Lesson 4-2

Area and the Riemann Sum

Objective

Students will…

- Be able to recognize and understand the sigma/summation notation.
- Be able to estimate the area underneath the curve by using the left and right Riemann Sums.

Sigma (Summation) Notation

We shift gears and refresh our memory on the **Sigma notation**, which represents **summation**.

Sigma Notation- The sum of n terms $a_1, a_2, a_3, a_4, \dots, a_n$ is written as

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + a_4 + \cdots + a_n$$

where i is the **index of summation**, a_i is the ith term of the sum, and **the upper and lower bound** of summation are $\boldsymbol{n}$ and $\mathbf{1}$.

Remember, i doesn't always equal 1 on the bottom. For example, we could take the sum of terms 2-8.

$$\sum_{i=2}^{8} a_i = a_2 + a_3 + \cdots + a_8$$

Examples

Expand the following summations.

a. $\sum_{i=1}^{6} i = 1+2+3+4+5+6 = 21$

b. $\sum_{i=0}^{5}(i+1) = 1+2+3+4+5+6 = 21 = \sum_{i=-1}^{4}(i+2)$

$= \sum_{i=2}^{7}(i-1)$

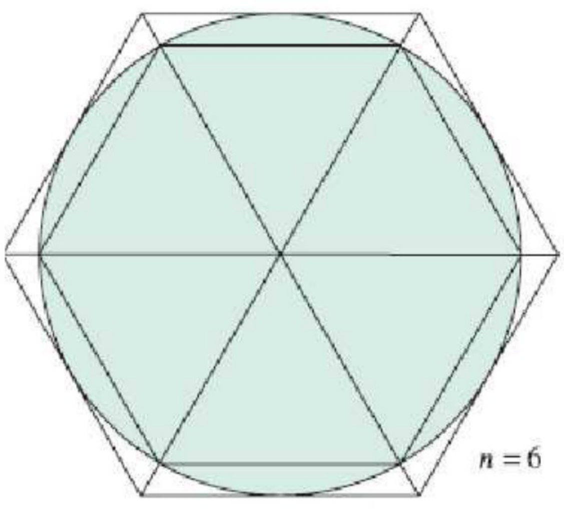
c. $\sum_{j=3}^{7} j^2 = 3^2+4^2+5^2+6^2+7^2$

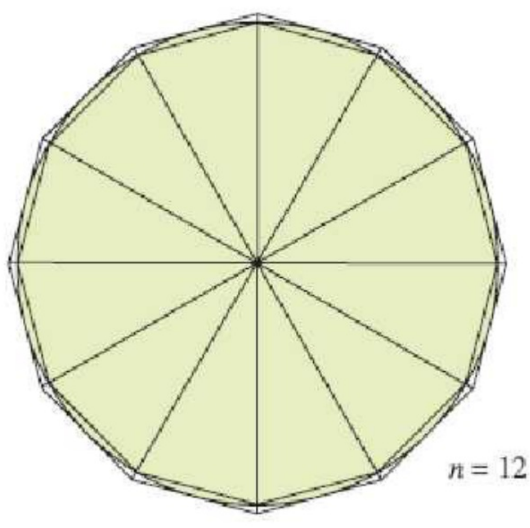
d. $\sum_{k=1}^{n}\frac{1}{n}(k^2+1) = \frac{1}{n}(1^2+1) + \frac{1}{n}(2^2+1) + \cdots + \frac{1}{n}(n^2+1)$

$= \frac{2}{n} + \frac{5}{n} + \frac{10}{n} + \cdots + \frac{n^2+1}{n}$

Area

So what does summation have to do with area? Well, let's consider the following problem. Suppose that you did not know the area formula of a circle. How would you find or at least estimate the area of a circle? A very prominent Greek mathematician, Archimedes used what we call today a method of "exhaustion" to figure it out during 3rd century B.C.

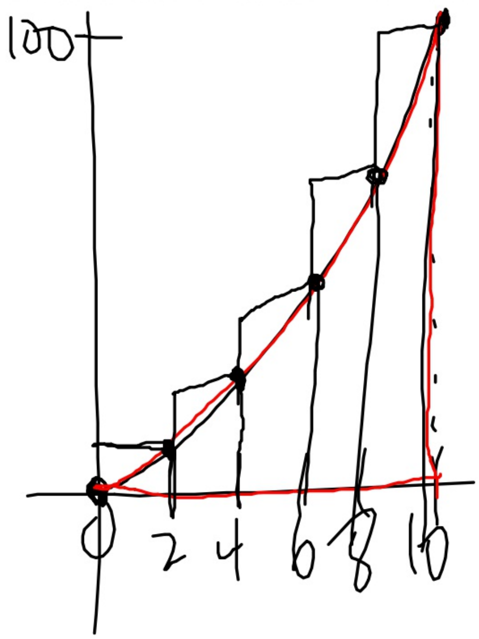




Area Underneath the Curve

One of the biggest applications of Calculus is the ability to find the area underneath or within a bounded region. Using summation approximation is a good way to estimate such area.

Ex. For $f(x) = x^2$, approximate the area of the region bounded by the x-axis between $x = 0$ and $x = 10$, using 5 rectangles.



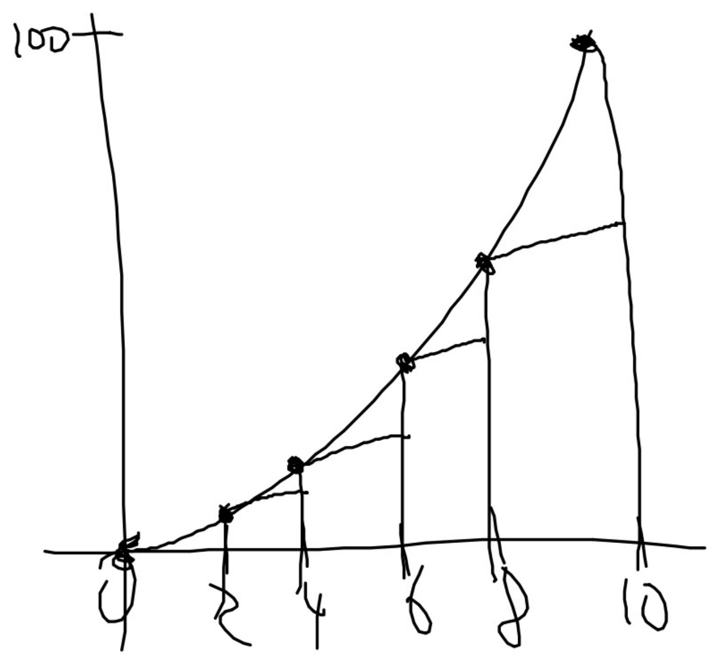
$A \approx (2 \cdot 4) + (2 \cdot 16) + (2 \cdot 36) + (2 \cdot 64) + (2 \cdot 100)$

$\approx 8 + 32 + 72 + 128 + 200$

≈ 440

Area Underneath the Curve

Ex. For $f(x) = x^2$, approximate the area of the region bounded by the x-axis between $x = 0$ and $x = 10$, using 5 rectangles.



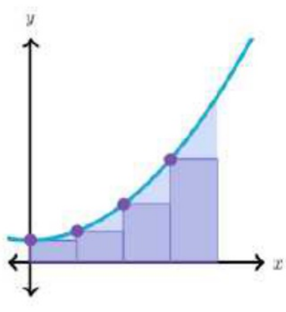
$A \approx 0 + (2\cdot 4) + (2\cdot 16) + (2\cdot 36)$
$+ (2\cdot 64) \approx 240$

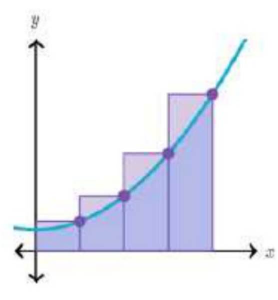
$240 < \text{Area} < 440$

Riemann Sum

As observed, one was clearly an underestimation, while the other was an overestimation. Although we do not have the precise area, we have a pretty decent idea as to what range it will fall under. This method is known as the **Riemann Sum**, named after Georg Riemann. The first method, fit the rectangles to the curve by the its right endpoints, while the second method, used the left endpoints. This is where we get the phrases, **The Left and the Right Riemann Sums**.

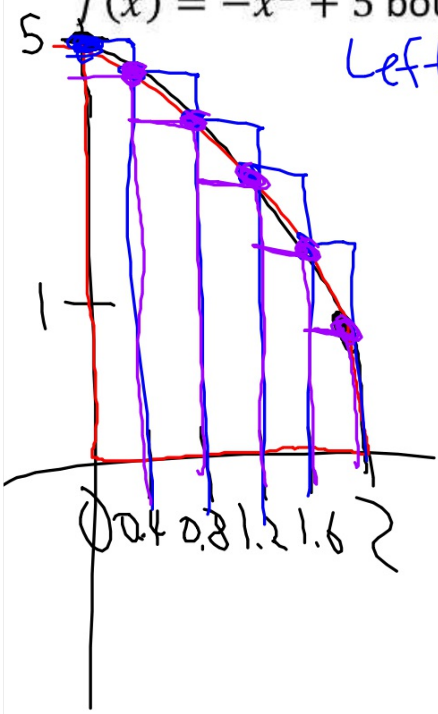
Note: The more rectangles you use, the more accurate the answer becomes.





Example

Use five rectangles to find the left and the right Riemann Sums of $f(x) = -x^2 + 5$ bounded by the x-axis and $x = 0$ and $x = 2$



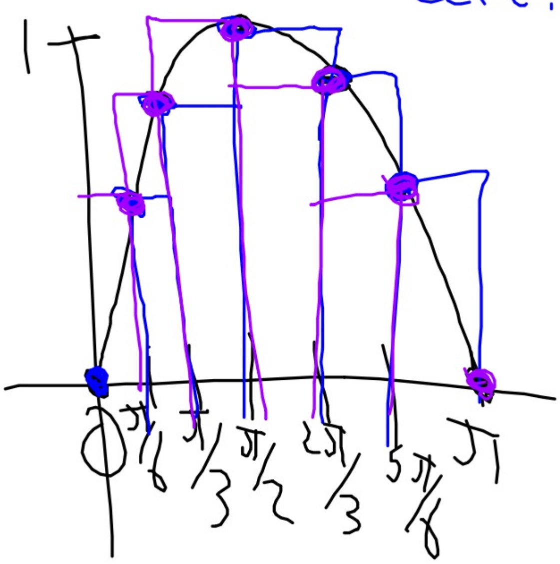
Left: $(0.4 \cdot 5) + (0.4 \cdot 4.84) + (0.4 \cdot 4.36) + (0.4 \cdot 3.56) + (0.4 \cdot 2.44) \approx 8.08$

Right: $(0.4 \cdot 4.84) + (0.4 \cdot 4.36) + (0.4 \cdot 3.56) + (0.4 \cdot 2.44) + (0.4 \cdot 1) \approx 6.48$

$6.48 < \text{Area} < 8.08$

Example

Use six rectangles to find the left and the right Riemann Sums of $f(x) = \sin x$ bounded by the x-axis and $x = 0$ and $x = \pi$



Left: $0 + \left(\frac{\pi}{6} \cdot \frac{1}{2}\right) + \left(\frac{\pi}{6} \cdot \frac{\sqrt{3}}{2}\right) + \left(\frac{\pi}{6} \cdot 1\right) + \left(\frac{\pi}{6} \cdot \frac{\sqrt{3}}{2}\right) + \left(\frac{\pi}{6} \cdot \frac{1}{2}\right)$

$= \frac{\pi}{12} + \frac{\pi\sqrt{3}}{12} + \frac{2\pi}{12} + \frac{\pi\sqrt{3}}{12} + \frac{\pi}{12}$

$= \boxed{\frac{4\pi + 2\pi\sqrt{3}}{12}}$

Homework 12/19

4.2- #27-36