $\int \frac{1}{x^2-5x+6} dx = \cancel{\frac{1}{x-3} - \frac{1}{x-2} + C}$

$\ln|x-3| - \ln|x-2| + C$

Partial Fractions

Now, suppose you were told that…

$$\frac{1}{x^2 - 5x + 6} = \frac{1}{x-3} - \frac{1}{x-2}$$

Then,

$$\int \frac{1}{x^2-5x+6}\,dx = \int \frac{1}{x-3}\,dx - \int \frac{1}{x-2}\,dx = \ln|x-3| - \ln|x-2| + C.$$

Partial Fractions

Partial Fraction technique allows us to "decompose" a fraction into a sum of two "easier" fractions. You have learned how to combine fractions like $\frac{(x+3)}{(x+3)}\frac{1}{x-2} - \frac{1}{x+3}\frac{(x-2)}{(x-2)} =$ $\frac{x+3-(x-2)}{(x+3)(x-2)} = \frac{5}{x^2+x-6}$

This method of partial fraction simply reverses this process:

$$\frac{5}{x^2+x-6} = \frac{5}{(x-2)(x+3)} = \frac{?}{x-2} + \frac{?}{x+3}$$

Guidelines for Partial Fractions

$$\frac{3}{x^2+4x+4}$$

$$\frac{3}{(x+2)^2}$$

$$= \frac{\quad}{(x+2)} + \frac{\quad}{(x+2)^2}$$

DECOMPOSITION OF $N(x)/D(x)$ INTO PARTIAL FRACTIONS

1. **Divide if improper:** If $N(x)/D(x)$ is an improper fraction (that is, if the degree of the numerator is greater than or equal to the degree of the denominator), divide the denominator into the numerator to obtain

$$\frac{N(x)}{D(x)} = (\text{a polynomial}) + \frac{N_1(x)}{D(x)}$$

where the degree of $N_1(x)$ is less than the degree of $D(x)$. Then apply Steps 2, 3, and 4 to the proper rational expression $N_1(x)/D(x)$.

2. **Factor denominator:** Completely factor the denominator into factors of the form

$$(px + q)^m \qquad \text{and} \qquad (ax^2 + bx + c)^n$$

where $ax^2 + bx + c$ is irreducible.

3. **Linear factors:** For each factor of the form $(px + q)^m$, the partial fraction decomposition must include the following sum of m fractions.

$$\frac{A_1}{(px+q)} + \frac{A_2}{(px+q)^2} + \cdots + \frac{A_m}{(px+q)^m}$$

4. **Quadratic factors:** For each factor of the form $(ax^2 + bx + c)^n$, the partial fraction decomposition must include the following sum of n fractions.

$$\frac{B_1x + C_1}{ax^2+bx+c} + \frac{B_2x + C_2}{(ax^2+bx+c)^2} + \cdots + \frac{B_nx + C_n}{(ax^2+bx+c)^n}$$

Examples

Write the partial fraction decomposition for $\frac{1}{x^2-5x+6} = \frac{1}{(x-3)(x-2)}$

$= \frac{A}{x-3} + \frac{B}{x-2}$

$A(x-2) + B(x-3) = 1$

Let $x=3$

$A(3-2) = 1$

$A = 1$

Let $x=2$

$B(2-3) = 1$

$-B = 1$

$B = -1$

$$= \boxed{\frac{1}{x-3} + \frac{-1}{x-2} = \frac{1}{x-3} - \frac{1}{x-2}}$$

$\frac{9}{(x+1)^2} = 9(x+1)^{-2}$ $u = x+1$, $du = 1\,dx$

Examples

$\frac{6}{x} = 6\frac{1}{x}$

Find $\int \frac{5x^2+20x+6}{x^3+2x^2+x}\,dx = \int \frac{5x^2+20x+6}{x(x^2+2x+1)}\,dx = \int \frac{5x^2+20x+6}{x(x+1)^2}\,dx$

$= \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$

$\boxed{\int \frac{6}{x} - \frac{1}{x+1} + \frac{9}{(x+1)^2}\,dx}$

$\Leftrightarrow A(x+1)^2 + B(x)(x+1) + Cx = 5x^2+20x+6$

$= \boxed{6\ln|x| - \ln|x+1| - 9(x+1)^{-1} + C}$

Let $x=0$

$A(0+1)^2 = 6$

$A = 6$

Let $x=-1$

$-C = -9$

$C = 9$

Let $x=1$

$6(1+1)^2 + B(1)(1+1) + 9(1) = 31$

$24 + 2B + 9 = 31$

$2B = -2$

$B = -1$

Examples

$$\frac{Cx+D}{x^2+4} = \frac{Cx}{x^2+4} + \frac{D}{x^2+4}$$

$u = x^2+4$

$du = 2x$

Find $\int \frac{2x^3-4x-8}{(x^2-x)(x^2+4)}\,dx = 2\int \frac{x^3-2x-4}{x(x-1)(x^2+4)}\,dx$

$$= \frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2+4} = x^3-2x-4$$

$$A(x-1)(x^2+4) + B(x)(x^2+4) + (Cx+D)(x)(x-1) = x^3-2x-4$$

let $x=0$

$-4A = -4$

$A = 1$

let $x=1$

$5B = -5$

$B = -1$

let $x=-1$

$-10+5+(D-C)(2) = -3$

$2(D-C) = 2$

$D-C = 1$

let $x=2$

$8-16+(2C+D)(2) = 0$

$2C+D = 4$

$-C+D = 1$

$3C = 3$

$C = 1$

$D = 2$

$$2\int \frac{1}{x} - \frac{1}{x-1} + \frac{x}{x^2+4} + \frac{2}{x^2+4}\,dx$$

$$= 2\left(\ln|x| - \ln|x-1| + \tfrac{1}{2}\ln|x^2+4| + 2\left(\tfrac{1}{2}\right)\arctan\left(\tfrac{x}{2}\right)\right)$$

$$\boxed{2\ln|x| - 2\ln|x-1| + \ln|x^2+4| + 2\arctan\left(\tfrac{x}{2}\right)}$$

Examples

Find $\int \frac{8x^3+13x}{(x^2+2)^2}\,dx$

$$\frac{Ax+B}{x^2+2} + \frac{Cx+D}{(x^2+2)^2} \Rightarrow \int \frac{8x}{x^2+2} - \frac{3x}{(x^2+2)^2}\,dx \Rightarrow \boxed{4\ln|x^2+2| - \frac{3}{2}(x^2+2)^{-1} + C}$$

$u = x^2+2$
$du = 2x\,dx$
$\frac{1}{2}du = x\,dx$

$u = x^2+2$
$du = 2x\,dx$
$\frac{1}{2}du = x\,dx$

$\frac{3}{2}\frac{1}{u^2}\,du = \frac{3}{2}u^{-2}\,du$

$-\frac{3}{2}u^{-1}$

$(Ax+B)(x^2+2) + Cx+D = 8x^3+13x$

Let $x=0$

$2B+D=0$

$D = -2B$

$\boxed{D=0}$

Let $x=1$

$3A+3B+C+D=21$

$3A+B+C=21$

$-3A+B-C=-21$

$\boxed{B=0}$

Let $x=-1$

$-3A+3B-C+D=-21$

$-3A+B-C=-21$

Let $x=2$

$12A+2C=90$

$2(3A+C)=(21)2$

$12A+2C=90$

$6A+2C=42$

$12A+2C=90$

$-6A=-48$

$\boxed{A=8}$ $\boxed{C=-3}$

Alternate

Examples

Find $\int \frac{8x^3+13x}{(x^2+2)^2}\,dx$

$$(Ax+B)(x^2+2)+Cx+D=8x^3+13x$$

$$Ax^3+2Ax+Bx^2+2B+Cx+D=8x^3+13x$$

$$Ax^3+Bx^2+(2A+C)x+2B+D=8x^3+13x=8x^3+0x^2+13x+0$$

$A=8$

$B=0$

$C=-3$

$2A+C=13$

$2B+D=0$

$D=0$

Let $x=1$

$A+(2A+C)=21$

$3A+C=21$

$2A+C=13$

Homework 3/23

8.5 #1-6, 7-31 (odd)