Lesson 4-5

Integration by "U"-Substitution

Objective

Students will…

- Be able to recognize that the technique of "U"-Substitution is the "anti-chain rule."
- Be able to use "U"-Substitution to find integrals.

Chain Rule

Chain Rule- If f and g are differentiable functions, then the composite function $f \circ g = f(g(x))$ is also differentiable such that …

$$\frac{d}{dx} f\big(g(x)\big) = f'(g(x)) \cdot g'(x)$$

In other words, the chain rule consists of first taking the derivative of the outer function, while leaving the inner function the same, multiplied by the derivative of the inner function.

The most challenging part of the chain rule being able to distinguish different functions that are being composed (i.e. outer and inner).

Anti-Chain rule

Based on the definition of chain rule, we then have that

$$\int [f'(g(x)) \cdot g'(x)\, dx] = f(g(x)) + C$$

Have you ever wondered what that dx means at the end of every integral? By definition, the dx represents the width of the rectangles (remember Riemann Sum?). In terms of the arithmetic, the dx simply represents the derivative of the function of x. This makes sense if you understand that every derivative rule is really the chain rule.

Power Rule: $\frac{d}{dx}x^a = ax^{a+1} = ax^{a+1} \cdot \frac{d}{dx}x = ax^{a+1} \cdot 1$

So, since integration can only be done on a function that is first differentiable whatever follows the integral is something that has been differentiated, hence $\int f(x)\, dx$.

U-Substitution

With that said, the way to reverse the chain rule is to find or identify the correct $g(x)$ so that you can **replace** $g'(x)\,dx$. Namely, if you were to set $u = g(x)$, then $du = g'(x)\,dx$ (chain rule). So, we end up with…

$$\int [f'(g(x)) \cdot g'(x)\,dx] = \int f'(u)\,du = f(u) = f(g(x))$$

Or using the F notation

$$\int [f(g(x)) \cdot g'(x)\,dx] = \int f(u)\,du = F(u) = F(g(x))$$

This technique is known as the **"U"-Substitution**. The challenge here is finding the $g(x)$. Unfortunately, you can only get better at this with practice!! (Gotta give you some challenge in life ☺). If you pick the right expression for u, then you should only end up with one variable, non-composite function with du.

Examples

Find $\int (x^2 + 1)^2 (2x)\, dx$

Examples

Find $\int 5 \cos 5x \, dx$

Example

Find $\int x(x^2 + 1)^2\, dx$

Example

Find $\int \sqrt{2x-1}\,dx$

Example

Find $\int \sin^2 3x \cos 3x \, dx$

Example

Find $\int x\sqrt{2x-1}\,dx$

$\int_a^b f(x)\,dx = F(b) - F(a)$

Definite Integration by U-Substitution

The way to integrate definite integrals follows the same logic, just with a few extra steps. There are two ways: direct way, and using a change of variable.

Direct way: Rewrite "u" back to the original and use the original bounds.

Change of Variable: Change the bounds using "u" and apply the First FTC without rewriting.

Example: Direct Way

Find $\int_0^1 x(x^2+3)^3\,dx$

$u = x^2 + 3$

$\frac{du}{2} = \frac{2x}{2}\,dx$

$\frac{1}{2}du = x\,dx$

$\Rightarrow \frac{1}{2}\int_0^1 u^3\,du = \frac{1}{2}\left(\frac{1}{4}u^4\Big|_0^1\right)$

$= \frac{1}{8}\left((x^2+3)^4\Big|_0^1\right)$

$= \frac{1}{8}(256-81)$

$= \boxed{\frac{175}{8}}$

Example: Using Change of Variable

Find $\int_0^1 x(x^2+3)^3\,dx$

$u = x^2 + 3$

$u_b = 1^2 + 3 = 4$

$u_a = 0^2 + 3 = 3$

$\Rightarrow \frac{1}{2}\left(\frac{1}{4}u^4\Big|_3^4\right)$

$= \frac{1}{8}\left(4^4 - 3^4\right) = \frac{1}{8}\left(256 - 81\right)$

$= \boxed{\frac{175}{8}}$

Example: Direct Way

Find $\int_1^5 \frac{x}{\sqrt{2x-1}}\,dx$

$u = 2x - 1 \Rightarrow x = \frac{1}{2}u + \frac{1}{2}$

$\frac{du}{2} = \frac{2\,dx}{2}$

$\Rightarrow \frac{1}{2}\int_1^5 \left(\frac{1}{2}u + \frac{1}{2}\right)\left(u^{-1/2}\right) du$

$= \frac{1}{2}\int_1^5 \frac{1}{2}u^{1/2} + \frac{1}{2}u^{-1/2}\, du = \frac{1}{4}\int_1^5 u^{1/2} + u^{-1/2}$

$= \frac{1}{4}\left(\frac{2}{3}u^{3/2} + 2u^{1/2}\right)\Big|_1^5$

$= \frac{1}{4}\left(\frac{2}{3}(2x-1)^{3/2} + 2(2x-1)^{1/2}\right)\Big|_1^5 =$

$= \frac{1}{4}\left(\left(18 + 6\right) - \left(\frac{8}{3}\right)\right)$

Example: Using Change of Variable

Find $\int_1^5 \frac{x}{\sqrt{2x-1}}\,dx$

$u = 2x - 1$

$u_b = 2(5) - 1 = 9$

$u_a = 2(1) - 1 = 1$

$$\Rightarrow \frac{1}{4}\left(\frac{2}{3}u^{3/2} + 2u^{1/2}\right)\Big|_1^9$$

$$= \frac{1}{4}\left(\left(\frac{2}{3}(27) + 2(3)\right) - \left(\frac{2}{3} + \frac{2}{1}\right)\right)$$

$$= \frac{1}{4}\left((18 + 6) - \frac{8}{3}\right) = \frac{6}{1} - \frac{2}{3}$$

$$= \frac{16}{3}$$

b;kj

h

Homework 1/30

4.5 #1-6, 7-19 (e.o.o), 29, 43-53 (e.o.o), 65, 67, 71-81 (odd)