Lesson 5-2, 5-4b

Exponential and Logarithmic Functions and their Integrals

Objective

Students will…

- Be able evaluate the integrals of exponential functions.
- Be able to evaluate the integrals of logarithmic functions.

Natural Exponential Functions

Recall, the number e. We define the natural exponential function as $f(x) = e^x$, where $e \approx 2.718281828459...$

The derivative of the natural exponential function is a simple chain rule:

$$\frac{d}{dx}[e^u] = e^u(u')$$

In other words, the derivative of e^u is e^u times the derivative of u.

Natural Exponential Functions

That being said, then, the integral of natural exponential functions is also quite simple:

Let u be a differentiable function of x.

1. $\int e^x \, dx = e^x + C$
2. $\int e^u du = e^u + C$

In other words, it's a simple "U"-Substitution.

Example

Find $\int e^{3x+1}dx$

$u = 3x+1$

$du = 3\,dx$

$\frac{1}{3}du = dx$

$$\Rightarrow \frac{1}{3}\int e^u\,du = \frac{1}{3}\left(e^u + C\right)$$

$$= \boxed{\frac{1}{3}e^{3x+1} + C}$$

Example

$= 5\int x e^{-x^2}$

Find $\int 5xe^{-x^2}\,dx$

$u = -x^2$

$du = -2x\,dx$

$-\frac{5}{2}du = +5x\,dx$

$\Rightarrow -\frac{5}{2}\int e^u\,du = \frac{-5}{2}\left(e^u + C\right)$

$= \boxed{\frac{-5}{2}e^{-x^2} + C}$

Example

Find $\int \frac{e^{\frac{1}{x}}}{x^2}\,dx = \int e^{x^{-1}} x^{-2}\,dx$

$u = x^{-1}$

$du = -x^{-2}\,dx$

$-du = x^{-2}\,dx$

$\Rightarrow -\int e^u\,du = -(e^u + C)$

$= -e^u + C$

$= -e^{x^{-1}} + C$

Example

Find $\int \sin x \, e^{\cos x} \, dx$ $= -\int e^u \, du = -(e^u + C)$

$u = \cos x$

$du = -\sin x \, dx$

$-du = \sin x \, dx$

$= \boxed{-e^{\cos x} + C}$

Natural Logarithmic Functions

Recall that the inverse function of e^x is known as the natural logarithmic function, or namely, $f(x) = \ln x$.

Remember, $b = e^a \qquad \rightarrow \qquad \ln b = a$

The derivative of the natural log function is also a chain rule as follows:

$$\frac{d}{dx}[\ln u] = \frac{1}{u}(u') = \frac{u'}{u}$$

So the derivative of $\ln u$ is 1 over u times the derivative of u.

$\ln(-a) = DNE$

Natural Logarithmic Functions

Then, the integration rule is as follows:

Let u be a differentiable function of x.

1. $\int \frac{1}{x} dx = \ln|x| + C$

2. $\int \frac{1}{u} du = \ln|u| + C$

3. $\int \frac{u'}{u} dx = \ln|u| + C$

It is also an application of the “U”-Substitution

Example

Find $\int \frac{2}{x}\,dx = 2\int \frac{1}{x}\,dx$

$u = x$

$du = 1\,dx$

$2\,du = 2\,dx$

$$\Rightarrow 2\int \frac{1}{u}\,du = 2\left(\ln|u| + C\right)$$

$$= \boxed{2\ln|x| + C}$$

Example

Find $\int \frac{1}{4x-1}\,dx$

$u = 4x - 1$

$du = 4\,dx$

$\frac{1}{4}du = 1dx$

$$\Rightarrow \frac{1}{4}\int \frac{1}{u}\,du = \frac{1}{4}\left(\ln|u| + C\right)$$

$$= \boxed{\frac{1}{4}\ln|4x-1| + C}$$

Example

$\int \frac{u'}{u}$

Find $\int \frac{3x^2+1}{x^3+x}\,dx = \boxed{\ln|x^3+x| + C}$

Example

$\int \frac{u'}{u}$

Find $\int \frac{\sec^2 x}{\tan x} dx = \boxed{\ln|\tan x| + C}$

Example

$\int \frac{1}{2}\frac{u'}{u}$

Find $\int \frac{x+1}{x^2+2x}\,dx$ $= \frac{1}{2}\left(\ln|x^2+2x|+C\right)$

$$= \boxed{\frac{1}{2}\ln|x^2+2x|+C}$$

$x+x \quad (x+1)^2 \neq x^2+1$

Example

Find $\int \frac{2x}{(x+1)^2}\,dx$

$u = x+1 \Rightarrow x = u-1$

$du = 1\,dx$

$\Rightarrow \int \frac{2(u-1)}{u^2}\,du = \int \frac{2u-2}{u^2}\,du$

$\frac{u}{u} \qquad 2\frac{1}{u^2}$

$= \int \boxed{\frac{2u}{u^2}} - \frac{2}{u^2}\,du$

$\boxed{\ln|u^2| - 2\ln|u^2| + C}$

$(x+1)^2 \qquad (x+1)^2$

Hi

Example

Find $\int \tan x \, dx$

Integrals of Six Basic Trig Functions

INTEGRALS OF THE SIX BASIC TRIGONOMETRIC FUNCTIONS

$$\int \sin u \, du = -\cos u + C \qquad \int \cos u \, du = \sin u + C$$

$$\int \tan u \, du = -\ln|\cos u| + C \qquad \int \cot u \, du = \ln|\sin u| + C$$

$$\int \sec u \, du = \ln|\sec u + \tan u| + C \qquad \int \csc u \, du = -\ln|\csc u + \cot u| + C$$

Homework 2/6

5.1 #1-23 (e.o.o), 25-35 (e.o.o), 47, 51

5.4 #85-105 (odd)