Lesson 8-1

Basic Integration Rules
(A.K.A harder integral problems)

Objective

Students will…

- Be able to rewrite a function using the power of algebra and other various techniques.
- Be able to use the basic integration rules to integrate.

Fitting Integrands to Basic Rules

Problems that the AP exam loves to contain are basic integral problems that are in "disguise." We need to be able to use our algebraic techniques to rewrite the integrand (function that you are taking the integral of) to our advantage. Consider the following three integral problems…

a. $\int \frac{4}{x^2+9} dx$ b. $\int \frac{4x}{x^2+9} dx$ c. $\int \frac{4x^2}{x^2+9} dx$

These problems look quite similar, but they are quite different in the world of integration.

$u = x^2+9$

$du = 2x$

Examples

arctan = $\tan^{-1}$

$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\arctan\left(\frac{u}{a}\right) + C$$

a. $\int \frac{4}{x^2+9}dx = 4\int \frac{1}{x^2+3^2}dx$

$$= 4\int \frac{dx}{3^2+x^2} = 4\left(\frac{1}{3}\arctan\left(\frac{x}{3}\right) + C\right) = \boxed{\frac{4}{3}\arctan\left(\frac{x}{3}\right) + C}$$

Examples

$u = x^2 + 9$

$du = \cancel{2}4x\,dx \quad 2\,du = 4x\,dx$

b. $\int \frac{4x}{x^2+9}\,dx = 2\int \frac{1}{u}\,du = 2(\ln u) + C$

$= \boxed{2\ln|x^2+9| + C}$

$\underline{ex}$ $\begin{array}{r} 1 \\ 3\overline{)4} \\ \underline{3} \\ 1 \end{array} \Rightarrow 1\frac{1}{3} = 1 + \frac{1}{3}$

Examples

c. $\int \frac{4x^2}{x^2+9}\,dx$

$$\begin{array}{r} 4 \\ x^2+9\overline{)4x^4+0} \\ \underline{\ominus\, 4x^2+36} \\ -36 \end{array}$$

$$= \int 4 + \frac{-36}{x^2+9}\,dx = \int 4\,dx - \int \frac{36}{\underset{3^2+x^2}{x^2+9}}\,dx$$

$$= 4x - 36\left(\frac{1}{3}\arctan\left(\frac{x}{3}\right)\right) + C = \boxed{4x - 12\arctan\left(\frac{x}{3}\right) + C}$$

$u = 4 - x^2$

$du = -2x\,dx \Rightarrow -\frac{1}{2}du = x\,dx$

Examples

$\int \frac{dx}{\sqrt{a^2-u^2}} = \arcsin$

Evaluate $\int_0^1 \frac{x+3}{\sqrt{4-x^2}}dx$ $= \int_0^1 \frac{x}{\sqrt{4-x^2}}\,dx + \int_0^1 \frac{3}{\sqrt{4-x^2}}\,dx$

$$= -\frac{1}{2}\int_0^1 u^{-1/2}\,du + 3\int_0^1 \frac{dx}{\sqrt{2^2-x^2}} = -\frac{1}{2}\left(2u^{1/2}\Big|_0^1\right) + 3\left(\arcsin\left(\frac{x}{2}\right)\Big|_0^1\right)$$

$$= -\left((4-x^2)^{1/2}\Big|_0^1\right) + 3\left(\arcsin\left(\frac{x}{2}\right)\Big|_0^1\right)$$

$$= -\left(\sqrt{3}-2\right) + 3\left(\frac{\pi}{6} - 0\right) = \boxed{2-\sqrt{3}+\frac{\pi}{2}}$$

Examples

$$u = x^3$$
$$du = 3x^2\,dx \Rightarrow \frac{1}{3}du = x^2\,dx$$

Find $\int \frac{x^2}{\sqrt{16-x^6}}\,dx = \int \frac{x^2}{\sqrt{4^2-(x^3)^2}}\,dx = \frac{1}{3}\int \frac{1}{\sqrt{4^2-u^2}}\,du$

$$= \frac{1}{3}\int \frac{du}{\sqrt{4^2-u^2}} = \frac{1}{3}\left(\arcsin\left(\frac{u}{4}\right)\right) + C = \boxed{\frac{1}{3}\arcsin\left(\frac{x^3}{4}\right) + C}$$

$u = 1 + e^x$

$du = e^x\,dx$

Examples

Find $\int \frac{1}{1+e^x}\,dx$ $= \int \frac{(1+e^x) - e^x}{1+e^x}\,dx = \int \frac{1+e^x}{1+e^x}\,dx - \int \frac{e^x}{1+e^x}\,dx$

$= \int 1\,dx - \int \frac{1}{u}\,du = x - \ln|u| + C$

$= \boxed{x - \ln|1+e^x| + C}$

$\int \ln x = ?$

Examples

Find $\int (\cot x)(\ln(\sin(x)))\,dx$

$u = \ln(\sin x)$

$du = \frac{1}{\sin x} \cdot \cos x \, dx$

$= \frac{\cos x}{\sin x} \, dx$

$du = \cot \, dx$

$\Rightarrow \int u \, du = \frac{1}{2}u^2 + C$

$= \boxed{\frac{1}{2}\left(\ln(\sin x)\right)^2 + C}$

$\int \tan x = \ln|\cdots|$

$\int \tan^2 x = ?$

Examples

$\frac{d}{dx} \tan x = \sec^2 x$

$\tan^2 x = -1 + \sec^2 x$

Find $\int \tan^2 2x\, dx$

$u = 2x$

$du = 2\, dx$

$\frac{1}{2} du = dx$

$$= \int -1 + \sec^2 2x\, dx = \int -1\, dx + \int \sec^2 2x\, dx$$

$$= \int -1\, dx + \frac{1}{2}\int \sec^2 u\, du = -x + \frac{1}{2}(\tan u) + C$$

$$= \boxed{-x + \frac{1}{2}\tan(2x) + C}$$

Homework 3/15

8.1 #1-4, 15-49 (e.o.o), 63, 65