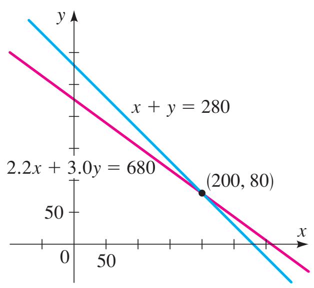
9 Systems of Equations and Inequalities

Chapter Overview

Many real-world situations have too many variables to be modeled by a *single* equation. For example, weather depends on many variables, including temperature, wind speed, air pressure, humidity, and so on. So to model (and forecast) the weather, scientists use many equations, each having many variables. Such systems of equations *work together* to describe the weather. Systems of equations with hundreds or even thousands of variables are also used extensively in the air travel and telecommunications industries to establish consistent airline schedules and to find efficient routing for telephone calls. To understand how such systems arise, let's consider the following simple example.



We can solve this system graphically. The point (200, 80) lies on the graph of each equation, so it satisfies both equations.

A gas station sells regular gas for \$2.20 per gallon and premium for \$3.00 per gallon. At the end of a business day 280 gallons of gas were sold and receipts totaled \$680. How many gallons of each type of gas were sold? If we let x and y be the number of gallons of regular and premium gasoline sold, respectively, we get the following system of two equations:

$$\begin{cases} x + y = 280 & \text{Gallons equation} \\ 2.20x + 3.00y = 680 & \text{Dollars equation} \end{cases}$$

These equations *work together* to help us find x and y; neither equation alone can tell us the value of x or y. The values $x = 200$ and $y = 80$ satisfy *both* equations, so they form a solution of the system. Thus, the station sold 200 gallons of regular and 80 gallons of premium.

We can also represent a linear system by a rectangular array of numbers called a matrix. The *augmented matrix* of the above system is:

$$\begin{array}{cl} \begin{bmatrix} 1 & 1 & 280 \\ 2.20 & 3.00 & 680 \end{bmatrix} & \begin{array}{l} \text{Gallons equation} \\ \text{Dollars equation} \end{array} \\ \begin{array}{cc} \;x\; & \;\;y\;\; \end{array} & \end{array}$$

The augmented matrix contains the same information as the system, but in a simpler form. One of the important ideas in this chapter is to think of a matrix as a single object, so we denote a matrix by a single letter, such as A, B, C, and so on. We can add, subtract, and multiply matrices, just as we do ordinary numbers. We will pay special attention to matrix multiplication—it's defined in a way (which may seem complicated at first) that makes it possible to write a linear system as a single *matrix equation*

$$AX = B$$

where X is the unknown matrix. As you will see, solving this matrix equation for the matrix X is analogous to solving the algebraic equation $ax = b$ for the number x.

In this chapter we consider many uses of matrices, including applications to population growth (*Will the Species Survive?* page 688) and to computer graphics (*Computer Graphics I*, page 700).

9.1 Systems of Equations

In this section we study how to solve systems of two equations in two unknowns. We learn three different methods of solving such systems: by substitution, by elimination, and graphically.

Systems of Equations and Their Solutions

A **system of equations** is a set of equations that involve the same variables. A **solution** of a system is an assignment of values for the variables that makes *each* equation in the system true. To **solve** a system means to find all solutions of the system.

Here is an example of a system of two equations in two variables:

$$\begin{cases} 2x - y = 5 & \text{Equation 1} \\ x + 4y = 7 & \text{Equation 2} \end{cases}$$

We can check that $x = 3$ and $y = 1$ is a solution of this system.

Equation 1	**Equation 2**
$2x - y = 5$	$x + 4y = 7$
$2(3) - 1 = 5$ ✓	$3 + 4(1) = 7$ ✓

The solution can also be written as the ordered pair $(3, 1)$.

Note that the graphs of Equations 1 and 2 are lines (see Figure 1). Since the solution $(3, 1)$ satisfies each equation, the point $(3, 1)$ lies on each line. So it is the point of intersection of the two lines.

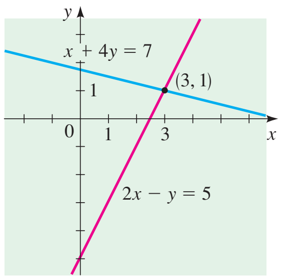


Figure 1

SUGGESTED TIME AND EMPHASIS

$\frac{1}{2}$–1 class.

Essential material. Can be combined with Section 9.2.

DRILL QUESTION

Solve this system of equations.

$$\begin{cases} x + y = 2 \\ 4x - 2y = -1 \end{cases}$$

Answer

$x = \frac{1}{2}, y = \frac{3}{2}$

POINT TO STRESS

Solving systems of equations with two variables, using the methods of substitution, elimination, and graphing.

Substitution Method

In the **substitution method** we start with one equation in the system and solve for one variable in terms of the other variable. The following box describes the procedure.

Substitution Method

1. **Solve for One Variable.** Choose one equation and solve for one variable in terms of the other variable.
2. **Substitute.** Substitute the expression you found in Step 1 into the other equation to get an equation in one variable, then solve for that variable.
3. **Back-Substitute.** Substitute the value you found in Step 2 back into the expression found in Step 1 to solve for the remaining variable.

Example 1 Substitution Method

Find all solutions of the system.

$$\begin{cases} 2x + y = 1 & \text{Equation 1} \\ 3x + 4y = 14 & \text{Equation 2} \end{cases}$$

Solution We solve for y in the first equation.

Solve for one variable

$$y = 1 - 2x \qquad \text{Solve for } y \text{ in Equation 1}$$

Now we substitute for y in the second equation and solve for x:

Substitute

$$\begin{aligned} 3x + 4(1 - 2x) &= 14 && \text{Substitute } y = 1 - 2x \text{ into Equation 2} \\ 3x + 4 - 8x &= 14 && \text{Expand} \\ -5x + 4 &= 14 && \text{Simplify} \\ -5x &= 10 && \text{Subtract 4} \\ x &= -2 && \text{Solve for } x \end{aligned}$$

Next we back-substitute $x = -2$ into the equation $y = 1 - 2x$:

Back-substitute

$$y = 1 - 2(-2) = 5 \qquad \text{Back-substitute}$$

Thus, $x = -2$ and $y = 5$, so the solution is the ordered pair $(-2, 5)$. Figure 2 shows that the graphs of the two equations intersect at the point $(-2, 5)$.

Check Your Answer

$x = -2, y = 5$:

$$\begin{cases} 2(-2) + 5 = 1 \\ 3(-2) + 4(5) = 14 \end{cases} \checkmark$$

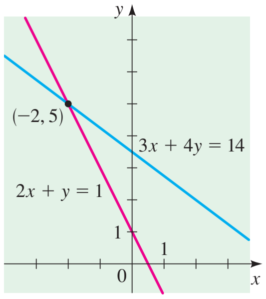


Figure 2

ALTERNATE EXAMPLE 1

Find all solutions of the system.

$$\begin{cases} 5x + y = 2 \\ 3x - 2y = 22 \end{cases}$$

ANSWER

$x = 2, y = -8$

IN-CLASS MATERIALS

This may be a good time to point out that a system of equations can have zero, one, more than one, or infinitely many solutions. The students will be able to solve systems that have one or more than one solution:

$$\begin{cases} 2x + y = 3 \\ -2x + 3y = 3 \end{cases} \qquad \begin{cases} x^2 = y \\ -x^2 + 8 = y \end{cases} \qquad \begin{cases} y = x^3 - x^2 - 4x \\ y = 4 \end{cases}$$

When demonstrating these systems, it is important to show the graphical solution as well as an analytic solution. Then have the students show, graphically, a system of equations with no solutions and then one with infinitely many solutions.

ALTERNATE EXAMPLE 2

Find all solutions of the system.

$$\begin{cases} x^2 + y^2 = 625 \\ 3x - y = -25 \end{cases}$$

ANSWER

(0, 25), (−15, −20)

Example 2 Substitution Method

Find all solutions of the system.

$$\begin{cases} x^2 + y^2 = 100 & \text{Equation 1} \\ 3x - y = 10 & \text{Equation 2} \end{cases}$$

Solution We start by solving for y in the second equation.

Solve for one variable

$$y = 3x - 10 \qquad \text{Solve for } y \text{ in Equation 2}$$

Next we substitute for y in the first equation and solve for x:

Substitute

$$\begin{aligned} x^2 + (3x - 10)^2 &= 100 && \text{Substitute } y = 3x - 10 \text{ into Equation 1} \\ x^2 + (9x^2 - 60x + 100) &= 100 && \text{Expand} \\ 10x^2 - 60x &= 0 && \text{Simplify} \\ 10x(x - 6) &= 0 && \text{Factor} \\ x = 0 \quad \text{or} \quad x &= 6 && \text{Solve for } x \end{aligned}$$

Now we back-substitute these values of x into the equation $y = 3x - 10$.

Back-substitute

$$\begin{aligned} &\text{For } x = 0: \quad y = 3(0) - 10 = -10 && \text{Back-substitute} \\ &\text{For } x = 6: \quad y = 3(6) - 10 = 8 && \text{Back-substitute} \end{aligned}$$

So we have two solutions: $(0, -10)$ and $(6, 8)$.

The graph of the first equation is a circle, and the graph of the second equation is a line; Figure 3 shows that the graphs intersect at the two points $(0, -10)$ and $(6, 8)$.

Check Your Answers

$x = 0, y = -10$:

$$\begin{cases} (0)^2 + (-10)^2 = 100 \\ 3(0) - (-10) = 10 \end{cases} \checkmark$$

$x = 6, y = 8$:

$$\begin{cases} (6)^2 + (8)^2 = 36 + 64 = 100 \\ 3(6) - (8) = 18 - 8 = 10 \end{cases} \checkmark$$

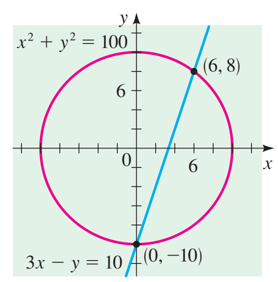


Figure 3

■

Elimination Method

To solve a system using the **elimination method**, we try to combine the equations using sums or differences so as to eliminate one of the variables.

SAMPLE QUESTION

Text Question

The textbook says that given the system

$$\begin{cases} 3x + 2y = 14 \\ x - 2y = 2 \end{cases}$$

we can add the equations to eliminate y, obtaining $4x = 16$, and thus determining $x = 4$ and $y = 1$. Could this system be solved using the substitution method? If not, why not? If so, would the answers be the same?

Answer

It could, and the answers would be the same.

Elimination Method

1. **Adjust the Coefficients.** Multiply one or more of the equations by appropriate numbers so that the coefficient of one variable in one equation is the negative of its coefficient in the other equation.
2. **Add the Equations.** Add the two equations to eliminate one variable, then solve for the remaining variable.
3. **Back-Substitute.** Substitute the value you found in Step 2 back into one of the original equations, and solve for the remaining variable.

Example 3 Elimination Method

Find all solutions of the system.

$$\begin{cases} 3x + 2y = 14 & \text{Equation 1} \\ x - 2y = 2 & \text{Equation 2} \end{cases}$$

Solution Since the coefficients of the y-terms are negatives of each other, we can add the equations to eliminate y.

$$\begin{array}{rll} \begin{cases} 3x + 2y = 14 \\ x - 2y = 2 \end{cases} & & \text{System} \\ 4x \qquad = 16 & & \text{Add} \\ x = 4 & & \text{Solve for } x \end{array}$$

Now we back-substitute $x = 4$ into one of the original equations and solve for y. Let's choose the second equation because it looks simpler.

$$\begin{array}{rl} x - 2y = 2 & \text{Equation 2} \\ 4 - 2y = 2 & \text{Back-substitute } x = 4 \text{ into Equation 2} \\ -2y = -2 & \text{Subtract 4} \\ y = 1 & \text{Solve for } y \end{array}$$

The solution is $(4, 1)$. Figure 4 shows that the graphs of the equations in the system intersect at the point $(4, 1)$. ■

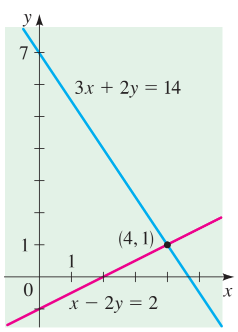


Figure 4

Example 4 Elimination Method

Find all solutions of the system.

$$\begin{cases} 3x^2 + 2y = 26 & \text{Equation 1} \\ 5x^2 + 7y = 3 & \text{Equation 2} \end{cases}$$

Solution We choose to eliminate the x-term, so we multiply the first equation by 5 and the second equation by -3. Then we add the two equations and solve for y.

$$\begin{array}{rl} \begin{cases} 15x^2 + 10y = 130 \\ -15x^2 - 21y = -9 \end{cases} & \begin{array}{l} 5 \times \text{Equation 1} \\ (-3) \times \text{Equation 2} \end{array} \\ -11y = 121 & \text{Add} \\ y = -11 & \text{Solve for } y \end{array}$$

ALTERNATE EXAMPLE 3

Find all solutions of the system.

$$\begin{cases} 5x + 4y = 54 \\ x - 4y = -18 \end{cases}$$

ANSWER

(6, 6)

ALTERNATE EXAMPLE 4

Find all solutions of the system.

$$\begin{cases} 3x^2 + 5y = -3 \\ 4x^2 + 3y = 18 \end{cases}$$

ANSWER

$(3, -6), (-3, -6)$

IN-CLASS MATERIALS

Many students will want to just learn one method of solving systems of equations, and stick with it. It is important that they are familiar with all three. One reason is that some systems are easier to solve with one method than another. Another reason is that if they take mathematics classes in the future, their teachers will use their own favorite method in class.

Either have the students try to come up with three sample systems, each lending itself to a different method, or use these:

$$\begin{array}{rl} y = 3x + 2 \\ \sqrt{y^2 - 8x^2 - 12x - 4} = 4 & \text{Substitution easiest} \\ 3x + \dfrac{5}{11}y = 2 \\ -3x + \dfrac{6}{11}y = 3 & \text{Elimination easiest} \\ y = x^3 - x & \text{Exact solution impossible—a} \\ y = e^x & \text{graph can give an approximation} \end{array}$$

EXAMPLE
A non-linear system with an integer solution:

$$\begin{cases} x^3 - y^2 = 23 \\ 2x - 3y = 0 \end{cases}$$

ANSWER
$x = 3, y = 2$

EXAMPLE
A non-linear system with a transcendental solution:

$$\begin{cases} e^x + 2y = 5 \\ -e^x + 5y = 2 \end{cases}$$

ANSWER
$x = \ln 3, y = 1$

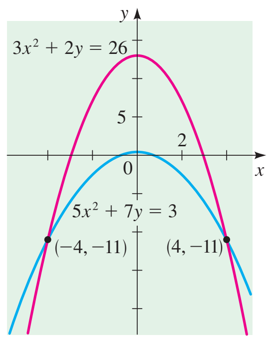


Figure 5
The graphs of quadratic functions $y = ax^2 + bx + c$ are called *parabolas*; see Section 2.5.

Now we back-substitute $y = -11$ into one of the original equations, say $3x^2 + 2y = 26$, and solve for x:

$$\begin{aligned} 3x^2 + 2(-11) &= 26 && \text{Back-substitute } y = -11 \text{ into Equation 1} \\ 3x^2 &= 48 && \text{Add 22} \\ x^2 &= 16 && \text{Divide by 3} \\ x = -4 \quad \text{or} \quad x &= 4 && \text{Solve for } x \end{aligned}$$

So we have two solutions: $(-4, -11)$ and $(4, -11)$.

The graphs of both equations are parabolas; Figure 5 shows that the graphs intersect at the two points $(-4, -11)$ and $(4, -11)$. ■

Check Your Answers

$x = -4, y = -11$:

$$\begin{cases} 3(-4)^2 + 2(-11) = 26 \\ 5(-4)^2 + 7(-11) = 3 \end{cases} \checkmark$$

$x = 4, y = -11$:

$$\begin{cases} 3(4)^2 + 2(-11) = 26 \\ 5(4)^2 + 7(-11) = 3 \end{cases} \checkmark$$

Graphical Method

In the **graphical method** we use a graphing device to solve the system of equations. Note that with many graphing devices, any equation must first be expressed in terms of one or more functions of the form $y = f(x)$ before we can use the calculator to graph it. Not all equations can be readily expressed in this way, so not all systems can be solved by this method.

Graphical Method

1. **Graph Each Equation.** Express each equation in a form suitable for the graphing calculator by solving for y as a function of x. Graph the equations on the same screen.
2. **Find the Intersection Points.** The solutions are the x- and y-coordinates of the points of intersection.

It may be more convenient to solve for x in terms of y in the equations. In that case, in Step 1 graph x as a function of y instead.

ALTERNATE EXAMPLE 5
Find all solutions of the system, correct to one decimal place.

$$\begin{cases} y = x^3 - x \\ x^2 + y^2 = 2 \end{cases}$$

ANSWER
$x \approx 1.2, y \approx 0.7$ and $x \approx -1.2, y \approx -0.7$

Example 5 Graphical Method

Find all solutions of the system.

$$\begin{cases} x^2 - y = 2 \\ 2x - y = -1 \end{cases}$$

Solution Solving for y in terms of x, we get the equivalent system

$$\begin{cases} y = x^2 - 2 \\ y = 2x + 1 \end{cases}$$

EXAMPLE
A straightforward (if contrived) word problem:

Retaining bricks for gardens are usually either 6 inches or 12 inches long. A 6-inch brick costs \$0.80 and a 12-inch brick costs \$1.20. Assume we have a garden with an 8-foot perimeter and we spend \$10.80. How many bricks of each type did we buy?

ANSWER
6 small bricks and 5 large ones

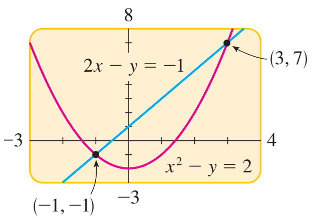


Figure 6

Figure 6 shows that the graphs of these equations intersect at two points. Zooming in, we see that the solutions are

$$(-1, -1) \qquad \text{and} \qquad (3, 7)$$

■

Check Your Answers

$x = -1, y = -1$:

$$\begin{cases} (-1)^2 - (-1) = 2 \\ 2(-1) - (-1) = -1 \end{cases} \quad ✓$$

$x = 3, y = 7$:

$$\begin{cases} 3^2 - 7 = 2 \\ 2(3) - 7 = -1 \end{cases} \quad ✓$$

Example 6 Solving a System of Equations Graphically

Find all solutions of the system, correct to one decimal place.

$$\begin{cases} x^2 + y^2 = 12 & \text{Equation 1} \\ y = 2x^2 - 5x & \text{Equation 2} \end{cases}$$

Solution The graph of the first equation is a circle and the second a parabola. To graph the circle on a graphing calculator, we must first solve for y in terms of x (see Section 2.3).

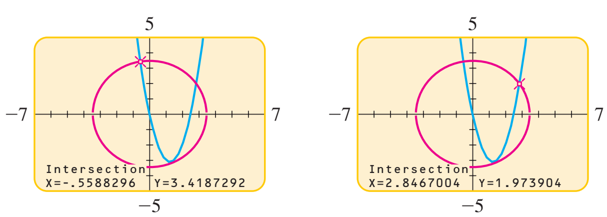
$$\begin{aligned} x^2 + y^2 &= 12 \\ y^2 &= 12 - x^2 && \text{Isolate } y^2 \text{ on LHS} \\ y &= \pm\sqrt{12 - x^2} && \text{Take square roots} \end{aligned}$$

To graph the circle, we must graph both functions:

$$y = \sqrt{12 - x^2} \qquad \text{and} \qquad y = -\sqrt{12 - x^2}$$

In Figure 7 the graph of the circle is shown in red and the parabola in blue. The graphs intersect in quadrants I and II. Zooming in, or using the `Intersect` command, we see that the intersection points are $(-0.559, 3.419)$ and $(2.847, 1.974)$. There also appears to be an intersection point in quadrant IV. However, when we zoom in, we see that the curves come close to each other but don't intersect (see Figure 8). Thus, the system has two solutions; correct to the nearest tenth, they are

$$(-0.6, 3.4) \qquad \text{and} \qquad (2.8, 2.0)$$



(a) (b)

Figure 7
$x^2 + y^2 = 12,\ y = 2x^2 - 5x$

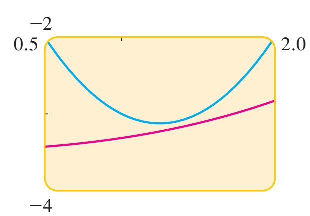


Figure 8
Zooming in

■

ALTERNATE EXAMPLE 6
Find all solutions of the system.

$$\begin{cases} x^2 - y = 1 \\ 3x - y = -3 \end{cases}$$

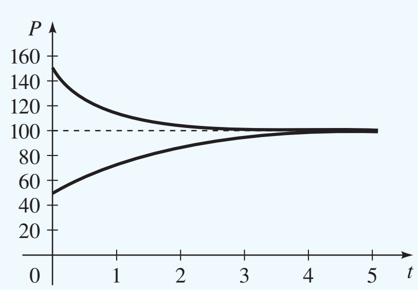
ANSWER
(4, 15), (−1, 0)

IN-CLASS MATERIALS

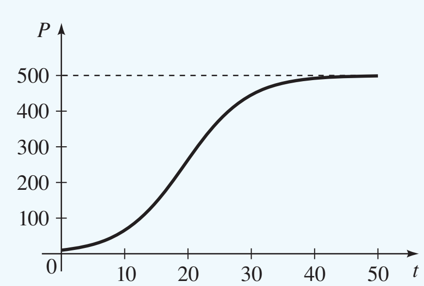
Populations with an initial population P_0 grow according to the logistic growth model

$$P = \frac{K}{1 + Ae^{-ct}}$$

where A, K and c are constants. $A = \dfrac{K - P_0}{P_0}$ and K is the "carrying capacity" of the environment. Draw a few sample graphs for the students:



$K = 100, c = 1, P_0 = 50, 100, 150$



$K = 500, c = 0.2, P_0 = 10$

Assume that we want to find out the carrying capacity of an environment ("How many trout will Big Island Lake support?"). We can find (or estimate) P_0, and then measure the population at two times (say, at $t = 1$ and $t = 6$ months). Now we have a system of two equations with two unknowns, and we are able to find K and c.

9.1 Exercises

1–8 ■ Use the substitution method to find all solutions of the system of equations.

1. $\begin{cases} x - y = 2 \\ 2x + 3y = 9 \end{cases}$ **2.** $\begin{cases} 2x + y = 7 \\ x + 2y = 2 \end{cases}$

3. $\begin{cases} y = x^2 \\ y = x + 12 \end{cases}$ **4.** $\begin{cases} x^2 + y^2 = 25 \\ y = 2x \end{cases}$

5. $\begin{cases} x^2 + y^2 = 8 \\ x + y = 0 \end{cases}$ **6.** $\begin{cases} x^2 + y = 9 \\ x - y + 3 = 0 \end{cases}$

7. $\begin{cases} x + y^2 = 0 \\ 2x + 5y^2 = 75 \end{cases}$ **8.** $\begin{cases} x^2 - y = 1 \\ 2x^2 + 3y = 17 \end{cases}$

9–16 ■ Use the elimination method to find all solutions of the system of equations.

9. $\begin{cases} x + 2y = 5 \\ 2x + 3y = 8 \end{cases}$ **10.** $\begin{cases} 4x - 3y = 11 \\ 8x + 4y = 12 \end{cases}$

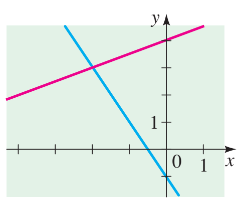
11. $\begin{cases} x^2 - 2y = 1 \\ x^2 + 5y = 29 \end{cases}$ **12.** $\begin{cases} 3x^2 + 4y = 17 \\ 2x^2 + 5y = 2 \end{cases}$

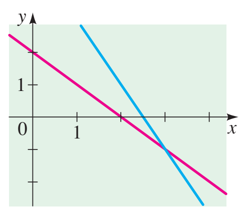
13. $\begin{cases} 3x^2 - y^2 = 11 \\ x^2 + 4y^2 = 8 \end{cases}$ **14.** $\begin{cases} 2x^2 + 4y = 13 \\ x^2 - y^2 = \frac{7}{2} \end{cases}$

15. $\begin{cases} x - y^2 + 3 = 0 \\ 2x^2 + y^2 - 4 = 0 \end{cases}$ **16.** $\begin{cases} x^2 - y^2 = 1 \\ 2x^2 - y^2 = x + 3 \end{cases}$

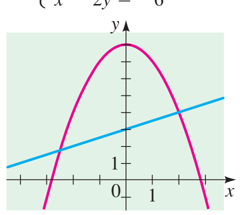
17–22 ■ Two equations and their graphs are given. Find the intersection point(s) of the graphs by solving the system.

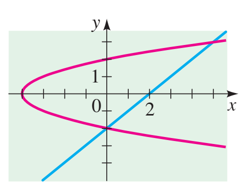
17. $\begin{cases} 2x + y = -1 \\ x - 2y = -8 \end{cases}$ **18.** $\begin{cases} x + y = 2 \\ 2x + y = 5 \end{cases}$



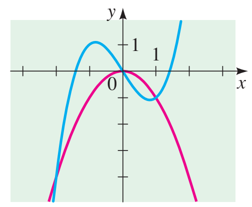


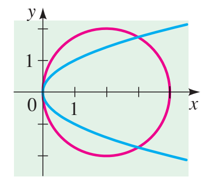
19. $\begin{cases} x^2 + y = 8 \\ x - 2y = -6 \end{cases}$ **20.** $\begin{cases} x - y^2 = -4 \\ x - y = 2 \end{cases}$





21. $\begin{cases} x^2 + y = 0 \\ x^3 - 2x - y = 0 \end{cases}$ **22.** $\begin{cases} x^2 + y^2 = 4x \\ x = y^2 \end{cases}$





23–36 ■ Find all solutions of the system of equations.

23. $\begin{cases} y + x^2 = 4x \\ y + 4x = 16 \end{cases}$ **24.** $\begin{cases} x - y^2 = 0 \\ y - x^2 = 0 \end{cases}$

25. $\begin{cases} x - 2y = 2 \\ y^2 - x^2 = 2x + 4 \end{cases}$ **26.** $\begin{cases} y = 4 - x^2 \\ y = x^2 - 4 \end{cases}$

27. $\begin{cases} x - y = 4 \\ xy = 12 \end{cases}$ **28.** $\begin{cases} xy = 24 \\ 2x^2 - y^2 + 4 = 0 \end{cases}$

29. $\begin{cases} x^2 y = 16 \\ x^2 + 4y + 16 = 0 \end{cases}$ **30.** $\begin{cases} x + \sqrt{y} = 0 \\ y^2 - 4x^2 = 12 \end{cases}$

31. $\begin{cases} x^2 + y^2 = 9 \\ x^2 - y^2 = 1 \end{cases}$ **32.** $\begin{cases} x^2 + 2y^2 = 2 \\ 2x^2 - 3y = 15 \end{cases}$

33. $\begin{cases} 2x^2 - 8y^3 = 19 \\ 4x^2 + 16y^3 = 34 \end{cases}$ **34.** $\begin{cases} x^4 - y^3 = 17 \\ 3x^4 + 5y^3 = 53 \end{cases}$

35. $\begin{cases} \dfrac{2}{x} - \dfrac{3}{y} = 1 \\ -\dfrac{4}{x} + \dfrac{7}{y} = 1 \end{cases}$ **36.** $\begin{cases} \dfrac{4}{x^2} + \dfrac{6}{y^4} = \dfrac{7}{2} \\ \dfrac{1}{x^2} - \dfrac{2}{y^4} = 0 \end{cases}$

37–46 ■ Use the graphical method to find all solutions of the system of equations, correct to two decimal places.

37. $\begin{cases} y = 2x + 6 \\ y = -x + 5 \end{cases}$ **38.** $\begin{cases} y = -2x + 12 \\ y = x + 3 \end{cases}$

39. $\begin{cases} y = x^2 + 8x \\ y = 2x + 16 \end{cases}$ **40.** $\begin{cases} y = x^2 - 4x \\ 2x - y = 2 \end{cases}$

41. $\begin{cases} x^2 + y^2 = 25 \\ x + 3y = 2 \end{cases}$ **42.** $\begin{cases} x^2 + y^2 = 17 \\ x^2 - 2x + y^2 = 13 \end{cases}$

43. $\begin{cases} \dfrac{x^2}{9} + \dfrac{y^2}{18} = 1 \\ y = -x^2 + 6x - 2 \end{cases}$ **44.** $\begin{cases} x^2 - y^2 = 3 \\ y = x^2 - 2x - 8 \end{cases}$

45. $\begin{cases} x^4 + 16y^4 = 32 \\ x^2 + 2x + y = 0 \end{cases}$ **46.** $\begin{cases} y = e^x + e^{-x} \\ y = 5 - x^2 \end{cases}$

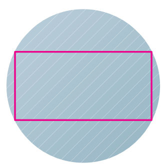
Applications

47. Dimensions of a Rectangle A rectangle has an area of 180 cm^2 and a perimeter of 54 cm. What are its dimensions?

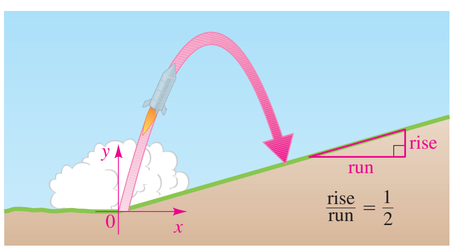
48. Legs of a Right Triangle A right triangle has an area of 84 ft^2 and a hypotenuse 25 ft long. What are the lengths of its other two sides?

49. Dimensions of a Rectangle The perimeter of a rectangle is 70 and its diagonal is 25. Find its length and width.

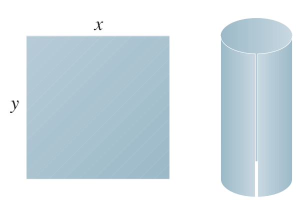
50. Dimensions of a Rectangle A circular piece of sheet metal has a diameter of 20 in. The edges are to be cut off to form a rectangle of area 160 in^2 (see the figure). What are the dimensions of the rectangle?



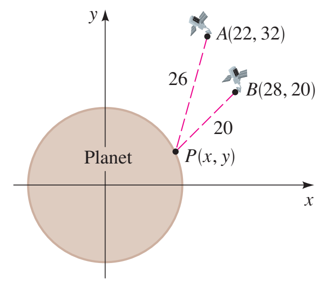
51. Flight of a Rocket A hill is inclined so that its "slope" is $\frac{1}{2}$, as shown in the figure. We introduce a coordinate system with the origin at the base of the hill and with the scales on the axes measured in meters. A rocket is fired from the base of the hill in such a way that its trajectory is the parabola $y = -x^2 + 401x$. At what point does the rocket strike the hillside? How far is this point from the base of the hill (to the nearest cm)?



52. Making a Stovepipe A rectangular piece of sheet metal with an area of 1200 in^2 is to be bent into a cylindrical length of stovepipe having a volume of 600 in^3. What are the dimensions of the sheet metal?



53. Global Positioning System (GPS) The Global Positioning System determines the location of an object from its distances to satellites in orbit around the earth. In the simplified, two-dimensional situation shown in the figure, determine the coordinates of P from the fact that P is 26 units from satellite A and 20 units from satellite B.



Discovery • Discussion

54. Intersection of a Parabola and a Line On a sheet of graph paper, or using a graphing calculator, draw the parabola $y = x^2$. Then draw the graphs of the linear equation $y = x + k$ on the same coordinate plane for various values of k. Try to choose values of k so that the line and the parabola intersect at two points for some of your k's, and not for others. For what value of k is there exactly one intersection point? Use the results of your experiment to make a conjecture about the values of k for which the following system has two solutions, one solution, and no solution. Prove your conjecture.

$$\begin{cases} y = x^2 \\ y = x + k \end{cases}$$

55. Some Trickier Systems Follow the hints and solve the systems.

(a) $\begin{cases} \log x + \log y = \frac{3}{2} \\ 2\log x - \log y = 0 \end{cases}$ [*Hint:* Add the equations.]

(b) $\begin{cases} 2^x + 2^y = 10 \\ 4^x + 4^y = 68 \end{cases}$ [*Hint:* Note that $4^x = 2^{2x} = (2^x)^2$.]

(c) $\begin{cases} x - y = 3 \\ x^3 - y^3 = 387 \end{cases}$ [*Hint:* Factor the left side of the second equation.]

(d) $\begin{cases} x^2 + xy = 1 \\ xy + y^2 = 3 \end{cases}$ [*Hint:* Add the equations and factor the result.]

SUGGESTED TIME AND EMPHASIS

$\frac{1}{2}$–1 class.

Essential material. Can be combined with Section 9.1.

POINTS TO STRESS

1. Solving systems of linear equations.
2. Inconsistent and dependent systems.
3. Using the modeling process to solve applied problems.

9.2 Systems of Linear Equations in Two Variables

Recall that an equation of the form $Ax + By = C$ is called linear because its graph is a line (see Section 1.10). In this section we study systems of two linear equations in two variables.

Systems of Linear Equations in Two Variables

A **system of two linear equations in two variables** has the form

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$$

We can use either the substitution method or the elimination method to solve such systems algebraically. But since the elimination method is usually easier for linear systems, we use elimination rather than substitution in our examples.

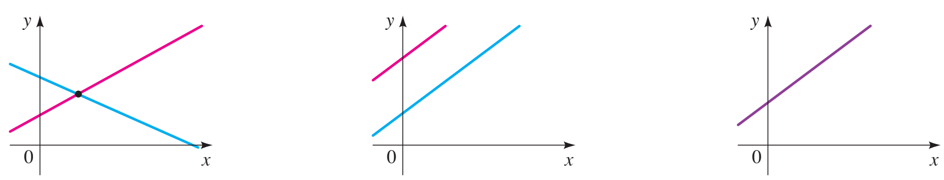
The graph of a linear system in two variables is a pair of lines, so to solve the system graphically, we must find the intersection point(s) of the lines. Two lines may intersect in a single point, they may be parallel, or they may coincide, as shown in Figure 1. So there are three possible outcomes when solving such a system.

Number of Solutions of a Linear System in Two Variables

For a system of linear equations in two variables, exactly one of the following is true. (See Figure 1.)

1. The system has exactly one solution.
2. The system has no solution.
3. The system has infinitely many solutions.

A system that has no solution is said to be **inconsistent**. A system with infinitely many solutions is called **dependent**.



(a) Linear system with one solution. Lines intersect at a single point.

(b) Linear system with no solution. Lines are parallel—they do not intersect.

(c) Linear system with infinitely many solutions. Lines coincide—equations are for the same line.

Figure 1

Example 1 A Linear System with One Solution

Solve the system and graph the lines.

$$\begin{cases} 3x - y = 0 & \text{Equation 1} \\ 5x + 2y = 22 & \text{Equation 2} \end{cases}$$

Solution We eliminate y from the equations and solve for x.

$$\begin{array}{rll} \begin{cases} 6x - 2y = 0 \\ 5x + 2y = 22 \end{cases} & & 2 \times \text{Equation 1} \\ 11x \qquad = 22 & & \text{Add} \\ x = 2 & & \text{Solve for } x \end{array}$$

Now we back-substitute into the first equation and solve for y:

$$\begin{array}{rl} 6(2) - 2y = 0 & \text{Back-substitute } x = 2 \\ -2y = -12 & \text{Subtract } 6 \times 2 = 12 \\ y = 6 & \text{Solve for } y \end{array}$$

The solution of the system is the ordered pair $(2, 6)$, that is,

$$x = 2, \qquad y = 6$$

The graph in Figure 2 shows that the lines in the system intersect at the point $(2, 6)$. ■

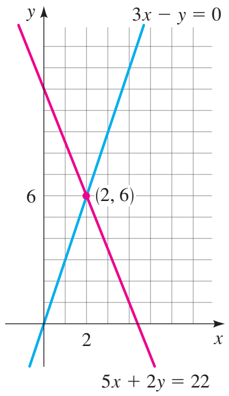


Figure 2

Check Your Answer

$x = 2, y = 6$:
$$\begin{cases} 3(2) - (6) = 0 \\ 5(2) + 2(6) = 22 \end{cases} \checkmark$$

Example 2 A Linear System with No Solution

Solve the system.

$$\begin{cases} 8x - 2y = 5 & \text{Equation 1} \\ -12x + 3y = 7 & \text{Equation 2} \end{cases}$$

Solution This time we try to find a suitable combination of the two equations to eliminate the variable y. Multiplying the first equation by 3 and the second by 2 gives

$$\begin{array}{rl} \begin{cases} 24x - 6y = 15 \\ -24x + 6y = 14 \end{cases} & \begin{array}{l} 3 \times \text{Equation 1} \\ 2 \times \text{Equation 2} \end{array} \\ 0 = 29 & \text{Add} \end{array}$$

Adding the two equations eliminates *both x and y* in this case, and we end up with $0 = 29$, which is obviously false. No matter what values we assign to x and y, we cannot make this statement true, so the system has *no solution*. Figure 3 shows that the lines in the system are parallel and do not intersect. The system is inconsistent. ■

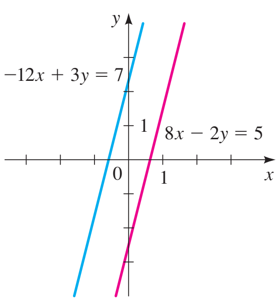


Figure 3

Example 3 A Linear System with Infinitely Many Solutions

Solve the system.

$$\begin{cases} 3x - 6y = 12 & \text{Equation 1} \\ 4x - 8y = 16 & \text{Equation 2} \end{cases}$$

ALTERNATE EXAMPLE 1

Solve the system.

$$\begin{cases} 4x - y = 0 \\ 5x + 2y = 52 \end{cases}$$

If the system is inconsistent, indicate this.

ANSWER

(4, 16)

SAMPLE QUESTION
Text Question

Why does the dependent system $x + y = 2$, $2x + 2y = 4$ have infinitely many solutions?

Answer

For any given x, there is a y that satisfies both equations.

ALTERNATE EXAMPLE 2

Solve the system.

$$\begin{cases} 12x - 3y = 7 \\ -20x + 5y = 4 \end{cases}$$

If the system is inconsistent, indicate this.

ANSWER

Inconsistent

ALTERNATE EXAMPLE 3

Solve the system.

$$\begin{cases} 2x - 6y = 24 \\ 3x - 9y = 36 \end{cases}$$

If the system is inconsistent, indicate this.

ANSWER

$$\left(x, \frac{1}{3}x - 4\right)$$

DRILL QUESTION

A woman rows a boat upstream from one point on a river to another point 4 mi away in 3 hours. The return trip, traveling with the current, takes 2 hours. How fast does she row relative to the water, and at what speed is the current flowing?

Answer

She rows at $\frac{5}{3}$ mi/h and the current flows at $\frac{1}{3}$ mi/h.

IN-CLASS MATERIALS

Notice that some systems that are not technically linear can be solved by similar means. For example, given

$$\begin{aligned} \sqrt{x} + 2^y &= 10 \\ 5\sqrt{x} - 3(2^y) &= -14 \end{aligned}$$

one can use elimination to obtain $\sqrt{x} = 2$, $2^y = 8 \Rightarrow x = 4$, $y = 3$.

EXAMPLE

A system of linear equations:

$$\begin{cases} 2x - 3y = 10 \\ x - 5y = 12 \end{cases}$$

ANSWER

$x = 2$, $y = -2$

Solution We multiply the first equation by 4 and the second by 3 to prepare for subtracting the equations to eliminate x. The new equations are

$$\begin{cases} 12x - 24y = 48 & 4 \times \text{Equation 1} \\ 12x - 24y = 48 & 3 \times \text{Equation 2} \end{cases}$$

We see that the two equations in the original system are simply different ways of expressing the equation of one single line. The coordinates of any point on this line give a solution of the system. Writing the equation in slope-intercept form, we have $y = \frac{1}{2}x - 2$. So if we let t represent any real number, we can write the solution as

$$x = t$$

$$y = \tfrac{1}{2}t - 2$$

We can also write the solution in ordered-pair form as

$$\left(t, \tfrac{1}{2}t - 2\right)$$

where t is any real number. The system has infinitely many solutions (see Figure 4). ■

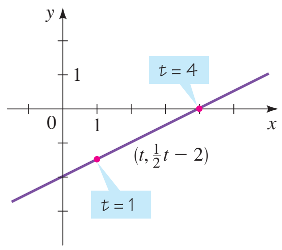


Figure 4

In Example 3, to get specific solutions we have to assign values to t. For instance, if $t = 1$, we get the solution $\left(1, -\frac{3}{2}\right)$. If $t = 4$, we get the solution $(4, 0)$. For every value of t we get a different solution. (See Figure 4.)

Modeling with Linear Systems

Frequently, when we use equations to solve problems in the sciences or in other areas, we obtain systems like the ones we've been considering. When modeling with systems of equations, we use the following guidelines, similar to those in Section 1.6.

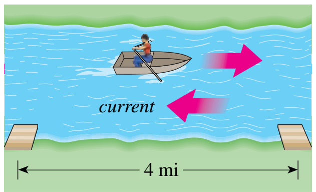
Guidelines for Modeling with Systems of Equations

1. **Identify the Variables.** Identify the quantities the problem asks you to find. These are usually determined by a careful reading of the question posed at the end of the problem. Introduce notation for the variables (call them x and y or some other letters).
2. **Express All Unknown Quantities in Terms of the Variables.** Read the problem again and express all the quantities mentioned in the problem in terms of the variables you defined in Step 1.
3. **Set Up a System of Equations.** Find the crucial facts in the problem that give the relationships between the expressions you found in Step 2. Set up a system of equations (or a model) that expresses these relationships.
4. **Solve the System and Interpret the Results.** Solve the system you found in Step 3, check your solutions, and state your final answer as a sentence that answers the question posed in the problem.

The next two examples illustrate how to model with systems of equations.

IN-CLASS MATERIALS

The text has shown examples of systems of linear equations with zero, one, and infinitely many solutions. Ask the students to try to come up with an example of a system with a different outcome, or to explain why it is not possible. Discussing graphical solutions to these systems will not only make the answer apparent, but it will also reinforce the nomenclature "linear equation."



Example 4 A Distance-Speed-Time Problem

A woman rows a boat upstream from one point on a river to another point 4 mi away in $1\frac{1}{2}$ hours. The return trip, traveling with the current, takes only 45 min. How fast does she row relative to the water, and at what speed is the current flowing?

Solution We are asked to find the rowing speed and the speed of the current, so we let

Identify the variables

$$x = \text{rowing speed (mi/h)}$$

$$y = \text{current speed (mi/h)}$$

The woman's speed when she rows upstream is her rowing speed minus the speed of the current; her speed downstream is her rowing speed plus the speed of the current. Now we translate this information into the language of algebra.

Express unknown quantities in terms of the variable

In Words	In Algebra
Rowing speed	x
Current speed	y
Speed upstream	$x - y$
Speed downstream	$x + y$

The distance upstream and downstream is 4 mi, so using the fact that speed $\times$ time $=$ distance for both legs of the trip, we get

speed upstream $\times$ time upstream $=$ distance traveled

speed downstream $\times$ time downstream $=$ distance traveled

In algebraic notation this translates into the following equations.

Set up a system of equations

$$(x - y)\tfrac{3}{2} = 4 \qquad \text{Equation 1}$$

$$(x + y)\tfrac{3}{4} = 4 \qquad \text{Equation 2}$$

Solve the system

(The times have been converted to hours, since we are expressing the speeds in miles per *hour*.) We multiply the equations by 2 and 4, respectively, to clear the denominators.

$$\begin{cases} 3x - 3y = 8 & 2 \times \text{Equation 1} \\ 3x + 3y = 16 & 4 \times \text{Equation 2} \end{cases}$$

$$6x = 24 \qquad \text{Add}$$

$$x = 4 \qquad \text{Solve for } x$$

Back-substituting this value of x into the first equation (the second works just as well) and solving for y gives

$$3(4) - 3y = 8 \qquad \text{Back-substitute } x = 4$$

$$-3y = 8 - 12 \qquad \text{Subtract 12}$$

$$y = \tfrac{4}{3} \qquad \text{Solve for } y$$

The woman rows at 4 mi/h and the current flows at $1\frac{1}{3}$ mi/h. ■

Check Your Answer

Speed upstream is

$$\frac{\text{distance}}{\text{time}} = \frac{4\text{ mi}}{1\frac{1}{2}\text{ h}} = 2\tfrac{2}{3}\text{ mi/h}$$

and this should equal

rowing speed $-$ current flow

$= 4\text{ mi/h} - \frac{4}{3}\text{ mi/h} = 2\frac{2}{3}\text{ mi/h}$

Speed downstream is

$$\frac{\text{distance}}{\text{time}} = \frac{4\text{ mi}}{\frac{3}{4}\text{ h}} = 5\tfrac{1}{3}\text{ mi/h}$$

and this should equal

rowing speed $+$ current flow

$= 4\text{ mi/h} + \frac{4}{3}\text{ mi/h} = 5\frac{1}{3}\text{ mi/h}$ ✓

ALTERNATE EXAMPLE 4

A woman rows a boat upstream from one point on a river to another point 7 mi away in $1\frac{1}{4}$ hours. The return trip, traveling with the current, takes only 50 min. How fast does she row relative to the water (in mi/h), and at what speed is the current flowing (in mi/h)?

ANSWER

$7\text{ mi/h}, \frac{7}{5}\text{ mi/h}$

IN-CLASS MATERIALS

Complex arithmetic can be reviewed at this time. Ask the students if this is a system of linear equations in two variables and (if so) to solve it:

$$(2 - i)\,x + 4y = 5$$
$$(4 - 3i)\,x - 2y = 3 + i$$

ALTERNATE EXAMPLE 5
A vintner fortifies wine that contains 10% alcohol by adding a 60% alcohol solution to it. The resulting mixture has an alcoholic strength of 12% and fills 1100 one-liter bottles. How many liters (L) of the wine and of the alcohol solution does he use?

ANSWER
1056, 44

Example 5 A Mixture Problem

A vintner fortifies wine that contains 10% alcohol by adding 70% alcohol solution to it. The resulting mixture has an alcoholic strength of 16% and fills 1000 one-liter bottles. How many liters (L) of the wine and of the alcohol solution does he use?

Identify the variables

Solution Since we are asked for the amounts of wine and alcohol, we let

x = amount of wine used (L)

y = amount of alcohol solution used (L)

From the fact that the wine contains 10% alcohol and the solution 70% alcohol, we get the following.

Express all unknown quantities in terms of the variable

In Words	In Algebra
Amount of wine used (L)	x
Amount of alcohol solution used (L)	y
Amount of alcohol in wine (L)	$0.10x$
Amount of alcohol in solution (L)	$0.70y$

The volume of the mixture must be the total of the two volumes the vintner is adding together, so

$$x + y = 1000$$

Also, the amount of alcohol in the mixture must be the total of the alcohol contributed by the wine and by the alcohol solution, that is

$$0.10x + 0.70y = (0.16)1000$$

$$0.10x + 0.70y = 160 \qquad \text{Simplify}$$

$$x + 7y = 1600 \qquad \text{Multiply by 10 to clear decimals}$$

Thus, we get the system

Set up a system of equations

$$\begin{cases} x + y = 1000 & \text{Equation 1} \\ x + 7y = 1600 & \text{Equation 2} \end{cases}$$

Subtracting the first equation from the second eliminates the variable x, and we get

Solve the system

$$6y = 600 \qquad \text{Subtract Equation 1 from Equation 2}$$

$$y = 100 \qquad \text{Solve for } y$$

We now back-substitute $y = 100$ into the first equation and solve for x:

$$x + 100 = 1000 \qquad \text{Back-substitute } y = 100$$

$$x = 900 \qquad \text{Solve for } x$$

The vintner uses 900 L of wine and 100 L of the alcohol solution. ■

EXAMPLE
A "mixture problem": Peanuts cost $4.00 per pound and cashews cost $7.50 per pound. If I buy a 5-pound bag consisting of peanuts and cashews, and I pay $23.50 for the bag, how many pounds of cashews are in it?

ANSWER
1 lb cashews, 4 lb peanuts

9.2 Exercises

1–6 ■ Graph each linear system, either by hand or using a graphing device. Use the graph to determine if the system has one solution, no solution, or infinitely many solutions. If there is exactly one solution, use the graph to find it.

1. $\begin{cases} x + y = 4 \\ 2x - y = 2 \end{cases}$

2. $\begin{cases} 2x + y = 11 \\ x - 2y = 4 \end{cases}$

3. $\begin{cases} 2x - 3y = 12 \\ -x + \frac{3}{2}y = 4 \end{cases}$

4. $\begin{cases} 2x + 6y = 0 \\ -3x - 9y = 18 \end{cases}$

5. $\begin{cases} -x + \frac{1}{2}y = -5 \\ 2x - y = 10 \end{cases}$

6. $\begin{cases} 12x + 15y = -18 \\ 2x + \frac{5}{2}y = -3 \end{cases}$

7–34 ■ Solve the system, or show that it has no solution. If the system has infinitely many solutions, express them in the ordered-pair form given in Example 3.

7. $\begin{cases} x + y = 4 \\ -x + y = 0 \end{cases}$

8. $\begin{cases} x - y = 3 \\ x + 3y = 7 \end{cases}$

9. $\begin{cases} 2x - 3y = 9 \\ 4x + 3y = 9 \end{cases}$

10. $\begin{cases} 3x + 2y = 0 \\ -x - 2y = 8 \end{cases}$

11. $\begin{cases} x + 3y = 5 \\ 2x - y = 3 \end{cases}$

12. $\begin{cases} x + y = 7 \\ 2x - 3y = -1 \end{cases}$

13. $\begin{cases} -x + y = 2 \\ 4x - 3y = -3 \end{cases}$

14. $\begin{cases} 4x - 3y = 28 \\ 9x - y = -6 \end{cases}$

15. $\begin{cases} x + 2y = 7 \\ 5x - y = 2 \end{cases}$

16. $\begin{cases} -4x + 12y = 0 \\ 12x + 4y = 160 \end{cases}$

17. $\begin{cases} \frac{1}{2}x + \frac{1}{3}y = 2 \\ \frac{1}{5}x - \frac{2}{3}y = 8 \end{cases}$

18. $\begin{cases} 0.2x - 0.2y = -1.8 \\ -0.3x + 0.5y = 3.3 \end{cases}$

19. $\begin{cases} 3x - 2y = 8 \\ -6x + 4y = 16 \end{cases}$

20. $\begin{cases} 4x + 2y = 16 \\ x - 5y = 70 \end{cases}$

21. $\begin{cases} x + 4y = 8 \\ 3x + 12y = 2 \end{cases}$

22. $\begin{cases} -3x + 5y = 2 \\ 9x - 15y = 6 \end{cases}$

23. $\begin{cases} 2x - 6y = 10 \\ -3x + 9y = -15 \end{cases}$

24. $\begin{cases} 2x - 3y = -8 \\ 14x - 21y = 3 \end{cases}$

25. $\begin{cases} 6x + 4y = 12 \\ 9x + 6y = 18 \end{cases}$

26. $\begin{cases} 25x - 75y = 100 \\ -10x + 30y = -40 \end{cases}$

27. $\begin{cases} 8s - 3t = -3 \\ 5s - 2t = -1 \end{cases}$

28. $\begin{cases} u - 30v = -5 \\ -3u + 80v = 5 \end{cases}$

29. $\begin{cases} \frac{1}{2}x + \frac{3}{5}y = 3 \\ \frac{5}{3}x + 2y = 10 \end{cases}$

30. $\begin{cases} \frac{3}{2}x - \frac{1}{3}y = \frac{1}{2} \\ 2x - \frac{1}{2}y = -\frac{1}{2} \end{cases}$

31. $\begin{cases} 0.4x + 1.2y = 14 \\ 12x - 5y = 10 \end{cases}$

32. $\begin{cases} 26x - 10y = -4 \\ -0.6x + 1.2y = 3 \end{cases}$

33. $\begin{cases} \frac{1}{3}x - \frac{1}{4}y = 2 \\ -8x + 6y = 10 \end{cases}$

34. $\begin{cases} -\frac{1}{10}x + \frac{1}{2}y = 4 \\ 2x - 10y = -80 \end{cases}$

35–38 ■ Use a graphing device to graph both lines in the same viewing rectangle. (Note that you must solve for y in terms of x before graphing if you are using a graphing calculator.) Solve the system correct to two decimal places, either by zooming in and using TRACE or by using `Intersect`.

35. $\begin{cases} 0.21x + 3.17y = 9.51 \\ 2.35x - 1.17y = 5.89 \end{cases}$

36. $\begin{cases} 18.72x - 14.91y = 12.33 \\ 6.21x - 12.92y = 17.82 \end{cases}$

37. $\begin{cases} 2371x - 6552y = 13{,}591 \\ 9815x + 992y = 618{,}555 \end{cases}$

38. $\begin{cases} -435x + 912y = 0 \\ 132x + 455y = 994 \end{cases}$

39–42 ■ Find x and y in terms of a and b.

39. $\begin{cases} x + y = 0 \\ x + ay = 1 \end{cases} \quad (a \neq 1)$

40. $\begin{cases} ax + by = 0 \\ x + y = 1 \end{cases} \quad (a \neq b)$

41. $\begin{cases} ax + by = 1 \\ bx + ay = 1 \end{cases} \quad (a^2 - b^2 \neq 0)$

42. $\begin{cases} ax + by = 0 \\ a^2x + b^2y = 1 \end{cases} \quad (a \neq 0, b \neq 0, a \neq b)$

Applications

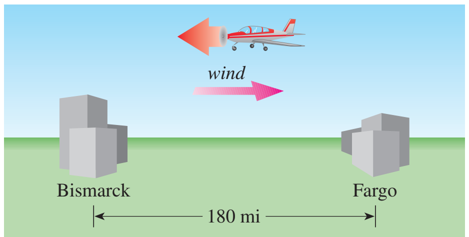
43. Number Problem Find two numbers whose sum is 34 and whose difference is 10.

44. Number Problem The sum of two numbers is twice their difference. The larger number is 6 more than twice the smaller. Find the numbers.

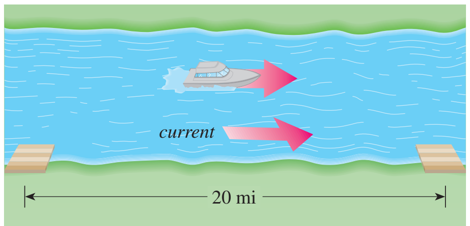
45. Value of Coins A man has 14 coins in his pocket, all of which are dimes and quarters. If the total value of his change is \$2.75, how many dimes and how many quarters does he have?

46. Admission Fees The admission fee at an amusement park is \$1.50 for children and \$4.00 for adults. On a certain day, 2200 people entered the park, and the admission fees collected totaled \$5050. How many children and how many adults were admitted?

47. Airplane Speed A man flies a small airplane from Fargo to Bismarck, North Dakota—a distance of 180 mi. Because he is flying into a head wind, the trip takes him 2 hours. On the way back, the wind is still blowing at the same speed, so the return trip takes only 1 h 12 min. What is his speed in still air, and how fast is the wind blowing?



48. Boat Speed A boat on a river travels downstream between two points, 20 mi apart, in one hour. The return trip against the current takes $2\frac{1}{2}$ hours. What is the boat's speed, and how fast does the current in the river flow?



49. Aerobic Exercise A woman keeps fit by bicycling and running every day. On Monday she spends $\frac{1}{2}$ hour at each activity, covering a total of $12\frac{1}{2}$ mi. On Tuesday, she runs for 12 min and cycles for 45 min, covering a total of 16 mi. Assuming her running and cycling speeds don't change from day to day, find these speeds.

50. Mixture Problem A biologist has two brine solutions, one containing 5% salt and another containing 20% salt. How many milliliters of each solution should he mix to obtain 1 L of a solution that contains 14% salt?

51. Nutrition A researcher performs an experiment to test a hypothesis that involves the nutrients niacin and retinol. She feeds one group of laboratory rats a daily diet of precisely 32 units of niacin and 22,000 units of retinol. She uses two types of commercial pellet foods. Food A contains 0.12 unit of niacin and 100 units of retinol per gram. Food B contains 0.20 unit of niacin and 50 units of retinol per gram. How many grams of each food does she feed this group of rats each day?

52. Coffee Mixtures A customer in a coffee shop purchases a blend of two coffees: Kenyan, costing \$3.50 a pound, and Sri Lankan, costing \$5.60 a pound. He buys 3 lb of the blend, which costs him \$11.55. How many pounds of each kind went into the mixture?

53. Mixture Problem A chemist has two large containers of sulfuric acid solution, with different concentrations of acid in each container. Blending 300 mL of the first solution and 600 mL of the second gives a mixture that is 15% acid, whereas 100 mL of the first mixed with 500 mL of the second gives a $12\frac{1}{2}\%$ acid mixture. What are the concentrations of sulfuric acid in the original containers?

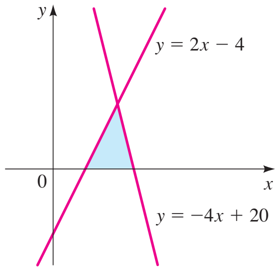
54. Investments A woman invests a total of \$20,000 in two accounts, one paying 5% and the other paying 8% simple interest per year. Her annual interest is \$1180. How much did she invest at each rate?

55. Investments A man invests his savings in two accounts, one paying 6% and the other paying 10% simple interest per year. He puts twice as much in the lower-yielding account because it is less risky. His annual interest is \$3520. How much did he invest at each rate?

56. Distance, Speed, and Time John and Mary leave their house at the same time and drive in opposite directions. John drives at 60 mi/h and travels 35 mi farther than Mary, who drives at 40 mi/h. Mary's trip takes 15 min longer than John's. For what length of time does each of them drive?

57. Number Problem The sum of the digits of a two-digit number is 7. When the digits are reversed, the number is increased by 27. Find the number.

58. Area of a Triangle Find the area of the triangle that lies in the first quadrant (with its base on the x-axis) and that is bounded by the lines $y = 2x - 4$ and $y = -4x + 20$.



Discovery • Discussion

59. The Least Squares Line The *least squares* line or *regression* line is the line that best fits a set of points in the plane. We studied this line in *Focus on Modeling* (see page 240). Using calculus, it can be shown that the line that best fits the n data points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ is the line $y = ax + b$, where the coefficients a and b satisfy the following pair of linear equations. [The notation $\sum_{k=1}^{n} x_k$ stands for the sum of all the x's. See Section 11.1 for a complete description of sigma (Σ) notation.]

$$\left(\sum_{k=1}^{n} x_k\right)a + nb = \sum_{k=1}^{n} y_k$$

$$\left(\sum_{k=1}^{n} x_k^2\right)a + \left(\sum_{k=1}^{n} x_k\right)b = \sum_{k=1}^{n} x_k y_k$$

Use these equations to find the least squares line for the following data points.

$$(1, 3), \quad (2, 5), \quad (3, 6), \quad (5, 6), \quad (7, 9)$$

Sketch the points and your line to confirm that the line fits these points well. If your calculator computes regression lines, see whether it gives you the same line as the formulas.

9.3 Systems of Linear Equations in Several Variables

A **linear equation in *n* variables** is an equation that can be put in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = c$$

where $a_1, a_2, \ldots, a_n$ and c are real numbers, and $x_1, x_2, \ldots, x_n$ are the variables. If we have only three or four variables, we generally use x, y, z, and w instead of x_1, x_2, x_3, and x_4. Such equations are called *linear* because if we have just two variables the equation is $a_1x + a_2y = c$, which is the equation of a line. Here are some examples of equations in three variables that illustrate the difference between linear and nonlinear equations.

Linear equations	Nonlinear equations	
$6x_1 - 3x_2 + \sqrt{5}x_3 = 10$	$x^2 + 3y - \sqrt{z} = 5$	Not linear because it contains the square and the square root of a variable.
$x + y + z = 2w - \frac{1}{2}$	$x_1x_2 + 6x_3 = -6$	Not linear because it contains a product of variables.

In this section we study systems of linear equations in three or more variables.

Solving a Linear System

The following are two examples of systems of linear equations in three variables. The second system is in **triangular form**; that is, the variable x doesn't appear in the second equation and the variables x and y do not appear in the third equation.

A system of linear equations

$$\begin{cases} x - 2y - z = 1 \\ -x + 3y + 3z = 4 \\ 2x - 3y + z = 10 \end{cases}$$

A system in triangular form

$$\begin{cases} x - 2y - z = 1 \\ y + 2z = 5 \\ z = 3 \end{cases}$$

It's easy to solve a system that is in triangular form using back-substitution. So, our goal in this section is to start with a system of linear equations and change it to a

SUGGESTED TIME AND EMPHASIS

1 class.
Essential material.

POINTS TO STRESS

1. Solving systems of linear equations using Gaussian elimination.
2. Using linear systems to solve applied problems.

IN-CLASS MATERIALS

Have the students go over the applied problems from their homework for the previous section. Ask them to try to come up with similar problems involving three or four variables. For example, Exercise 49 in Section 9.2 can be expanded this way:

> A woman keeps fit by bicycling, running, and swimming every day. On Monday she spends 1/2 hour at each activity, covering a total of 14.25 mi. On Tuesday she runs for 12 min., cycles for 45 min., and swims for 30 min., covering a total of 15.45 mi. On Wednesday she runs for 30 min., cycles for 10 min., and swims for 30 min., covering a total of 9.25 mi. Assuming her running and cycling speeds don't change from day to day, find these speeds.

Make sure to point out that, for the problem to be solvable, they now need three pieces of information instead of two (that is, if there is a Monday and a Tuesday in the problem, we now need a Wednesday). Also, they need to make sure that their problem has a solution. In two dimensions, it is very easy to get a consistent system, using random numbers. In three dimensions, it is a bit tougher.

Pierre de Fermat (1601–1665) was a French lawyer who became interested in mathematics at the age of 30. Because of his job as a magistrate, Fermat had little time to write complete proofs of his discoveries and often wrote them in the margin of whatever book he was reading at the time. After his death, his copy of Diophantus' *Arithmetica* (see page 20) was found to contain a particularly tantalizing comment. Where Diophantus discusses the solutions of $x^2 + y^2 = z^2$ (for example, $x = 3$, $y = 4$, $z = 5$), Fermat states in the margin that for $n \geq 3$ there are no natural number solutions to the equation $x^n + y^n = z^n$. In other words, it's impossible for a cube to equal the sum of two cubes, a fourth power to equal the sum of two fourth powers, and so on. Fermat writes "I have discovered a truly wonderful proof for this but the margin is too small to contain it." All the other margin comments in Fermat's copy of *Arithmetica* have been proved. This one, however, remained unproved, and it came to be known as "Fermat's Last Theorem."

In 1994, Andrew Wiles of Princeton University announced a proof of Fermat's Last Theorem, an astounding 350 years after it was conjectured. His proof is one of the most widely reported mathematical results in the popular press.

system in triangular form that has the same solutions as the original system. We begin by showing how to use back-substitution to solve a system that is already in triangular form.

Example 1 Solving a Triangular System Using Back-Substitution

Solve the system using back-substitution:

$$\begin{cases} x - 2y - z = 1 & \text{Equation 1} \\ y + 2z = 5 & \text{Equation 2} \\ z = 3 & \text{Equation 3} \end{cases}$$

Solution From the last equation we know that $z = 3$. We back-substitute this into the second equation and solve for y.

$$\begin{aligned} y + 2(3) &= 5 && \text{Back-substitute } z = 3 \text{ into Equation 2} \\ y &= -1 && \text{Solve for } y \end{aligned}$$

Then we back-substitute $y = -1$ and $z = 3$ into the first equation and solve for x.

$$\begin{aligned} x - 2(-1) - (3) &= 1 && \text{Back-substitute } y = -1 \text{ and } z = 3 \text{ into Equation 1} \\ x &= 2 && \text{Solve for } x \end{aligned}$$

The solution of the system is $x = 2$, $y = -1$, $z = 3$. We can also write the solution as the ordered triple $(2, -1, 3)$. ■

To change a system of linear equations to an **equivalent system** (that is, a system with the same solutions as the original system), we use the elimination method. This means we can use the following operations.

Operations That Yield an Equivalent System

1. Add a nonzero multiple of one equation to another.
2. Multiply an equation by a nonzero constant.
3. Interchange the positions of two equations.

To solve a linear system, we use these operations to change the system to an equivalent triangular system. Then we use back-substitution as in Example 1. This process is called **Gaussian elimination**.

Example 2 Solving a System of Three Equations in Three Variables

Solve the system using Gaussian elimination.

$$\begin{cases} x - 2y + 3z = 1 & \text{Equation 1} \\ x + 2y - z = 13 & \text{Equation 2} \\ 3x + 2y - 5z = 3 & \text{Equation 3} \end{cases}$$

ALTERNATE EXAMPLE 1

Solve the system using back-substitution.

$$\begin{cases} x - 5y - z = 22 \\ y + 4z = 8 \\ z = 3 \end{cases}$$

If the system is dependent or inconsistent, indicate this.

ANSWER

$(5, -4, 3)$

DRILL QUESTION

Solve this system using Gaussian elimination.

$$\begin{cases} x - 2y + z = 1 \\ -x + y - z = 4 \\ -x + 2y - 4z = 8 \end{cases}$$

Answer

$x = -6, y = -5, z = -3$

ALTERNATE EXAMPLE 2

Solve the system using Gaussian elimination.

$$\begin{cases} x - 2y + 3z = 1 \\ x + 2y - z = 13 \\ 2x + 4y - 7z = 11 \end{cases}$$

If the system is dependent or inconsistent, indicate this.

ANSWER

$(4, 6, 3)$

Solution We need to change this to a triangular system, so we begin by eliminating the x-term from the second equation.

$$\begin{array}{rl} x + 2y - z = 13 & \text{Equation 2} \\ \underline{x - 2y + 3z = 1} & \text{Equation 1} \\ 4y - 4z = 12 & \text{Equation 2} + (-1) \times \text{Equation 1} = \text{new Equation 2} \end{array}$$

This gives us a new, equivalent system that is one step closer to triangular form:

$$\begin{cases} x - 2y + 3z = 1 & \text{Equation 1} \\ 4y - 4z = 12 & \text{Equation 2} \\ 3x + 2y - 5z = 3 & \text{Equation 3} \end{cases}$$

Now we eliminate the x-term from the third equation.

$$\begin{array}{r} 3x + 2y - 5z = 3 \\ \underline{-3x + 6y - 9z = -3} \\ 8y - 14z = 0 \end{array}$$

$$\begin{cases} x - 2y + 3z = 1 \\ 4y - 4z = 12 \\ 8y - 14z = 0 & \text{Equation 3} + (-3) \times \text{Equation 1} = \text{new Equation 3} \end{cases}$$

Then we eliminate the y-term from the third equation.

$$\begin{array}{r} 8y - 14z = 0 \\ \underline{-8y + 8z = -24} \\ -6z = -24 \end{array}$$

$$\begin{cases} x - 2y + 3z = 1 \\ 4y - 4z = 12 \\ -6z = -24 & \text{Equation 3} + (-2) \times \text{Equation 1} = \text{new Equation 3} \end{cases}$$

The system is now in triangular form, but it will be easier to work with if we divide the second and third equations by the common factors of each term.

$$\begin{cases} x - 2y + 3z = 1 \\ y - z = 3 & \frac{1}{4} \times \text{Equation 2} = \text{new Equation 2} \\ z = 4 & -\frac{1}{6} \times \text{Equation 3} = \text{new Equation 3} \end{cases}$$

Now we use back-substitution to solve the system. From the third equation we get $z = 4$. We back-substitute this into the second equation and solve for y.

$$\begin{array}{rl} y - (4) = 3 & \text{Back-substitute } z = 4 \text{ into Equation 2} \\ y = 7 & \text{Solve for } y \end{array}$$

Then we back-substitute $y = 7$ and $z = 4$ into the first equation and solve for x.

$$\begin{array}{rl} x - 2(7) + 3(4) = 1 & \text{Back-substitute } y = 7 \text{ and } z = 4 \text{ into Equation 1} \\ x = 3 & \text{Solve for } x \end{array}$$

The solution of the system is $x = 3$, $y = 7$, $z = 4$, which we can write as the ordered triple $(3, 7, 4)$. ■

Check Your Answer

We must check that the answer satisfies *all three* equations, $x = 3$, $y = 7$, $z = 4$:

$$\begin{array}{r} (3) - 2(7) + 3(4) = 1 \\ (3) + 2(7) - (4) = 13 \\ 3(3) + 2(7) - 5(4) = 3 \end{array} \quad ✓$$

SAMPLE QUESTION
Text Question

Consider this system of three equations:

$$\begin{cases} x - 2y + 3z = 1 \\ x + 2y - z = 13 \\ 3x + 2y - 5z = 3 \end{cases}$$

Is the following system equivalent to the first one? Why or why not?

$$\begin{cases} x - 2y + 3z = 1 \\ 100x + 200y - 100z = 1300 \\ 3x + 2y - 5z = 3 \end{cases}$$

Answer

It is. The second equation was simply multiplied by a nonzero constant.

EXAMPLE

A consistent 3×3 system:

$$\begin{cases} 3x + 2y + z = 4 \\ x - y - z = 1 \\ 2x - 4y - z = -1 \end{cases}$$

ANSWER

$x = 1, y = 1, z = -1$

EXAMPLE

An inconsistent 3×3 system:

$$\begin{cases} x - y + z = 3 \\ 2x - y + 2z = 4 \\ 3x - 2y + 3z = 8 \end{cases}$$

EXAMPLE

A dependent 3×3 system:

$$\begin{cases} x + 5y - z = 4 \\ 2x + 3y + 4z = 0 \\ 3x + 8y + 3z = 4 \end{cases}$$

The Number of Solutions of a Linear System

Just as in the case of two variables, a system of equations in several variables may have one solution, no solution, or infinitely many solutions. The graphical interpretation of the solutions of a linear system is analogous to that for systems of equations in two variables (see the margin note).

Number of Solutions of a Linear System

For a system of linear equations, exactly one of the following is true.

1. The system has exactly one solution.
2. The system has no solution.
3. The system has infinitely many solutions.

A system with no solutions is said to be **inconsistent**, and a system with infinitely many solutions is said to be **dependent**. As we see in the next example, a linear system has no solution if we end up with a *false equation* after applying Gaussian elimination to the system.

Example 3 A System with No Solution

Solve the following system.

$$\begin{cases} x + 2y - 2z = 1 & \text{Equation 1} \\ 2x + 2y - z = 6 & \text{Equation 2} \\ 3x + 4y - 3z = 5 & \text{Equation 3} \end{cases}$$

Solution To put this in triangular form, we begin by eliminating the x-terms from the second equation and the third equation.

$$\begin{cases} x + 2y - 2z = 1 \\ -2y + 3z = 4 & \text{Equation 2} + (-2) \times \text{Equation 1} = \text{new Equation 2} \\ 3x + 4y - 3z = 5 \end{cases}$$

$$\begin{cases} x + 2y - 2z = 1 \\ -2y + 3z = 4 \\ -2y + 3z = 2 & \text{Equation 3} + (-3) \times \text{Equation 1} = \text{new Equation 3} \end{cases}$$

Now we eliminate the y-term from the third equation.

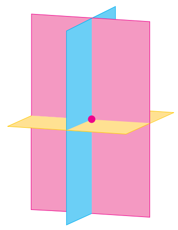
$$\begin{cases} x + 2y - 2z = 1 \\ -2y + 3z = 4 \\ 0 = -2 & \text{Equation 3} + (-1) \times \text{Equation 2} = \text{new Equation 3} \end{cases}$$

The system is now in triangular form, but the third equation says $0 = -2$, which is false. No matter what values we assign x, y, and z, the third equation will never be true. This means the system has *no solution*. ■

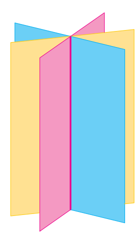
Intersection of Three Planes

When you study calculus or linear algebra, you will learn that the graph of a linear equation in three variables is a *plane* in a three-dimensional coordinate system. For a system of three equations in three variables, the following situations arise:

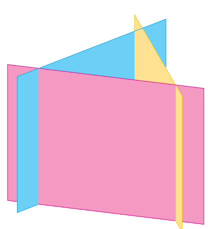
1. The three planes intersect in a single point.
   The system has a unique solution.



2. The three planes intersect in more than one point.
   The system has infinitely many solutions.



3. The three planes have no point in common.
   The system has no solution.



ALTERNATE EXAMPLE 3

Solve the following system.

$$\begin{cases} x + 7y - 7z = 3 \\ 3x + 14y - 16z = 17 \\ 2x + 7y - 9z = 10 \end{cases}$$

If the system is dependent or inconsistent, indicate this.

ANSWER

Inconsistent

IN-CLASS MATERIALS

If students have learned to use their calculators to solve linear systems, they may not realize the calculator's limitations. Have them attempt to solve a 5×5 system using their calculators, and record the length of time that it takes. Then have them attempt a 10×10 system. It turns out that the length of time required to solve an arbitrary linear system grows very quickly with the number of variables involved. When doing a system by hand, it is possible to take advantage of properties of the particular system in question. For example, show them this system:

$$\begin{cases} -v - 2y + z = 1 \\ v + 2y = 3 \\ v + w - x + 2y = 2 \\ -v + x - 2y = 7 \\ v + w + x + y + z = 26 \end{cases}$$

One can go through the traditional algorithm (or have a calculator do it), and it will take some time. (The time it takes to enter it into a calculator counts!) But, by looking at the individual equations, one can get w, x, and z instantly (add equations 1 and 2, 2 and 4, and 3 and 4) and then v and y come easily as well.

Example 4 A System with Infinitely Many Solutions

Solve the following system.

$$\begin{cases} x - y + 5z = -2 & \text{Equation 1} \\ 2x + y + 4z = 2 & \text{Equation 2} \\ 2x + 4y - 2z = 8 & \text{Equation 3} \end{cases}$$

Solution To put this in triangular form, we begin by eliminating the x-terms from the second equation and the third equation.

$$\begin{cases} x - y + 5z = -2 & \\ 3y - 6z = 6 & \text{Equation 2} + (-2) \times \text{Equation 1} = \text{new Equation 2} \\ 2x + 4y - 2z = 8 & \end{cases}$$

$$\begin{cases} x - y + 5z = -2 & \\ 3y - 6z = 6 & \\ 6y - 12z = 12 & \text{Equation 3} + (-2) \times \text{Equation 1} = \text{new Equation 3} \end{cases}$$

Now we eliminate the y-term from the third equation.

$$\begin{cases} x - y + 5z = -2 & \\ 3y - 6z = 6 & \\ 0 = 0 & \text{Equation 3} + (-2) \times \text{Equation 2} = \text{new Equation 3} \end{cases}$$

The new third equation is true, but it gives us no new information, so we can drop it from the system. Only two equations are left. We can use them to solve for x and y in terms of z, but z can take on any value, so there are infinitely many solutions.

To find the complete solution of the system we begin by solving for y in terms of z, using the new second equation.

$$\begin{aligned} 3y - 6z &= 6 && \text{Equation 2} \\ y - 2z &= 2 && \text{Multiply by } \tfrac{1}{3} \\ y &= 2z + 2 && \text{Solve for } y \end{aligned}$$

Then we solve for x in terms of z, using the first equation.

$$\begin{aligned} x - (2z + 2) + 5z &= -2 && \text{Substitute } y = 2z + 2 \text{ into Equation 1} \\ x + 3z - 2 &= -2 && \text{Simplify} \\ x &= -3z && \text{Solve for } x \end{aligned}$$

To describe the complete solution, we let t represent any real number. The solution is

$$\begin{aligned} x &= -3t \\ y &= 2t + 2 \\ z &= t \end{aligned}$$

We can also write this as the ordered triple $(-3t, 2t + 2, t)$. ■

ALTERNATE EXAMPLE 4

Solve the following system.

$$\begin{cases} x - y + 5z = -2 \\ 3x + y + 3z = 2 \\ 3x + 9y - 21z = 18 \end{cases}$$

If the system is dependent or inconsistent, indicate this.

ANSWER

$(-2t, 3t + 2, t)$

IN-CLASS MATERIALS

One can discuss the geometry of systems of three variables as an extension of the geometry of two variables, replacing lines by planes. Ask the class how they could think of a four-variable system. This is not a trivial question. We are looking at the intersection of three hyperplanes in four-dimensional space. Even though you may not get a satisfactory answer (but then again, you may) there is value in trying to come up with visual interpretations for complex, abstract mathematical concepts.

In the solution of Example 4 the variable t is called a **parameter**. To get a specific solution, we give a specific value to the parameter t. For instance, if we set $t = 2$, we get

$$x = -3(2) = -6$$
$$y = 2(2) + 2 = 6$$
$$z = 2$$

Thus, $(-6, 6, 2)$ is a solution of the system. Here are some other solutions of the system obtained by substituting other values for the parameter t.

Parameter t	Solution $(-3t, 2t + 2, t)$
-1	$(3, 0, -1)$
0	$(0, 2, 0)$
3	$(-9, 8, 3)$
10	$(-30, 22, 10)$

You should check that these points satisfy the original equations. There are infinitely many choices for the parameter t, so the system has infinitely many solutions.

Modeling Using Linear Systems

Linear systems are used to model situations that involve several varying quantities. In the next example we consider an application of linear systems to finance.

Example 5 Modeling a Financial Problem Using a Linear System

John receives an inheritance of \$50,000. His financial advisor suggests that he invest this in three mutual funds: a money-market fund, a blue-chip stock fund, and a high-tech stock fund. The advisor estimates that the money-market fund will return 5% over the next year, the blue-chip fund 9%, and the high-tech fund 16%. John wants a total first-year return of \$4000. To avoid excessive risk, he decides to invest three times as much in the money-market fund as in the high-tech stock fund. How much should he invest in each fund?

Solution Let

x = amount invested in the money-market fund

y = amount invested in the blue-chip stock fund

z = amount invested in the high-tech stock fund

We convert each fact given in the problem into an equation.

$$x + y + z = 50{,}000 \qquad \text{Total amount invested is \$50,000}$$
$$0.05x + 0.09y + 0.16z = 4000 \qquad \text{Total investment return is \$4000}$$
$$x = 3z \qquad \text{Money-market amount is } 3 \times \text{high-tech amount}$$

Mathematics in the Modern World

Courtesy of NASA

Global Positioning System (GPS)

On a cold, foggy day in 1707, a British naval fleet was sailing home at a fast clip. The fleet's navigators didn't know it, but the fleet was only a few yards from the rocky shores of England. In the ensuing disaster the fleet was totally destroyed. This tragedy could have been avoided had the navigators known their positions. In those days latitude was determined by the position of the North Star (and this could only be done at night in good weather) and longitude by the position of the sun relative to where it would be in England *at that same time*. So navigation required an accurate method of telling time on ships. (The invention of the spring-loaded clock brought about the eventual solution.)

Since then, several different methods have been developed to determine position, and all rely heavily on mathematics (see LORAN, page 768). The latest method, called the Global Positioning System, uses triangulation. In this system 24 primary satellites are strategically located above the surface of the earth. A hand-held GPS device measures distance from a satellite using the travel

(*continued*)

ALTERNATE EXAMPLE 5

John receives an inheritance of \$55,000. His financial advisor suggests that he invest this in three mutual funds: a money-market fund, a blue-chip stock fund, and a high-tech stock fund. The advisor estimates that the money-market fund will return 5% over the next year, the blue-chip fund 9%, and the high-tech fund 18%. John wants a total first-year return of \$4650. To avoid excessive risk, he decides to invest three times as much in the money-market fund as in the high-tech stock fund. How much should he invest in each fund?

ANSWER

\$30,000, \$15,000, \$10,000

time of radio signals emitted from the satellite. Knowing the distance to three different satellites tells us that we are at the point of intersection of three different spheres. This uniquely determines our position (see Exercise 53, page 643).

Multiplying the second equation by 100 and rewriting the third gives the following system, which we solve using Gaussian elimination.

$$\begin{cases} x + y + z = 50{,}000 & \\ 5x + 9y + 16z = 400{,}000 & \quad 100 \times \text{Equation 2} \\ x - 3z = 0 & \quad \text{Subtract } 3z \end{cases}$$

$$\begin{cases} x + y + z = 50{,}000 & \\ 4y + 11z = 150{,}000 & \quad \text{Equation 2} + (-5) \times \text{Equation 1} = \text{new Equation 2} \\ -y - 4z = -50{,}000 & \quad \text{Equation 3} + (-1) \times \text{Equation 1} = \text{new Equation 3} \end{cases}$$

$$\begin{cases} x + y + z = 50{,}000 & \\ -5z = -50{,}000 & \quad \text{Equation 2} + 4 \times \text{Equation 3} = \text{new Equation 2} \\ -y - 4z = -50{,}000 & \end{cases}$$

$$\begin{cases} x + y + z = 50{,}000 & \\ z = 10{,}000 & \quad (-\tfrac{1}{5}) \times \text{Equation 2} \\ y + 4z = 50{,}000 & \quad (-1) \times \text{Equation 3} \end{cases}$$

$$\begin{cases} x + y + z = 50{,}000 & \\ y + 4z = 50{,}000 & \quad \text{Interchange Equations 2 and 3} \\ z = 10{,}000 & \end{cases}$$

Now that the system is in triangular form, we use back-substitution to find that $x = 30{,}000$, $y = 10{,}000$, and $z = 10{,}000$. This means that John should invest

$30,000 in the money market fund

$10,000 in the blue-chip stock fund

$10,000 in the high-tech stock fund ■

9.3 Exercises

1–4 ■ State whether the equation or system of equations is linear.

1. $6x - \sqrt{3}y + \frac{1}{2}z = 0$

2. $x^2 + y^2 + z^2 = 4$

3. $\begin{cases} xy - 3y + z = 5 \\ x - y^2 + 5z = 0 \\ 2x + yz = 3 \end{cases}$

4. $\begin{cases} x - 2y + 3z = 10 \\ 2x + 5y = 2 \\ y + 2z = 4 \end{cases}$

5–10 ■ Use back-substitution to solve the triangular system.

5. $\begin{cases} x - 2y + 4z = 3 \\ y + 2z = 7 \\ z = 2 \end{cases}$

6. $\begin{cases} x + y - 3z = 8 \\ y - 3z = 5 \\ z = -1 \end{cases}$

7. $\begin{cases} x + 2y + z = 7 \\ -y + 3z = 9 \\ 2z = 6 \end{cases}$

8. $\begin{cases} x - 2y + 3z = 10 \\ 2y - z = 2 \\ 3z = 12 \end{cases}$

9. $\begin{cases} 2x - y + 6z = 5 \\ y + 4z = 0 \\ -2z = 1 \end{cases}$

10. $\begin{cases} 4x + 3z = 10 \\ 2y - z = -6 \\ \frac{1}{2}z = 4 \end{cases}$

11–14 ■ Perform an operation on the given system that eliminates the indicated variable. Write the new equivalent system.

11. $\begin{cases} x - 2y - z = 4 \\ x - y + 3z = 0 \\ 2x + y + z = 0 \end{cases}$ Eliminate the x-term from the second equation.

12. $\begin{cases} x + y - 3z = 3 \\ -2x + 3y + z = 2 \\ x - y + 2z = 0 \end{cases}$ Eliminate the x-term from the second equation.

13. $\begin{cases} 2x - y + 3z = 2 \\ x + 2y - z = 4 \\ -4x + 5y + z = 10 \end{cases}$ Eliminate the x-term from the third equation.

14. $\begin{cases} x - 4y + z = 3 \\ y - 3z = 10 \\ 3y - 8z = 24 \end{cases}$ Eliminate the y-term from the third equation.

15–32 ■ Find the complete solution of the linear system, or show that it is inconsistent.

15. $\begin{cases} x + y + z = 4 \\ x + 3y + 3z = 10 \\ 2x + y - z = 3 \end{cases}$

16. $\begin{cases} x + y + z = 0 \\ -x + 2y + 5z = 3 \\ 3x - y = 6 \end{cases}$

17. $\begin{cases} x - 4z = 1 \\ 2x - y - 6z = 4 \\ 2x + 3y - 2z = 8 \end{cases}$

18. $\begin{cases} x - y + 2z = 2 \\ 3x + y + 5z = 8 \\ 2x - y - 2z = -7 \end{cases}$

19. $\begin{cases} 2x + 4y - z = 2 \\ x + 2y - 3z = -4 \\ 3x - y + z = 1 \end{cases}$

20. $\begin{cases} 2x + y - z = -8 \\ -x + y + z = 3 \\ -2x + 4z = 18 \end{cases}$

21. $\begin{cases} y - 2z = 0 \\ 2x + 3y = 2 \\ -x - 2y + z = -1 \end{cases}$

22. $\begin{cases} 2y + z = 3 \\ 5x + 4y + 3z = -1 \\ x - 3y = -2 \end{cases}$

23. $\begin{cases} x + 2y - z = 1 \\ 2x + 3y - 4z = -3 \\ 3x + 6y - 3z = 4 \end{cases}$

24. $\begin{cases} -x + 2y + 5z = 4 \\ x - 2z = 0 \\ 4x - 2y - 11z = 2 \end{cases}$

25. $\begin{cases} 2x + 3y - z = 1 \\ x + 2y = 3 \\ x + 3y + z = 4 \end{cases}$

26. $\begin{cases} x - 2y - 3z = 5 \\ 2x + y - z = 5 \\ 4x - 3y - 7z = 5 \end{cases}$

27. $\begin{cases} x + y - z = 0 \\ x + 2y - 3z = -3 \\ 2x + 3y - 4z = -3 \end{cases}$

28. $\begin{cases} x - 2y + z = 3 \\ 2x - 5y + 6z = 7 \\ 2x - 3y - 2z = 5 \end{cases}$

29. $\begin{cases} x + 3y - 2z = 0 \\ 2x + 4z = 4 \\ 4x + 6y = 4 \end{cases}$

30. $\begin{cases} 2x + 4y - z = 3 \\ x + 2y + 4z = 6 \\ x + 2y - 2z = 0 \end{cases}$

31. $\begin{cases} x + z + 2w = 6 \\ y - 2z = -3 \\ x + 2y - z = -2 \\ 2x + y + 3z - 2w = 0 \end{cases}$

32. $\begin{cases} x + y + z + w = 0 \\ x + y + 2z + 2w = 0 \\ 2x + 2y + 3z + 4w = 1 \\ 2x + 3y + 4z + 5w = 2 \end{cases}$

Applications

33–34 ■ Finance An investor has \$100,000 to invest in three types of bonds: short-term, intermediate-term, and long-term. How much should she invest in each type to satisfy the given conditions?

33. Short-term bonds pay 4% annually, intermediate-term bonds pay 5%, and long-term bonds pay 6%. The investor wishes to realize a total annual income of 5.1%, with equal amounts invested in short- and intermediate-term bonds.

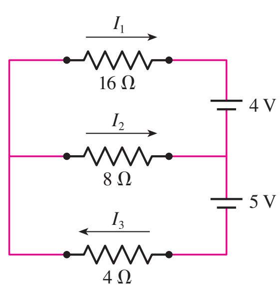
34. Short-term bonds pay 4% annually, intermediate-term bonds pay 6%, and long-term bonds pay 8%. The investor wishes to have a total annual return of \$6700 on her investment, with equal amounts invested in intermediate- and long-term bonds.

35. Nutrition A biologist is performing an experiment on the effects of various combinations of vitamins. She wishes to feed each of her laboratory rabbits a diet that contains exactly 9 mg of niacin, 14 mg of thiamin, and 32 mg of riboflavin. She has available three different types of commercial rabbit pellets; their vitamin content (per ounce) is given in the table. How many ounces of each type of food should each rabbit be given daily to satisfy the experiment requirements?

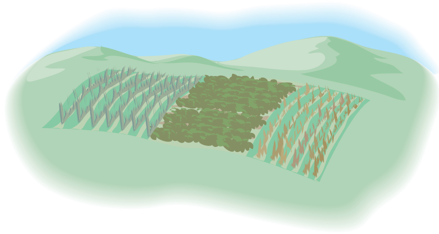
	Type A	Type B	Type C
Niacin (mg)	2	3	1
Thiamin (mg)	3	1	3
Riboflavin (mg)	8	5	7

36. Electricity Using Kirchhoff's Laws, it can be shown that the currents I_1, I_2, and I_3 that pass through the three branches of the circuit in the figure satisfy the given linear system. Solve the system to find I_1, I_2, and I_3.

$$\begin{cases} I_1 + I_2 - I_3 = 0 \\ 16I_1 - 8I_2 = 4 \\ 8I_2 + 4I_3 = 5 \end{cases}$$



37. Agriculture A farmer has 1200 acres of land on which he grows corn, wheat, and soybeans. It costs \$45 per acre to grow corn, \$60 for wheat, and \$50 for soybeans. Because of market demand he will grow twice as many acres of wheat as of corn. He has allocated \$63,750 for the cost of growing his crops. How many acres of each crop should he plant?



38. Stock Portfolio An investor owns three stocks: A, B, and C. The closing prices of the stocks on three successive trading days are given in the table.

	Stock A	Stock B	Stock C
Monday	\$10	\$25	\$29
Tuesday	\$12	\$20	\$32
Wednesday	\$16	\$15	\$32

Despite the volatility in the stock prices, the total value of the investor's stocks remained unchanged at \$74,000 at the end of each of these three days. How many shares of each stock does the investor own?

Discovery • Discussion

39. Can a Linear System Have Exactly Two Solutions?

(a) Suppose that (x_0, y_0, z_0) and (x_1, y_1, z_1) are solutions of the system

$$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{cases}$$

Show that $\left(\dfrac{x_0 + x_1}{2}, \dfrac{y_0 + y_1}{2}, \dfrac{z_0 + z_1}{2}\right)$ is also a solution.

(b) Use the result of part (a) to prove that if the system has two different solutions, then it has infinitely many solutions.

DISCOVERY PROJECT

Best Fit versus Exact Fit

Given several points in the plane, we can find the line that best fits them (see the *Focus on Modeling*, page 239). Of course, not all the points will necessarily lie on the line. We can also find the quadratic polynomial that best fits the points. Again, not every point will necessarily lie on the graph of the polynomial.

However, if we are given just two points, we can find a line of *exact* fit, that is, a line that actually passes through both points. Similarly, given three points (not all on the same line), we can find the quadratic polynomial of *exact* fit. For example, suppose we are given the following three points:

$$(-1, 6), \quad (1, 2), \quad (2, 3)$$

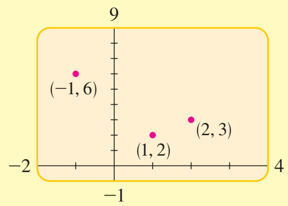


Figure 1

From Figure 1 we see that the points do not lie on a line. Let's find the quadratic polynomial that fits these points exactly. The polynomial must have the form

$$y = ax^2 + bx + c$$

We need to find values for a, b, and c so that the graph of the resulting polynomial contains the given points. Substituting the given points into the equation, we get the following.

Point	Substitute	Equation
$(-1, 6)$	$x = -1, \quad y = 6$	$6 = a(-1)^2 + b(-1) + c$
$(1, 2)$	$x = 1, \quad y = 2$	$2 = a(1)^2 + b(1) + c$
$(2, 3)$	$x = 2, \quad y = 3$	$3 = a(2)^2 + b(2) + c$

These three equations simplify into the following system.

$$\begin{cases} a - b + c = 6 \\ a + b + c = 2 \\ 4a + 2b + c = 3 \end{cases}$$

Using Gaussian elimination we obtain the solution $a = 1$, $b = -2$, and $c = 3$. So the required quadratic polynomial is

$$y = x^2 - 2x + 3$$

From Figure 2 we see that the graph of the polynomial passes through the given points.

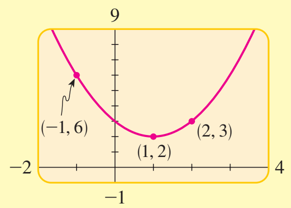


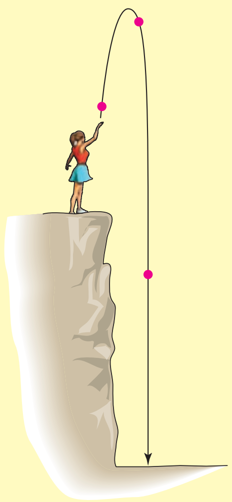
Figure 2

1. Find the quadratic polynomial $y = ax^2 + bx + c$ whose graph passes through the given points.

(a) $(-2, 3),\quad (-1, 1),\quad (1, 9)$
(b) $(-1, -3),\quad (2, 0),\quad (3, -3)$

2. Find the cubic polynomial $y = ax^3 + bx^2 + cx + d$ whose graph passes through the given points.

(a) $(-1, -4),\quad (1, 2),\quad (2, 11),\quad (3, 32)$
(b) $(-2, 10),\quad (-1, 1),\quad (1, -1),\quad (3, 45)$



3. A stone is thrown upward with velocity v from a height h. Its elevation d above the ground at time t is given by

$$d = at^2 + vt + h$$

The elevation is measured at three different times as shown.

Time (s)	1.0	2.0	6.0
Elevation (ft)	144	192	64

(a) Find the constants a, v, and h.
(b) Find the elevation of the stone when $t = 4$ s.

4. (a) Find the quadratic function $y = ax^2 + bx + c$ whose graph passes through the given points. (This is the quadratic curve of *exact* fit.) Graph the points and the quadratic curve that you found.

$$(-2, 10),\quad (1, -5),\quad (2, -6),\quad (4, -2)$$

(b) Now use the `QuadReg` command on your calculator to find the quadratic curve that *best* fits the points in part (a). How does this compare to the function you found in part (a)?

(c) Show that no quadratic function passes through the points

$$(-2, 11),\quad (1, -6),\quad (2, -5),\quad (4, -1)$$

(d) Use the `QuadReg` command on your calculator to find the quadratic curve that best fits the points in part (b). Graph the points and the quadratic curve that you found.

(e) Explain how the curve of exact fit differs from the curve of best fit.

SUGGESTED TIME AND EMPHASIS
2 classes.
Recommended material.

9.4 Systems of Linear Equations: Matrices

In this section we express a linear system as a rectangular array of numbers, called a matrix. Matrices* provide us with an efficient way of solving linear systems.

Matrices

We begin by defining the various elements that make up a matrix.

Definition of Matrix

An $m \times n$ **matrix** is a rectangular array of numbers with m **rows** and n **columns**.

$$\begin{array}{c} \left[\begin{array}{ccccc} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{array}\right] \begin{array}{l} \leftarrow \\ \leftarrow \\ \leftarrow \\ \\ \leftarrow \end{array} \Bigg\} m \text{ rows} \\ \underbrace{\uparrow \quad \uparrow \quad \uparrow \qquad\quad \uparrow}_{n \text{ columns}} \end{array}$$

We say that the matrix has **dimension** $m \times n$. The numbers a_{ij} are the **entries** of the matrix. The subscript on the entry a_{ij} indicates that it is in the ith row and the jth column.

Here are some examples of matrices.

Matrix	Dimension	
$\begin{bmatrix} 1 & 3 & 0 \\ 2 & 4 & -1 \end{bmatrix}$	2×3	2 rows by 3 columns
$\begin{bmatrix} 6 & -5 & 0 & 1 \end{bmatrix}$	1×4	1 row by 4 columns

The Augmented Matrix of a Linear System

We can write a system of linear equations as a matrix, called the **augmented matrix** of the system, by writing only the coefficients and constants that appear in the equations. Here is an example.

Linear system	Augmented matrix
$\begin{cases} 3x - 2y + z = 5 \\ x + 3y - z = 0 \\ -x + 4z = 11 \end{cases}$	$\begin{bmatrix} 3 & -2 & 1 & 5 \\ 1 & 3 & -1 & 0 \\ -1 & 0 & 4 & 11 \end{bmatrix}$

* The plural of *matrix* is *matrices*.

POINTS TO STRESS

1. Definitions: matrix, dimension, row, column, row-echelon form, reduced row-echelon form.
2. Finding the augmented matrix of a linear system, and manipulating it using row operations.
3. Gaussian and Gauss-Jordan elimination, including inconsistent and dependent systems.

Notice that a missing variable in an equation corresponds to a 0 entry in the augmented matrix.

Example 1 **Finding the Augmented Matrix of a Linear System**

Write the augmented matrix of the system of equations.

$$\begin{cases} 6x - 2y - z = 4 \\ x + 3z = 1 \\ 7y + z = 5 \end{cases}$$

Solution First we write the linear system with the variables lined up in columns.

$$\begin{cases} 6x - 2y - z = 4 \\ x + 3z = 1 \\ 7y + z = 5 \end{cases}$$

The augmented matrix is the matrix whose entries are the coefficients and the constants in this system.

$$\begin{bmatrix} 6 & -2 & -1 & 4 \\ 1 & 0 & 3 & 1 \\ 0 & 7 & 1 & 5 \end{bmatrix}$$

■

Elementary Row Operations

The operations that we used in Section 9.3 to solve linear systems correspond to operations on the rows of the augmented matrix of the system. For example, adding a multiple of one equation to another corresponds to adding a multiple of one row to another.

Elementary Row Operations

1. Add a multiple of one row to another.
2. Multiply a row by a nonzero constant.
3. Interchange two rows.

Note that performing any of these operations on the augmented matrix of a system does not change its solution. We use the following notation to describe the elementary row operations:

Symbol	Description
$R_i + kR_j \rightarrow R_i$	Change the ith row by adding k times row j to it, then put the result back in row i.
kR_i	Multiply the ith row by k.
$R_i \leftrightarrow R_j$	Interchange the ith and jth rows.

ALTERNATE EXAMPLE 1

Write the augmented matrix of the system of equations.

$$\begin{cases} 7x - 2y - z = 4 \\ x + 3z = 6 \\ 4y + z = 7 \end{cases}$$

ANSWER

$$\begin{bmatrix} 7 & -2 & -1 & 4 \\ 1 & 0 & 3 & 6 \\ 0 & 4 & 1 & 7 \end{bmatrix}$$

ALTERNATE EXAMPLE 2
Solve the system using its matrix form.

$$\begin{cases} x - y + 5z = 3 \\ x + 2y - 6z = 7 \\ 4x - y + 8z = 15 \end{cases}$$

If the system is dependent or inconsistent, indicate this.

ANSWER
(3, 5, 1)

In the next example we compare the two ways of writing systems of linear equations.

Example 2 Using Elementary Row Operations to Solve a Linear System

Solve the system of linear equations.

$$\begin{cases} x - y + 3z = 4 \\ x + 2y - 2z = 10 \\ 3x - y + 5z = 14 \end{cases}$$

Solution Our goal is to eliminate the x-term from the second equation and the x- and y-terms from the third equation. For comparison, we write both the system of equations and its augmented matrix.

	System		Augmented matrix
	$\begin{cases} x - y + 3z = 4 \\ x + 2y - 2z = 10 \\ 3x - y + 5z = 14 \end{cases}$		$\begin{bmatrix} 1 & -1 & 3 & 4 \\ 1 & 2 & -2 & 10 \\ 3 & -1 & 5 & 14 \end{bmatrix}$
Add $(-1) \times$ Equation 1 to Equation 2. Add $(-3) \times$ Equation 1 to Equation 3.	$\begin{cases} x - y + 3z = 4 \\ 3y - 5z = 6 \\ 2y - 4z = 2 \end{cases}$	$\xrightarrow[R_3 - 3R_1 \to R_3]{R_2 - R_1 \to R_2}$	$\begin{bmatrix} 1 & -1 & 3 & 4 \\ 0 & 3 & -5 & 6 \\ 0 & 2 & -4 & 2 \end{bmatrix}$
Multiply Equation 3 by $\frac{1}{2}$.	$\begin{cases} x - y + 3z = 4 \\ 3y - 5z = 6 \\ y - 2z = 1 \end{cases}$	$\xrightarrow{\frac{1}{2}R_3}$	$\begin{bmatrix} 1 & -1 & 3 & 4 \\ 0 & 3 & -5 & 6 \\ 0 & 1 & -2 & 1 \end{bmatrix}$
Add $(-3) \times$ Equation 3 to Equation 2 (to eliminate y from Equation 2).	$\begin{cases} x - y + 3z = 4 \\ z = 3 \\ y - 2z = 1 \end{cases}$	$\xrightarrow{R_2 - 3R_3 \to R_2}$	$\begin{bmatrix} 1 & -1 & 3 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 1 & -2 & 1 \end{bmatrix}$
Interchange Equations 2 and 3.	$\begin{cases} x - y + 3z = 4 \\ y - 2z = 1 \\ z = 3 \end{cases}$	$\xrightarrow{R_2 \leftrightarrow R_3}$	$\begin{bmatrix} 1 & -1 & 3 & 4 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$

Now we use back-substitution to find that $x = 2$, $y = 7$, and $z = 3$. The solution is $(2, 7, 3)$. ■

Gaussian Elimination

In general, to solve a system of linear equations using its augmented matrix, we use elementary row operations to arrive at a matrix in a certain form. This form is described in the following box.

Row-Echelon Form and Reduced Row-Echelon Form of a Matrix

A matrix is in **row-echelon form** if it satisfies the following conditions.

1. The first nonzero number in each row (reading from left to right) is 1. This is called the **leading entry**.
2. The leading entry in each row is to the right of the leading entry in the row immediately above it.
3. All rows consisting entirely of zeros are at the bottom of the matrix.

A matrix is in **reduced row-echelon form** if it is in row-echelon form and also satisfies the following condition.

4. Every number above and below each leading entry is a 0.

In the following matrices the first matrix is in reduced row-echelon form, but the second one is just in row-echelon form. The third matrix is not in row-echelon form. The entries in red are the leading entries.

Reduced row-echelon form

$$\begin{bmatrix} 1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Leading 1's have 0's above and below them.

Row-echelon form

$$\begin{bmatrix} 1 & 3 & -6 & 10 & 0 \\ 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Leading 1's shift to the right in successive rows.

Not in row-echelon form

$$\begin{bmatrix} 0 & 1 & -\frac{1}{2} & 0 & 7 \\ 1 & 0 & 3 & 4 & -5 \\ 0 & 0 & 0 & 1 & 0.4 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

Leading 1's do not shift to the right in successive rows.

Here is a systematic way to put a matrix in row-echelon form using elementary row operations:

- Start by obtaining 1 in the top left corner. Then obtain zeros below that 1 by adding appropriate multiples of the first row to the rows below it.
- Next, obtain a leading 1 in the next row, and then obtain zeros below that 1.
- At each stage make sure that every leading entry is to the right of the leading entry in the row above it—rearrange the rows if necessary.
- Continue this process until you arrive at a matrix in row-echelon form.

This is how the process might work for a 3×4 matrix:

$$\begin{bmatrix} 1 & \blacksquare & \blacksquare & \blacksquare \\ 0 & \blacksquare & \blacksquare & \blacksquare \\ 0 & \blacksquare & \blacksquare & \blacksquare \end{bmatrix} \to \begin{bmatrix} 1 & \blacksquare & \blacksquare & \blacksquare \\ 0 & 1 & \blacksquare & \blacksquare \\ 0 & 0 & \blacksquare & \blacksquare \end{bmatrix} \to \begin{bmatrix} 1 & \blacksquare & \blacksquare & \blacksquare \\ 0 & 1 & \blacksquare & \blacksquare \\ 0 & 0 & 1 & \blacksquare \end{bmatrix}$$

Once an augmented matrix is in row-echelon form, we can solve the corresponding linear system using back-substitution. This technique is called **Gaussian elimination**, in honor of its inventor, the German mathematician C. F. Gauss (see page 294).

DRILL QUESTION

Consider this system of equations:

$$\begin{cases} x + y = -1 \\ 2x - 3y = 8 \end{cases}$$

(a) Find the augmented matrix of this system.

(b) Put the matrix in reduced row-echelon form.

Answers

$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 8 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

IN-CLASS MATERIALS

Point out that as noted in the text, the row-echelon form of a given matrix is not unique, but the reduced row-echelon form is unique. In fact, it can be shown that if two matrices have the same reduced row-echelon form, you can transform one into the other via elementary row operations.

ALTERNATE EXAMPLE 3
Solve the system of linear equations using Gaussian elimination.

$$\begin{cases} 11x + 22y - 11z = 22 \\ 7x + 14y + 5z = -10 \\ -6x + y + 58z = -51 \end{cases}$$

If the system is dependent or inconsistent, indicate this.

ANSWER
$(-10, 5, -2)$

Solving a System Using Gaussian Elimination

1. **Augmented Matrix.** Write the augmented matrix of the system.
2. **Row-Echelon Form.** Use elementary row operations to change the augmented matrix to row-echelon form.
3. **Back-Substitution.** Write the new system of equations that corresponds to the row-echelon form of the augmented matrix and solve by back-substitution.

Example 3 Solving a System Using Row-Echelon Form

Solve the system of linear equations using Gaussian elimination.

$$\begin{cases} 4x + 8y - 4z = 4 \\ 3x + 8y + 5z = -11 \\ -2x + y + 12z = -17 \end{cases}$$

Solution We first write the augmented matrix of the system, and then use elementary row operations to put it in row-echelon form.

Need a 1 here.

$$\left[\begin{array}{rrr|r} 4 & 8 & -4 & 4 \\ 3 & 8 & 5 & -11 \\ -2 & 1 & 12 & -17 \end{array}\right]$$

Need 0's here.

$$\xrightarrow{\frac{1}{4}R_1} \left[\begin{array}{rrr|r} 1 & 2 & -1 & 1 \\ 3 & 8 & 5 & -11 \\ -2 & 1 & 12 & -17 \end{array}\right]$$

Need a 1 here.

$$\xrightarrow[R_3 + 2R_1 \to R_3]{R_2 - 3R_1 \to R_2} \left[\begin{array}{rrr|r} 1 & 2 & -1 & 1 \\ 0 & 2 & 8 & -14 \\ 0 & 5 & 10 & -15 \end{array}\right]$$

Need a 0 here.

$$\xrightarrow{\frac{1}{2}R_2} \left[\begin{array}{rrr|r} 1 & 2 & -1 & 1 \\ 0 & 1 & 4 & -7 \\ 0 & 5 & 10 & -15 \end{array}\right]$$

Need a 1 here.

$$\xrightarrow{R_3 - 5R_2 \to R_3} \left[\begin{array}{rrr|r} 1 & 2 & -1 & 1 \\ 0 & 1 & 4 & -7 \\ 0 & 0 & -10 & 20 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{10}R_3} \left[\begin{array}{rrr|r} 1 & 2 & -1 & 1 \\ 0 & 1 & 4 & -7 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

SAMPLE QUESTIONS

Text Questions

Which of the following matrices are in row-echelon form? Which are in reduced row-echelon form?

(a) $\left[\begin{array}{rrrr|r} 1 & 4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & \pi \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ **(b)** $\left[\begin{array}{rrrr|r} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{array}\right]$ **(c)** $\left[\begin{array}{rrrr|r} 1 & 4 & 3 & 0 & 5 \\ 0 & 0 & 1 & 2 & -5 \\ 0 & 0 & 0 & 1 & \pi \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

Answers

(a) and (c) are in row-echelon form; (a) is in reduced row-echelon form.

We now have an equivalent matrix in row-echelon form, and the corresponding system of equations is

$$\begin{cases} x + 2y - z = 1 \\ y + 4z = -7 \\ z = -2 \end{cases}$$

We use back-substitution to solve the system.

$$\begin{aligned} y + 4(-2) &= -7 && \text{Back-substitute } z = -2 \text{ into Equation 2} \\ y &= 1 && \text{Solve for } y \\ x + 2(1) - (-2) &= 1 && \text{Back-substitute } y = 1 \text{ and } z = -2 \text{ into Equation 1} \\ x &= -3 && \text{Solve for } x \end{aligned}$$

So the solution of the system is $(-3, 1, -2)$. ■

```
ref([A])
   [[1 2 -1  1 ]
    [0 1 2  -3]
    [0 0 1  -2]]
```

Figure 1

Graphing calculators have a "row-echelon form" command that puts a matrix in row-echelon form. (On the TI-83 this command is `ref`.) For the augmented matrix in Example 3, the `ref` command gives the output shown in Figure 1. Notice that the row-echelon form obtained by the calculator differs from the one we got in Example 3. This is because the calculator used different row operations than we did. You should check that your calculator's row-echelon form leads to the same solution as ours.

Gauss-Jordan Elimination

If we put the augmented matrix of a linear system in *reduced* row-echelon form, then we don't need to back-substitute to solve the system. To put a matrix in reduced row-echelon form, we use the following steps.

- Use the elementary row operations to put the matrix in row-echelon form.
- Obtain zeros above each leading entry by adding multiples of the row containing that entry to the rows above it. Begin with the last leading entry and work up.

Here is how the process works for a 3×4 matrix:

$$\begin{bmatrix} 1 & \blacksquare & \blacksquare & \blacksquare \\ 0 & 1 & \blacksquare & \blacksquare \\ 0 & 0 & 1 & \blacksquare \end{bmatrix} \rightarrow \begin{bmatrix} 1 & \blacksquare & 0 & \blacksquare \\ 0 & 1 & 0 & \blacksquare \\ 0 & 0 & 1 & \blacksquare \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & \blacksquare \\ 0 & 1 & 0 & \blacksquare \\ 0 & 0 & 1 & \blacksquare \end{bmatrix}$$

Using the reduced row-echelon form to solve a system is called **Gauss-Jordan elimination**. We illustrate this process in the next example.

Example 4 Solving a System Using Reduced Row-Echelon Form

Solve the system of linear equations, using Gauss-Jordan elimination.

$$\begin{cases} 4x + 8y - 4z = 4 \\ 3x + 8y + 5z = -11 \\ -2x + y + 12z = -17 \end{cases}$$

Solution In Example 3 we used Gaussian elimination on the augmented matrix of this system to arrive at an equivalent matrix in row-echelon form. We continue

ALTERNATE EXAMPLE 4

Solve the system of linear equations, using Gauss-Jordan elimination.

$$\begin{cases} 6x + 18y - 6z = 24 \\ 5x + 15y + 3z = -20 \\ -3x + y + 33z = -42 \end{cases}$$

If the system is dependent or inconsistent, indicate this.

ANSWER

$(-37, 12, -5)$

using elementary row operations on the last matrix in Example 3 to arrive at an equivalent matrix in reduced row-echelon form.

$$\begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & 4 & -7 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

Need 0's here.

Need a 0 here.

$$\xrightarrow[R_1 + R_3 \to R_1]{R_2 - 4R_3 \to R_2} \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$$\xrightarrow{R_1 - 2R_2 \to R_1} \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

We now have an equivalent matrix in reduced row-echelon form, and the corresponding system of equations is

Since the system is in reduced row-echelon form, back-substitution is not required to get the solution.

$$\begin{cases} x = -3 \\ y = 1 \\ z = -2 \end{cases}$$

Hence we immediately arrive at the solution $(-3, 1, -2)$. ■

```
rref([A])
    [[1 0 0 -3]
     [0 1 0 1 ]
     [0 0 1 -2]]
```

Figure 2

Graphing calculators also have a command that puts a matrix in reduced row-echelon form. (On the TI-83 this command is `rref`.) For the augmented matrix in Example 4, the `rref` command gives the output shown in Figure 2. The calculator gives the same reduced row-echelon form as the one we got in Example 4. This is because every matrix has a *unique* reduced row-echelon form.

Inconsistent and Dependent Systems

The systems of linear equations that we considered in Examples 1–4 had exactly one solution. But as we know from Section 9.3 a linear system may have one solution, no solution, or infinitely many solutions. Fortunately, the row-echelon form of a system allows us to determine which of these cases applies, as described in the following box.

First we need some terminology. A **leading variable** in a linear system is one that corresponds to a leading entry in the row-echelon form of the augmented matrix of the system.

IN-CLASS MATERIALS

It is possible to introduce the concept of homogeneous systems as a way of getting the students to think about dependent and inconsistent systems. Define a homogeneous system as one where the equations are all equal to zero:

$$\begin{cases} 3x - 2y - z = 0 \\ 4x + 5y - 4z = 0 \\ 2x - 8y - z = 0 \end{cases}$$

Start by asking the class some simple questions: Are homogeneous systems easier or harder to solve than arbitrary systems? If so, why? Then ask them to find and solve a dependent homogenous system, a homogeneous system with a unique solution, and finally, one with no solution. (Give them some time to do this—a lot of learning will take place while they go through the process of creating and solving problems both forward and backward.) When students or groups of students finish early, ask them to articulate why there cannot be an inconsistent homogeneous system.

After the students have thought about this type of system, bring them all together. Point out that it is clear that $x = 0$, $y = 0$, $z = 0$ is always a solution to a homogeneous system, and so there cannot be an inconsistent one. The only possible solutions sets are $(0, 0, 0)$ and a set with infinitely many points, one of which is $(0, 0, 0)$.

The Solutions of a Linear System in Row-Echelon Form

Suppose the augmented matrix of a system of linear equations has been transformed by Gaussian elimination into row-echelon form. Then exactly one of the following is true.

1. **No solution.** If the row-echelon form contains a row that represents the equation $0 = c$ where c is not zero, then the system has no solution. A system with no solution is called **inconsistent**.
2. **One solution.** If each variable in the row-echelon form is a leading variable, then the system has exactly one solution, which we find using back-substitution or Gauss-Jordan elimination.
3. **Infinitely many solutions.** If the variables in the row-echelon form are not all leading variables, and if the system is not inconsistent, then it has infinitely many solutions. In this case, the system is called **dependent**. We solve the system by putting the matrix in reduced row-echelon form and then expressing the leading variables in terms of the nonleading variables. The nonleading variables may take on any real numbers as their values.

The matrices below, all in row-echelon form, illustrate the three cases described in the box.

No solution

$$\begin{bmatrix} 1 & 2 & 5 & 7 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Last equation says $0 = 1$.

One solution

$$\begin{bmatrix} 1 & 6 & -1 & 3 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 1 & 8 \end{bmatrix}$$

Each variable is a leading variable.

Infinitely many solutions

$$\begin{bmatrix} 1 & 2 & -3 & 1 \\ 0 & 1 & 5 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

z is not a leading variable.

Example 5 A System with No Solution

Solve the system.

$$\begin{cases} x - 3y + 2z = 12 \\ 2x - 5y + 5z = 14 \\ x - 2y + 3z = 20 \end{cases}$$

Solution We transform the system into row-echelon form.

$$\begin{bmatrix} 1 & -3 & 2 & 12 \\ 2 & -5 & 5 & 14 \\ 1 & -2 & 3 & 20 \end{bmatrix} \xrightarrow[\mathrm{R}_3 - \mathrm{R}_1 \to \mathrm{R}_3]{\mathrm{R}_2 - 2\mathrm{R}_1 \to \mathrm{R}_2} \begin{bmatrix} 1 & -3 & 2 & 12 \\ 0 & 1 & 1 & -10 \\ 0 & 1 & 1 & 8 \end{bmatrix}$$

$$\xrightarrow{\mathrm{R}_3 - \mathrm{R}_2 \to \mathrm{R}_3} \begin{bmatrix} 1 & -3 & 2 & 12 \\ 0 & 1 & 1 & -10 \\ 0 & 0 & 0 & 18 \end{bmatrix} \xrightarrow{\frac{1}{18}\mathrm{R}_3} \begin{bmatrix} 1 & -3 & 2 & 12 \\ 0 & 1 & 1 & -10 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

ALTERNATE EXAMPLE 5

Solve the system.

$$\begin{cases} x - 8y + 4z = 11 \\ 2x - 12y + 12z = 17 \\ x - 4y + 8z = 22 \end{cases}$$

If the system is dependent or inconsistent, indicate this.

ANSWER

Inconsistent

```
ref([A])
[[1 -2.5 2.5 7  ]
 [0 1    1   -10]
 [0 0    0   1  ]]
```

Figure 3

This last matrix is in row-echelon form, so we can stop the Gaussian elimination process. Now if we translate the last row back into equation form, we get $0x + 0y + 0z = 1$, or $0 = 1$, which is false. No matter what values we pick for x, y, and z, the last equation will never be a true statement. This means the system *has no solution*. ■

Figure 3 shows the row-echelon form produced by a TI-83 calculator for the augmented matrix in Example 5. You should check that this gives the same solution.

Example 6 A System with Infinitely Many Solutions

Find the complete solution of the system.

$$\begin{cases} -3x - 5y + 36z = 10 \\ -x + 7z = 5 \\ x + y - 10z = -4 \end{cases}$$

Solution We transform the system into reduced row-echelon form.

$$\begin{bmatrix} -3 & -5 & 36 & 10 \\ -1 & 0 & 7 & 5 \\ 1 & 1 & -10 & -4 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 1 & -10 & -4 \\ -1 & 0 & 7 & 5 \\ -3 & -5 & 36 & 10 \end{bmatrix}$$

$$\xrightarrow[R_3 + 3R_1 \to R_3]{R_2 + R_1 \to R_2} \begin{bmatrix} 1 & 1 & -10 & -4 \\ 0 & 1 & -3 & 1 \\ 0 & -2 & 6 & -2 \end{bmatrix} \xrightarrow{R_3 + 2R_2 \to R_3} \begin{bmatrix} 1 & 1 & -10 & -4 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1 - R_2 \to R_1} \begin{bmatrix} 1 & 0 & -7 & -5 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The third row corresponds to the equation $0 = 0$. This equation is always true, no matter what values are used for x, y, and z. Since the equation adds no new information about the variables, we can drop it from the system. So the last matrix corresponds to the system

$$\begin{cases} x - 7z = -5 & \text{Equation 1} \\ y - 3z = 1 & \text{Equation 2} \end{cases}$$

Leading variables

Reduced row-echelon form on the TI-83 calculator:

```
rref([A])
   [[1 0 -7 -5]
    [0 1 -3 1 ]
    [0 0 0  0 ]]
```

Now we solve for the leading variables x and y in terms of the nonleading variable z:

$$x = 7z - 5 \qquad \text{Solve for } x \text{ in Equation 1}$$

$$y = 3z + 1 \qquad \text{Solve for } y \text{ in Equation 2}$$

ALTERNATE EXAMPLE 6

Find the complete solution of the system.

$$\begin{cases} -5x - 10y + 75z = 10 \\ -x + 7z = 4 \\ x + y - 11z = -3 \end{cases}$$

ANSWER

$(7k - 4, 4k + 1, k)$

EXAMPLE

A system with infinitely many solutions:

$$\begin{cases} x + 2y - 3z = 12 \\ -2x + y + 4z = 2 \\ x + 7y - 5z = 38 \end{cases}$$

ANSWER

$x = \frac{11}{5}t + \frac{8}{5}$, $y = \frac{2}{5}t + \frac{26}{5}$, $z = t$

Note: The representation of this set of solutions is not unique.

IN-CLASS MATERIALS

Point out that the techniques in this section are extensible in a way that some *ad hoc* techniques are not. One can solve an 8×8 or even a 100×100 system (in theory) using this method. If you want to pursue this line earlier, it is interesting to estimate the complexity of using this technique. Have the students solve a 2×2 system, keeping track of every multiplication they do, and every addition, then have them do the same for a 3×3. They can then do the same for a 4×4—not necessarily bothering to actually do all the additions and multiplications, just doing the count. Notice that the increase in complexity grows faster than a linear function.

To obtain the complete solution, we let t represent any real number, and we express x, y, and z in terms of t:

$$x = 7t - 5$$
$$y = 3t + 1$$
$$z = t$$

We can also write the solution as the ordered triple $(7t - 5, 3t + 1, t)$, where t is any real number. ■

In Example 6, to get specific solutions we give a specific value to t. For example, if $t = 1$, then

$$x = 7(1) - 5 = 2$$
$$y = 3(1) + 1 = 4$$
$$z = 1$$

Here are some other solutions of the system obtained by substituting other values for the parameter t.

Parameter t	Solution $(7t - 5, 3t + 1, t)$
-1	$(-12, -2, -1)$
0	$(-5, 1, 0)$
2	$(9, 7, 2)$
5	$(30, 16, 5)$

Example 7 A System with Infinitely Many Solutions

Find the complete solution of the system.

$$\begin{cases} x + 2y - 3z - 4w = 10 \\ x + 3y - 3z - 4w = 15 \\ 2x + 2y - 6z - 8w = 10 \end{cases}$$

Solution We transform the system into reduced row-echelon form.

$$\begin{bmatrix} 1 & 2 & -3 & -4 & 10 \\ 1 & 3 & -3 & -4 & 15 \\ 2 & 2 & -6 & -8 & 10 \end{bmatrix} \xrightarrow[R_3 - 2R_1 \to R_3]{R_2 - R_1 \to R_2} \begin{bmatrix} 1 & 2 & -3 & -4 & 10 \\ 0 & 1 & 0 & 0 & 5 \\ 0 & -2 & 0 & 0 & -10 \end{bmatrix}$$

$$\xrightarrow{R_3 + 2R_2 \to R_3} \begin{bmatrix} 1 & 2 & -3 & -4 & 10 \\ 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 - 2R_2 \to R_1} \begin{bmatrix} 1 & 0 & -3 & -4 & 0 \\ 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This is in reduced row-echelon form. Since the last row represents the equation $0 = 0$, we may discard it. So the last matrix corresponds to the system

$$\begin{cases} x \quad\;\; - 3z - 4w = 0 \\ \quad\; y \qquad\qquad\;\; = 5 \end{cases}$$

Leading variables

ALTERNATE EXAMPLE 7
Find the complete solution of the system.

$$\begin{cases} x + 2y - 4z - 5w = 14 \\ x + 3y - 4z - 5w = 21 \\ 2x + 2y - 8z - 10w = 14 \end{cases}$$

ANSWER
$(4m + 5n, 7, m, n)$

EXAMPLE
A pet shop has 100 puppies, kittens, and turtles. A puppy costs \$30, a kitten costs \$20, and a turtle costs \$5. If there are twice as many kittens as puppies, and if the stock is worth \$1050, how many of each type of animal is there?

ANSWER

$$p + k + t = 100$$
$$30p + 20k + 5t = 1050$$
$$2p - k + 0t = 0$$

10 puppies, 20 kittens, 70 turtles

To obtain the complete solution, we solve for the leading variables x and y in terms of the nonleading variables z and w, and we let z and w be any real numbers. Thus, the complete solution is

$$x = 3s + 4t$$
$$y = 5$$
$$z = s$$
$$w = t$$

where s and t are any real numbers.

We can also express the answer as the ordered quadruple $(3s + 4t, 5, s, t)$. ■

Note that s and t do *not* have to be the *same* real number in the solution for Example 7. We can choose arbitrary values for each if we wish to construct a specific solution to the system. For example, if we let $s = 1$ and $t = 2$, then we get the solution $(11, 5, 1, 2)$. You should check that this does indeed satisfy all three of the original equations in Example 7.

Examples 6 and 7 illustrate this general fact: If a system in row-echelon form has n nonzero equations in m variables $(m > n)$, then the complete solution will have $m - n$ nonleading variables. For instance, in Example 6 we arrived at *two* nonzero equations in the *three* variables x, y, and z, which gave us $3 - 2 = 1$ nonleading variable.


Courtesy of Caltech

Olga Taussky-Todd (1906–1995) was instrumental in developing applications of Matrix Theory. Described as "in love with anything matrices can do," she successfully applied matrices to aerodynamics, a field used in the design of airplanes and rockets. Taussky-Todd was also famous for her work in Number Theory, which deals with prime numbers and divisibility. Although Number Theory was once considered the least applicable branch of mathematics, it is now used in significant ways throughout the computer industry.

Taussky-Todd studied mathematics at a time when young women rarely aspired to be mathematicians. She said, "When I entered university I had no idea what it meant to study mathematics." One of the most respected mathematicians of her day, she was for many years a professor of mathematics at Caltech in Pasadena.

Modeling with Linear Systems

Linear equations, often containing hundreds or even thousands of variables, occur frequently in the applications of algebra to the sciences and to other fields. For now, let's consider an example that involves only three variables.

Example 8 Nutritional Analysis Using a System of Linear Equations

A nutritionist is performing an experiment on student volunteers. He wishes to feed one of his subjects a daily diet that consists of a combination of three commercial diet foods: MiniCal, LiquiFast, and SlimQuick. For the experiment it's important that the subject consume exactly 500 mg of potassium, 75 g of protein, and 1150 units of vitamin D every day. The amounts of these nutrients in one ounce of each food are given in the table. How many ounces of each food should the subject eat every day to satisfy the nutrient requirements exactly?

	MiniCal	LiquiFast	SlimQuick
Potassium (mg)	50	75	10
Protein (g)	5	10	3
Vitamin D (units)	90	100	50

Solution Let x, y, and z represent the number of ounces of MiniCal, LiquiFast, and SlimQuick, respectively, that the subject should eat every day. This means that he will get $50x$ mg of potassium from MiniCal, $75y$ mg from LiquiFast, and $10z$ mg from SlimQuick, for a total of $50x + 75y + 10z$ mg potassium in all. Since the

ALTERNATE EXAMPLE 8
A nutritionist is performing an experiment on student volunteers. He wishes to feed one of his subjects a daily diet that consists of a combination of three commercial diet foods: MiniCal, SloStarve, and SlimQuick. For the experiment it's important that the subject consume exactly 310 mg of potassium, 77 g of protein, and 1520 units of vitamin D every day. The amounts of these nutrients in one ounce of each food are given in the table. How many ounces of each food should the subject eat every day to satisfy the nutrient requirements exactly?

	Mini Cal	Slo Starve	Slim Quick
Potassium (mg)	25	75	10
Protein (g)	8	15	2
Vitamin D (units)	130	100	70

ANSWER
(5, 1, 11)

EXAMPLE
I have $6.50 in nickels, dimes, and quarters. I have twice as many nickels as dimes. That's a lot of nickels. In fact, if you add the number of dimes I have to twice the number of quarters I have, you get the number of nickels I have. How many nickels do I have?

ANSWER

$$\begin{aligned} 5n + 10d + 25q &= 650 \\ n - 2d + 0q &= 0 \\ -n + d + 2q &= 0 \end{aligned}$$

40 nickels, 20 dimes, 10 quarters

```
rref([A])
   [[1 0 0 5 ]
    [0 1 0 2 ]
    [0 0 1 10]]
```

Figure 4

Check Your Answer

$x = 5, y = 2, z = 10$:

$$\begin{cases} 10(5) + 15(2) + 2(10) = 100 \\ 5(5) + 10(2) + 3(10) = 75 \\ 9(5) + 10(2) + 5(10) = 115 \end{cases} \checkmark$$

potassium requirement is 500 mg, we get the first equation below. Similar reasoning for the protein and vitamin D requirements leads to the system

$$\begin{cases} 50x + 75y + 10z = 500 & \text{Potassium} \\ 5x + 10y + 3z = 75 & \text{Protein} \\ 90x + 100y + 50z = 1150 & \text{Vitamin D} \end{cases}$$

Dividing the first equation by 5 and the third one by 10 gives the system

$$\begin{cases} 10x + 15y + 2z = 100 \\ 5x + 10y + 3z = 75 \\ 9x + 10y + 5z = 115 \end{cases}$$

We can solve this system using Gaussian elimination, or we can use a graphing calculator to find the reduced row-echelon form of the augmented matrix of the system. Using the `rref` command on the TI-83, we get the output in Figure 4. From the reduced row-echelon form we see that $x = 5$, $y = 2$, $z = 10$. The subject should be fed 5 oz of MiniCal, 2 oz of LiquiFast, and 10 oz of SlimQuick every day. ■

A more practical application might involve dozens of foods and nutrients rather than just three. Such problems lead to systems with large numbers of variables and equations. Computers or graphing calculators are essential for solving such large systems.

9.4 Exercises

1–6 ■ State the dimension of the matrix.

1. $\begin{bmatrix} 2 & 7 \\ 0 & -1 \\ 5 & -3 \end{bmatrix}$ **2.** $\begin{bmatrix} -1 & 5 & 4 & 0 \\ 0 & 2 & 11 & 3 \end{bmatrix}$ **3.** $\begin{bmatrix} 12 \\ 35 \end{bmatrix}$

4. $\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$ **5.** $\begin{bmatrix} 1 & 4 & 7 \end{bmatrix}$ **6.** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

7–14 ■ A matrix is given.

(a) Determine whether the matrix is in row-echelon form.

(b) Determine whether the matrix is in reduced row-echelon form.

(c) Write the system of equations for which the given matrix is the augmented matrix.

7. $\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 5 \end{bmatrix}$ **8.** $\begin{bmatrix} 1 & 3 & -3 \\ 0 & 1 & 5 \end{bmatrix}$

9. $\begin{bmatrix} 1 & 2 & 8 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ **10.** $\begin{bmatrix} 1 & 0 & -7 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

11. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 5 & 1 \end{bmatrix}$ **12.** $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$

13. $\begin{bmatrix} 1 & 3 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ **14.** $\begin{bmatrix} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$

15–24 ■ The system of linear equations has a unique solution. Find the solution using Gaussian elimination or Gauss-Jordan elimination.

15. $\begin{cases} x - 2y + z = 1 \\ y + 2z = 5 \\ x + y + 3z = 8 \end{cases}$ **16.** $\begin{cases} x + y + 6z = 3 \\ x + y + 3z = 3 \\ x + 2y + 4z = 7 \end{cases}$

17. $\begin{cases} x + y + z = 2 \\ 2x - 3y + 2z = 4 \\ 4x + y - 3z = 1 \end{cases}$ **18.** $\begin{cases} x + y + z = 4 \\ -x + 2y + 3z = 17 \\ 2x - y = -7 \end{cases}$

19. $\begin{cases} x + 2y - z = -2 \\ x + z = 0 \\ 2x - y - z = -3 \end{cases}$ **20.** $\begin{cases} 2y + z = 4 \\ x + y = 4 \\ 3x + 3y - z = 10 \end{cases}$

21. $\begin{cases} x_1 + 2x_2 - x_3 = 9 \\ 2x_1 - x_3 = -2 \\ 3x_1 + 5x_2 + 2x_3 = 22 \end{cases}$ **22.** $\begin{cases} 2x_1 + x_2 = 7 \\ 2x_1 - x_2 + x_3 = 6 \\ 3x_1 - 2x_2 + 4x_3 = 11 \end{cases}$

23. $\left\{\begin{array}{l} 2x - 3y - z = 13 \\ -x + 2y - 5z = 6 \\ 5x - y - z = 49 \end{array}\right.$

24. $\left\{\begin{array}{l} 10x + 10y - 20z = 60 \\ 15x + 20y + 30z = -25 \\ -5x + 30y - 10z = 45 \end{array}\right.$

25–34 ■ Determine whether the system of linear equations is inconsistent or dependent. If it is dependent, find the complete solution.

25. $\left\{\begin{array}{l} x + y + z = 2 \\ y - 3z = 1 \\ 2x + y + 5z = 0 \end{array}\right.$

26. $\left\{\begin{array}{l} x + 3z = 3 \\ 2x + y - 2z = 5 \\ -y + 8z = 8 \end{array}\right.$

27. $\left\{\begin{array}{l} 2x - 3y - 9z = -5 \\ x + 3z = 2 \\ -3x + y - 4z = -3 \end{array}\right.$

28. $\left\{\begin{array}{l} x - 2y + 5z = 3 \\ -2x + 6y - 11z = 1 \\ 3x - 16y + 20z = -26 \end{array}\right.$

29. $\left\{\begin{array}{l} x - y + 3z = 3 \\ 4x - 8y + 32z = 24 \\ 2x - 3y + 11z = 4 \end{array}\right.$

30. $\left\{\begin{array}{l} -2x + 6y - 2z = -12 \\ x - 3y + 2z = 10 \\ -x + 3y + 2z = 6 \end{array}\right.$

31. $\left\{\begin{array}{l} x + 4y - 2z = -3 \\ 2x - y + 5z = 12 \\ 8x + 5y + 11z = 30 \end{array}\right.$

32. $\left\{\begin{array}{l} 3r + 2s - 3t = 10 \\ r - s - t = -5 \\ r + 4s - t = 20 \end{array}\right.$

33. $\left\{\begin{array}{l} 2x + y - 2z = 12 \\ -x - \frac{1}{2}y + z = -6 \\ 3x + \frac{3}{2}y - 3z = 18 \end{array}\right.$

34. $\left\{\begin{array}{l} y - 5z = 7 \\ 3x + 2y = 12 \\ 3x + 10z = 80 \end{array}\right.$

35–46 ■ Solve the system of linear equations.

35. $\left\{\begin{array}{l} 4x - 3y + z = -8 \\ -2x + y - 3z = -4 \\ x - y + 2z = 3 \end{array}\right.$

36. $\left\{\begin{array}{l} 2x - 3y + 5z = 14 \\ 4x - y - 2z = -17 \\ -x - y + z = 3 \end{array}\right.$

37. $\left\{\begin{array}{l} x + 2y - 3z = -5 \\ -2x - 4y - 6z = 10 \\ 3x + 7y - 2z = -13 \end{array}\right.$

38. $\left\{\begin{array}{l} 3x - y + 2z = -1 \\ 4x - 2y + z = -7 \\ -x + 3y - 2z = -1 \end{array}\right.$

39. $\left\{\begin{array}{l} -x + 2y + z - 3w = 3 \\ 3x - 4y + z + w = 9 \\ -x - y + z + w = 0 \\ 2x + y + 4z - 2w = 3 \end{array}\right.$

40. $\left\{\begin{array}{l} x + y - z - w = 6 \\ 2x + z - 3w = 8 \\ x - y + 4w = -10 \\ 3x + 5y - z - w = 20 \end{array}\right.$

41. $\left\{\begin{array}{l} x + y + 2z - w = -2 \\ 3y + z + 2w = 2 \\ x + y + 3w = 2 \\ -3x + z + 2w = 5 \end{array}\right.$

42. $\left\{\begin{array}{l} x - 3y + 2z + w = -2 \\ x - 2y - 2w = -10 \\ z + 5w = 15 \\ 3x + 2z + w = -3 \end{array}\right.$

43. $\left\{\begin{array}{l} x + z + w = 4 \\ y - z = -4 \\ x - 2y + 3z + w = 12 \\ 2x - 2z + 5w = -1 \end{array}\right.$

44. $\left\{\begin{array}{l} y - z + 2w = 0 \\ 3x + 2y + w = 0 \\ 2x + 4w = 12 \\ -2x - 2z + 5w = 6 \end{array}\right.$

45. $\left\{\begin{array}{l} x - y + w = 0 \\ 3x - z + 2w = 0 \\ x - 4y + z + 2w = 0 \end{array}\right.$

46. $\left\{\begin{array}{l} 2x - y + 2z + w = 5 \\ -x + y + 4z - w = 3 \\ 3x - 2y - z = 0 \end{array}\right.$

Applications

47. Nutrition A doctor recommends that a patient take 50 mg each of niacin, riboflavin, and thiamin daily to alleviate a vitamin deficiency. In his medicine chest at home, the patient finds three brands of vitamin pills. The amounts of the relevant vitamins per pill are given in the table. How many pills of each type should he take every day to get 50 mg of each vitamin?

	VitaMax	Vitron	VitaPlus
Niacin (mg)	5	10	15
Riboflavin (mg)	15	20	0
Thiamin (mg)	10	10	10

48. Mixtures A chemist has three acid solutions at various concentrations. The first is 10% acid, the second is 20%, and the third is 40%. How many milliliters of each should he use to make 100 mL of 18% solution, if he has to use four times as much of the 10% solution as the 40% solution?

49. Distance, Speed, and Time Amanda, Bryce, and Corey enter a race in which they have to run, swim, and cycle over a marked course. Their average speeds are given in the table. Corey finishes first with a total time of 1 h 45 min. Amanda comes in second with a time of 2 h 30 min. Bryce finishes last with a time of 3 h. Find the distance (in miles) for each part of the race.

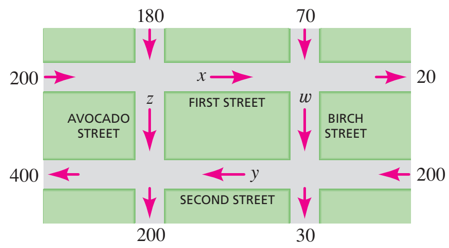
	Average speed (mi/h) Running	Swimming	Cycling
Amanda	10	4	20
Bryce	$7\frac{1}{2}$	6	15
Corey	15	3	40

50. Classroom Use A small school has 100 students who occupy three classrooms: A, B, and C. After the first period of the school day, half the students in room A move to room B, one-fifth of the students in room B move to room C, and one-third of the students in room C move to room A. Nevertheless, the total number of students in each room is the same for both periods. How many students occupy each room?

51. Manufacturing Furniture A furniture factory makes wooden tables, chairs, and armoires. Each piece of furniture requires three operations: cutting the wood, assembling, and finishing. Each operation requires the number of hours (h) given in the table. The workers in the factory can provide 300 hours of cutting, 400 hours of assembling, and 590 hours of finishing each work week. How many tables, chairs, and armoires should be produced so that all available labor-hours are used? Or is this impossible?

	Table	Chair	Armoire
Cutting (h)	$\frac{1}{2}$	1	1
Assembling (h)	$\frac{1}{2}$	$1\frac{1}{2}$	1
Finishing (h)	1	$1\frac{1}{2}$	2

52. Traffic Flow A section of a city's street network is shown in the figure. The arrows indicate one-way streets, and the numbers show how many cars enter or leave this section of the city via the indicated street in a certain one-hour period. The variables x, y, z, and w represent the number of cars that travel along the portions of First, Second, Avocado, and Birch Streets during this period. Find x, y, z, and w, assuming that none of the cars stop or park on any of the streets shown.



Discovery • Discussion

53. Polynomials Determined by a Set of Points We all know that two points uniquely determine a line $y = ax + b$ in the coordinate plane. Similarly, three points uniquely determine a quadratic (second-degree) polynomial

$$y = ax^2 + bx + c$$

four points uniquely determine a cubic (third-degree) polynomial

$$y = ax^3 + bx^2 + cx + d$$

and so on. (Some exceptions to this rule are if the three points actually lie on a line, or the four points lie on a quadratic or line, and so on.) For the following set of five points, find the line that contains the first two points, the quadratic that contains the first three points, the cubic that contains the first four points, and the fourth-degree polynomial that contains all five points.

$$(0, 0), \quad (1, 12), \quad (2, 40), \quad (3, 6), \quad (-1, -14)$$

Graph the points and functions in the same viewing rectangle using a graphing device.

9.5 The Algebra of Matrices

Thus far we've used matrices simply for notational convenience when solving linear systems. Matrices have many other uses in mathematics and the sciences, and for most of these applications a knowledge of matrix algebra is essential. Like numbers, matrices can be added, subtracted, multiplied, and divided. In this section we learn how to perform these algebraic operations on matrices.

SUGGESTED TIME AND EMPHASIS

1 class.
Recommended material.

POINTS TO STRESS

1. Matrix addition.
2. Scalar and matrix multiplication.

Equality of Matrices

Two matrices are equal if they have the same entries in the same positions.

Equal Matrices

$$\begin{bmatrix} \sqrt{4} & 2^2 & e^0 \\ 0.5 & 1 & 1-1 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 1 \\ \frac{1}{2} & \frac{2}{2} & 0 \end{bmatrix}$$

Unequal Matrices

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \neq \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

Equality of Matrices

The matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are **equal** if and only if they have the same dimension $m \times n$, and corresponding entries are equal, that is,

$$a_{ij} = b_{ij}$$

for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

ALTERNATE EXAMPLE 1

Find $a, b, c,$ and d if

$$\begin{bmatrix} a & 2 \\ 3 & b \end{bmatrix} = \begin{bmatrix} 4 & c \\ d & 2 \end{bmatrix}$$

ANSWER

$a = 4, b = 2, c = 2, d = 3$

Example 1 Equal Matrices

Find $a, b, c,$ and d, if

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 5 & 2 \end{bmatrix}$$

Solution Since the two matrices are equal, corresponding entries must be the same. So we must have $a = 1$, $b = 3$, $c = 5$, and $d = 2$. ■

Addition, Subtraction, and Scalar Multiplication of Matrices

Two matrices can be added or subtracted if they have the same dimension. (Otherwise, their sum or difference is undefined.) We add or subtract the matrices by adding or subtracting corresponding entries. To multiply a matrix by a number, we multiply every element of the matrix by that number. This is called the *scalar product*.

Sum, Difference, and Scalar Product of Matrices

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be matrices of the same dimension $m \times n$, and let c be any real number.

1. The **sum** $A + B$ is the $m \times n$ matrix obtained by adding corresponding entries of A and B.

$$A + B = [a_{ij} + b_{ij}]$$

2. The **difference** $A - B$ is the $m \times n$ matrix obtained by subtracting corresponding entries of A and B.

$$A - B = [a_{ij} - b_{ij}]$$

3. The **scalar product** cA is the $m \times n$ matrix obtained by multiplying each entry of A by c.

$$cA = [ca_{ij}]$$

SAMPLE QUESTIONS

Text Questions

True or false:

(a) $\begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} -1 & 5 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -3 & 20 \\ 1 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} -1 & 5 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 9 \\ 2 & 2 \end{bmatrix}$

Answers

(a) False
(b) True

Example 2 Performing Algebraic Operations on Matrices

Let $A = \begin{bmatrix} 2 & -3 \\ 0 & 5 \\ 7 & -\frac{1}{2} \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 & 0 \\ -3 & 1 \\ 2 & 2 \end{bmatrix}$

$C = \begin{bmatrix} 7 & -3 & 0 \\ 0 & 1 & 5 \end{bmatrix}$ $\quad D = \begin{bmatrix} 6 & 0 & -6 \\ 8 & 1 & 9 \end{bmatrix}$

Carry out each indicated operation, or explain why it cannot be performed.

(a) $A + B$ (b) $C - D$ (c) $C + A$ (d) $5A$

Solution

(a) $$A + B = \begin{bmatrix} 2 & -3 \\ 0 & 5 \\ 7 & -\frac{1}{2} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ -3 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ -3 & 6 \\ 9 & \frac{3}{2} \end{bmatrix}$$

(b) $$C - D = \begin{bmatrix} 7 & -3 & 0 \\ 0 & 1 & 5 \end{bmatrix} - \begin{bmatrix} 6 & 0 & -6 \\ 8 & 1 & 9 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 6 \\ -8 & 0 & -4 \end{bmatrix}$$

(c) $C + A$ is undefined because we can't add matrices of different dimensions.

(d) $$5A = 5\begin{bmatrix} 2 & -3 \\ 0 & 5 \\ 7 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 10 & -15 \\ 0 & 25 \\ 35 & -\frac{5}{2} \end{bmatrix}$$ ■

The properties in the box follow from the definitions of matrix addition and scalar multiplication, and the corresponding properties of real numbers.

Properties of Addition and Scalar Multiplication of Matrices

Let A, B, and C be $m \times n$ matrices and let c and d be scalars.

$A + B = B + A$	Commutative Property of Matrix Addition
$(A + B) + C = A + (B + C)$	Associative Property of Matrix Addition
$c(dA) = (cd)A$	Associative Property of Scalar Multiplication
$(c + d)A = cA + dA$	Distributive Properties of Scalar
$c(A + B) = cA + cB$	Multiplication

Example 3 Solving a Matrix Equation

Solve the matrix equation

$$2X - A = B$$

for the unknown matrix X, where

$$A = \begin{bmatrix} 2 & 3 \\ -5 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & -1 \\ 1 & 3 \end{bmatrix}$$

ALTERNATE EXAMPLE 2

Let $A = \begin{bmatrix} 6 & 0 \\ -2 & 4 \\ 1 & 2 \end{bmatrix}$

$B = \begin{bmatrix} 4 & 2 \\ 0 & 0 \\ 12 & -2 \end{bmatrix}$

$C = \begin{bmatrix} 3 & 2 & 5 \\ 1 & 2 & 9 \end{bmatrix}$

$D = \begin{bmatrix} -5 & 4 & 1 \\ 1 & -6 & 2 \end{bmatrix}$

Carry out each indicated operation or explain why it can't be performed

(a) $C + D$
(b) $B - C$
(c) $A + 2B$
(d) $3B$

ANSWERS

(a) $C + D = \begin{bmatrix} -2 & 6 & 6 \\ 2 & -4 & 11 \end{bmatrix}$

(b) Not possible; the dimensions are not the same.

(c) $A + 2B = \begin{bmatrix} 14 & 4 \\ -2 & 4 \\ 25 & -2 \end{bmatrix}$

(d) $3B = \begin{bmatrix} 12 & 6 \\ 0 & 0 \\ 36 & -6 \end{bmatrix}$

ALTERNATE EXAMPLE 3

Solve the matrix equation
$3X - 4A = B$
for the unknown matrix X where

$A = \begin{bmatrix} 4 & -1 \\ 3 & 2 \end{bmatrix}$ and

$B = \begin{bmatrix} -13 & 1 \\ -3 & -2 \end{bmatrix}$.

ANSWER

$X = \begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix}$.

IN-CLASS MATERIALS

The text describes which properties of real number addition and multiplication carry over to matrix addition, matrix multiplication, and scalar multiplication. Discuss how some of their consequences carry over as well. Ask the students if they believe that $(A + B)(A + B) = AA + 2AB + BB$. Let them discuss and argue. It turns out that this is false, because of commutativity. It is true that $(A + B)(A + B) = AA + AB + BA + BB$. Have the class look at $(A + B)(A - B)$ next.


The National Academy of Sciences

Julia Robinson (1919–1985) was born in St. Louis, Missouri, and grew up at Point Loma, California. Due to an illness, Robinson missed two years of school but later, with the aid of a tutor, she completed fifth, sixth, seventh, and eighth grades, all in one year. Later at San Diego State University, reading biographies of mathematicians in E. T. Bell's *Men of Mathematics* awakened in her what became a lifelong passion for mathematics. She said, "I cannot overemphasize the importance of such books . . . in the intellectual life of a student." Robinson is famous for her work on Hilbert's tenth problem (page 708), which asks for a general procedure for determining whether an equation has integer solutions. Her ideas led to a complete answer to the problem. Interestingly, the answer involved certain properties of the Fibonacci numbers (page 826) discovered by the then 22-year-old Russian mathematician Yuri Matijasevič. As a result of her brilliant work on Hilbert's tenth problem, Robinson was offered a professorship at the University of California, Berkeley, and became the first woman mathematician elected to the National Academy of Sciences. She also served as president of the American Mathematical Society.

Solution We use the properties of matrices to solve for X.

$$
\begin{aligned}
2X - A &= B && \text{Given equation}\\
2X &= B + A && \text{Add the matrix } A \text{ to each side}\\
X &= \tfrac{1}{2}(B + A) && \text{Multiply each side by the scalar } \tfrac{1}{2}
\end{aligned}
$$

So
$$
\begin{aligned}
X &= \frac{1}{2}\left(\begin{bmatrix} 4 & -1 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ -5 & 1 \end{bmatrix}\right) && \text{Substitute the matrices } A \text{ and } B\\
&= \frac{1}{2}\begin{bmatrix} 6 & 2 \\ -4 & 4 \end{bmatrix} && \text{Add matrices}\\
&= \begin{bmatrix} 3 & 1 \\ -2 & 2 \end{bmatrix} && \text{Multiply by the scalar } \tfrac{1}{2}
\end{aligned}
$$
■

Multiplication of Matrices

Multiplying two matrices is more difficult to describe than other matrix operations. In later examples we will see why taking the matrix product involves a rather complex procedure, which we now describe.

First, the product AB (or $A \cdot B$) of two matrices A and B is defined only when the number of columns in A is equal to the number of rows in B. This means that if we write their dimensions side by side, the two inner numbers must match:

Matrices	A	B
Dimensions	$m \times n$	$n \times k$
	Columns in A	Rows in B

If the dimensions of A and B match in this fashion, then the product AB is a matrix of dimension $m \times k$. Before describing the procedure for obtaining the elements of AB, we define the *inner product* of a row of A and a column of B.

If $[a_1 \quad a_2 \cdots a_n]$ is a row of A, and if $\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ is a column of B, then

their **inner product** is the number $a_1b_1 + a_2b_2 + \cdots + a_nb_n$. For example, taking

the inner product of $[2 \quad -1 \quad 0 \quad 4]$ and $\begin{bmatrix} 5 \\ 4 \\ -3 \\ \frac{1}{2} \end{bmatrix}$ gives

$$2 \cdot 5 + (-1) \cdot 4 + 0 \cdot (-3) + 4 \cdot \tfrac{1}{2} = 8$$

IN-CLASS MATERIALS

Let A and B be 2×2 matrices. Although AB may not be equal to BA, there are special matrices for which $AB = BA$. One necessary (but not sufficient) condition for this to happen is that $a_{12}b_{21} = a_{21}b_{12}$. It is relatively simple to show this condition, by explicitly multiplying out AB and BA. After demonstrating this condition, challenge the students to find a pair of distinct matrices, without zero elements, such that $AB = BA$. The process of searching for them will give students good practice multiplying matrices. One example that works is

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} -1 & 2 \\ 3 & 2 \end{bmatrix}$$

We now define the **product** AB of two matrices.

Matrix Multiplication

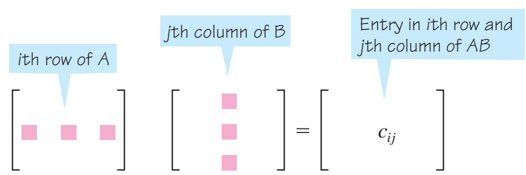
If $A = [a_{ij}]$ is an $m \times n$ matrix and $B = [b_{ij}]$ an $n \times k$ matrix, then their product is the $m \times k$ matrix

$$C = [c_{ij}]$$

where c_{ij} is the inner product of the ith row of A and the jth column of B. We write the product as

$$C = AB$$

This definition of matrix product says that each entry in the matrix AB is obtained from a *row* of A and a *column* of B as follows: The entry c_{ij} in the ith row and jth column of the matrix AB is obtained by multiplying the entries in the ith row of A with the corresponding entries in the jth column of B and adding the results.



Example 4 Multiplying Matrices

Let

$$A = \begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & 5 & 2 \\ 0 & 4 & 7 \end{bmatrix}$$

Calculate, if possible, the products AB and BA.

Solution Since A has dimension 2×2 and B has dimension 2×3, the product AB is defined and has dimension 2×3. We can thus write

Inner numbers match, so product is defined.

$2 \times 2 \quad 2 \times 3$

Outer numbers give dimension of product: 2×3.

$$AB = \begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 5 & 2 \\ 0 & 4 & 7 \end{bmatrix} = \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \end{bmatrix}$$

where the question marks must be filled in using the rule defining the product of two matrices. If we define $C = AB = [c_{ij}]$, then the entry c_{11} is the inner product of the first row of A and the first column of B:

$$\begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 5 & 2 \\ 0 & 4 & 7 \end{bmatrix} \qquad 1 \cdot (-1) + 3 \cdot 0 = -1$$

ALTERNATE EXAMPLE 4

Let $A = \begin{bmatrix} 4 & 1 \\ -3 & 0 \\ 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix}$.

Calculate, if possible, the products AB and BA.

ANSWER

AB has dimension 3×2.

$$AB = \begin{bmatrix} -1 & 9 \\ 3 & -6 \\ -5 & 3 \end{bmatrix}$$

BA is not defined.

IN-CLASS MATERIALS

This is a good time to discuss permutation matrices. A permutation matrix is a matrix that is all zeros except for a single 1 in each row and each column.

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Have the class figure out if the sum of two permutation matrices must always be a permutation matrix (no) and if the product of two permutation matrices must always be a permutation matrix (yes). Finally, have the students multiply arbitrary matrices by permutation matrices, to see what happens to them (the rows or columns get rearranged, depending on whether the permutation matrix was multiplied on the left or on the right).

DRILL QUESTION

Compute $\begin{bmatrix} 5 & 9 & 2 \\ 6 & 5 & 3 \end{bmatrix}\begin{bmatrix} 5 \\ -1 \\ 1 \end{bmatrix}$.

Answer

$\begin{bmatrix} 18 \\ 28 \end{bmatrix}$

Similarly, we calculate the remaining entries of the product as follows.

Entry	Inner product of:	Value	Product matrix
c_{12}	$\begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 5 & 2 \\ 0 & 4 & 7 \end{bmatrix}$	$1 \cdot 5 + 3 \cdot 4 = 17$	$\begin{bmatrix} -1 & 17 & \\ & & \end{bmatrix}$
c_{13}	$\begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 5 & 2 \\ 0 & 4 & 7 \end{bmatrix}$	$1 \cdot 2 + 3 \cdot 7 = 23$	$\begin{bmatrix} -1 & 17 & 23 \\ & & \end{bmatrix}$
c_{21}	$\begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 5 & 2 \\ 0 & 4 & 7 \end{bmatrix}$	$(-1) \cdot (-1) + 0 \cdot 0 = 1$	$\begin{bmatrix} -1 & 17 & 23 \\ 1 & & \end{bmatrix}$
c_{22}	$\begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 5 & 2 \\ 0 & 4 & 7 \end{bmatrix}$	$(-1) \cdot 5 + 0 \cdot 4 = -5$	$\begin{bmatrix} -1 & 17 & 23 \\ 1 & -5 & \end{bmatrix}$
c_{23}	$\begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 5 & 2 \\ 0 & 4 & 7 \end{bmatrix}$	$(-1) \cdot 2 + 0 \cdot 7 = -2$	$\begin{bmatrix} -1 & 17 & 23 \\ 1 & -5 & -2 \end{bmatrix}$

Thus, we have

$$AB = \begin{bmatrix} -1 & 17 & 23 \\ 1 & -5 & -2 \end{bmatrix}$$

The product BA is not defined, however, because the dimensions of B and A are

$$2 \times 3 \qquad \text{and} \qquad 2 \times 2$$

Not equal, so product not defined.

$2 \times 3 \quad 2 \times 2$

The inner two numbers are not the same, so the rows and columns won't match up when we try to calculate the product. ■

```
[A]*[B]
    [[-1 17 23]
     [1  -5 -2]]
```

Figure 1

Graphing calculators and computers are capable of performing matrix algebra. For instance, if we enter the matrices in Example 4 into the matrix variables [A] and [B] on a TI-83 calculator, then the calculator finds their product as shown in Figure 1.

Properties of Matrix Multiplication

Although matrix multiplication is not commutative, it does obey the Associative and Distributive Properties.

Properties of Matrix Multiplication

Let A, B, and C be matrices for which the following products are defined. Then

$A(BC) = (AB)C$	Associative Property
$A(B + C) = AB + AC$ $(B + C)A = BA + CA$	Distributive Property

IN-CLASS MATERIALS

Foreshadow the next section by having students compute

$$\begin{bmatrix} 2 & 5 & 1 \\ 4 & 2 & 1 \\ 5 & 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

The next example shows that even when both AB and BA are defined, they aren't necessarily equal. This result proves that matrix multiplication is *not* commutative.

Example 5 **Matrix Multiplication Is Not Commutative**

Let $A = \begin{bmatrix} 5 & 7 \\ -3 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 9 & -1 \end{bmatrix}$

Calculate the products AB and BA.

Solution Since both matrices A and B have dimension 2×2, both products AB and BA are defined, and each product is also a 2×2 matrix.

$$AB = \begin{bmatrix} 5 & 7 \\ -3 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 9 & -1 \end{bmatrix} = \begin{bmatrix} 5\cdot 1 + 7\cdot 9 & 5\cdot 2 + 7\cdot(-1) \\ (-3)\cdot 1 + 0\cdot 9 & (-3)\cdot 2 + 0\cdot(-1) \end{bmatrix}$$

$$= \begin{bmatrix} 68 & 3 \\ -3 & -6 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 2 \\ 9 & -1 \end{bmatrix}\begin{bmatrix} 5 & 7 \\ -3 & 0 \end{bmatrix} = \begin{bmatrix} 1\cdot 5 + 2\cdot(-3) & 1\cdot 7 + 2\cdot 0 \\ 9\cdot 5 + (-1)\cdot(-3) & 9\cdot 7 + (-1)\cdot 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 7 \\ 48 & 63 \end{bmatrix}$$

This shows that, in general, $AB \neq BA$. In fact, in this example AB and BA don't even have an entry in common. ■

ALTERNATE EXAMPLE 5

Let $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 0 \\ 2 & 3 \end{bmatrix}$. Calculate the products AB and BA. Does $AB = BA$?

ANSWER

$AB = \begin{bmatrix} 10 & 6 \\ 2 & 12 \end{bmatrix}$

$BA = \begin{bmatrix} 6 & 12 \\ -1 & 16 \end{bmatrix}$

They are not equal.

Applications of Matrix Multiplication

We now consider some applied examples that give some indication of why mathematicians chose to define the matrix product in such an apparently bizarre fashion. Example 6 shows how our definition of matrix product allows us to express a system of linear equations as a single matrix equation.

Example 6 **Writing a Linear System as a Matrix Equation**

Show that the following matrix equation is equivalent to the system of equations in Example 2 of Section 9.4.

$$\begin{bmatrix} 1 & -1 & 3 \\ 1 & 2 & -2 \\ 3 & -1 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \\ 14 \end{bmatrix}$$

Matrix equations like this one are described in more detail on page 694.

Solution If we perform matrix multiplication on the left side of the equation, we get

$$\begin{bmatrix} x - y + 3z \\ x + 2y - 2z \\ 3x - y + 5z \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \\ 14 \end{bmatrix}$$

ALTERNATE EXAMPLE 6

Find a system of equations that is equivalent to the following matrix equation:

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & 0 \\ 2 & 6 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

ANSWER

$$\begin{aligned} x + y + z &= 1 \\ 2x - 3y &= 2 \\ 2x + 6y + 2z &= 5 \end{aligned}$$

EXAMPLES

Let $A = \begin{bmatrix} 3 & 1 \\ -1 & 0 \\ 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 1 \\ 1 & 2 \\ -6 & -1 \end{bmatrix}$, and $C = \begin{bmatrix} 3 & 1 \\ 2 & 5 \end{bmatrix}$.

$A + B = \begin{bmatrix} 8 & 2 \\ 0 & 2 \\ -3 & 1 \end{bmatrix}$ $B - A = \begin{bmatrix} 2 & 0 \\ 2 & 2 \\ -9 & -3 \end{bmatrix}$ $AC = \begin{bmatrix} 11 & 8 \\ -3 & -1 \\ 13 & 13 \end{bmatrix}$ $BC = \begin{bmatrix} 17 & 10 \\ 7 & 11 \\ -20 & -11 \end{bmatrix}$ $(A + B)C = \begin{bmatrix} 28 & 18 \\ 4 & 10 \\ -7 & 2 \end{bmatrix}$

If using these examples, occasionally throw in an undefined operation such as $A + C$ or AB.

Because two matrices are equal only if their corresponding entries are equal, we equate entries to get

$$\begin{cases} x - y + 3z = 4 \\ x + 2y - 2z = 10 \\ 3x - y + 5z = 14 \end{cases}$$

This is exactly the system of equations in Example 2 of Section 9.4. ■

Mathematics in the Modern World

Fair Voting Methods

The methods of mathematics have recently been applied to problems in the social sciences. For example, how do we find fair voting methods? You may ask, what is the problem with how we vote in elections? Well, suppose candidates A, B, and C are running for president. The final vote tally is as follows: A gets 40%, B gets 39%, and C gets 21%. So candidate A wins. But 60% of the voters didn't want A. Moreover, you voted for C, but you dislike A so much that you would have been willing to change your vote to B to avoid having A win. Most of the voters who voted for C feel the same way you do, so we have a situation where most of the voters prefer B over A, but A wins. So is that fair?

In the 1950s Kenneth Arrow showed mathematically that no democratic method of voting can be completely fair, and later won a Nobel Prize for his work. Mathematicians continue to work on finding fairer voting systems. The system most often used in federal, state, and local elections is called plurality voting (the candidate with the most votes wins). Other systems include majority voting (if no candidate gets a majority, a runoff is held between the top two vote-getters), approval voting (each voter can vote for as many candidates as he or she approves of), preference voting (each voter orders the candidates according to his or her preference), and cumulative voting (each voter gets as many votes as there are candidates

(continued)

ALTERNATE EXAMPLE 7

In a certain city the proportion of voters in each age group who are registered as Democrats, Republicans, or Independents is given by the following matrix.

$$\begin{array}{r c} & \begin{array}{ccc} & \text{Age} & \\ \text{18–30} & \text{31–50} & \text{Over 50} \end{array} \\ \begin{array}{r} \text{Democrat} \\ \text{Republican} \\ \text{Independent} \end{array} & \begin{bmatrix} 0.35 & 0.55 & 0.50 \\ 0.50 & 0.20 & 0.25 \\ 0.15 & 0.25 & 0.25 \end{bmatrix} = A \end{array}$$

The next matrix gives the distribution, by age and sex, of the voting population of this city.

$$\begin{array}{r c} & \begin{array}{cc} \text{Male} & \text{Female} \end{array} \\ \text{Age}\ \begin{array}{r} \text{18–30} \\ \text{31–50} \\ \text{Over 50} \end{array} & \begin{bmatrix} 7{,}000 & 8{,}000 \\ 15{,}000 & 18{,}000 \\ 18{,}000 & 19{,}000 \end{bmatrix} = B \end{array}$$

For this problem, let's make the assumption that within each age group, political preference is not related to gender. That is, the percentage of Democratic males in the 18–30 group, for example, is the same as the percentage of Democratic females in this group. Find how many females are registered as Independents in this city.

ANSWER

10,450

Example 7 Representing Demographic Data by Matrices

In a certain city the proportion of voters in each age group who are registered as Democrats, Republicans, or Independents is given by the following matrix.

$$\begin{array}{r c} & \begin{array}{ccc} & \text{Age} & \\ \text{18–30} & \text{31–50} & \text{Over 50} \end{array} \\ \begin{array}{r} \text{Democrat} \\ \text{Republican} \\ \text{Independent} \end{array} & \begin{bmatrix} 0.30 & 0.60 & 0.50 \\ 0.50 & 0.35 & 0.25 \\ 0.20 & 0.05 & 0.25 \end{bmatrix} = A \end{array}$$

The next matrix gives the distribution, by age and sex, of the voting population of this city.

$$\begin{array}{r c} & \begin{array}{cc} \text{Male} & \text{Female} \end{array} \\ \text{Age}\ \begin{array}{r} \text{18–30} \\ \text{31–50} \\ \text{Over 50} \end{array} & \begin{bmatrix} 5{,}000 & 6{,}000 \\ 10{,}000 & 12{,}000 \\ 12{,}000 & 15{,}000 \end{bmatrix} = B \end{array}$$

For this problem, let's make the (highly unrealistic) assumption that within each age group, political preference is not related to gender. That is, the percentage of Democrat males in the 18–30 group, for example, is the same as the percentage of Democrat females in this group.

(a) Calculate the product AB.

(b) How many males are registered as Democrats in this city?

(c) How many females are registered as Republicans?

Solution

(a) $$AB = \begin{bmatrix} 0.30 & 0.60 & 0.50 \\ 0.50 & 0.35 & 0.25 \\ 0.20 & 0.05 & 0.25 \end{bmatrix} \begin{bmatrix} 5{,}000 & 6{,}000 \\ 10{,}000 & 12{,}000 \\ 12{,}000 & 15{,}000 \end{bmatrix} = \begin{bmatrix} 13{,}500 & 16{,}500 \\ 9{,}000 & 10{,}950 \\ 4{,}500 & 5{,}550 \end{bmatrix}$$

(b) When we take the inner product of a row in A with a column in B, we are adding the number of people in each age group who belong to the category in question. For example, the entry c_{21} of AB (the 9000) is obtained by taking the inner product of the Republican row in A with the Male column in B. This and can give all of his or her votes to one candidate or distribute them among the candidates as he or she sees fit). This last system is often used to select corporate boards of directors. Each system of voting has both advantages and disadvantages.

number is therefore the total number of male Republicans in this city. We can label the rows and columns of AB as follows.

$$\begin{array}{r c} & \begin{array}{cc} \text{Male} & \text{Female} \end{array} \\ \begin{array}{r} \text{Democrat} \\ \text{Republican} \\ \text{Independent} \end{array} & \begin{bmatrix} 13{,}500 & 16{,}500 \\ 9{,}000 & 10{,}950 \\ 4{,}500 & 5{,}550 \end{bmatrix} = AB \end{array}$$

Thus, 13,500 males are registered as Democrats in this city.

(c) There are 10,950 females registered as Republicans. ■

In Example 7, the entries in each column of A add up to 1. (Can you see why this has to be true, given what the matrix describes?) A matrix with this property is called **stochastic**. Stochastic matrices are used extensively in statistics, where they arise frequently in situations like the one described here.

Computer Graphics

One important use of matrices is in the digital representation of images. A digital camera or a scanner converts an image into a matrix by dividing the image into a rectangular array of elements called pixels. Each pixel is assigned a value that represents the color, brightness, or some other feature of that location. For example, in a 256-level gray-scale image each pixel is assigned a value between 0 and 255, where 0 represents white, 255 black, and the numbers in between increasing gradations of gray. The gradations of a much simpler 8-level gray scale are shown in Figure 2. We use this 8-level gray scale to illustrate the process.

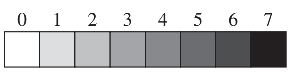
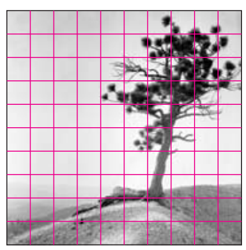


Figure 2

E. O. Hoppé/Corbis

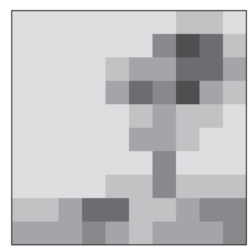
(a) Original image



(b) 10×10 grid

$$\begin{bmatrix} 1&1&1&1&1&1&1&2&2&1 \\ 1&1&1&1&1&1&4&6&5&2 \\ 1&1&1&1&2&3&3&5&5&3 \\ 1&1&1&1&3&5&4&6&3&2 \\ 1&1&1&1&1&2&3&2&2&1 \\ 1&1&1&1&1&3&3&2&1&1 \\ 1&1&1&1&1&1&4&1&1&1 \\ 1&1&1&1&2&2&4&2&2&2 \\ 2&2&3&5&5&2&2&3&4&4 \\ 3&3&3&4&3&2&3&3&3&4 \end{bmatrix}$$

(c) Matrix representation



(d) Digital image

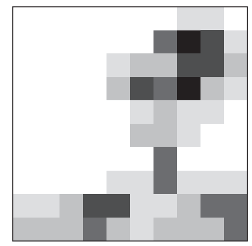
Figure 3

To digitize the black and white image in Figure 3(a), we place a grid over the picture as shown in Figure 3(b). Each cell in the grid is compared to the gray scale, and then assigned a value between 0 and 7 depending on which gray square in the scale most closely matches the “darkness” of the cell. (If the cell is not uniformly gray, an average value is assigned.) The values are stored in the matrix shown in Figure 3(c). The digital image corresponding to this matrix is shown in Figure 3(d). Obviously the grid that we have used is far too coarse to provide good image resolution. In practice, currently available high-resolution digital cameras use matrices with dimensions 2048×2048 or larger.

Once the image is stored as a matrix, it can be manipulated using matrix operations. For example, to darken the image, we add a constant to each entry in the matrix; to lighten the image, we subtract. To increase the contrast, we darken the darker areas and lighten the lighter areas, so we could add 1 to each entry that is 4, 5, or 6 and subtract 1 from each entry that is 1, 2, or 3. (Note that we cannot darken an entry of 7 or lighten a 0.) Applying this process to the matrix in Figure 3(c) produces the new matrix in Figure 4(a). This generates the high-contrast image shown in Figure 4(b).

$$\begin{bmatrix} 0&0&0&0&0&0&0&1&1&0\\ 0&0&0&0&0&0&5&7&6&1\\ 0&0&0&0&1&2&2&6&6&2\\ 0&0&0&0&2&6&5&7&2&1\\ 0&0&0&0&0&1&2&1&1&0\\ 0&0&0&0&0&2&2&1&0&0\\ 0&0&0&0&0&0&5&0&0&0\\ 0&0&0&0&1&1&5&1&1&1\\ 1&1&2&6&6&1&1&2&5&5\\ 2&2&2&5&2&1&2&2&2&5 \end{bmatrix}$$

(a) Matrix modified to increase contrast



(b) High-contrast image

Figure 4

Other ways of representing and manipulating images using matrices are discussed in the *Discovery Projects* on pages 700 and 792.

9.5 Exercises

1–2 ■ Determine whether the matrices A and B are equal.

1. $A = \begin{bmatrix} 1 & -2 & 0 \\ \frac{1}{2} & 6 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -2 \\ \frac{1}{2} & 6 \end{bmatrix}$

2. $A = \begin{bmatrix} \frac{1}{4} & \ln 1 \\ 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 0.25 & 0 \\ \sqrt{4} & \frac{6}{2} \end{bmatrix}$

3–10 ■ Perform the matrix operation, or if it is impossible, explain why.

3. $\begin{bmatrix} 2 & 6 \\ -5 & 3 \end{bmatrix} + \begin{bmatrix} -1 & -3 \\ 6 & 2 \end{bmatrix}$

4. $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} - \begin{bmatrix} 2 & 1 & -1 \\ 1 & 3 & -2 \end{bmatrix}$

5. $3\begin{bmatrix} 1 & 2 \\ 4 & -1 \\ 1 & 0 \end{bmatrix}$

6. $2\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}$

7. $\begin{bmatrix} 2 & 6 \\ 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 3 & 6 \\ -2 & 0 \end{bmatrix}$

8. $\begin{bmatrix} 2 & 1 & 2 \\ 6 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 3 & 6 \\ -2 & 0 \end{bmatrix}$

9. $\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \\ 2 & 2 & -1 \end{bmatrix}$

10. $\begin{bmatrix} 2 & -3 \\ 0 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

11–16 ■ Solve the matrix equation for the unknown matrix X, or explain why no solution exists.

$$A = \begin{bmatrix} 4 & 6 \\ 1 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & 3 \\ 1 & 0 \\ 0 & 2 \end{bmatrix} \qquad D = \begin{bmatrix} 10 & 20 \\ 30 & 20 \\ 10 & 0 \end{bmatrix}$$

11. $2X + A = B$

12. $3X - B = C$

13. $2(B - X) = D$

14. $5(X - C) = D$

15. $\frac{1}{5}(X + D) = C$

16. $2A = B - 3X$

17–38 ■ The matrices A, B, C, D, E, F, and G are defined as follows.

$$A = \begin{bmatrix} 2 & -5 \\ 0 & 7 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & \frac{1}{2} & 5 \\ 1 & -1 & 3 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & -\frac{5}{2} & 0 \\ 0 & 2 & -3 \end{bmatrix}$$

$$D = \begin{bmatrix} 7 & 3 \end{bmatrix} \qquad E = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad G = \begin{bmatrix} 5 & -3 & 10 \\ 6 & 1 & 0 \\ -5 & 2 & 2 \end{bmatrix}$$

Carry out the indicated algebraic operation, or explain why it cannot be performed.

17. $B + C$

18. $B + F$

19. $C - B$

20. $5A$

21. $3B + 2C$

22. $C - 5A$

23. $2C - 6B$

24. DA

25. AD

26. BC

27. BF

28. GF

29. $(DA)B$

30. $D(AB)$

31. GE

32. A^2

33. A^3

34. $DB + DC$

35. B^2

36. F^2

37. $BF + FE$

38. ABE

39–42 ■ Solve for x and y.

39. $\begin{bmatrix} x & 2y \\ 4 & 6 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 2x & -6y \end{bmatrix}$

40. $3\begin{bmatrix} x & y \\ y & x \end{bmatrix} = \begin{bmatrix} 6 & -9 \\ -9 & 6 \end{bmatrix}$

41. $2\begin{bmatrix} x & y \\ x + y & x - y \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ -2 & 6 \end{bmatrix}$

42. $\begin{bmatrix} x & y \\ -y & x \end{bmatrix} - \begin{bmatrix} y & x \\ x & -y \end{bmatrix} = \begin{bmatrix} 4 & -4 \\ -6 & 6 \end{bmatrix}$

43–46 ■ Write the system of equations as a matrix equation (see Example 6).

43. $\begin{cases} 2x - 5y = 7 \\ 3x + 2y = 4 \end{cases}$

44. $\begin{cases} 6x - y + z = 12 \\ 2x + z = 7 \\ \phantom{6x - {}} y - 2z = 4 \end{cases}$

45. $\begin{cases} 3x_1 + 2x_2 - x_3 + x_4 = 0 \\ x_1 - x_3 = 5 \\ \phantom{3x_1 + {}} 3x_2 + x_3 - x_4 = 4 \end{cases}$

46. $\begin{cases} x - y + z = 2 \\ 4x - 2y - z = 2 \\ x + y + 5z = 2 \\ -x - y - z = 2 \end{cases}$

47. Let

$$A = \begin{bmatrix} 1 & 0 & 6 & -1 \\ 2 & \frac{1}{2} & 4 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 7 & -9 & 2 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 \\ 0 \\ -1 \\ -2 \end{bmatrix}$$

Determine which of the following products are defined, and calculate the ones that are:

ABC	ACB	BAC
BCA	CAB	CBA

48. **(a)** Prove that if A and B are 2×2 matrices, then

$$(A + B)^2 = A^2 + AB + BA + B^2$$

(b) If A and B are 2×2 matrices, is it necessarily true that

$$(A + B)^2 \stackrel{?}{=} A^2 + 2AB + B^2$$

Applications

49. Fast-Food Sales A small fast-food chain with restaurants in Santa Monica, Long Beach, and Anaheim sells only hamburgers, hot dogs, and milk shakes. On a certain day, sales were distributed according to the following matrix.

Number of items sold

$$\begin{array}{r} \\ \text{Hamburgers} \\ \text{Hot dogs} \\ \text{Milk shakes} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Santa Monica} & \text{Long Beach} & \text{Anaheim} \end{array} \\ \begin{bmatrix} 4000 & 1000 & 3500 \\ 400 & 300 & 200 \\ 700 & 500 & 9000 \end{bmatrix} \end{array} = A$$

The price of each item is given by the following matrix.

$$\begin{array}{ccc} \text{Hamburger} & \text{Hot dog} & \text{Milk Shake} \\ [\$0.90 & \$0.80 & \$1.10] \end{array} = B$$

(a) Calculate the product BA.

(b) Interpret the entries in the product matrix BA.

50. Car-Manufacturing Profits A specialty-car manufacturer has plants in Auburn, Biloxi, and Chattanooga. Three models are produced, with daily production given in the following matrix.

Cars produced each day

$$\begin{array}{r} \\ \text{Auburn} \\ \text{Biloxi} \\ \text{Chattanooga} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Model K} & \text{Model R} & \text{Model W} \end{array} \\ \begin{bmatrix} 12 & 10 & 0 \\ 4 & 4 & 20 \\ 8 & 9 & 12 \end{bmatrix} \end{array} = A$$

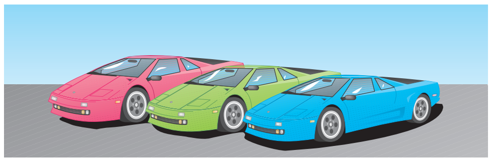
Because of a wage increase, February profits are less than January profits. The profit per car is tabulated by model in the following matrix.

$$\begin{array}{r} \\ \text{Model K} \\ \text{Model R} \\ \text{Model W} \end{array} \begin{array}{c} \begin{array}{cc} \text{January} & \text{February} \end{array} \\ \begin{bmatrix} \$1000 & \$500 \\ \$2000 & \$1200 \\ \$1500 & \$1000 \end{bmatrix} \end{array} = B$$

(a) Calculate AB.

(b) Assuming all cars produced were sold, what was the daily profit in January from the Biloxi plant?

(c) What was the total daily profit (from all three plants) in February?



51. Canning Tomato Products Jaeger Foods produces tomato sauce and tomato paste, canned in small, medium, large, and giant sized tins. The matrix A gives the size (in ounces) of each container.

$$\begin{array}{rcccc} & \text{Small} & \text{Medium} & \text{Large} & \text{Giant} \\ \text{Ounces} & [6 & 10 & 14 & 28] \end{array} = A$$

The matrix B tabulates one day's production of tomato sauce and tomato paste.

$$\begin{array}{r} \\ \text{Small} \\ \text{Medium} \\ \text{Large} \\ \text{Giant} \end{array} \begin{array}{c} \begin{array}{cc} \text{Cans of sauce} & \text{Cans of paste} \end{array} \\ \begin{bmatrix} 2000 & 2500 \\ 3000 & 1500 \\ 2500 & 1000 \\ 1000 & 500 \end{bmatrix} \end{array} = B$$

(a) Calculate the product of AB.

(b) Interpret the entries in the product matrix AB.

52. Produce Sales A farmer's three children, Amy, Beth, and Chad, run three roadside produce stands during the summer months. One weekend they all sell watermelons, yellow squash, and tomatoes. The matrices A and B tabulate the number of pounds of each product sold by each sibling on Saturday and Sunday.

Saturday

$$\begin{array}{l} \\ \text{Amy}\\ \text{Beth}\\ \text{Chad}\end{array}\begin{array}{c}\begin{array}{ccc}\text{Melons} & \text{Squash} & \text{Tomatoes}\end{array}\\ \begin{bmatrix} 120 & 50 & 60\\ 40 & 25 & 30\\ 60 & 30 & 20\end{bmatrix}\end{array} = A$$

Sunday

$$\begin{array}{l} \\ \text{Amy}\\ \text{Beth}\\ \text{Chad}\end{array}\begin{array}{c}\begin{array}{ccc}\text{Melons} & \text{Squash} & \text{Tomatoes}\end{array}\\ \begin{bmatrix} 100 & 60 & 30\\ 35 & 20 & 20\\ 60 & 25 & 30\end{bmatrix}\end{array} = B$$

The matrix C gives the price per pound (in dollars) for each type of produce that they sell.

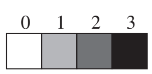
Price per pound

$$\begin{array}{r}\text{Melons}\\ \text{Squash}\\ \text{Tomatoes}\end{array}\begin{bmatrix}0.10\\ 0.50\\ 1.00\end{bmatrix} = C$$

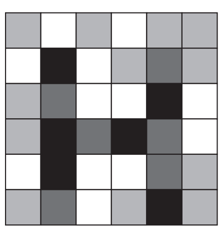
Perform the following matrix operations, and interpret the entries in each result.

(a) AC **(b)** BC **(c)** $A + B$ **(d)** $(A + B)C$

53. Digital Images A four-level gray scale is shown below.



(a) Use the gray scale to find a 6×6 matrix that digitally represents the image in the figure.



(b) Find a matrix that represents a darker version of the image in the figure.

(c) The **negative** of an image is obtained by reversing light and dark, as in the negative of a photograph. Find the matrix that represents the negative of the image in the figure. How do you change the elements of the matrix to create the negative?

(d) Increase the contrast of the image by changing each 1 to a 0 and each 2 to a 3 in the matrix you found in part (b). Draw the image represented by the resulting matrix. Does this clarify the image?

(e) Draw the image represented by the matrix I. Can you recognize what this is? If you don't, try increasing the contrast.

$$I = \begin{bmatrix} 1 & 2 & 3 & 3 & 2 & 0\\ 0 & 3 & 0 & 1 & 0 & 1\\ 1 & 3 & 2 & 3 & 0 & 0\\ 0 & 3 & 0 & 1 & 0 & 1\\ 1 & 3 & 3 & 2 & 3 & 0\\ 0 & 1 & 0 & 1 & 0 & 1\end{bmatrix}$$

Discovery • Discussion

54. When Are Both Products Defined? What must be true about the dimensions of the matrices A and B if both products AB and BA are defined?

55. Powers of a Matrix Let

$$A = \begin{bmatrix} 1 & 1\\ 0 & 1\end{bmatrix}$$

Calculate $A^2, A^3, A^4, \ldots$ until you detect a pattern. Write a general formula for A^n.

56. Powers of a Matrix Let $A = \begin{bmatrix} 1 & 1\\ 1 & 1\end{bmatrix}$. Calculate $A^2, A^3, A^4, \ldots$ until you detect a pattern. Write a general formula for A^n.

57. Square Roots of Matrices A square root of a matrix B is a matrix A with the property that $A^2 = B$. (This is the same definition as for a square root of a number.) Find as many square roots as you can of each matrix:

$$\begin{bmatrix} 4 & 0\\ 0 & 9\end{bmatrix} \qquad \begin{bmatrix} 1 & 5\\ 0 & 9\end{bmatrix}$$

[*Hint:* If $A = \begin{bmatrix} a & b\\ c & d\end{bmatrix}$, write the equations that a, b, c, and d would have to satisfy if A is the square root of the given matrix.]

DISCOVERY PROJECT

Will the Species Survive?

To study how species survive, mathematicians model their populations by observing the different stages in their life. They consider, for example, the stage at which the animal is fertile, the proportion of the population that reproduces, and the proportion of the young that survive each year. For a certain species, there are three stages: immature, juvenile, and adult. An animal is considered immature for the first year of its life, juvenile for the second year, and an adult from then on. Conservation biologists have collected the following field data for this species:

$$A = \begin{array}{c} \begin{array}{ccc} \text{Immature} & \text{Juvenile} & \text{Adult} \end{array} \\ \begin{bmatrix} 0 & 0 & 0.4 \\ 0.1 & 0 & 0 \\ 0 & 0.3 & 0.8 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Immature} \\ \text{Juvenile} \\ \text{Adult} \end{array} \qquad X_0 = \begin{bmatrix} 600 \\ 400 \\ 3500 \end{bmatrix} \begin{array}{l} \text{Immature} \\ \text{Juvenile} \\ \text{Adult} \end{array}$$

Art Wolfe/Stone/Getty Images

The entries in the matrix A indicate the proportion of the population that survives *to the next year*. For example, the first column describes what happens to the immature population: None remain immature, 10% survive to become juveniles, and of course none become adults. The second column describes what happens to the juvenile population: None become immature or remain juvenile, and 30% survive to adulthood. The third column describes the adult population: The number of their new offspring is 40% of the adult population, no adults become juveniles, and 80% survive to live another year. The entries in the population matrix X_0 indicate the current population (year 0) of immature, juvenile, and adult animals.

Let $X_1 = AX_0$, $X_2 = AX_1$, $X_3 = AX_2$, and so on.

1. Explain why X_1 gives the population in year 1, X_2 the population in year 2, and so on.

2. Find the population matrix for years 1, 2, 3, and 4. (Round fractional entries to the nearest whole number.) Do you see any trend?

3. Show that $X_2 = A^2X_0$, $X_3 = A^3X_0$, and so on.

4. Find the population after 50 years—that is, find X_{50}. (Use your results in Problem 3 and a graphing calculator.) Does it appear that the species will survive?

5. Suppose the environment has improved so that the proportion of immatures that become juveniles each year increases to 0.1 from 0.3, the proportion of juveniles that become adults increases to 0.3 from 0.7, and the proportion of adults that survives to the next year increases from 0.8 to 0.95. Find the population after 50 years with the new matrix A. Does it appear that the species will survive under these new conditions?

6. The survival-rate matrix A given above is called a **transition matrix**. Such matrices occur in many applications of matrix algebra. The following transition matrix T predicts the calculus grades of a class of college students who must take a four-semester sequence of calculus courses. The first column of the matrix, for instance, indicates that of those students who get an A in one course, 70% will get an A in the following course, 15% will get a B, and 10% will get a C. (Students who receive D or F are not permitted to go on to the next course, so are not included in the matrix.) The entries in the matrix Y_0 give the number of incoming students who got A, B, and C, respectively, in their final high school mathematics course.

Let $Y_1 = TY_0$, $Y_2 = TY_1$, $Y_3 = TY_2$, and $Y_4 = TY_3$. Calculate and interpret the entries of Y_1, Y_2, Y_3, and Y_4.

$$T = \begin{array}{c} \begin{matrix} \text{A} & \text{B} & \text{C} \end{matrix} \\ \begin{bmatrix} 0.70 & 0.25 & 0.05 \\ 0.15 & 0.50 & 0.25 \\ 0.05 & 0.15 & 0.45 \end{bmatrix} \end{array} \begin{matrix} \text{A} \\ \text{B} \\ \text{C} \end{matrix} \qquad Y_0 = \begin{bmatrix} 140 \\ 320 \\ 400 \end{bmatrix} \begin{matrix} \text{A} \\ \text{B} \\ \text{C} \end{matrix}$$

9.6 Inverses of Matrices and Matrix Equations

In the preceding section we saw that, when the dimensions are appropriate, matrices can be added, subtracted, and multiplied. In this section we investigate division of matrices. With this operation we can solve equations that involve matrices.

The Inverse of a Matrix

First, we define *identity matrices*, which play the same role for matrix multiplication as the number 1 does for ordinary multiplication of numbers; that is, $1 \cdot a = a \cdot 1 = a$ for all numbers a. In the following definition the term **main diagonal** refers to the entries of a square matrix whose row and column numbers are the same. These entries stretch diagonally down the matrix, from top left to bottom right.

The **identity matrix** I_n is the $n \times n$ matrix for which each main diagonal entry is a 1 and for which all other entries are 0.

Thus, the 2×2, 3×3, and 4×4 identity matrices are

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Identity matrices behave like the number 1 in the sense that

$$A \cdot I_n = A \qquad \text{and} \qquad I_n \cdot B = B$$

whenever these products are defined.

SUGGESTED TIME AND EMPHASIS

1 class.

Recommended material.

DRILL QUESTION

Find the inverse of the matrix $\begin{bmatrix} 3 & 0 \\ 0 & -\frac{5}{8} \end{bmatrix}$.

Answer

$\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & -\frac{8}{5} \end{bmatrix}$

POINTS TO STRESS

1. The identity matrix.

2. Definition and computation of the inverse of a matrix.

ALTERNATE EXAMPLE 1

$$\begin{bmatrix} 1 & 2 \\ 3 & -1 \\ 5 & 8 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & -1 \\ 5 & 8 \end{bmatrix}$$

SAMPLE QUESTION
Text Question

If $AB = I$, where I is the identity matrix, is it necessarily true that $BA = I$?

Answer

Yes

ALTERNATE EXAMPLE 2

Verify that B is the inverse of A.

$A = \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}$ and

$B = \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix}$

ANSWER

$AB = BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Example 1 Identity Matrices

The following matrix products show how multiplying a matrix by an identity matrix of the appropriate dimension leaves the matrix unchanged.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 5 & 6 \\ -1 & 2 & 7 \end{bmatrix} = \begin{bmatrix} 3 & 5 & 6 \\ -1 & 2 & 7 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 7 & \frac{1}{2} \\ 12 & 1 & 3 \\ -2 & 0 & 7 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 7 & \frac{1}{2} \\ 12 & 1 & 3 \\ -2 & 0 & 7 \end{bmatrix}$$ ■

If A and B are $n \times n$ matrices, and if $AB = BA = I_n$, then we say that B is the *inverse* of A, and we write $B = A^{-1}$. The concept of the inverse of a matrix is analogous to that of the reciprocal of a real number.

Inverse of a Matrix

Let A be a square $n \times n$ matrix. If there exists an $n \times n$ matrix A^{-1} with the property that

$$AA^{-1} = A^{-1}A = I_n$$

then we say that A^{-1} is the **inverse** of A.

Example 2 Verifying That a Matrix Is an Inverse

Verify that B is the inverse of A, where

$$A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$$

Solution We perform the matrix multiplications to show that $AB = I$ and $BA = I$:

$$\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 2\cdot 3 + 1(-5) & 2(-1) + 1\cdot 2 \\ 5\cdot 3 + 3(-5) & 5(-1) + 3\cdot 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 3\cdot 2 + (-1)5 & 3\cdot 1 + (-1)3 \\ (-5)2 + 2\cdot 5 & (-5)1 + 2\cdot 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ ■

Finding the Inverse of a 2 × 2 Matrix

The following rule provides a simple way for finding the inverse of a 2×2 matrix, when it exists. For larger matrices, there's a more general procedure for finding inverses, which we consider later in this section.

Inverse of a 2 × 2 Matrix

$$\text{If } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{then} \quad A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $ad - bc = 0$, then A has no inverse.

IN-CLASS MATERIALS

Perhaps take this opportunity to talk about the inverse of a complex number: How do we find $\dfrac{1}{3 + 4i}$? The technique, multiplying by $\dfrac{3 - 4i}{3 - 4i}$, is not as important as the concept that given a real, complex, or matrix quantity it is often possible to find an inverse that will reduce it to unity. One can also add "inverse functions" to this discussion—in this case $f(x) = x$ is the identity function, so-called because it leaves inputs unchanged (analogous to multiplying by 1). Try to get the students to see the conceptual similarities in solving the three following equations:

$$3x = 2$$

$$(3 + i)x = 2 - 4i$$

$$\begin{bmatrix} 2 & 1 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Example 3 Finding the Inverse of a 2 × 2 Matrix

Let A be the matrix

$$A = \begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix}$$

Find A^{-1} and verify that $AA^{-1} = A^{-1}A = I_2$.

Solution Using the rule for the inverse of a 2×2 matrix, we get

$$A^{-1} = \frac{1}{4 \cdot 3 - 5 \cdot 2}\begin{bmatrix} 3 & -5 \\ -2 & 4 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 3 & -5 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & -\frac{5}{2} \\ -1 & 2 \end{bmatrix}$$

To verify that this is indeed the inverse of A, we calculate AA^{-1} and $A^{-1}A$:

$$AA^{-1} = \begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} \frac{3}{2} & -\frac{5}{2} \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 4 \cdot \frac{3}{2} + 5(-1) & 4\left(-\frac{5}{2}\right) + 5 \cdot 2 \\ 2 \cdot \frac{3}{2} + 3(-1) & 2\left(-\frac{5}{2}\right) + 3 \cdot 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^{-1}A = \begin{bmatrix} \frac{3}{2} & -\frac{5}{2} \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \cdot 4 + \left(-\frac{5}{2}\right)2 & \frac{3}{2} \cdot 5 + \left(-\frac{5}{2}\right)3 \\ (-1)4 + 2 \cdot 2 & (-1)5 + 2 \cdot 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ ■

The quantity $ad - bc$ that appears in the rule for calculating the inverse of a 2×2 matrix is called the **determinant** of the matrix. If the determinant is 0, then the matrix does not have an inverse (since we cannot divide by 0).

Finding the Inverse of an $n \times n$ Matrix

For 3×3 and larger square matrices, the following technique provides the most efficient way to calculate their inverses. If A is an $n \times n$ matrix, we first construct the $n \times 2n$ matrix that has the entries of A on the left and of the identity matrix I_n on the right:

$$\left[\begin{array}{cccc|cccc} a_{11} & a_{12} & \cdots & a_{1n} & 1 & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & 0 & 0 & \cdots & 1 \end{array}\right]$$

We then use the elementary row operations on this new large matrix to change the left side into the identity matrix. (This means that we are changing the large matrix to reduced row-echelon form.) The right side is transformed automatically into A^{-1}. (We omit the proof of this fact.)

Example 4 Finding the Inverse of a 3 × 3 Matrix

Let A be the matrix

$$A = \begin{bmatrix} 1 & -2 & -4 \\ 2 & -3 & -6 \\ -3 & 6 & 15 \end{bmatrix}$$

(a) Find A^{-1}.

(b) Verify that $AA^{-1} = A^{-1}A = I_3$.

ALTERNATE EXAMPLE 3

Find the inverse of $\begin{bmatrix} 6 & 4 \\ 4 & 3 \end{bmatrix}$.

ANSWER

$$\frac{1}{2}\begin{bmatrix} 3 & -4 \\ -4 & 6 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & -2 \\ -2 & 3 \end{bmatrix}$$

ALTERNATE EXAMPLE 4

Find the inverse of the following matrix:

$$\begin{bmatrix} 3 & -8 & 1 \\ 1 & -1 & 0 \\ 2 & 1 & 1 \end{bmatrix}$$

ANSWER

$$\begin{bmatrix} -\frac{1}{8} & \frac{9}{8} & \frac{1}{8} \\ -\frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\ \frac{3}{8} & -\frac{19}{8} & \frac{5}{8} \end{bmatrix}$$

IN-CLASS MATERIALS

It is straightforward to demonstrate that $(A - 1)^{-1} = A$ for specific 2×2 or 3×3 matrices. Students can pick up why it should be true, given the definition of inverse. A general algebraic proof is a little messy:

$$(A^{-1})^{-1} = \begin{bmatrix} \dfrac{d}{ad - bc} & -\dfrac{b}{ad - bc} \\ -\dfrac{c}{ad - bc} & \dfrac{a}{ad - bc} \end{bmatrix}^{-1}$$

$$= \frac{1}{\left(\dfrac{a}{ad - bc}\right)\left(\dfrac{d}{ad - bc}\right) - \left(\dfrac{b}{ad - bc}\right)\left(\dfrac{c}{ad - bc}\right)}\begin{bmatrix} \dfrac{a}{ad - bc} & -\dfrac{b}{ad - bc} \\ -\dfrac{c}{ad - bc} & \dfrac{d}{ad - bc} \end{bmatrix} = \frac{1}{1/(ad - bc)} \cdot \frac{1}{ad - bc} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = A$$

Note that there is a formula for 3×3 inverses, just as there is a formula for 2×2 inverses. Unfortunately, it is so complicated that it is easier to do 3×3 inverses manually than to use a formula.

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}^{-1} = \frac{1}{afh - aei + bdi - bfg + ceg - cdh}\begin{bmatrix} fh - ei & bi - ch & ce - bf \\ di - fg & cg - ai & af - cd \\ eg - dh & ah - bg & bd - ae \end{bmatrix}$$

EXAMPLES

- Nonsingular real matrices

$$\begin{bmatrix} -2 & 1 \\ 3 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{1}{3} & \frac{1}{9} \\ \frac{1}{3} & \frac{2}{9} \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 1 \\ 4 & -1 & 1 \\ 4 & 2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{1}{5} & \frac{1}{15} & \frac{2}{15} \\ 0 & -\frac{1}{3} & \frac{1}{3} \\ \frac{4}{5} & \frac{2}{5} & -\frac{1}{5} \end{bmatrix}$$

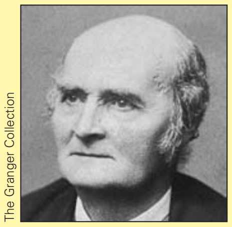
$$\begin{bmatrix} 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 1 & 0 & 6 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{5} & 0 \\ -\frac{1}{12} & 0 & 0 & \frac{1}{6} \end{bmatrix}$$

- Singular real matrices

$$\begin{bmatrix} 5 & 4 \\ 15 & 24 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ -1 & 3 & -2 \\ -1 & 8 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 8 & 3 & 5 & -8 \\ -7 & 6 & 0 & -6 \\ -5 & -3 & 7 & 9 \\ -4 & 6 & 12 & -5 \end{bmatrix}$$



The Granger Collection

Arthur Cayley (1821–1895) was an English mathematician who was instrumental in developing the theory of matrices. He was the first to use a single symbol such as A to represent a matrix, thereby introducing the idea that a matrix is a single entity rather than just a collection of numbers. Cayley practiced law until the age of 42, but his primary interest from adolescence was mathematics, and he published almost 200 articles on the subject in his spare time. In 1863 he accepted a professorship in mathematics at Cambridge, where he taught until his death. Cayley's work on matrices was of purely theoretical interest in his day, but in the 20th century many of his results found application in physics, the social sciences, business, and other fields. One of the most common uses of matrices today is in computers, where matrices are employed for data storage, error correction, image manipulation, and many other purposes. These applications have made matrix algebra more useful than ever.

Solution

(a) We begin with the 3×6 matrix whose left half is A and whose right half is the identity matrix.

$$\left[\begin{array}{rrr|rrr} 1 & -2 & -4 & 1 & 0 & 0 \\ 2 & -3 & -6 & 0 & 1 & 0 \\ -3 & 6 & 15 & 0 & 0 & 1 \end{array}\right]$$

We then transform the left half of this new matrix into the identity matrix by performing the following sequence of elementary row operations on the *entire* new matrix:

$$\xrightarrow[R_3 + 3R_1 \to R_3]{R_2 - 2R_1 \to R_2} \left[\begin{array}{rrr|rrr} 1 & -2 & -4 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 3 & 3 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{3}R_3} \left[\begin{array}{rrr|rrr} 1 & -2 & -4 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & \frac{1}{3} \end{array}\right]$$

$$\xrightarrow{R_1 + 2R_2 \to R_1} \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & -3 & 2 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & \frac{1}{3} \end{array}\right]$$

$$\xrightarrow{R_2 - 2R_3 \to R_2} \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & -3 & 2 & 0 \\ 0 & 1 & 0 & -4 & 1 & -\frac{2}{3} \\ 0 & 0 & 1 & 1 & 0 & \frac{1}{3} \end{array}\right]$$

We have now transformed the left half of this matrix into an identity matrix. (This means we've put the entire matrix in reduced row-echelon form.) Note that to do this in as systematic a fashion as possible, we first changed the elements below the main diagonal to zeros, just as we would if we were using Gaussian elimination. We then changed each main diagonal element to a 1 by multiplying by the appropriate constant(s). Finally, we completed the process by changing the remaining entries on the left side to zeros.

The right half is now A^{-1}.

$$A^{-1} = \begin{bmatrix} -3 & 2 & 0 \\ -4 & 1 & -\frac{2}{3} \\ 1 & 0 & \frac{1}{3} \end{bmatrix}$$

(b) We calculate AA^{-1} and $A^{-1}A$, and verify that both products give the identity matrix I_3.

$$AA^{-1} = \begin{bmatrix} 1 & -2 & -4 \\ 2 & -3 & -6 \\ -3 & 6 & 15 \end{bmatrix} \begin{bmatrix} -3 & 2 & 0 \\ -4 & 1 & -\frac{2}{3} \\ 1 & 0 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1}A = \begin{bmatrix} -3 & 2 & 0 \\ -4 & 1 & -\frac{2}{3} \\ 1 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & -2 & -4 \\ 2 & -3 & -6 \\ -3 & 6 & 15 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

■

- A complex matrix and its inverse

$$\begin{bmatrix} 1 & 2i \\ 1 - i & 1 \end{bmatrix}^{-1} = \frac{1}{-1 - 2i} \begin{bmatrix} 1 & -2i \\ -1 + i & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{5} + \frac{2}{5}i & \frac{4}{5} + \frac{2}{5}i \\ -\frac{1}{5} - \frac{3}{5}i & -\frac{1}{5} + \frac{2}{5}i \end{bmatrix}$$

```
[A]-1▶Frac
   [[-3 2 0    ]
    [-4 1 -2/3]
    [1  0 1/3 ]]
```

Figure 1

Graphing calculators are also able to calculate matrix inverses. On the TI-82 and TI-83 calculators, matrices are stored in memory using names such as `[A]`, `[B]`, `[C]`, ... To find the inverse of `[A]`, we key in

`[A]` `x⁻¹` `ENTER`

For the matrix of Example 4, this results in the output shown in Figure 1 (where we have also used the `▶Frac` command to display the output in fraction form rather than in decimal form).

The next example shows that not every square matrix has an inverse.

Example 5 A Matrix That Does Not Have an Inverse

Find the inverse of the matrix.

$$\begin{bmatrix} 2 & -3 & -7 \\ 1 & 2 & 7 \\ 1 & 1 & 4 \end{bmatrix}$$

Solution We proceed as follows.

$$\left[\begin{array}{ccc|ccc} 2 & -3 & -7 & 1 & 0 & 0 \\ 1 & 2 & 7 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 7 & 0 & 1 & 0 \\ 2 & -3 & -7 & 1 & 0 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_3 - R_1 \to R_3]{R_2 - 2R_1 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 7 & 0 & 1 & 0 \\ 0 & -7 & -21 & 1 & -2 & 0 \\ 0 & -1 & -3 & 0 & -1 & 1 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{7}R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 7 & 0 & 1 & 0 \\ 0 & 1 & 3 & -\frac{1}{7} & \frac{2}{7} & 0 \\ 0 & -1 & -3 & 0 & -1 & 1 \end{array}\right]$$

$$\xrightarrow[R_1 - 2R_2 \to R_1]{R_3 + R_2 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & \frac{2}{7} & \frac{3}{7} & 0 \\ 0 & 1 & 3 & -\frac{1}{7} & \frac{2}{7} & 0 \\ 0 & 0 & 0 & -\frac{1}{7} & -\frac{5}{7} & 1 \end{array}\right]$$

At this point, we would like to change the 0 in the $(3, 3)$ position of this matrix to a 1, without changing the zeros in the $(3, 1)$ and $(3, 2)$ positions. But there is no way to accomplish this, because no matter what multiple of rows 1 and/or 2 we add to row 3, we can't change the third zero in row 3 without changing the first or second zero as well. Thus, we cannot change the left half to the identity matrix, so the original matrix doesn't have an inverse. ■

```
ERR:SINGULAR MAT
1:Quit
2:Goto
```

Figure 2

If we encounter a row of zeros on the left when trying to find an inverse, as in Example 5, then the original matrix does not have an inverse. If we try to calculate the inverse of the matrix from Example 5 on a TI-83 calculator, we get the error message shown in Figure 2. (A matrix that has no inverse is called *singular*.)

ALTERNATE EXAMPLE 5
Show that the following matrix does not have an inverse.

$$\begin{bmatrix} 3 & -8 & 1 \\ 1 & -1 & 0 \\ 5 & -15 & 2 \end{bmatrix}$$

ANSWER
When we try to find the inverse, we wind up with a row of zeros.

IN-CLASS MATERIALS

Example 5 is particularly important, because Section 9.4 presented a straightforward method of solving an $n \times n$ system. The advantage of finding the inverse matrix really kicks in when solving a series of systems with the same coefficient matrix.

Matrix Equations

We saw in Example 6 in Section 9.5 that a system of linear equations can be written as a single matrix equation. For example, the system

$$\begin{cases} x - 2y - 4z = 7 \\ 2x - 3y - 6z = 5 \\ -3x + 6y + 15z = 0 \end{cases}$$

is equivalent to the matrix equation

$$\underbrace{\begin{bmatrix} 1 & -2 & -4 \\ 2 & -3 & -6 \\ -3 & 6 & 15 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 7 \\ 5 \\ 0 \end{bmatrix}}_{B}$$

If we let

$$A = \begin{bmatrix} 1 & -2 & -4 \\ 2 & -3 & -6 \\ -3 & 6 & 15 \end{bmatrix} \qquad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad B = \begin{bmatrix} 7 \\ 5 \\ 0 \end{bmatrix}$$

then this matrix equation can be written as

$$AX = B$$

The matrix A is called the **coefficient matrix**.

We solve this matrix equation by multiplying each side by the inverse of A (provided this inverse exists):

Solving the matrix equation $AX = B$ is very similar to solving the simple real-number equation

$$3x = 12$$

which we do by multiplying each side by the reciprocal (or inverse) of 3:

$$\tfrac{1}{3}(3x) = \tfrac{1}{3}(12)$$

$$x = 4$$

$$\begin{aligned} AX &= B \\ A^{-1}(AX) &= A^{-1}B && \text{Multiply both sides of equation on the left by } A^{-1} \\ (A^{-1}A)X &= A^{-1}B && \text{Associative Property} \\ I_3X &= A^{-1}B && \text{Property of inverses} \\ X &= A^{-1}B && \text{Property of identity matrix} \end{aligned}$$

In Example 4 we showed that

$$A^{-1} = \begin{bmatrix} -3 & 2 & 0 \\ -4 & 1 & -\frac{2}{3} \\ 1 & 0 & \frac{1}{3} \end{bmatrix}$$

So, from $X = A^{-1}B$ we have

$$\underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} -3 & 2 & 0 \\ -4 & 1 & -\frac{2}{3} \\ 1 & 0 & \frac{1}{3} \end{bmatrix}}_{A^{-1}} \underbrace{\begin{bmatrix} 7 \\ 5 \\ 0 \end{bmatrix}}_{B} = \begin{bmatrix} -11 \\ -23 \\ 7 \end{bmatrix}$$

Thus, $x = -11$, $y = -23$, $z = 7$ is the solution of the original system.

IN-CLASS MATERIALS

After doing a standard example or two, throw a singular matrix on the board before defining singularity. "Unexpectedly" run into trouble and thus discover, with your class, that not every matrix has an inverse.

We have proved that the matrix equation $AX = B$ can be solved by the following method.

Solving a Matrix Equation

If A is a square $n \times n$ matrix that has an inverse A^{-1}, and if X is a variable matrix and B a known matrix, both with n rows, then the solution of the matrix equation

$$AX = B$$

is given by

$$X = A^{-1}B$$

Example 6 Solving a System Using a Matrix Inverse

(a) Write the system of equations as a matrix equation.

(b) Solve the system by solving the matrix equation.

$$\begin{cases} 2x - 5y = 15 \\ 3x - 6y = 36 \end{cases}$$

Solution

(a) We write the system as a matrix equation of the form $AX = B$:

$$\underbrace{\begin{bmatrix} 2 & -5 \\ 3 & -6 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{X} \underset{=}{=} \underbrace{\begin{bmatrix} 15 \\ 36 \end{bmatrix}}_{B}$$

(b) Using the rule for finding the inverse of a 2×2 matrix, we get

$$A^{-1} = \begin{bmatrix} 2 & -5 \\ 3 & -6 \end{bmatrix}^{-1} = \frac{1}{2(-6) - (-5)3}\begin{bmatrix} -6 & -(-5) \\ -3 & 2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -6 & 5 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} -2 & \frac{5}{3} \\ -1 & \frac{2}{3} \end{bmatrix}$$

Multiplying each side of the matrix equation by this inverse matrix, we get

$$\underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{X} \underset{=}{=} \underbrace{\begin{bmatrix} -2 & \frac{5}{3} \\ -1 & \frac{2}{3} \end{bmatrix}}_{A^{-1}} \underbrace{\begin{bmatrix} 15 \\ 36 \end{bmatrix}}_{B} = \begin{bmatrix} 30 \\ 9 \end{bmatrix}$$

So $x = 30$ and $y = 9$. ■

ALTERNATE EXAMPLE 6

Consider the following system of equations:

$$\begin{cases} 4x + y = 14 \\ 12x - y = 2 \end{cases}$$

(a) Write the system as a matrix equation.

(b) Solve the system by solving the matrix equation.

ANSWERS

(a) $\begin{bmatrix} 4 & 1 \\ 12 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 14 \\ 2 \end{bmatrix}$

(b) $x = 1, y = 10$ $\left(\text{alternatively, } \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \end{bmatrix}\right)$

Mathematics in the Modern World

Volvox/Index Stock

Mathematical Ecology

In the 1970s humpback whales became a center of controversy. Environmentalists believed that whaling threatened the whales with imminent extinction; whalers saw their livelihood threatened by any attempt to stop whaling. Are whales really threatened to extinction by whaling? What level of whaling is safe to guarantee survival of the whales? These questions motivated mathematicians to study population patterns of whales and other species more closely.

As early as the 1920s Alfred J. Lotka and Vito Volterra had founded the field of mathematical biology by creating predator-prey models. Their models, which draw on a branch of mathematics called differential equations, take into account the rates at which predator eats prey and the rates of growth of each population. Notice that as predator eats prey, the prey population decreases; this means less food supply for the predators, so their population begins to decrease; with fewer predators the prey population begins to increase, and so on. Normally, a state of equilibrium develops, and the two populations alternate between a minimum and a maximum. Notice that if the predators eat the prey too fast they will be left without food and ensure their own extinction.

(continued)

Applications

Suppose we need to solve several systems of equations with the same coefficient matrix. Then converting the systems to matrix equations provides an efficient way to obtain the solutions, because we only need to find the inverse of the coefficient matrix once. This procedure is particularly convenient if we use a graphing calculator to perform the matrix operations, as in the next example.

Example 7 Modeling Nutritional Requirements Using Matrix Equations

A pet-store owner feeds his hamsters and gerbils different mixtures of three types of rodent food: KayDee Food, Pet Pellets, and Rodent Chow. He wishes to feed his animals the correct amount of each brand to satisfy their daily requirements for protein, fat, and carbohydrates exactly. Suppose that hamsters require 340 mg of protein, 280 mg of fat, and 440 mg of carbohydrates, and gerbils need 480 mg of protein, 360 mg of fat, and 680 mg of carbohydrates each day. The amount of each nutrient (in mg) in one gram of each brand is given in the following table. How many grams of each food should the storekeeper feed his hamsters and gerbils daily to satisfy their nutrient requirements?

	KayDee Food	Pet Pellets	Rodent Chow
Protein (mg)	10	0	20
Fat (mg)	10	20	10
Carbohydrates (mg)	5	10	30

Solution We let x_1, x_2, and x_3 be the respective amounts (in grams) of KayDee Food, Pet Pellets, and Rodent Chow that the hamsters should eat and y_1, y_2, and y_3 be the corresponding amounts for the gerbils. Then we want to solve the matrix equations

$$\begin{bmatrix} 10 & 0 & 20 \\ 10 & 20 & 10 \\ 5 & 10 & 30 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 340 \\ 280 \\ 440 \end{bmatrix} \qquad \text{Hamster equation}$$

$$\begin{bmatrix} 10 & 0 & 20 \\ 10 & 20 & 10 \\ 5 & 10 & 30 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 480 \\ 360 \\ 680 \end{bmatrix} \qquad \text{Gerbil equation}$$

Let

$$A = \begin{bmatrix} 10 & 0 & 20 \\ 10 & 20 & 10 \\ 5 & 10 & 30 \end{bmatrix}, \quad B = \begin{bmatrix} 340 \\ 280 \\ 440 \end{bmatrix}, \quad C = \begin{bmatrix} 480 \\ 360 \\ 680 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

ALTERNATE EXAMPLE 7

Consider the situation given in the text, only with three different brands of rodent food:

	Rodent Kibble	Small Animal Food	Pet Choice
Protein (mg)	12	2	18
Fat (mg)	8	22	8
Carbohydrates (mg)	6	10	28

Now determine the amount of grams of each of the new brands of food the storekeeper should feed his hamsters.

ANSWER

(A graphing calculator was used to solve the matrix equations.)

For the gerbils: 8.46 g of Rodent Kibble, 5.88 g of Small Animal Food, 20.37 g of Pet Choice. For the hamsters: 9.67 g of Rodent Kibble, 4.88 g of Small Animal Food, 11.90 g of Pet Choice.

Since Lotka and Volterra's time, more detailed mathematical models of animal populations have been developed. For many species the population is divided into several stages—immature, juvenile, adult, and so on. The proportion of each stage that survives or reproduces in a given time period is entered into a matrix (called a transition matrix); matrix multiplication is then used to predict the population in succeeding time periods. (See the *Discovery Project*, page 688.)

As you can see, the power of mathematics to model and predict is an invaluable tool in the ongoing debate over the environment.

Then we can write these matrix equations as

$$AX = B \qquad \text{Hamster equation}$$
$$AY = C \qquad \text{Gerbil equation}$$

We want to solve for X and Y, so we multiply both sides of each equation by A^{-1}, the inverse of the coefficient matrix. We could find A^{-1} by hand, but it is more convenient to use a graphing calculator as shown in Figure 3.

```
[A]⁻¹*[B]
            [[10]
             [3 ]
             [12]]
```

```
[A]⁻¹*[C]
            [[8 ]
             [4 ]
             [20]]
```

Figure 3 (a) (b)

From the calculator displays, we see that

$$X = A^{-1}B = \begin{bmatrix} 10 \\ 3 \\ 12 \end{bmatrix}, \qquad Y = A^{-1}C = \begin{bmatrix} 8 \\ 4 \\ 20 \end{bmatrix}$$

Thus, each hamster should be fed 10 g of KayDee Food, 3 g of Pet Pellets, and 12 g of Rodent Chow, and each gerbil should be fed 8 g of KayDee Food, 4 g of Pet Pellets, and 20 g of Rodent Chow daily. ■

9.6 Exercises

1–4 ■ Calculate the products AB and BA to verify that B is the inverse of A.

1. $A = \begin{bmatrix} 4 & 1 \\ 7 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -1 \\ -7 & 4 \end{bmatrix}$

2. $A = \begin{bmatrix} 2 & -3 \\ 4 & -7 \end{bmatrix}, \quad B = \begin{bmatrix} \frac{7}{2} & -\frac{3}{2} \\ 2 & -1 \end{bmatrix}$

3. $A = \begin{bmatrix} 1 & 3 & -1 \\ 1 & 4 & 0 \\ -1 & -3 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 8 & -3 & 4 \\ -2 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$

4. $A = \begin{bmatrix} 3 & 2 & 4 \\ 1 & 1 & -6 \\ 2 & 1 & 12 \end{bmatrix}, \quad B = \begin{bmatrix} 9 & -10 & -8 \\ -12 & 14 & 11 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

5–6 ■ Find the inverse of the matrix and verify that $A^{-1}A = AA^{-1} = I_2$ and $B^{-1}B = BB^{-1} = I_3$.

5. $A = \begin{bmatrix} 7 & 4 \\ 3 & 2 \end{bmatrix}$

6. $B = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 2 \\ -2 & -1 & 0 \end{bmatrix}$

7–22 ■ Find the inverse of the matrix if it exists.

7. $\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$

8. $\begin{bmatrix} 3 & 4 \\ 7 & 9 \end{bmatrix}$

9. $\begin{bmatrix} 2 & 5 \\ -5 & -13 \end{bmatrix}$

10. $\begin{bmatrix} -7 & 4 \\ 8 & -5 \end{bmatrix}$

11. $\begin{bmatrix} 6 & -3 \\ -8 & 4 \end{bmatrix}$

12. $\begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ 5 & 4 \end{bmatrix}$

13. $\begin{bmatrix} 0.4 & -1.2 \\ 0.3 & 0.6 \end{bmatrix}$

14. $\begin{bmatrix} 4 & 2 & 3 \\ 3 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix}$

15. $\begin{bmatrix} 2 & 4 & 1 \\ -1 & 1 & -1 \\ 1 & 4 & 0 \end{bmatrix}$

16. $\begin{bmatrix} 5 & 7 & 4 \\ 3 & -1 & 3 \\ 6 & 7 & 5 \end{bmatrix}$

17. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & -1 \\ 1 & -1 & -10 \end{bmatrix}$

18. $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 4 \\ 2 & 1 & 2 \end{bmatrix}$

19. $\begin{bmatrix} 0 & -2 & 2 \\ 3 & 1 & 3 \\ 1 & -2 & 3 \end{bmatrix}$

20. $\begin{bmatrix} 3 & -2 & 0 \\ 5 & 1 & 1 \\ 2 & -2 & 0 \end{bmatrix}$

21. $\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 2 & 0 & 2 \end{bmatrix}$

22. $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

23–30 ■ Solve the system of equations by converting to a matrix equation and using the inverse of the coefficient matrix, as in Example 6. Use the inverses from Exercises 7–10, 15, 16, 19, and 21.

23. $\begin{cases} 5x + 3y = 4 \\ 3x + 2y = 0 \end{cases}$

24. $\begin{cases} 3x + 4y = 10 \\ 7x + 9y = 20 \end{cases}$

25. $\begin{cases} 2x + 5y = 2 \\ -5x - 13y = 20 \end{cases}$

26. $\begin{cases} -7x + 4y = 0 \\ 8x - 5y = 100 \end{cases}$

27. $\begin{cases} 2x + 4y + z = 7 \\ -x + y - z = 0 \\ x + 4y = -2 \end{cases}$

28. $\begin{cases} 5x + 7y + 4z = 1 \\ 3x - y + 3z = 1 \\ 6x + 7y + 5z = 1 \end{cases}$

29. $\begin{cases} -2y + 2z = 12 \\ 3x + y + 3z = -2 \\ x - 2y + 3z = 8 \end{cases}$

30. $\begin{cases} x + 2y + 3w = 0 \\ y + z + w = 1 \\ y + w = 2 \\ x + 2y + 2w = 3 \end{cases}$

31–36 ■ Use a calculator that can perform matrix operations to solve the system, as in Example 7.

31. $\begin{cases} x + y - 2z = 3 \\ 2x + 5z = 11 \\ 2x + 3y = 12 \end{cases}$

32. $\begin{cases} 3x + 4y - z = 2 \\ 2x - 3y + z = -5 \\ 5x - 2y + 2z = -3 \end{cases}$

33. $\begin{cases} 12x + \frac{1}{2}y - 7z = 21 \\ 11x - 2y + 3z = 43 \\ 13x + y - 4z = 29 \end{cases}$

34. $\begin{cases} x + \frac{1}{2}y - \frac{1}{3}z = 4 \\ x - \frac{1}{4}y + \frac{1}{6}z = 7 \\ x + y - z = -6 \end{cases}$

35. $\begin{cases} x + y - 3w = 0 \\ x - 2z = 8 \\ 2y - z + w = 5 \\ 2x + 3y - 2w = 13 \end{cases}$

36. $\begin{cases} x + y + z + w = 15 \\ x - y + z - w = 5 \\ x + 2y + 3z + 4w = 26 \\ x - 2y + 3z - 4w = 2 \end{cases}$

37–38 ■ Solve the matrix equation by multiplying each side by the appropriate inverse matrix.

37. $\begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} x & y & z \\ u & v & w \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \end{bmatrix}$

38. $\begin{bmatrix} 0 & -2 & 2 \\ 3 & 1 & 3 \\ 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x & u \\ y & v \\ z & w \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 6 & 12 \\ 0 & 0 \end{bmatrix}$

39–40 ■ Find the inverse of the matrix.

39. $\begin{bmatrix} a & -a \\ a & a \end{bmatrix}$ $(a \neq 0)$

40. $\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{bmatrix}$ $(abcd \neq 0)$

41–46 ■ Find the inverse of the matrix. For what value(s) of x, if any, does the matrix have no inverse?

41. $\begin{bmatrix} 2 & x \\ x & x^2 \end{bmatrix}$

42. $\begin{bmatrix} e^x & -e^{2x} \\ e^{2x} & e^{3x} \end{bmatrix}$

43. $\begin{bmatrix} 1 & e^x & 0 \\ e^x & -e^{2x} & 0 \\ 0 & 0 & 2 \end{bmatrix}$

44. $\begin{bmatrix} x & 1 \\ -x & \dfrac{1}{x-1} \end{bmatrix}$

45. $\begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}$

46. $\begin{bmatrix} \sec x & \tan x \\ \tan x & \sec x \end{bmatrix}$

Applications

47. Nutrition A nutritionist is studying the effects of the nutrients folic acid, choline, and inositol. He has three types of food available, and each type contains the following amounts of these nutrients per ounce:

	Type A	Type B	Type C
Folic acid (mg)	3	1	3
Choline (mg)	4	2	4
Inositol (mg)	3	2	4

(a) Find the inverse of the matrix

$$\begin{bmatrix} 3 & 1 & 3 \\ 4 & 2 & 4 \\ 3 & 2 & 4 \end{bmatrix}$$

and use it to solve the remaining parts of this problem.

(b) How many ounces of each food should the nutritionist feed his laboratory rats if he wants their daily diet to contain 10 mg of folic acid, 14 mg of choline, and 13 mg of inositol?

(c) How much of each food is needed to supply 9 mg of folic acid, 12 mg of choline, and 10 mg of inositol?

(d) Will any combination of these foods supply 2 mg of folic acid, 4 mg of choline, and 11 mg of inositol?

48. Nutrition Refer to Exercise 47. Suppose food type C has been improperly labeled, and it actually contains 4 mg of folic acid, 6 mg of choline, and 5 mg of inositol per ounce. Would it still be possible to use matrix inversion to solve parts (b), (c), and (d) of Exercise 47? Why or why not?

49. Sales Commissions An encyclopedia saleswoman works for a company that offers three different grades of bindings for its encyclopedias: standard, deluxe, and leather. For each set she sells, she earns a commission based on the set's binding grade. One week she sells one standard, one deluxe, and two leather sets and makes \$675 in commission. The next week she sells two standard, one deluxe, and one leather set for a \$600 commission. The third week she sells one standard, two deluxe, and one leather set, earning \$625 in commission.

(a) Let x, y, and z represent the commission she earns on standard, deluxe, and leather sets, respectively. Translate the given information into a system of equations in x, y, and z.

(b) Express the system of equations you found in part (a) as a matrix equation of the form $AX = B$.

(c) Find the inverse of the coefficient matrix A and use it to solve the matrix equation in part (b). How much commission does the saleswoman earn on a set of encyclopedias in each grade of binding?

Discovery • Discussion

50. No Zero-Product Property for Matrices We have used the Zero-Product Property to solve algebraic equations. Matrices do *not* have this property. Let O represent the **2 × 2 zero matrix:**

$$O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Find 2×2 matrices $A \neq O$ and $B \neq O$ such that $AB = O$. Can you find a matrix $A \neq O$ such that $A^2 = O$?

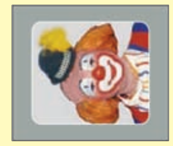
DISCOVERY PROJECT

Computer Graphics I

Matrix algebra is the basic tool used in computer graphics to manipulate images on a computer screen. We will see how matrix multiplication can be used to "move" a point in the plane to a prescribed location. Combining such moves enables us to stretch, compress, rotate, and otherwise transform a figure, as we see in the images below.


Image


Compressed


Rotated


Sheared

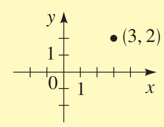
Moving Points in the Plane

Let's represent the point (x, y) in the plane by a 2×1 matrix:

$$(x, y) \quad \leftrightarrow \quad \begin{bmatrix} x \\ y \end{bmatrix}$$

For example, the point $(3, 2)$ in the figure is represented by the matrix

$$P = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

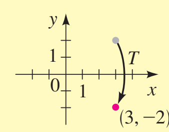


Multiplying by a 2×2 matrix *moves* the point in the plane. For example, if

$$T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

then multiplying P by T we get

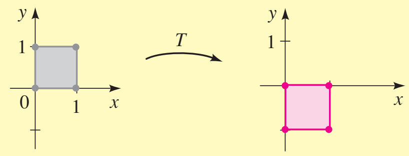
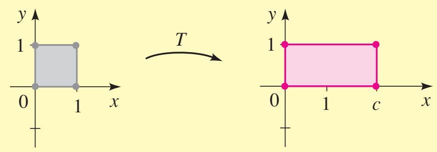
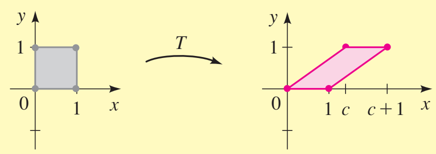
$$TP = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$



We see that the point $(3, 2)$ has been moved to the point $(3, -2)$. In general, multiplication by this matrix T reflects points in the x-axis. If every point in an image is multiplied by this matrix, then the entire image will be flipped upside down about the x-axis. Matrix multiplication "transforms" a point to a new point in the plane. For this reason, a matrix used in this way is called a **transformation**.

Table 1 gives some standard transformations and their effects on the gray square in the first quadrant.

Table 1

Transformation matrix	Effect
$T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ Reflection in x-axis	
$T = \begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix}$ Expansion (or contraction) in the x-direction	
$T = \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}$ Shear in x-direction	

Moving Images in the Plane

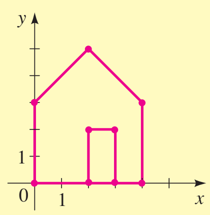


Figure 1

Simple line drawings such as the house in Figure 1 consist of a collection of vertex points and connecting line segments. The entire image in Figure 1 can be represented in a computer by the 2×11 **data matrix**

$$D = \begin{bmatrix} 2 & 0 & 0 & 2 & 4 & 4 & 3 & 3 & 2 & 2 & 3 \\ 0 & 0 & 3 & 5 & 3 & 0 & 0 & 2 & 2 & 0 & 0 \end{bmatrix}$$

The columns of D represent the vertex points of the image. To draw the house, we connect successive points (columns) in D by line segments. Now we can transform the whole house by multiplying D by an appropriate transformation matrix. For example, if we apply the shear transformation $T = \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix}$, we get the following matrix.

$$TD = \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 & 2 & 4 & 4 & 3 & 3 & 2 & 2 & 3 \\ 0 & 0 & 3 & 5 & 3 & 0 & 0 & 2 & 2 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 1.5 & 4.5 & 5.5 & 4 & 3 & 4 & 3 & 2 & 3 \\ 0 & 0 & 3 & 5 & 3 & 0 & 0 & 2 & 2 & 0 & 0 \end{bmatrix}$$

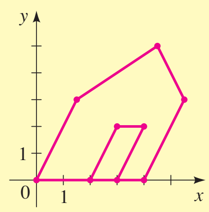
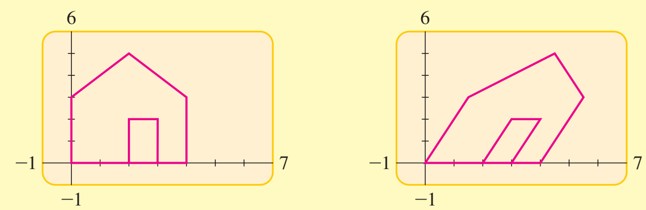


Figure 2

```
PROGRAM:IMAGE
:For(N,1,10)
:Line([A](1,N),
 [A](2,N),[A](1,N+1),
 [A](2,N+1))
:End
```

To draw the image represented by TD, we start with the point $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$, connect it by a line segment to the point $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, then follow that by a line segment to $\begin{bmatrix} 1.5 \\ 3 \end{bmatrix}$, and so on. The resulting tilted house is shown in Figure 2.

A convenient way to draw an image corresponding to a given data matrix is to use a graphing calculator. The TI-83 program in the margin converts a data matrix stored in `[A]` into the corresponding image, as shown in Figure 3. (To use this program for a data matrix with m columns, store the matrix in `[A]` and change the "10" in the `For` command to $m - 1$.)



(a) House with data matrix D

(b) Tilted house with data matrix TD

Figure 3

We will revisit computer graphics in the *Discovery Project* on page 792, where we will find matrices that rotate an image by any given angle.

1. The gray square in Table 1 has the following vertices.

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Apply each of the three transformations given in Table 1 to these vertices and sketch the result, to verify that each transformation has the indicated effect. Use $c = 2$ in the expansion matrix and $c = 1$ in the shear matrix.

2. Verify that multiplication by the given matrix has the indicated effect when applied to the gray square in the table. Use $c = 3$ in the expansion matrix and $c = 1$ in the shear matrix.

$$T_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \qquad T_2 = \begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix} \qquad T_3 = \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}$$

Reflection in y-axis — Expansion (or contraction) in y-direction — Shear in y-direction

3. Let $T = \begin{bmatrix} 1 & 1.5 \\ 0 & 1 \end{bmatrix}$.

(a) What effect does T have on the gray square in the Table 1?

(b) Find T^{-1}.

(c) What effect does T^{-1} have on the gray square?

(d) What happens to the square if we first apply T, then T^{-1}?

4. (a) Let $T = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$. What effect does T have on the gray square in Table 1?

(b) Let $S = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. What effect does S have on the gray square in Table 1?

(c) Apply S to the vertices of the square, and then apply T to the result. What is the effect of the combined transformation?

(d) Find the product matrix $W = TS$.

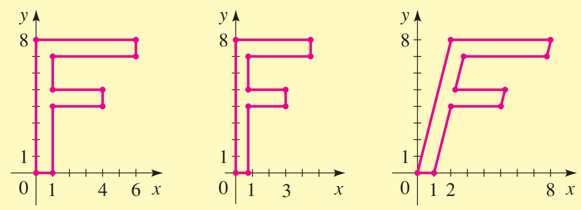
(e) Apply the transformation W to the square. Compare to your final result in part (c). What do you notice?

5. The figure shows three outline versions of the letter **F**. The second one is obtained from the first by shrinking horizontally by a factor of 0.75, and the third is obtained from the first by shearing horizontally by a factor of 0.25.

(a) Find a data matrix D for the first letter **F**.

(b) Find the transformation matrix T that transforms the first **F** into the second. Calculate TD and verify that this is a data matrix for the second **F**.

(c) Find the transformation matrix S that transforms the first **F** into the third. Calculate SD and verify that this is a data matrix for the third **F**.



6. Here is a data matrix for a line drawing.

$$D = \begin{bmatrix} 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 4 & 4 & 0 \end{bmatrix}$$

(a) Draw the image represented by D.

(b) Let $T = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$. Calculate the matrix product TD and draw the image represented by this product. What is the effect of the transformation T?

(c) Express T as a product of a shear matrix and a reflection matrix. (See problem 2.)

9.7 Determinants and Cramer's Rule

If a matrix is **square** (that is, if it has the same number of rows as columns), then we can assign to it a number called its *determinant*. Determinants can be used to solve systems of linear equations, as we will see later in this section. They are also useful in determining whether a matrix has an inverse.

Determinant of a 2 × 2 Matrix

We denote the determinant of a square matrix A by the symbol $\det(A)$ or $|A|$. We first define $\det(A)$ for the simplest cases. If $A = [a]$ is a 1×1 matrix, then $\det(A) = a$. The following box gives the definition of a 2×2 determinant.

We will use both notations, $\det(A)$ and $|A|$, for the determinant of A. Although the symbol $|A|$ looks like the absolute value symbol, it will be clear from the context which meaning is intended.

Determinant of a 2 × 2 Matrix

The **determinant** of the 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is

$$\det(A) = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Example 1 Determinant of a 2 × 2 Matrix

Evaluate $|A|$ for $A = \begin{bmatrix} 6 & -3 \\ 2 & 3 \end{bmatrix}$.

Solution

To evaluate a 2×2 determinant, we take the product of the diagonal from top left to bottom right, and subtract the product from top right to bottom left, as indicated by the arrows.

$$\begin{vmatrix} 6 & -3 \\ 2 & 3 \end{vmatrix} = 6 \cdot 3 - (-3)2 = 18 - (-6) = 24$$ ■

Determinant of an $n \times n$ Matrix

To define the concept of determinant for an arbitrary $n \times n$ matrix, we need the following terminology.

Let A be an $n \times n$ matrix.

1. The **minor** M_{ij} of the element a_{ij} is the determinant of the matrix obtained by deleting the ith row and jth column of A.
2. The **cofactor** A_{ij} of the element a_{ij} is

$$A_{ij} = (-1)^{i+j} M_{ij}$$

SUGGESTED TIME AND EMPHASIS

1 class.
Recommended material: determinants; optional material: Cramer's Rule.

POINTS TO STRESS

1. Definition and computation of determinants, including row operations.
2. The relationship between determinants and singularity.
3. Cramer's Rule.

ALTERNATE EXAMPLE 1

Evaluate $|A|$.

$A = \begin{bmatrix} 5 & -9 \\ 1 & 2 \end{bmatrix}$

ANSWER

19

IN-CLASS MATERIALS

Discuss computational complexity. If a 2×2 determinant requires 3 arithmetical operations (two multiplications and a subtraction), then a 3×3 determinant requires 12, and a 4×4 requires 52. If we let $f(n)$ be the number of arithmetical operations for an $n \times n$ matrix, we get the formula $f(n) = nf(n-1) + n$: there are n determinants of $(n-1) \times (n-1)$ matrices and n extra multiplications (when expanding by a row). So we can generate the following table:

n	Number of Operations	n	Number of Operations
2	3	9	804,969
3	12	10	8,049,700
4	52	11	88,546,711
5	265	.	.
6	1596	.	.
7	11,179	.	.
8	89,440	20	5,396,862,315,159,760,000

Students can observe this rapid growth on their calculators; it will take the calculator ten times as long to do a 10×10 determinant as it takes to do a 9×9. In fact, $f(n)$ grows a little more rapidly than $n!$.

For example, if A is the matrix

$$\begin{bmatrix} 2 & 3 & -1 \\ 0 & 2 & 4 \\ -2 & 5 & 6 \end{bmatrix}$$

then the minor M_{12} is the determinant of the matrix obtained by deleting the first row and second column from A. Thus

$$M_{12} = \begin{vmatrix} \cancel{2} & \cancel{3} & \cancel{-1} \\ 0 & \cancel{2} & 4 \\ -2 & \cancel{5} & 6 \end{vmatrix} = \begin{vmatrix} 0 & 4 \\ -2 & 6 \end{vmatrix} = 0(6) - 4(-2) = 8$$

So, the cofactor $A_{12} = (-1)^{1+2}M_{12} = -8$. Similarly

$$M_{33} = \begin{vmatrix} 2 & 3 & \cancel{-1} \\ 0 & 2 & \cancel{4} \\ \cancel{-2} & \cancel{5} & \cancel{6} \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 0 & 2 \end{vmatrix} = 2 \cdot 2 - 3 \cdot 0 = 4$$

So, $A_{33} = (-1)^{3+3}M_{33} = 4$.

Note that the cofactor of a_{ij} is simply the minor of a_{ij} multiplied by either 1 or -1, depending on whether $i + j$ is even or odd. Thus, in a 3×3 matrix we obtain the cofactor of any element by prefixing its minor with the sign obtained from the following checkerboard pattern:

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

We are now ready to define the determinant of any square matrix.

The Determinant of a Square Matrix

If A is an $n \times n$ matrix, then the **determinant** of A is obtained by multiplying each element of the first row by its cofactor, and then adding the results. In symbols,

$$\det(A) = |A| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n}$$

Example 2 Determinant of a 3×3 Matrix

Evaluate the determinant of the matrix.

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 0 & 2 & 4 \\ -2 & 5 & 6 \end{bmatrix}$$

SAMPLE QUESTION
Text Question
Name one application of determinants.

Answer
Any of the following: determining the singularity of a matrix, solving systems with Cramer's Rule, finding the area of a triangle.

DRILL QUESTION

Find $\begin{vmatrix} 2 & 1 & 3 \\ 1 & 0 & 0 \\ 0 & 2 & -1 \end{vmatrix}$.

Answer
7

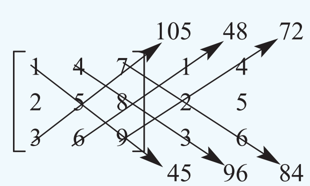
ALTERNATE EXAMPLE 2
Evaluate the determinant of the matrix.

$$A = \begin{bmatrix} 9 & 3 & -1 \\ 0 & 3 & 2 \\ -4 & 2 & 4 \end{bmatrix}$$

ANSWER
36

IN-CLASS MATERIALS

Show the students the "basket" method of computing a 3×3 determinant. Stress that this method does not generalize to higher dimensions. One rewrites the first two columns of the matrix, and then multiplies along the diagonals, adding the top-to-bottom diagonals and subtracting the bottom-to-top ones. In the example below, we calculate the determinant of $\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$ to be $105 + 48 + 72 - (45 + 96 + 84) = 0$.



Solution

$$\det(A) = \begin{vmatrix} 2 & 3 & -1 \\ 0 & 2 & 4 \\ -2 & 5 & 6 \end{vmatrix} = 2\begin{vmatrix} 2 & 4 \\ 5 & 6 \end{vmatrix} - 3\begin{vmatrix} 0 & 4 \\ -2 & 6 \end{vmatrix} + (-1)\begin{vmatrix} 0 & 2 \\ -2 & 5 \end{vmatrix}$$

$$= 2(2\cdot 6 - 4\cdot 5) - 3[0\cdot 6 - 4(-2)] - [0\cdot 5 - 2(-2)]$$

$$= -16 - 24 - 4$$

$$= -44$$

■

In our definition of the determinant we used the cofactors of elements in the first row only. This is called **expanding the determinant by the first row**. In fact, *we can expand the determinant by any row or column in the same way, and obtain the same result in each case* (although we won't prove this). The next example illustrates this principle.

Example 3 Expanding a Determinant about a Row and a Column

Let A be the matrix of Example 2. Evaluate the determinant of A by expanding

(a) by the second row

(b) by the third column

Verify that each expansion gives the same value.

Solution

(a) Expanding by the second row, we get

$$\det(A) = \begin{vmatrix} 2 & 3 & -1 \\ 0 & 2 & -4 \\ -2 & 5 & 6 \end{vmatrix} = -0\begin{vmatrix} 3 & -1 \\ 5 & 6 \end{vmatrix} + 2\begin{vmatrix} 2 & -1 \\ -2 & 6 \end{vmatrix} - 4\begin{vmatrix} 2 & 3 \\ -2 & 5 \end{vmatrix}$$

$$= 0 + 2[2\cdot 6 - (-1)(-2)] - 4[2\cdot 5 - 3(-2)]$$

$$= 0 + 20 - 64 = -44$$

(b) Expanding by the third column gives

$$\det(A) = \begin{vmatrix} 2 & 3 & -1 \\ 0 & 2 & 4 \\ -2 & 5 & 6 \end{vmatrix}$$

$$= -1\begin{vmatrix} 0 & 2 \\ -2 & 5 \end{vmatrix} - 4\begin{vmatrix} 2 & 3 \\ -2 & 5 \end{vmatrix} + 6\begin{vmatrix} 2 & 3 \\ 0 & 2 \end{vmatrix}$$

$$= -[0\cdot 5 - 2(-2)] - 4[2\cdot 5 - 3(-2)] + 6(2\cdot 2 - 3\cdot 0)$$

$$= -4 - 64 + 24 = -44$$

In both cases, we obtain the same value for the determinant as when we expanded by the first row in Example 2. ■

The following criterion allows us to determine whether a square matrix has an inverse without actually calculating the inverse. This is one of the most important uses of the determinant in matrix algebra, and it is the reason for the name *determinant*.

Graphing calculators are capable of computing determinants. Here is the output when the TI-83 is used to calculate the determinant in Example 3.

```
[A]
        [[2  3 -1]
         [0  2  4 ]
         [-2 5  6 ]]
det([A])
                -44
```

ALTERNATE EXAMPLE 3

Evaluate the determinant of the matrix by expanding by the second row and by expanding by the third column.

$$A = \begin{bmatrix} 6 & 6 & -1 \\ 0 & 9 & 8 \\ -2 & 2 & 9 \end{bmatrix}$$

ANSWER

276, 276

EXAMPLE

A 4×4 determinant:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & -1 & 0 & 5 \\ 0 & 8 & 1 & 2 \\ 0 & 2 & 1 & 3 \end{bmatrix}$$

Expand by Row 3:

$$D = -8\begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 5 \\ 0 & 1 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 5 \\ 0 & 2 & 3 \end{bmatrix} - 2\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$$= -8(-9) + (-15) - 2(1) = 55$$

IN-CLASS MATERIALS

Make sure the students see the fundamentally recursive nature of determinants. Draw a 6×6 matrix on the board and expand by a row, showing how you would then have to compute six 5×5 determinants, which would require the computation of thirty 4×4 determinants, and so forth.

Invertibility Criterion

If A is a square matrix, then A has an inverse if and only if $\det(A) \neq 0$.

We will not prove this fact, but from the formula for the inverse of a 2×2 matrix (page 704), you can see why it is true in the 2×2 case.

Example 4 Using the Determinant to Show That a Matrix Is Not Invertible

Show that the matrix A has no inverse.

$$A = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 3 \\ 5 & 6 & 2 & 6 \\ 2 & 4 & 0 & 9 \end{bmatrix}$$

Solution We begin by calculating the determinant of A. Since all but one of the elements of the second row is zero, we expand the determinant by the second row. If we do this, we see from the following equation that only the cofactor A_{24} will have to be calculated.

$$\det(A) = \begin{vmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 3 \\ 5 & 6 & 2 & 6 \\ 2 & 4 & 0 & 9 \end{vmatrix}$$

$$= -0 \cdot A_{21} + 0 \cdot A_{22} - 0 \cdot A_{23} + 3 \cdot A_{24} = 3A_{24}$$

$$= 3\begin{vmatrix} 1 & 2 & 0 \\ 5 & 6 & 2 \\ 2 & 4 & 0 \end{vmatrix} \qquad \text{Expand this by column 3}$$

$$= 3(-2)\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix}$$

$$= 3(-2)(1 \cdot 4 - 2 \cdot 2) = 0$$

Since the determinant of A is zero, A cannot have an inverse, by the Invertibility Criterion. ■

Row and Column Transformations

The preceding example shows that if we expand a determinant about a row or column that contains many zeros, our work is reduced considerably because we don't have to evaluate the cofactors of the elements that are zero. The following principle often simplifies the process of finding a determinant by introducing zeros into it without changing its value.

ALTERNATE EXAMPLE 4

Does the matrix A have an inverse?

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 8 & 2 & 1 & 5 \\ 4 & 8 & 0 & 9 \end{bmatrix}$$

ANSWER

No

IN-CLASS MATERIALS

Determinants have many nice properties. For example, if a row or column consists of all zeros, it is trivial to prove that the determinant is zero. The row/column transformation rule given in the text then lets us conclude that if two rows of a matrix are identical, the determinant is zero. Also, the determinant of a product is the product of the determinants.

ALTERNATE EXAMPLE 5

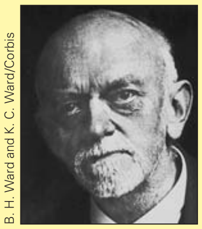
Evaluate the determinant of the matrix.

$$A = \begin{bmatrix} 6 & 2 & 1 & -3 \\ 1 & 2 & 6 & 9 \\ 18 & 6 & 7 & -9 \\ 18 & 8 & 3 & -9 \end{bmatrix}$$

Does the matrix A have an inverse?

ANSWER

-456, Yes



B. H. Ward and K. C. Ward/Corbis

David Hilbert (1862–1943) was born in Königsberg, Germany, and became a professor at Göttingen University. He is considered by many to be the greatest mathematician of the 20th century. At the International Congress of Mathematicians held in Paris in 1900, Hilbert set the direction of mathematics for the about-to-dawn 20th century by posing 23 problems he believed to be of crucial importance. He said that "these are problems whose solutions we expect from the future." Most of Hilbert's problems have now been solved (see Julia Robinson, page 678 and Alan Turing, page 103), and their solutions have led to important new areas of mathematical research. Yet as we enter the new millennium, some of his problems remain unsolved. In his work, Hilbert emphasized structure, logic, and the foundations of mathematics. Part of his genius lay in his ability to see the most general possible statement of a problem. For instance, Euler proved that every whole number is the sum of four squares; Hilbert proved a similar statement for all powers of positive integers.

Row and Column Transformations of a Determinant

If A is a square matrix, and if the matrix B is obtained from A by adding a multiple of one row to another, or a multiple of one column to another, then $\det(A) = \det(B)$.

Example 5 Using Row and Column Transformations to Calculate a Determinant

Find the determinant of the matrix A. Does it have an inverse?

$$A = \begin{bmatrix} 8 & 2 & -1 & -4 \\ 3 & 5 & -3 & 11 \\ 24 & 6 & 1 & -12 \\ 2 & 2 & 7 & -1 \end{bmatrix}$$

Solution If we add -3 times row 1 to row 3, we change all but one element of row 3 to zeros:

$$\begin{bmatrix} 8 & 2 & -1 & -4 \\ 3 & 5 & -3 & 11 \\ 0 & 0 & 4 & 0 \\ 2 & 2 & 7 & -1 \end{bmatrix}$$

This new matrix has the same determinant as A, and if we expand its determinant by the third row, we get

$$\det(A) = 4\begin{vmatrix} 8 & 2 & -4 \\ 3 & 5 & 11 \\ 2 & 2 & -1 \end{vmatrix}$$

Now, adding 2 times column 3 to column 1 in this determinant gives us

$$\det(A) = 4\begin{vmatrix} 0 & 2 & -4 \\ 25 & 5 & 11 \\ 0 & 2 & -1 \end{vmatrix} \qquad \text{Expand this by column 1}$$

$$= 4(-25)\begin{vmatrix} 2 & -4 \\ 2 & -1 \end{vmatrix}$$

$$= 4(-25)[2(-1) - (-4)2] = -600$$

Since the determinant of A is not zero, A does have an inverse. ■

Cramer's Rule

The solutions of linear equations can sometimes be expressed using determinants. To illustrate, let's solve the following pair of linear equations for the variable x.

$$\begin{cases} ax + by = r \\ cx + dy = s \end{cases}$$

To eliminate the variable y, we multiply the first equation by d and the second by b, and subtract.

$$\begin{array}{r} adx + bdy = rd \\ bcx + bdy = bs \\ \hline adx - bcx = rd - bs \end{array}$$

Factoring the left-hand side, we get $(ad - bc)x = rd - bs$. Assuming that $ad - bc \neq 0$, we can now solve this equation for x:

$$x = \frac{rd - bs}{ad - bc}$$

Similarly, we find

$$y = \frac{as - cr}{ad - bc}$$

The numerator and denominator of the fractions for x and y are determinants of 2×2 matrices. So we can express the solution of the system using determinants as follows.

Cramer's Rule for Systems in Two Variables

The linear system

$$\begin{cases} ax + by = r \\ cx + dy = s \end{cases}$$

has the solution

$$x = \frac{\begin{vmatrix} r & b \\ s & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a & r \\ c & s \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

provided $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$.

Using the notation

$$D = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad D_x = \begin{bmatrix} r & b \\ s & d \end{bmatrix} \qquad D_y = \begin{bmatrix} a & r \\ c & s \end{bmatrix}$$

Coefficient matrix

Replace first column of D by r and s.

Replace second column of D by r and s.

we can write the solution of the system as

$$x = \frac{|D_x|}{|D|} \quad \text{and} \quad y = \frac{|D_y|}{|D|}$$

EXAMPLE

Cramer's Rule:

$$\begin{aligned} x - y + z &= 8 \\ -x - y - z &= -9 \\ x - 2y - 4z &= 5 \end{aligned}$$

$$D = \begin{bmatrix} 1 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -2 & -4 \end{bmatrix} = 10,$$

$$D_x = \begin{bmatrix} 8 & -1 & 1 \\ -9 & -1 & -1 \\ 5 & -2 & -4 \end{bmatrix} = 80,$$

$$D_y = \begin{bmatrix} 1 & 8 & 1 \\ -1 & -9 & -1 \\ 1 & 5 & -4 \end{bmatrix} = 5,$$

and

$$D_z = \begin{bmatrix} 1 & -1 & 8 \\ -1 & -1 & -9 \\ 1 & -2 & 5 \end{bmatrix} = 5,$$

so we have $x = 8$, $y = \frac{1}{2}$, and $z = \frac{1}{2}$.

ALTERNATE EXAMPLE 6

Use Cramer's Rule to solve the system.

$$\begin{cases} 3x + 10y = -4 \\ x + 4y = -1 \end{cases}$$

If the equations of the system are dependent, or if a system is inconsistent, so indicate.

ANSWER

$\left(-3, \frac{1}{2}\right)$

The Granger Collection

Emmy Noether (1882–1935) was one of the foremost mathematicians of the early 20th century. Her groundbreaking work in abstract algebra provided much of the foundation for this field, and her work in Invariant Theory was essential in the development of Einstein's theory of general relativity. Although women weren't allowed to study at German universities at that time, she audited courses unofficially and went on to receive a doctorate at Erlangen *summa cum laude*, despite the opposition of the academic senate, which declared that women students would "overthrow all academic order." She subsequently taught mathematics at Göttingen, Moscow, and Frankfurt. In 1933 she left Germany to escape Nazi persecution, accepting a position at Bryn Mawr College in suburban Philadelphia. She lectured there and at the Institute for Advanced Study in Princeton, New Jersey, until her untimely death in 1935.

Example 6 Using Cramer's Rule to Solve a System with Two Variables

Use Cramer's Rule to solve the system.

$$\begin{cases} 2x + 6y = -1 \\ x + 8y = 2 \end{cases}$$

Solution For this system we have

$$|D| = \begin{vmatrix} 2 & 6 \\ 1 & 8 \end{vmatrix} = 2 \cdot 8 - 6 \cdot 1 = 10$$

$$|D_x| = \begin{vmatrix} -1 & 6 \\ 2 & 8 \end{vmatrix} = (-1)8 - 6 \cdot 2 = -20$$

$$|D_y| = \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} = 2 \cdot 2 - (-1)1 = 5$$

The solution is

$$x = \frac{|D_x|}{|D|} = \frac{-20}{10} = -2$$

$$y = \frac{|D_y|}{|D|} = \frac{5}{10} = \frac{1}{2}$$

■

Cramer's Rule can be extended to apply to any system of n linear equations in n variables in which the determinant of the coefficient matrix is not zero. As we saw in the preceding section, any such system can be written in matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

By analogy with our derivation of Cramer's Rule in the case of two equations in two unknowns, we let D be the coefficient matrix in this system, and D_{x_i} be the matrix obtained by replacing the ith column of D by the numbers $b_1, b_2, \dots, b_n$ that appear to the right of the equal sign. The solution of the system is then given by the following rule.

Cramer's Rule

If a system of n linear equations in the n variables $x_1, x_2, \dots, x_n$ is equivalent to the matrix equation $DX = B$, and if $|D| \neq 0$, then its solutions are

$$x_1 = \frac{|D_{x_1}|}{|D|}, \quad x_2 = \frac{|D_{x_2}|}{|D|}, \quad \dots, \quad x_n = \frac{|D_{x_n}|}{|D|}$$

where D_{x_i} is the matrix obtained by replacing the ith column of D by the $n \times 1$ matrix B.

IN-CLASS MATERIALS

When solving a single $n \times n$ system, Cramer's Rule doesn't have much of an advantage over Gaussian elimination. There are, however, some circumstances when Cramer's Rule is vastly superior. One such situation is when there is a large system (say 100 equations with 100 unknowns) and we are interested in only one variable. Gaussian elimination requires us to do the work to solve the complete system. Finding the inverse of the matrix also requires us to do all the work necessary to find all the variables. Cramer's Rule, however, allows us to find two (admittedly large) determinants to get our answer.

Another situation is when the coefficient matrix is sparse—a large percentage of the entries are zero. It is usually very quick to find determinants of sparse matrices, and in that case Cramer's Rule can be very quick.

Example 7 **Using Cramer's Rule to Solve a System with Three Variables**

Use Cramer's Rule to solve the system.

$$\begin{cases} 2x - 3y + 4z = 1 \\ x + 6z = 0 \\ 3x - 2y = 5 \end{cases}$$

Solution First, we evaluate the determinants that appear in Cramer's Rule. Note that D is the coefficient matrix and that D_x, D_y, and D_z are obtained by replacing the first, second, and third columns of D by the constant terms.

$$|D| = \begin{vmatrix} 2 & -3 & 4 \\ 1 & 0 & 6 \\ 3 & -2 & 0 \end{vmatrix} = -38 \qquad |D_x| = \begin{vmatrix} 1 & -3 & 4 \\ 0 & 0 & 6 \\ 5 & -2 & 0 \end{vmatrix} = -78$$

$$|D_y| = \begin{vmatrix} 2 & 1 & 4 \\ 1 & 0 & 6 \\ 3 & 5 & 0 \end{vmatrix} = -22 \qquad |D_z| = \begin{vmatrix} 2 & -3 & 1 \\ 1 & 0 & 0 \\ 3 & -2 & 5 \end{vmatrix} = 13$$

Now we use Cramer's Rule to get the solution:

$$x = \frac{|D_x|}{|D|} = \frac{-78}{-38} = \frac{39}{19}$$

$$y = \frac{|D_y|}{|D|} = \frac{-22}{-38} = \frac{11}{19}$$

$$z = \frac{|D_z|}{|D|} = \frac{13}{-38} = -\frac{13}{38}$$

■

Solving the system in Example 7 using Gaussian elimination would involve matrices whose elements are fractions with fairly large denominators. Thus, in cases like Examples 6 and 7, Cramer's Rule gives us an efficient way to solve systems of linear equations. But in systems with more than three equations, evaluating the various determinants involved is usually a long and tedious task (unless you are using a graphing calculator). Moreover, the rule doesn't apply if $|D| = 0$ or if D is not a square matrix. So, Cramer's Rule is a useful alternative to Gaussian elimination, but only in some situations.

Areas of Triangles Using Determinants

Determinants provide a simple way to calculate the area of a triangle in the coordinate plane.

ALTERNATE EXAMPLE 7

Use Cramer's Rule to solve the system.

$$\begin{cases} 9x - 8y + 5z = 8 \\ x + 6z = 0 \\ 6x - 4y = 3 \end{cases}$$

If the equations of the system are dependent, or if a system is inconsistent, so indicate.

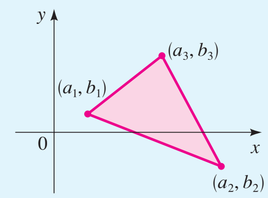
ANSWER

$$\left(-\frac{12}{23}, -\frac{141}{92}, \frac{2}{23}\right)$$

Area of a Triangle

If a triangle in the coordinate plane has vertices (a_1, b_1), (a_2, b_2), and (a_3, b_3), then its area is

$$\text{area} = \pm\tfrac{1}{2}\begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix}$$



where the sign is chosen to make the area positive.

You are asked to prove this formula in Exercise 59.

ALTERNATE EXAMPLE 8
Find the area of a triangle with vertices (0, 2), (3, 5), and (−4, 0).

ANSWER

$$\pm\frac{1}{2}\begin{vmatrix} 0 & 2 & 1 \\ 3 & 5 & 1 \\ -4 & 0 & 1 \end{vmatrix} = \pm 3,$$

so the area is 3 square units.

Example 8 Area of a Triangle

Find the area of the triangle shown in Figure 1.

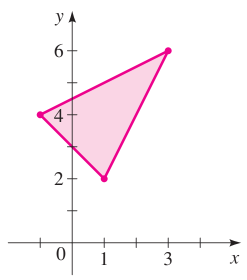
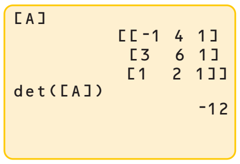


Figure 1

We can calculate the determinant by hand or by using a graphing calculator.



Solution The vertices are $(-1, 4)$, $(3, 6)$, and $(1, 2)$. Using the formula in the preceding box, we get:

$$\text{area} = \pm\tfrac{1}{2}\begin{vmatrix} -1 & 4 & 1 \\ 3 & 6 & 1 \\ 1 & 2 & 1 \end{vmatrix} = \pm\tfrac{1}{2}(-12)$$

To make the area positive, we choose the negative sign in the formula. Thus, the area of the triangle is

$$\text{area} = -\tfrac{1}{2}(-12) = 6$$

■

9.7 Exercises

1–8 ■ Find the determinant of the matrix, if it exists.

1. $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

2. $\begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}$

3. $\begin{bmatrix} 4 & 5 \\ 0 & -1 \end{bmatrix}$

4. $\begin{bmatrix} -2 & 1 \\ 3 & -2 \end{bmatrix}$

5. $\begin{bmatrix} 2 & 5 \end{bmatrix}$

6. $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$

7. $\begin{bmatrix} \frac{1}{2} & \frac{1}{8} \\ 1 & \frac{1}{2} \end{bmatrix}$

8. $\begin{bmatrix} 2.2 & -1.4 \\ 0.5 & 1.0 \end{bmatrix}$

9–14 ■ Evaluate the minor and cofactor using the matrix A.

$$A = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ -3 & 5 & 2 \\ 0 & 0 & 4 \end{bmatrix}$$

9. M_{11}, A_{11}

10. M_{33}, A_{33}

11. M_{12}, A_{12}

12. M_{13}, A_{13}

13. M_{23}, A_{23}

14. M_{32}, A_{32}

15–22 ■ Find the determinant of the matrix. Determine whether the matrix has an inverse, but don't calculate the inverse.

15. $\begin{bmatrix} 2 & 1 & 0 \\ 0 & -2 & 4 \\ 0 & 1 & -3 \end{bmatrix}$

16. $\begin{bmatrix} 0 & -1 & 0 \\ 2 & 6 & 4 \\ 1 & 0 & 3 \end{bmatrix}$

17. $\begin{bmatrix} 1 & 3 & 7 \\ 2 & 0 & -1 \\ 0 & 2 & 6 \end{bmatrix}$

18. $\begin{bmatrix} -2 & -\frac{3}{2} & \frac{1}{2} \\ 2 & 4 & 0 \\ \frac{1}{2} & 2 & 1 \end{bmatrix}$

19. $\begin{bmatrix} 30 & 0 & 20 \\ 0 & -10 & -20 \\ 40 & 0 & 10 \end{bmatrix}$

20. $\begin{bmatrix} 1 & 2 & 5 \\ -2 & -3 & 2 \\ 3 & 5 & 3 \end{bmatrix}$

21. $\begin{bmatrix} 1 & 3 & 3 & 0 \\ 0 & 2 & 0 & 1 \\ -1 & 0 & 0 & 2 \\ 1 & 6 & 4 & 1 \end{bmatrix}$

22. $\begin{bmatrix} 1 & 2 & 0 & 2 \\ 3 & -4 & 0 & 4 \\ 0 & 1 & 6 & 0 \\ 1 & 0 & 2 & 0 \end{bmatrix}$

23–26 ■ Evaluate the determinant, using row or column operations whenever possible to simplify your work.

23. $\begin{vmatrix} 0 & 0 & 4 & 6 \\ 2 & 1 & 1 & 3 \\ 2 & 1 & 2 & 3 \\ 3 & 0 & 1 & 7 \end{vmatrix}$

24. $\begin{vmatrix} -2 & 3 & -1 & 7 \\ 4 & 6 & -2 & 3 \\ 7 & 7 & 0 & 5 \\ 3 & -12 & 4 & 0 \end{vmatrix}$

25. $\begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 2 & 4 & 6 & 8 \\ 0 & 0 & 3 & 6 & 9 \\ 0 & 0 & 0 & 4 & 8 \\ 0 & 0 & 0 & 0 & 5 \end{vmatrix}$

26. $\begin{vmatrix} 2 & -1 & 6 & 4 \\ 7 & 2 & -2 & 5 \\ 4 & -2 & 10 & 8 \\ 6 & 1 & 1 & 4 \end{vmatrix}$

27. Let

$$B = \begin{bmatrix} 4 & 1 & 0 \\ -2 & -1 & 1 \\ 4 & 0 & 3 \end{bmatrix}$$

(a) Evaluate $\det(B)$ by expanding by the second row.
(b) Evaluate $\det(B)$ by expanding by the third column.
(c) Do your results in parts (a) and (b) agree?

28. Consider the system

$$\begin{cases} x + 2y + 6z = 5 \\ -3x - 6y + 5z = 8 \\ 2x + 6y + 9z = 7 \end{cases}$$

(a) Verify that $x = -1$, $y = 0$, $z = 1$ is a solution of the system.
(b) Find the determinant of the coefficient matrix.
(c) Without solving the system, determine whether there are any other solutions.
(d) Can Cramer's Rule be used to solve this system? Why or why not?

29–44 ■ Use Cramer's Rule to solve the system.

29. $\begin{cases} 2x - y = -9 \\ x + 2y = 8 \end{cases}$

30. $\begin{cases} 6x + 12y = 33 \\ 4x + 7y = 20 \end{cases}$

31. $\begin{cases} x - 6y = 3 \\ 3x + 2y = 1 \end{cases}$

32. $\begin{cases} \frac{1}{2}x + \frac{1}{3}y = 1 \\ \frac{1}{4}x - \frac{1}{6}y = -\frac{3}{2} \end{cases}$

33. $\begin{cases} 0.4x + 1.2y = 0.4 \\ 1.2x + 1.6y = 3.2 \end{cases}$

34. $\begin{cases} 10x - 17y = 21 \\ 20x - 31y = 39 \end{cases}$

35. $\begin{cases} x - y + 2z = 0 \\ 3x + z = 11 \\ -x + 2y = 0 \end{cases}$

36. $\begin{cases} 5x - 3y + z = 6 \\ 4y - 6z = 22 \\ 7x + 10y = -13 \end{cases}$

37. $\begin{cases} 2x_1 + 3x_2 - 5x_3 = 1 \\ x_1 + x_2 - x_3 = 2 \\ 2x_2 + x_3 = 8 \end{cases}$

38. $\begin{cases} -2a + c = 2 \\ a + 2b - c = 9 \\ 3a + 5b + 2c = 22 \end{cases}$

39. $\begin{cases} \frac{1}{3}x - \frac{1}{5}y + \frac{1}{2}z = \frac{7}{10} \\ -\frac{2}{3}x + \frac{2}{5}y + \frac{3}{2}z = \frac{11}{10} \\ x - \frac{4}{5}y + z = \frac{9}{5} \end{cases}$

40. $\begin{cases} 2x - y = 5 \\ 5x + 3z = 19 \\ 4y + 7z = 17 \end{cases}$

41. $\begin{cases} 3y + 5z = 4 \\ 2x - z = 10 \\ 4x + 7y = 0 \end{cases}$

42. $\begin{cases} 2x - 5y = 4 \\ x + y - z = 8 \\ 3x + 5z = 0 \end{cases}$

43. $\begin{cases} x + y + z + w = 0 \\ 2x + w = 0 \\ y - z = 0 \\ x + 2z = 1 \end{cases}$ **44.** $\begin{cases} x + y = 1 \\ y + z = 2 \\ z + w = 3 \\ w - x = 4 \end{cases}$

45–46 ■ Evaluate the determinants.

45. $\begin{vmatrix} a & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 \\ 0 & 0 & 0 & d & 0 \\ 0 & 0 & 0 & 0 & e \end{vmatrix}$ **46.** $\begin{vmatrix} a & a & a & a & a \\ 0 & a & a & a & a \\ 0 & 0 & a & a & a \\ 0 & 0 & 0 & a & a \\ 0 & 0 & 0 & 0 & a \end{vmatrix}$

47–50 ■ Solve for x.

47. $\begin{vmatrix} x & 12 & 13 \\ 0 & x-1 & 23 \\ 0 & 0 & x-2 \end{vmatrix} = 0$ **48.** $\begin{vmatrix} x & 1 & 1 \\ 1 & 1 & x \\ x & 1 & x \end{vmatrix} = 0$

49. $\begin{vmatrix} 1 & 0 & x \\ x^2 & 1 & 0 \\ x & 0 & 1 \end{vmatrix} = 0$ **50.** $\begin{vmatrix} a & b & x-a \\ x & x+b & x \\ 0 & 1 & 1 \end{vmatrix} = 0$

51–54 ■ Sketch the triangle with the given vertices and use a determinant to find its area.

51. $(0, 0), (6, 2), (3, 8)$

52. $(1, 0), (3, 5), (-2, 2)$

53. $(-1, 3), (2, 9), (5, -6)$

54. $(-2, 5), (7, 2), (3, -4)$

55. Show that $\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = (x - y)(y - z)(z - x)$

Applications

56. Buying Fruit A roadside fruit stand sells apples at 75¢ a pound, peaches at 90¢ a pound, and pears at 60¢ a pound. Muriel buys 18 pounds of fruit at a total cost of \$13.80. Her peaches and pears together cost \$1.80 more than her apples.

(a) Set up a linear system for the number of pounds of apples, peaches, and pears that she bought.

(b) Solve the system using Cramer's Rule.

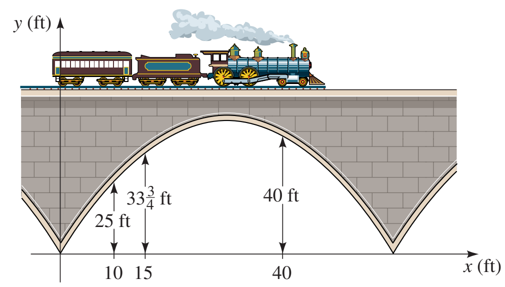
57. The Arch of a Bridge The opening of a railway bridge over a roadway is in the shape of a parabola. A surveyor measures the heights of three points on the bridge, as shown in the figure. He wishes to find an equation of the form

$$y = ax^2 + bx + c$$

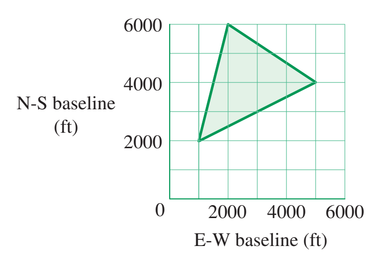
to model the shape of the arch.

(a) Use the surveyed points to set up a system of linear equations for the unknown coefficients a, b, and c.

(b) Solve the system using Cramer's Rule.



58. A Triangular Plot of Land An outdoors club is purchasing land to set up a conservation area. The last remaining piece they need to buy is the triangular plot shown in the figure. Use the determinant formula for the area of a triangle to find the area of the plot.



Discovery • Discussion

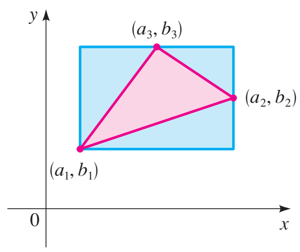
59. Determinant Formula for the Area of a Triangle The figure shows a triangle in the plane with vertices (a_1, b_1), (a_2, b_2), and (a_3, b_3).

(a) Find the coordinates of the vertices of the surrounding rectangle and find its area.

(b) Find the area of the red triangle by subtracting the areas of the three blue triangles from the area of the rectangle.

(c) Use your answer to part (b) to show that the area of the red triangle is given by

$$\text{area} = \pm\tfrac{1}{2}\begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix}$$



60. Collinear Points and Determinants

(a) If three points lie on a line, what is the area of the "triangle" that they determine? Use the answer to this question, together with the determinant formula for the area of a triangle, to explain why the points (a_1, b_1), (a_2, b_2), and (a_3, b_3) are collinear if and only if

$$\begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix} = 0$$

(b) Use a determinant to check whether each set of points is collinear. Graph them to verify your answer.

(i) $(-6, 4), (2, 10), (6, 13)$

(ii) $(-5, 10), (2, 6), (15, -2)$

61. Determinant Form for the Equation of a Line

(a) Use the result of Exercise 60(a) to show that the equation of the line containing the points (x_1, y_1) and (x_2, y_2) is

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

(b) Use the result of part (a) to find an equation for the line containing the points $(20, 50)$ and $(-10, 25)$.

62. Matrices with Determinant Zero Use the definition of determinant and the elementary row and column operations to explain why matrices of the following types have determinant 0.

(a) A matrix with a row or column consisting entirely of zeros

(b) A matrix with two rows the same or two columns the same

(c) A matrix in which one row is a multiple of another row, or one column is a multiple of another column

63. Solving Linear Systems Suppose you have to solve a linear system with five equations and five variables without the assistance of a calculator or computer. Which method would you prefer: Cramer's Rule or Gaussian elimination? Write a short paragraph explaining the reasons for your answer.

9.8 Partial Fractions

SUGGESTED TIME AND EMPHASIS

$\frac{1}{2}$–1 class.

Optional material.

To write a sum or difference of fractional expressions as a single fraction, we bring them to a common denominator. For example,

$$\frac{1}{x-1} + \frac{1}{2x+1} = \frac{(2x+1) + (x-1)}{(x-1)(2x+1)} = \frac{3x}{2x^2 - x - 1}$$

Common denominator →

$$\frac{1}{x-1} + \frac{1}{2x+1} = \frac{3x}{2x^2 - x - 1}$$

← Partial fractions

But for some applications of algebra to calculus, we must reverse this process—that is, we must express a fraction such as $3x/(2x^2 - x - 1)$ as the sum of the simpler fractions $1/(x-1)$ and $1/(2x+1)$. These simpler fractions are called *partial fractions*; we learn how to find them in this section.

POINT TO STRESS

Decomposing a rational function into partial fractions.

Let r be the rational function

$$r(x) = \frac{P(x)}{Q(x)}$$

where the degree of P is less than the degree of Q. By the Linear and Quadratic Factors Theorem in Section 3.4, every polynomial with real coefficients can be factored completely into linear and irreducible quadratic factors, that is, factors of the form $ax + b$ and $ax^2 + bx + c$, where a, b, and c are real numbers. For instance,

$$x^4 - 1 = (x^2 - 1)(x^2 + 1) = (x - 1)(x + 1)(x^2 + 1)$$

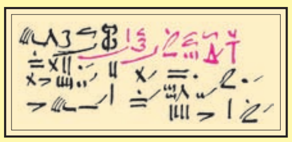
After we have completely factored the denominator Q of r, we can express $r(x)$ as a sum of **partial fractions** of the form

$$\frac{A}{(ax + b)^i} \quad \text{and} \quad \frac{Ax + B}{(ax^2 + bx + c)^j}$$

This sum is called the **partial fraction decomposition** of r. Let's examine the details of the four possible cases.

The Rhind papyrus is the oldest known mathematical document. It is an Egyptian scroll written in 1650 B.C. by the scribe Ahmes, who explains that it is an exact copy of a scroll written 200 years earlier. Ahmes claims that his papyrus contains "a thorough study of all things, insight into all that exists, knowledge of all obscure secrets." Actually, the document contains rules for doing arithmetic, including multiplication and division of fractions and several exercises with solutions. The exercise shown below reads: A heap and its seventh make nineteen; how large is the heap? In solving problems of this sort, the Egyptians used partial fractions because their number system required all fractions to be written as sums of reciprocals of whole numbers. For example, $\frac{7}{12}$ would be written as $\frac{1}{3} + \frac{1}{4}$.

The papyrus gives a correct formula for the volume of a truncated pyramid (page 143). It also gives the formula $A = \left(\frac{8}{9}d\right)^2$ for the area of a circle with diameter d. How close is this to the actual area?



Case 1: The Denominator Is a Product of Distinct Linear Factors

Suppose that we can factor $Q(x)$ as

$$Q(x) = (a_1x + b_1)(a_2x + b_2) \cdots (a_nx + b_n)$$

with no factor repeated. In this case, the partial fraction decomposition of $P(x)/Q(x)$ takes the form

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \cdots + \frac{A_n}{a_nx + b_n}$$

The constants $A_1, A_2, \ldots, A_n$ are determined as in the following example.

Example 1 Distinct Linear Factors

Find the partial fraction decomposition of $\dfrac{5x + 7}{x^3 + 2x^2 - x - 2}$.

Solution The denominator factors as follows:

$$\begin{aligned} x^3 + 2x^2 - x - 2 &= x^2(x + 2) - (x + 2) = (x^2 - 1)(x + 2) \\ &= (x - 1)(x + 1)(x + 2) \end{aligned}$$

This gives us the partial fraction decomposition

$$\frac{5x + 7}{x^3 + 2x^2 - x - 2} = \frac{A}{x - 1} + \frac{B}{x + 1} + \frac{C}{x + 2}$$

SAMPLE QUESTION
Text Question

What does it mean to express a rational expression as partial fractions?

Answer

It means to write the expression as a sum of fractions with simpler denominators.

EXAMPLE

Case 1:

$$\frac{x - 7}{(x - 2)(x + 3)} = \frac{-1}{x - 2} + \frac{2}{x + 3}$$

DRILL QUESTION

Express $\dfrac{2}{x^2 - 1}$ as a sum of partial fractions.

Answer

$$\frac{2}{x^2 - 1} = \frac{1}{x - 1} - \frac{1}{x + 1}$$

ALTERNATE EXAMPLE 1

Find the partial fraction decomposition of $\dfrac{13x + 17}{x^3 + 2x^2 - x - 2}$.

ANSWER

$$\frac{5}{x - 1} - \frac{2}{x + 1} - \frac{3}{x + 2}$$

IN-CLASS MATERIALS

Remind students of the process of polynomial division, perhaps by rewriting $\dfrac{2x^3 + 3x^2 + 7x + 4}{2x + 1}$ as $x^2 + x + 3 + \dfrac{1}{2x + 1}$. Be sure to indicate that, in order to use partial fractions, the degree of the numerator has to be less than the degree of the denominator.

Multiplying each side by the common denominator, $(x-1)(x+1)(x+2)$, we get

$$\begin{aligned} 5x+7 &= A(x+1)(x+2) + B(x-1)(x+2) + C(x-1)(x+1) \\ &= A(x^2+3x+2) + B(x^2+x-2) + C(x^2-1) && \text{Expand} \\ &= (A+B+C)x^2 + (3A+B)x + (2A-2B-C) && \text{Combine like terms} \end{aligned}$$

If two polynomials are equal, then their coefficients are equal. Thus, since $5x+7$ has no x^2-term, we have $A+B+C=0$. Similarly, by comparing the coefficients of x, we see that $3A+B=5$, and by comparing constant terms, we get $2A-2B-C=7$. This leads to the following system of linear equations for A, B, and C.

$$\begin{cases} A + B + C = 0 & \text{Equation 1: Coefficients of } x^2 \\ 3A + B = 5 & \text{Equation 2: Coefficients of } x \\ 2A - 2B - C = 7 & \text{Equation 3: Constant coefficients} \end{cases}$$

We use Gaussian elimination to solve this system.

$$\begin{cases} A + B + C = 0 & \\ -2B - 3C = 5 & \text{Equation 2} + (-3) \times \text{Equation 1} \\ -4B - 3C = 7 & \text{Equation 3} + (-2) \times \text{Equation 1} \end{cases}$$

$$\begin{cases} A + B + C = 0 & \\ -2B - 3C = 5 & \\ 3C = -3 & \text{Equation 3} + (-2) \times \text{Equation 2} \end{cases}$$

From the third equation we get $C=-1$. Back-substituting we find that $B=-1$ and $A=2$. So, the partial fraction decomposition is

$$\frac{5x+7}{x^3+2x^2-x-2} = \frac{2}{x-1} + \frac{-1}{x+1} + \frac{-1}{x+2}$$ ■

The same approach works in the remaining cases. We set up the partial fraction decomposition with the unknown constants, $A, B, C, \ldots$. Then we multiply each side of the resulting equation by the common denominator, simplify the right-hand side of the equation, and equate coefficients. This gives a set of linear equations that will always have a unique solution (provided that the partial fraction decomposition has been set up correctly).

Case 2: The Denominator Is a Product of Linear Factors, Some of Which Are Repeated

Suppose the complete factorization of $Q(x)$ contains the linear factor $ax+b$ repeated k times; that is, $(ax+b)^k$ is a factor of $Q(x)$. Then, corresponding to each such factor, the partial fraction decomposition for $P(x)/Q(x)$ contains

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_k}{(ax+b)^k}$$

EXAMPLE

Case 2:

$$\frac{3x^2-x-3}{x^2(x+1)} = \frac{1}{x+1} + \frac{2}{x} - \frac{3}{x^2}$$

IN-CLASS MATERIALS

Find the coefficients for the partial fraction decomposition for $\dfrac{x+3}{(x-2)(x-1)}$ in two different ways: first using two linear equations, and then using the method of creating zeros [setting $x=1$ and then $x=-2$ in $x+3 = A(x+2) + B(x-1)$].

ALTERNATE EXAMPLE 2
Find the partial fraction decomposition of $\dfrac{6x^2+6}{x(x-1)^3}$.

ANSWER
$\dfrac{6}{x}+\dfrac{6}{x-1}+\dfrac{12}{(x-1)^3}$

EXAMPLE
Case 3:
$$\frac{3}{(x-1)(x^2+2)}=\frac{1}{x-1}-\frac{x+1}{x^2+2}$$

ALTERNATE EXAMPLE 3
Find the partial fraction decomposition of $\dfrac{11x^2-2x+24}{x^3+4x}$.

ANSWER
$\dfrac{6}{x}+\dfrac{5x-2}{x^2+4}$

Example 2 Repeated Linear Factors

Find the partial fraction decomposition of $\dfrac{x^2+1}{x(x-1)^3}$.

Solution Because the factor $x-1$ is repeated three times in the denominator, the partial fraction decomposition has the form

$$\frac{x^2+1}{x(x-1)^3}=\frac{A}{x}+\frac{B}{x-1}+\frac{C}{(x-1)^2}+\frac{D}{(x-1)^3}$$

Multiplying each side by the common denominator, $x(x-1)^3$, gives

$$\begin{aligned}x^2+1&=A(x-1)^3+Bx(x-1)^2+Cx(x-1)+Dx\\&=A(x^3-3x^2+3x-1)+B(x^3-2x^2+x)+C(x^2-x)+Dx&&\text{Expand}\\&=(A+B)x^3+(-3A-2B+C)x^2+(3A+B-C+D)x-A&&\text{Combine like terms}\end{aligned}$$

Equating coefficients, we get the equations

$$\begin{cases}A+B=0 & \text{Coefficients of } x^3\\ -3A-2B+C=1 & \text{Coefficients of } x^2\\ 3A+B-C+D=0 & \text{Coefficients of } x\\ -A=1 & \text{Constant coefficients}\end{cases}$$

If we rearrange these equations by putting the last one in the first position, we can easily see (using substitution) that the solution to the system is $A=-1$, $B=1$, $C=0$, $D=2$, and so the partial fraction decomposition is

$$\frac{x^2+1}{x(x-1)^3}=\frac{-1}{x}+\frac{1}{x-1}+\frac{2}{(x-1)^3}$$

■

Case 3: The Denominator Has Irreducible Quadratic Factors, None of Which Is Repeated

Suppose the complete factorization of $Q(x)$ contains the quadratic factor ax^2+bx+c (which can't be factored further). Then, corresponding to this, the partial fraction decomposition of $P(x)/Q(x)$ will have a term of the form

$$\frac{Ax+B}{ax^2+bx+c}$$

Example 3 Distinct Quadratic Factors

Find the partial fraction decomposition of $\dfrac{2x^2-x+4}{x^3+4x}$.

Solution Since $x^3+4x=x(x^2+4)$, which can't be factored further, we write

$$\frac{2x^2-x+4}{x^3+4x}=\frac{A}{x}+\frac{Bx+C}{x^2+4}$$

IN-CLASS MATERIALS

Point out that the quadratic in the denominator of $f(x)=\dfrac{1}{x^2+x-6}$ is not irreducible. It can be factored into the two linear terms $x-2$ and $x+3$, and so the partial fraction decomposition is found by writing $\dfrac{1}{x^2+x-6}=\dfrac{A}{x+2}+\dfrac{B}{x-3}$ and solving for A and B. Therefore it is important, when factoring the denominator, to make sure all quadratics are irreducible.

Multiplying by $x(x^2 + 4)$, we get

$$\begin{aligned} 2x^2 - x + 4 &= A(x^2 + 4) + (Bx + C)x \\ &= (A + B)x^2 + Cx + 4A \end{aligned}$$

Equating coefficients gives us the equations

$$\begin{cases} A + B = 2 & \text{Coefficients of } x^2 \\ C = -1 & \text{Coefficients of } x \\ 4A = 4 & \text{Constant coefficients} \end{cases}$$

and so $A = 1$, $B = 1$, and $C = -1$. The required partial fraction decomposition is

$$\frac{2x^2 - x + 4}{x^3 + 4x} = \frac{1}{x} + \frac{x - 1}{x^2 + 4}$$

Case 4: The Denominator Has a Repeated Irreducible Quadratic Factor

Suppose the complete factorization of $Q(x)$ contains the factor $(ax^2 + bx + c)^k$, where $ax^2 + bx + c$ can't be factored further. Then the partial fraction decomposition of $P(x)/Q(x)$ will have the terms

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_kx + B_k}{(ax^2 + bx + c)^k}$$

EXAMPLE

Case 4:

$$\frac{1}{(x^2 + 1)^2 x} = \frac{1}{x} - \frac{x}{x^2 + 1} - \frac{x}{(x^2 + 1)^2}$$

Example 4 Repeated Quadratic Factors

Write the form of the partial fraction decomposition of

$$\frac{x^5 - 3x^2 + 12x - 1}{x^3(x^2 + x + 1)(x^2 + 2)^3}$$

Solution

$$\begin{aligned} &\frac{x^5 - 3x^2 + 12x - 1}{x^3(x^2 + x + 1)(x^2 + 2)^3} \\ &= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{Dx + E}{x^2 + x + 1} + \frac{Fx + G}{x^2 + 2} + \frac{Hx + I}{(x^2 + 2)^2} + \frac{Jx + K}{(x^2 + 2)^3} \end{aligned}$$

ALTERNATE EXAMPLE 4

Write the form of the partial fraction decomposition of the following:

$$\frac{x^2 + 12}{(x + 1)^2 (x^2 + 2x + 30)^3 (x - 2)}$$

ANSWER

$$\frac{A}{x - 2} + \frac{B}{(x + 1)} + \frac{C}{(x + 1)^2} + \frac{Dx + E}{(x^2 + 2x + 30)} + \frac{Fx + G}{(x^2 + 2x + 30)^2} + \frac{Hx + I}{(x^2 + 2x + 30)^3}$$

To find the values of $A, B, C, D, E, F, G, H, I, J$, and K in Example 4, we would have to solve a system of 11 linear equations. Although possible, this would certainly involve a great deal of work!

The techniques we have described in this section apply only to rational functions $P(x)/Q(x)$ in which the degree of P is less than the degree of Q. If this isn't the case, we must first use long division to divide Q into P.

ALTERNATE EXAMPLE 5a
Find the partial fraction decomposition of
$$\frac{3x^4 + 9x^3 - 12x^2 - 27x + 22}{x^3 + 3x^2 - 4x - 12}$$

ANSWER
$$3x + \frac{2}{x-2} - \frac{1}{x+2} - \frac{1}{x+3}.$$

ALTERNATE EXAMPLE 5b
Prepare the following for partial fraction decomposition:
$$\frac{x^5 + 3x^2 - 5}{x^3 - 4x}$$

ANSWER
We write
$$\frac{x^5 + 3x^2 - 5}{x^3 - 4x} = x^2 + 4 + \frac{3x^2 + 16x - 5}{x(x+2)(x-2)}$$

Example 5 Using Long Division to Prepare for Partial Fractions

Find the partial fraction decomposition of
$$\frac{2x^4 + 4x^3 - 2x^2 + x + 7}{x^3 + 2x^2 - x - 2}$$

Solution Since the degree of the numerator is larger than the degree of the denominator, we use long division to obtain

$$\begin{array}{r} 2x \\ x^3 + 2x^2 - x - 2 \overline{)2x^4 + 4x^3 - 2x^2 + x + 7} \\ \underline{2x^4 + 4x^3 - 2x^2 - 4x} \\ 5x + 7 \end{array}$$

$$\frac{2x^4 + 4x^3 - 2x^2 + x + 7}{x^3 + 2x^2 - x - 2} = 2x + \frac{5x + 7}{x^3 + 2x^2 - x - 2}$$

The remainder term now satisfies the requirement that the degree of the numerator is less than the degree of the denominator. At this point we proceed as in Example 1 to obtain the decomposition

$$\frac{2x^4 + 4x^3 - 2x^2 + x + 7}{x^3 + 2x^2 - x - 2} = 2x + \frac{2}{x-1} + \frac{-1}{x+1} + \frac{-1}{x+2}$$ ■

9.8 Exercises

1–10 ■ Write the form of the partial fraction decomposition of the function (as in Example 4). Do not determine the numerical values of the coefficients.

1. $\frac{1}{(x-1)(x+2)}$
2. $\frac{x}{x^2 + 3x - 4}$
3. $\frac{x^2 - 3x + 5}{(x-2)^2(x+4)}$
4. $\frac{1}{x^4 - x^3}$
5. $\frac{x^2}{(x-3)(x^2+4)}$
6. $\frac{1}{x^4 - 1}$
7. $\frac{x^3 - 4x^2 + 2}{(x^2+1)(x^2+2)}$
8. $\frac{x^4 + x^2 + 1}{x^2(x^2+4)^2}$
9. $\frac{x^3 + x + 1}{x(2x-5)^3(x^2+2x+5)^2}$
10. $\frac{1}{(x^3-1)(x^2-1)}$

11–42 ■ Find the partial fraction decomposition of the rational function.

11. $\frac{2}{(x-1)(x+1)}$
12. $\frac{2x}{(x-1)(x+1)}$
13. $\frac{5}{(x-1)(x+4)}$
14. $\frac{x+6}{x(x+3)}$
15. $\frac{12}{x^2 - 9}$
16. $\frac{x - 12}{x^2 - 4x}$
17. $\frac{4}{x^2 - 4}$
18. $\frac{2x + 1}{x^2 + x - 2}$
19. $\frac{x + 14}{x^2 - 2x - 8}$
20. $\frac{8x - 3}{2x^2 - x}$
21. $\frac{x}{8x^2 - 10x + 3}$
22. $\frac{7x - 3}{x^3 + 2x^2 - 3x}$
23. $\frac{9x^2 - 9x + 6}{2x^3 - x^2 - 8x + 4}$
24. $\frac{-3x^2 - 3x + 27}{(x+2)(2x^2+3x-9)}$
25. $\frac{x^2 + 1}{x^3 + x^2}$
26. $\frac{3x^2 + 5x - 13}{(3x+2)(x^2-4x+4)}$
27. $\frac{2x}{4x^2 + 12x + 9}$
28. $\frac{x - 4}{(2x-5)^2}$
29. $\frac{4x^2 - x - 2}{x^4 + 2x^3}$
30. $\frac{x^3 - 2x^2 - 4x + 3}{x^4}$
31. $\frac{-10x^2 + 27x - 14}{(x-1)^3(x+2)}$
32. $\frac{-2x^2 + 5x - 1}{x^4 - 2x^3 + 2x - 1}$
33. $\frac{3x^3 + 22x^2 + 53x + 41}{(x+2)^2(x+3)^2}$
34. $\frac{3x^2 + 12x - 20}{x^4 - 8x^2 + 16}$
35. $\frac{x - 3}{x^3 + 3x}$
36. $\frac{3x^2 - 2x + 8}{x^3 - x^2 + 2x - 2}$
37. $\frac{2x^3 + 7x + 5}{(x^2+x+2)(x^2+1)}$
38. $\frac{x^2 + x + 1}{2x^4 + 3x^2 + 1}$

IN-CLASS MATERIALS

It is possible to cover this section without covering every single case. For example, one might just cover the idea of linear factors (Cases 1 and 2) and mention that it is also possible to work with irreducible quadratic factors. Notice that just because every rational expression can be decomposed in theory doesn't mean it is always possible in practice, because there is no closed-form formula for factoring a polynomial of degree 5 or higher.

IN-CLASS MATERIALS

Show the class how a complicated partial fractions problem would be set up, without trying to solve it.

39. $\dfrac{x^4 + x^3 + x^2 - x + 1}{x(x^2 + 1)^2}$ **40.** $\dfrac{2x^2 - x + 8}{(x^2 + 4)^2}$

41. $\dfrac{x^5 - 2x^4 + x^3 + x + 5}{x^3 - 2x^2 + x - 2}$

42. $\dfrac{x^5 - 3x^4 + 3x^3 - 4x^2 + 4x + 12}{(x - 2)^2(x^2 + 2)}$

43. Determine A and B in terms of a and b:

$$\frac{ax + b}{x^2 - 1} = \frac{A}{x - 1} + \frac{B}{x + 1}$$

44. Determine A, B, C, and D in terms of a and b:

$$\frac{ax^3 + bx^2}{(x^2 + 1)^2} = \frac{Ax + B}{x^2 + 1} + \frac{Cx + D}{(x^2 + 1)^2}$$

Discovery • Discussion

45. Recognizing Partial Fraction Decompositions For each expression, determine whether it is already a partial fraction decomposition, or whether it can be decomposed further.

(a) $\dfrac{x}{x^2 + 1} + \dfrac{1}{x + 1}$ **(b)** $\dfrac{x}{(x + 1)^2}$

(c) $\dfrac{1}{x + 1} + \dfrac{2}{(x + 1)^2}$ **(d)** $\dfrac{x + 2}{(x^2 + 1)^2}$

46. Assembling and Disassembling Partial Fractions The following expression is a partial fraction decomposition:

$$\frac{2}{x - 1} + \frac{1}{(x - 1)^2} + \frac{1}{x + 1}$$

Use a common denominator to combine the terms into one fraction. Then use the techniques of this section to find its partial fraction decomposition. Did you get back the original expression?

9.9 Systems of Inequalities

SUGGESTED TIME AND EMPHASIS
1 class.
Recommended material.

In this section we study systems of inequalities in two variables from a graphical point of view.

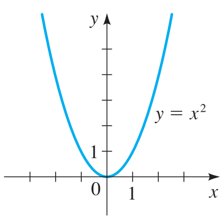


Figure 1

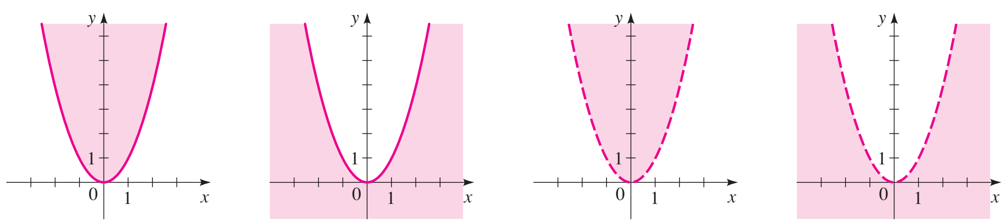
Graphing an Inequality

We begin by considering the graph of a single inequality. We already know that the graph of $y = x^2$, for example, is the *parabola* in Figure 1. If we replace the equal sign by the symbol $\geq$, we obtain the *inequality*

$$y \geq x^2$$

Its graph consists of not just the parabola in Figure 1, but also every point whose y-coordinate is *larger* than x^2. We indicate the solution in Figure 2(a) by shading the points *above* the parabola.

Similarly, the graph of $y \leq x^2$ in Figure 2(b) consists of all points on and *below* the parabola. However, the graphs of $y > x^2$ and $y < x^2$ do not include the points on the parabola itself, as indicated by the dashed curves in Figures 2(c) and 2(d).



(a) $y \geq x^2$ (b) $y \leq x^2$ (c) $y > x^2$ (d) $y < x^2$

Figure 2

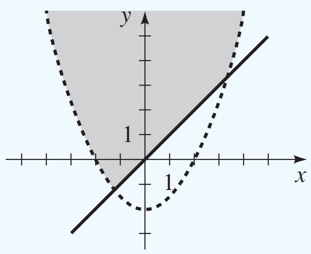
POINT TO STRESS
Graphing inequalities and systems of inequalities by graphing the border and testing points in the defined regions.

DRILL QUESTION

Graph the solution set of the system of inequalities.

$$-\frac{1}{2}x^2 + y \geq -2$$
$$x - y < 0$$

Answer



ALTERNATE EXAMPLE 1a

For the graph of the inequality $x^2 + y^2 < 81$ determine whether the inside or the outside of the circle satisfies the inequality.

ANSWER
Inside

ALTERNATE EXAMPLE 1b

For the graph of the inequality $x + 4y \geq 9$ determine whether the points above or below the boundary line satisfy the inequality.

ANSWER
Above

The graph of an inequality, in general, consists of a region in the plane whose boundary is the graph of the equation obtained by replacing the inequality sign ($\geq$, $\leq$, $>$, or $<$) with an equal sign. To determine which side of the graph gives the solution set of the inequality, we need only check **test points**.

Graphing Inequalities

To graph an inequality, we carry out the following steps.

1. **Graph Equation.** Graph the equation corresponding to the inequality. Use a dashed curve for $>$ or $<$, and a solid curve for $\leq$ or $\geq$.
2. **Test Points.** Test one point in each region formed by the graph in Step 1. If the point satisfies the inequality, then all the points in that region satisfy the inequality. (In that case, shade the region to indicate it is part of the graph.) If the test point does not satisfy the inequality, then the region isn't part of the graph.

Example 1 Graphs of Inequalities

Graph each inequality.

(a) $x^2 + y^2 < 25$ (b) $x + 2y \geq 5$

Solution

(a) The graph of $x^2 + y^2 = 25$ is a circle of radius 5 centered at the origin. The points on the circle itself do not satisfy the inequality because it is of the form $<$, so we graph the circle with a dashed curve, as shown in Figure 3.

To determine whether the inside or the outside of the circle satisfies the inequality, we use the test points $(0, 0)$ on the inside and $(6, 0)$ on the outside. To do this, we substitute the coordinates of each point into the inequality and check if the result satisfies the inequality. (Note that *any* point inside or outside the circle can serve as a test point. We have chosen these points for simplicity.)

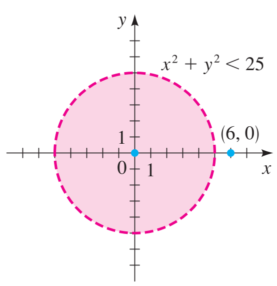


Figure 3

Test point	$x^2 + y^2 < 25$	Conclusion
$(0, 0)$	$0^2 + 0^2 = 0 < 25$	Part of graph
$(6, 0)$	$6^2 + 0^2 = 36 \not< 25$	Not part of graph

Thus, the graph of $x^2 + y^2 < 25$ is the set of all points *inside* the circle (see Figure 3).

(b) The graph of $x + 2y = 5$ is the line shown in Figure 4. We use the test points $(0, 0)$ and $(5, 5)$ on opposite sides of the line.

Test point	$x + 2y \geq 5$	Conclusion
$(0, 0)$	$0 + 2(0) = 0 \not\geq 5$	Not part of graph
$(5, 5)$	$5 + 2(5) = 15 \geq 5$	Part of graph

IN-CLASS MATERIALS

Surprisingly, many students are not able to correctly answer this question: "True or false: If $a < b$ then $a \leq b$." Perhaps take a minute or two to remind the students of the basics of inequalities. From straightforward statements involving inequalities, make the transition to some very simple regions such as $x > 2$, $y \leq 3$, etc. The idea is to make sure that the students are crystal-clear on the objects they are working with before working with them.

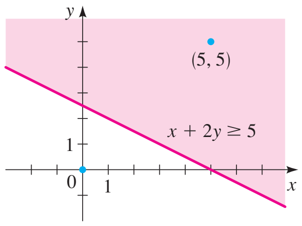


Figure 4

Our check shows that the points *above* the line satisfy the inequality.

Alternatively, we could put the inequality into slope-intercept form and graph it directly:

$$x + 2y \geq 5$$
$$2y \geq -x + 5$$
$$y \geq -\tfrac{1}{2}x + \tfrac{5}{2}$$

From this form we see that the graph includes all points whose y-coordinates are *greater* than those on the line $y = -\frac{1}{2}x + \frac{5}{2}$; that is, the graph consists of the points *on or above* this line, as shown in Figure 4. ■

Systems of Inequalities

We now consider *systems* of inequalities. The solution of such a system is the set of all points in the coordinate plane that satisfy every inequality in the system.

Example 2 A System of Two Inequalities

Graph the solution of the system of inequalities.

$$\begin{cases} x^2 + y^2 < 25 \\ x + 2y \geq 5 \end{cases}$$

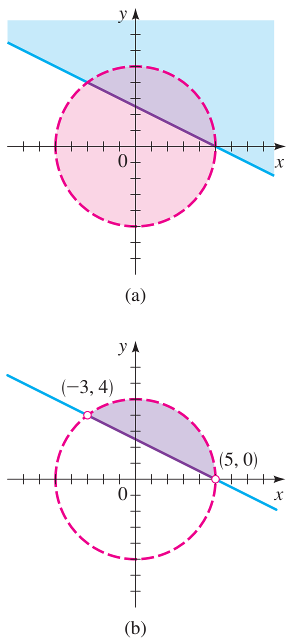


Figure 5

$$\begin{cases} x^2 + y^2 < 25 \\ x + 2y \geq 5 \end{cases}$$

Solution These are the two inequalities of Example 1. In this example we wish to graph only those points that simultaneously satisfy both inequalities. The solution consists of the intersection of the graphs in Example 1. In Figure 5(a) we show the two regions on the same coordinate plane (in different colors), and in Figure 5(b) we show their intersection.

VERTICES The points $(-3, 4)$ and $(5, 0)$ in Figure 5(b) are the **vertices** of the solution set. They are obtained by solving the system of *equations*

$$\begin{cases} x^2 + y^2 = 25 \\ x + 2y = 5 \end{cases}$$

We solve this system of equations by substitution. Solving for x in the second equation gives $x = 5 - 2y$, and substituting this into the first equation gives

$$(5 - 2y)^2 + y^2 = 25 \quad \text{Substitute } x = 5 - 2y$$
$$(25 - 20y + 4y^2) + y^2 = 25 \quad \text{Expand}$$
$$-20y + 5y^2 = 0 \quad \text{Simplify}$$
$$-5y(4 - y) = 0 \quad \text{Factor}$$

Thus, $y = 0$ or $y = 4$. When $y = 0$, we have $x = 5 - 2(0) = 5$, and when $y = 4$, we have $x = 5 - 2(4) = -3$. So the points of intersection of these curves are $(5, 0)$ and $(-3, 4)$.

Note that in this case the vertices are not part of the solution set, since they don't satisfy the inequality $x^2 + y^2 < 25$ (and so they are graphed as open circles in the figure). They simply show where the "corners" of the solution set lie. ■

SAMPLE QUESTION

Text Question

Consider the system of inequalities.

$$x^2 + y^2 < 25$$
$$x + 2y \geq 5$$

Which of the following are true:

1. The solution set of this system is all the points that satisfy at least one of these inequalities.
2. The solution set of this system is all the points that satisfy exactly one of these inequalities.
3. The solution set of this system is all the points that satisfy both of these inequalities.

Answer

Only 3 is true.

DRILL QUESTION

Find the vertices of the graph of the solution set of the system.

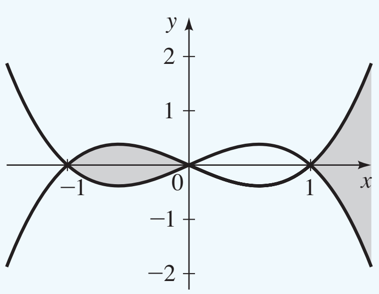
$$\begin{cases} x^2 + y^2 < 25 \\ x + 3y \geq 5 \end{cases}$$

Answer

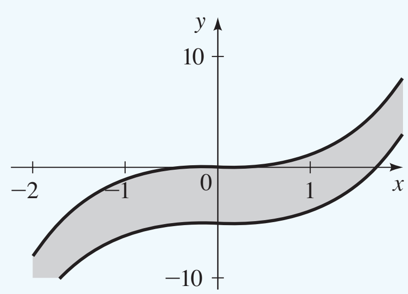
(−4, 3), (5, 0)

IN-CLASS MATERIALS

Have the students come up with examples of systems of two cubic inequalities with varying numbers of separate solution spaces.



$\{y \leq x^3 - x, y \geq -x^3 + x\}$, two solution spaces



$\{y \leq x^3, y \geq x^3 - 5\}$, one solution space

ALTERNATE EXAMPLE 3
Find the vertices for the graph of the solution set of the system.

$$\begin{cases} x + 5y \le 15 \\ x + y \le 11 \\ x \ge 0 \\ y \ge 0 \end{cases}$$

ANSWER
(10, 1), (0, 3), (11, 0)

Systems of Linear Inequalities

An inequality is **linear** if it can be put into one of the following forms:

$$ax + by \ge c \qquad ax + by \le c \qquad ax + by > c \qquad ax + by < c$$

In the next example we graph the solution set of a system of linear inequalities.

Example 3 A System of Four Linear Inequalities

Graph the solution set of the system, and label its vertices.

$$\begin{cases} x + 3y \le 12 \\ x + y \le 8 \\ x \ge 0 \\ y \ge 0 \end{cases}$$

Solution In Figure 6 we first graph the lines given by the equations that correspond to each inequality. To determine the graphs of the linear inequalities, we only need to check one test point. For simplicity let's use the point $(0, 0)$.

Inequality	Test point (0, 0)	Conclusion
$x + 3y \le 12$	$0 + 3(0) = 0 \le 12$	Satisfies inequality
$x + y \le 8$	$0 + 0 = 0 \le 8$	Satisfies inequality

Since $(0, 0)$ is below the line $x + 3y = 12$, our check shows that the region on or below the line must satisfy the inequality. Likewise, since $(0, 0)$ is below the line $x + y = 8$, our check shows that the region on or below this line must satisfy the inequality. The inequalities $x \ge 0$ and $y \ge 0$ say that x and y are nonnegative. These regions are sketched in Figure 6(a), and the intersection—the solution set—is sketched in Figure 6(b).

VERTICES The coordinates of each vertex are obtained by simultaneously solving the equations of the lines that intersect at that vertex. From the system

$$\begin{cases} x + 3y = 12 \\ x + y = 8 \end{cases}$$

we get the vertex $(6, 2)$. The other vertices are the x- and y-intercepts of the corresponding lines, $(8, 0)$ and $(0, 4)$, and the origin $(0, 0)$. In this case, all the vertices *are* part of the solution set.

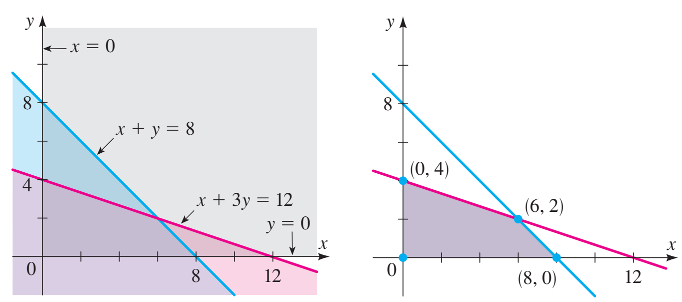


Figure 6 (a) (b) ■

IN-CLASS MATERIALS

If areas are discussed, then the idea of infinite regions with finite areas can be discussed. It is interesting that these regions have finite area:

$$x \ge 1 \quad x \ge 1 \quad x \ge 1 \quad y \le e^{-x} \quad y \le \frac{1}{x^2} \quad y \le \frac{1}{x^{1.05}}$$

While these do not:

$$x \ge 1 \quad x \ge 1 \quad x \ge 1 \quad y \le \frac{1}{x} \quad y \le \frac{1}{x\ln(x + 1)} \quad y \le \frac{x}{100(x^2 + x)}$$

Example 4 A System of Linear Inequalities

Graph the solution set of the system.

$$\begin{cases} x + 2y \geq 8 \\ -x + 2y \leq 4 \\ 3x - 2y \leq 8 \end{cases}$$

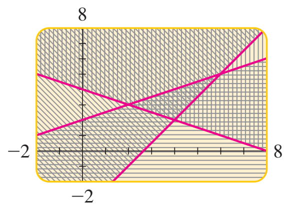


Figure 7

Solution We must graph the lines that correspond to these inequalities and then shade the appropriate regions, as in Example 3. We will use a graphing calculator, so we must first isolate y on the left-hand side of each inequality.

$$\begin{cases} y \geq -\frac{1}{2}x + 4 \\ y \leq \frac{1}{2}x + 2 \\ y \geq \frac{3}{2}x - 4 \end{cases}$$

Using the shading feature of the calculator, we obtain the graph in Figure 7. The solution set is the triangular region that is shaded in all three patterns. We then use TRACE or the `Intersect` command to find the vertices of the region. The solution set is graphed in Figure 8. ■

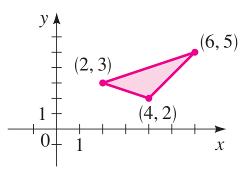


Figure 8

When a region in the plane can be covered by a (sufficiently large) circle, it is said to be **bounded**. A region that is not bounded is called **unbounded**. For example, the regions graphed in Figures 3, 5(b), 6(b), and 8 are bounded, whereas those in Figures 2 and 4 are unbounded. An unbounded region cannot be "fenced in"—it extends infinitely far in at least one direction.

Application: Feasible Regions

Many applied problems involve *constraints* on the variables. For instance, a factory manager has only a certain number of workers that can be assigned to perform jobs on the factory floor. A farmer deciding what crops to cultivate has only a certain amount of land that can be seeded. Such constraints or limitations can usually be expressed as systems of inequalities. When dealing with applied inequalities, we usually refer to the solution set of a system as a *feasible region*, because the points in the solution set represent feasible (or possible) values for the quantities being studied.

Example 5 Restricting Pollutant Outputs

A factory produces two agricultural pesticides, A and B. For every barrel of A, the factory emits 0.25 kg of carbon monoxide (CO) and 0.60 kg of sulfur dioxide (SO_2), and for every barrel of B, it emits 0.50 kg of CO and 0.20 kg of SO_2. Pollution laws restrict the factory's output of CO to a maximum of 75 kg and SO_2 to a maximum of 90 kg per day.

(a) Find a system of inequalities that describes the number of barrels of each pesticide the factory can produce and still satisfy the pollution laws. Graph the feasible region.

(b) Would it be legal for the factory to produce 100 barrels of A and 80 barrels of B per day?

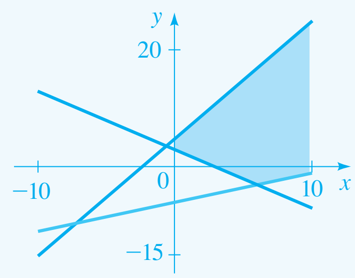
(c) Would it be legal for the factory to produce 60 barrels of A and 160 barrels of B per day?

ALTERNATE EXAMPLE 4

Graph the solution set of the system.

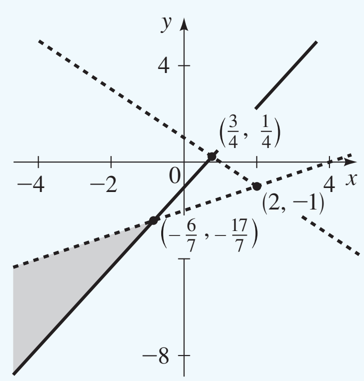
$$\begin{aligned} x + y &\geq 3 \\ -2x + y &\leq 5 \\ x - 2y &\leq 12 \end{aligned}$$

ANSWER

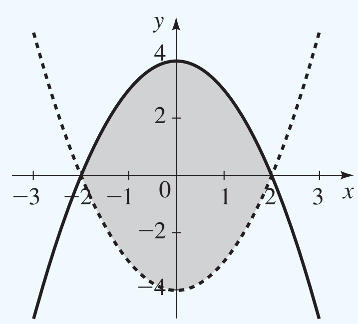


EXAMPLES

- $\{5x - 3y < 3, x - 2y > 4, x + y \leq 1\}$



- $\{y > x^2 - 4, y \leq 4 - x^2\}$



IN-CLASS MATERIALS

This section allows one to foreshadow the concept of area. For example, students should be able to compute the area of the regions defined by the following systems:

$$\{x \geq 1, x \leq 3, y \leq 2, y \geq 0\}$$
$$\{y \geq 2, x \geq 1, y + 2x \geq 0\}$$
$$\{y \geq 2x, (x - 3)^2 + (y - 6)^2 \leq 4\}$$

Solution

(a) To set up the required inequalities, it's helpful to organize the given information into a table.

	A	B	Maximum
CO (kg)	0.25	0.50	75
SO_2 (kg)	0.60	0.20	90

We let

x = number of barrels of A produced per day

y = number of barrels of B produced per day

From the data in the table and the fact that x and y can't be negative, we obtain the following inequalities.

$$\begin{cases} 0.25x + 0.50y \le 75 & \text{CO inequality} \\ 0.60x + 0.20y \le 90 & \text{SO}_2 \text{ inequality} \\ x \ge 0, \quad y \ge 0 \end{cases}$$

Multiplying the first inequality by 4 and the second by 5 simplifies this to

$$\begin{cases} x + 2y \le 300 \\ 3x + y \le 450 \\ x \ge 0, \quad y \ge 0 \end{cases}$$

The feasible region is the solution of this system of inequalities, shown in Figure 9.

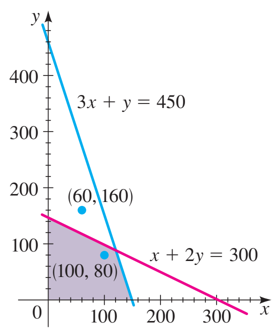


Figure 9

(b) Since the point (100, 80) lies inside the feasible region, this production plan is legal (see Figure 9).

(c) Since the point (60, 160) lies outside the feasible region, this production plan is not legal. It violates the CO restriction, although it does not violate the SO_2 restriction (see Figure 9). ■

9.9 Exercises

1–14 ■ Graph the inequality.

1. $x < 3$

2. $y \ge -2$

3. $y > x$

4. $y < x + 2$

5. $y \le 2x + 2$

6. $y < -x + 5$

7. $2x - y \le 8$

8. $3x + 4y + 12 > 0$

9. $4x + 5y < 20$

10. $-x^2 + y \ge 10$

11. $y > x^2 + 1$

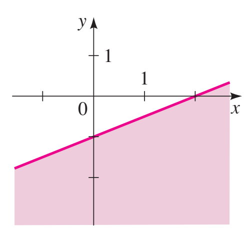
12. $x^2 + y^2 \ge 9$

13. $x^2 + y^2 \le 25$

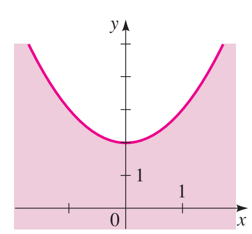
14. $x^2 + (y - 1)^2 \le 1$

15–18 ■ An equation and its graph are given. Find an inequality whose solution is the shaded region.

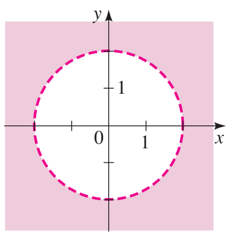
15. $y = \frac{1}{2}x - 1$



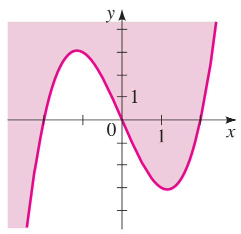
16. $y = x^2 + 2$



17. $x^2 + y^2 = 4$



18. $y = x^3 - 4x$



19–40 ■ Graph the solution of the system of inequalities. Find the coordinates of all vertices, and determine whether the solution set is bounded.

19. $\begin{cases} x + y \le 4 \\ y \ge x \end{cases}$

20. $\begin{cases} 2x + 3y > 12 \\ 3x - y < 21 \end{cases}$

21. $\begin{cases} y < \frac{1}{4}x + 2 \\ y \ge 2x - 5 \end{cases}$

22. $\begin{cases} x - y > 0 \\ 4 + y \le 2x \end{cases}$

23. $\begin{cases} x \ge 0 \\ y \ge 0 \\ 3x + 5y \le 15 \\ 3x + 2y \le 9 \end{cases}$

24. $\begin{cases} x > 2 \\ y < 12 \\ 2x - 4y > 8 \end{cases}$

25. $\begin{cases} y < 9 - x^2 \\ y \ge x + 3 \end{cases}$

26. $\begin{cases} y \ge x^2 \\ x + y \ge 6 \end{cases}$

27. $\begin{cases} x^2 + y^2 \le 4 \\ x - y > 0 \end{cases}$

28. $\begin{cases} x > 0 \\ y > 0 \\ x + y < 10 \\ x^2 + y^2 > 9 \end{cases}$

29. $\begin{cases} x^2 - y \le 0 \\ 2x^2 + y \le 12 \end{cases}$

30. $\begin{cases} x^2 + y^2 < 9 \\ 2x + y^2 \ge 1 \end{cases}$

31. $\begin{cases} x + 2y \le 14 \\ 3x - y \ge 0 \\ x - y \ge 2 \end{cases}$

32. $\begin{cases} y < x + 6 \\ 3x + 2y \ge 12 \\ x - 2y \le 2 \end{cases}$

33. $\begin{cases} x \ge 0 \\ y \ge 0 \\ x \le 5 \\ x + y \le 7 \end{cases}$

34. $\begin{cases} x \ge 0 \\ y \ge 0 \\ y \le 4 \\ 2x + y \le 8 \end{cases}$

35. $\begin{cases} y > x + 1 \\ x + 2y \le 12 \\ x + 1 > 0 \end{cases}$

36. $\begin{cases} x + y > 12 \\ y < \frac{1}{2}x - 6 \\ 3x + y < 6 \end{cases}$

37. $\begin{cases} x^2 + y^2 \le 8 \\ x \ge 2 \\ y \ge 0 \end{cases}$

38. $\begin{cases} x^2 - y \ge 0 \\ x + y < 6 \\ x - y < 6 \end{cases}$

39. $\begin{cases} x^2 + y^2 < 9 \\ x + y > 0 \\ x \le 0 \end{cases}$

40. $\begin{cases} y \ge x^3 \\ y \le 2x + 4 \\ x + y \ge 0 \end{cases}$

41–44 ■ Use a graphing calculator to graph the solution of the system of inequalities. Find the coordinates of all vertices, correct to one decimal place.

41. $\begin{cases} y \ge x - 3 \\ y \ge -2x + 6 \\ y \le 8 \end{cases}$

42. $\begin{cases} x + y \ge 12 \\ 2x + y \le 24 \\ x - y \ge -6 \end{cases}$

43. $\begin{cases} y \le 6x - x^2 \\ x + y \ge 4 \end{cases}$

44. $\begin{cases} y \ge x^3 \\ 2x + y \ge 0 \\ y \le 2x + 6 \end{cases}$

Applications

45. Publishing Books A publishing company publishes a total of no more than 100 books every year. At least 20 of these are nonfiction, but the company always publishes at least as much fiction as nonfiction. Find a system of inequalities that describes the possible numbers of fiction and nonfiction books that the company can produce each year consistent with these policies. Graph the solution set.

46. Furniture Manufacturing A man and his daughter manufacture unfinished tables and chairs. Each table requires 3 hours of sawing and 1 hour of assembly. Each chair requires 2 hours of sawing and 2 hours of assembly. The two of them can put in up to 12 hours of sawing and 8 hours of assembly work each day. Find a system of inequalities that describes all possible combinations of tables and chairs that they can make daily. Graph the solution set.

47. Coffee Blends A coffee merchant sells two different coffee blends. The Standard blend uses 4 oz of arabica and 12 oz of robusta beans per package; the Deluxe blend uses 10 oz of arabica and 6 oz of robusta beans per package. The merchant has 80 lb of arabica and 90 lb of robusta beans available. Find a system of inequalities that describes the possible number of Standard and Deluxe packages he can make. Graph the solution set.

48. Nutrition A cat food manufacturer uses fish and beef by-products. The fish contains 12 g of protein and 3 g of fat per ounce. The beef contains 6 g of protein and 9 g of fat per ounce. Each can of cat food must contain at least 60 g of protein and 45 g of fat. Find a system of inequalities that describes the possible number of ounces of fish and beef that can be used in each can to satisfy these minimum requirements. Graph the solution set.

Discovery • Discussion

49. Shading Unwanted Regions To graph the solution of a system of inequalities, we have shaded the solution of each inequality in a different color; the solution of the system is the region where all the shaded parts overlap. Here is a different method: For each inequality, shade the region that does *not* satisfy the inequality. Explain why the part of the plane that is left unshaded is the solution of the system. Solve the following system by both methods. Which do you prefer?

$$\begin{cases} x + 2y > 4 \\ -x + y < 1 \\ x + 3y < 9 \\ x < 3 \end{cases}$$

9 Review

Concept Check

1. Suppose you are asked to solve a system of two equations (not necessarily linear) in two variables. Explain how you would solve the system
   (a) by the substitution method
   (b) by the elimination method
   (c) graphically

2. Suppose you are asked to solve a system of two *linear* equations in two variables.
   (a) Would you prefer to use the substitution method or the elimination method?
   (b) How many solutions are possible? Draw diagrams to illustrate the possibilities.

3. What operations can be performed on a linear system that result in an equivalent system?

4. Explain how Gaussian elimination works. Your explanation should include a discussion of the steps used to obtain a system in triangular form, and back-substitution.

5. What does it mean to say that A is a matrix with dimension $m \times n$?

6. What is the augmented matrix of a system? Describe the role of elementary row operations, row-echelon form, back-substitution, and leading variables when solving a system in matrix form.

7. **(a)** What is meant by an inconsistent system?
   (b) What is meant by a dependent system?

8. Suppose you have used Gaussian elimination to transform the augmented matrix of a linear system into row-echelon form. How can you tell if the system has
   (a) exactly one solution?
   (b) no solution?
   (c) infinitely many solutions?

9. How can you tell if a matrix is in reduced row-echelon form?

10. How do Gaussian elimination and Gauss-Jordan elimination differ? What advantage does Gauss-Jordan elimination have?

11. If A and B are matrices with the same dimension and k is a real number, how do you find $A + B$, $A - B$, and kA?

12. **(a)** What must be true of the dimensions of A and B for the product AB to be defined?
    (b) If the product AB is defined, how do you calculate it?

13. **(a)** What is the identity matrix I_n?
    (b) If A is a square $n \times n$ matrix, what is its inverse matrix?
    (c) Write a formula for the inverse of a 2×2 matrix.
    (d) Explain how you would find the inverse of a 3×3 matrix.

14. **(a)** Explain how to express a linear system as a matrix equation of the form $AX = B$.
    (b) If A has an inverse, how would you solve the matrix equation $AX = B$?

15. Suppose A is an $n \times n$ matrix.
    (a) What is the minor M_{ij} of the element a_{ij}?
    (b) What is the cofactor A_{ij}?
    (c) How do you find the determinant of A?
    (d) How can you tell if A has an inverse?

16. State Cramer's Rule for solving a system of linear equations in terms of determinants. Do you prefer to use Cramer's Rule or Gaussian elimination? Explain.

17. Explain how to find the partial fraction decomposition of a rational expression. Include in your explanation a discussion of each of the four cases that arise.

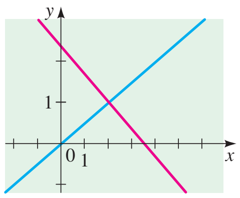
18. How do you graph an inequality in two variables?

19. How do you graph the solution set of a system of inequalities?

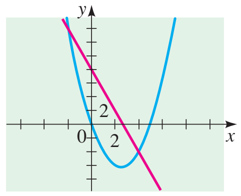
Exercises

1–4 ■ Two equations and their graphs are given. Find the intersection point(s) of the graphs by solving the system.

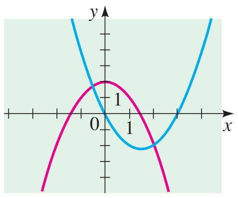
1. $\begin{cases} 2x + 3y = 7 \\ x - 2y = 0 \end{cases}$



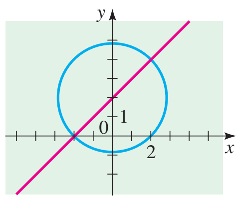
2. $\begin{cases} 3x + y = 8 \\ y = x^2 - 5x \end{cases}$



3. $\begin{cases} x^2 + y = 2 \\ x^2 - 3x - y = 0 \end{cases}$



4. $\begin{cases} x - y = -2 \\ x^2 + y^2 - 4y = 4 \end{cases}$



5–10 ■ Solve the system of equations and graph the lines.

5. $\begin{cases} 3x - y = 5 \\ 2x + y = 5 \end{cases}$

6. $\begin{cases} y = 2x + 6 \\ y = -x + 3 \end{cases}$

7. $\begin{cases} 2x - 7y = 28 \\ y = \frac{2}{7}x - 4 \end{cases}$

8. $\begin{cases} 6x - 8y = 15 \\ -\frac{3}{2}x + 2y = -4 \end{cases}$

9. $\begin{cases} 2x - y = 1 \\ x + 3y = 10 \\ 3x + 4y = 15 \end{cases}$

10. $\begin{cases} 2x + 5y = 9 \\ -x + 3y = 1 \\ 7x - 2y = 14 \end{cases}$

11–14 ■ Solve the system of equations.

11. $\begin{cases} y = x^2 + 2x \\ y = 6 + x \end{cases}$

12. $\begin{cases} x^2 + y^2 = 8 \\ y = x + 2 \end{cases}$

13. $\begin{cases} 3x + \dfrac{4}{y} = 6 \\ x - \dfrac{8}{y} = 4 \end{cases}$

14. $\begin{cases} x^2 + y^2 = 10 \\ x^2 + 2y^2 - 7y = 0 \end{cases}$

15–18 ■ Use a graphing device to solve the system, correct to the nearest hundredth.

15. $\begin{cases} 0.32x + 0.43y = 0 \\ 7x - 12y = 341 \end{cases}$

16. $\begin{cases} \sqrt{12}x - 3\sqrt{2}y = 660 \\ 7137x + 3931y = 20{,}000 \end{cases}$

17. $\begin{cases} x - y^2 = 10 \\ x = \frac{1}{22}y + 12 \end{cases}$

18. $\begin{cases} y = 5^x + x \\ y = x^5 + 5 \end{cases}$

19–24 ■ A matrix is given.

(a) State the dimension of the matrix.

(b) Is the matrix in row-echelon form?

(c) Is the matrix in reduced row-echelon form?

(d) Write the system of equations for which the given matrix is the augmented matrix.

19. $\begin{bmatrix} 1 & 2 & -5 \\ 0 & 1 & 3 \end{bmatrix}$

20. $\begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & 0 \end{bmatrix}$

21. $\begin{bmatrix} 1 & 0 & 8 & 0 \\ 0 & 1 & 5 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

22. $\begin{bmatrix} 1 & 3 & 6 & 2 \\ 2 & 1 & 0 & 5 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

23. $\begin{bmatrix} 0 & 1 & -3 & 4 \\ 1 & 1 & 0 & 7 \\ 1 & 2 & 1 & 2 \end{bmatrix}$

24. $\begin{bmatrix} 1 & 8 & 6 & -4 \\ 0 & 1 & -3 & 5 \\ 0 & 0 & 2 & -7 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

25–46 ■ Find the complete solution of the system, or show that the system has no solution.

25. $\begin{cases} x + y + 2z = 6 \\ 2x + 5z = 12 \\ x + 2y + 3z = 9 \end{cases}$

26. $\begin{cases} x - 2y + 3z = 1 \\ x - 3y - z = 0 \\ 2x - 6z = 6 \end{cases}$

27. $\begin{cases} x - 2y + 3z = 1 \\ 2x - y + z = 3 \\ 2x - 7y + 11z = 2 \end{cases}$

28. $\begin{cases} x + y + z + w = 2 \\ 2x - 3z = 5 \\ x - 2y + 4w = 9 \\ x + y + 2z + 3w = 5 \end{cases}$

29. $\begin{cases} x + 2y + 2z = 6 \\ x - y = -1 \\ 2x + y + 3z = 7 \end{cases}$

30. $\begin{cases} x - y + z = 2 \\ x + y + 3z = 6 \\ 2y + 3z = 5 \end{cases}$

31. $\left\{\begin{array}{l} x - 2y + 3z = -2 \\ 2x - y + z = 2 \\ 2x - 7y + 11z = -9 \end{array}\right.$

32. $\left\{\begin{array}{l} x - y + z = 2 \\ x + y + 3z = 6 \\ 3x - y + 5z = 10 \end{array}\right.$

33. $\left\{\begin{array}{l} x + y + z + w = 0 \\ x - y - 4z - w = -1 \\ x - 2y + 4w = -7 \\ 2x + 2y + 3z + 4w = -3 \end{array}\right.$

34. $\left\{\begin{array}{l} x + 3z = -1 \\ y - 4w = 5 \\ 2y + z + w = 0 \\ 2x + y + 5z - 4w = 4 \end{array}\right.$

35. $\left\{\begin{array}{l} x - 3y + z = 4 \\ 4x - y + 15z = 5 \end{array}\right.$

36. $\left\{\begin{array}{l} 2x - 3y + 4z = 3 \\ 4x - 5y + 9z = 13 \\ 2x + 7z = 0 \end{array}\right.$

37. $\left\{\begin{array}{l} -x + 4y + z = 8 \\ 2x - 6y + z = -9 \\ x - 6y - 4z = -15 \end{array}\right.$

38. $\left\{\begin{array}{l} x - z + w = 2 \\ 2x + y - 2w = 12 \\ 3y + z + w = 4 \\ x + y - z = 10 \end{array}\right.$

39. $\left\{\begin{array}{l} x - y + 3z = 2 \\ 2x + y + z = 2 \\ 3x + 4z = 4 \end{array}\right.$

40. $\left\{\begin{array}{l} x - y = 1 \\ x + y + 2z = 3 \\ x - 3y - 2z = -1 \end{array}\right.$

41. $\left\{\begin{array}{l} x - y + z - w = 0 \\ 3x - y - z - w = 2 \end{array}\right.$

42. $\left\{\begin{array}{l} x - y = 3 \\ 2x + y = 6 \\ x - 2y = 9 \end{array}\right.$

43. $\left\{\begin{array}{l} x - y + z = 0 \\ 3x + 2y - z = 6 \\ x + 4y - 3z = 3 \end{array}\right.$

44. $\left\{\begin{array}{l} x + 2y + 3z = 2 \\ 2x - y - 5z = 1 \\ 4x + 3y + z = 6 \end{array}\right.$

45. $\left\{\begin{array}{l} x + y - z - w = 2 \\ x - y + z - w = 0 \\ 2x + 2w = 2 \\ 2x + 4y - 4z - 2w = 6 \end{array}\right.$

46. $\left\{\begin{array}{l} x - y - 2z + 3w = 0 \\ y - z + w = 1 \\ 3x - 2y - 7z + 10w = 2 \end{array}\right.$

47. A man invests his savings in two accounts, one paying 6% interest per year and the other paying 7%. He has twice as much invested in the 7% account as in the 6% account, and his annual interest income is \$600. How much is invested in each account?

48. A piggy bank contains 50 coins, all of them nickels, dimes, or quarters. The total value of the coins is \$5.60, and the value of the dimes is five times the value of the nickels. How many coins of each type are there?

49. Clarisse invests \$60,000 in money-market accounts at three different banks. Bank A pays 2% interest per year, bank B pays 2.5%, and bank C pays 3%. She decides to invest twice as much in bank B as in the other two banks. After one year, Clarisse has earned \$1575 in interest. How much did she invest in each bank?

50. A commercial fisherman fishes for haddock, sea bass, and red snapper. He is paid \$1.25 a pound for haddock, \$0.75 a pound for sea bass, and \$2.00 a pound for red snapper. Yesterday he caught 560 lb of fish worth \$575. The haddock and red snapper together are worth \$320. How many pounds of each fish did he catch?

51–62 ■ Let

$$A = \begin{bmatrix} 2 & 0 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 4 \\ -2 & 1 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} \frac{1}{2} & 3 \\ 2 & \frac{3}{2} \\ -2 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 4 \\ 0 & -1 \\ 2 & 0 \end{bmatrix}$$

$$E = \begin{bmatrix} 2 & -1 \\ -\frac{1}{2} & 1 \end{bmatrix} \qquad F = \begin{bmatrix} 4 & 0 & 2 \\ -1 & 1 & 0 \\ 7 & 5 & 0 \end{bmatrix}$$

$$G = \begin{bmatrix} 5 \end{bmatrix}$$

Carry out the indicated operation, or explain why it cannot be performed.

51. $A + B$ **52.** $C - D$ **53.** $2C + 3D$

54. $5B - 2C$ **55.** GA **56.** AG

57. BC **58.** CB **59.** BF

60. FC **61.** $(C + D)E$ **62.** $F(2C - D)$

63–64 ■ Verify that the matrices A and B are inverses of each other by calculating the products AB and BA.

63. $A = \begin{bmatrix} 2 & -5 \\ -2 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & \frac{5}{2} \\ 1 & 1 \end{bmatrix}$

64. $A = \begin{bmatrix} 2 & -1 & 3 \\ 2 & -2 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -\frac{3}{2} & 2 & \frac{5}{2} \\ -1 & 1 & 2 \\ 1 & -1 & -1 \end{bmatrix}$

65–70 ■ Solve the matrix equation for the unknown matrix, X, or show that no solution exists, where

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 3 \\ -2 & 4 & 0 \end{bmatrix}$$

65. $A + 3X = B$ **66.** $\frac{1}{2}(X - 2B) = A$

67. $2(X - A) = 3B$ **68.** $2X + C = 5A$

69. $AX = C$ **70.** $AX = B$

71–78 ■ Find the determinant and, if possible, the inverse of the matrix.

71. $\begin{bmatrix} 1 & 4 \\ 2 & 9 \end{bmatrix}$ **72.** $\begin{bmatrix} 2 & 2 \\ 1 & -3 \end{bmatrix}$

73. $\begin{bmatrix} 4 & -12 \\ -2 & 6 \end{bmatrix}$ **74.** $\begin{bmatrix} 2 & 4 & 0 \\ -1 & 1 & 2 \\ 0 & 3 & 2 \end{bmatrix}$

75. $\begin{bmatrix} 3 & 0 & 1 \\ 2 & -3 & 0 \\ 4 & -2 & 1 \end{bmatrix}$ **76.** $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 2 & 5 & 6 \end{bmatrix}$

77. $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}$ **78.** $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & 2 & 1 & 2 \end{bmatrix}$

79–82 ■ Express the system of linear equations as a matrix equation. Then solve the matrix equation by multiplying each side by the inverse of the coefficient matrix.

79. $\begin{cases} 12x - 5y = 10 \\ 5x - 2y = 17 \end{cases}$ **80.** $\begin{cases} 6x - 5y = 1 \\ 8x - 7y = -1 \end{cases}$

81. $\begin{cases} 2x + y + 5z = \frac{1}{3} \\ x + 2y + 2z = \frac{1}{4} \\ x + 3z = \frac{1}{6} \end{cases}$ **82.** $\begin{cases} 2x + 3z = 5 \\ x + y + 6z = 0 \\ 3x - y + z = 5 \end{cases}$

83–86 ■ Solve the system using Cramer's Rule.

83. $\begin{cases} 2x + 7y = 13 \\ 6x + 16y = 30 \end{cases}$

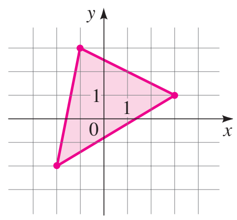
84. $\begin{cases} 12x - 11y = 140 \\ 7x + 9y = 20 \end{cases}$

85. $\begin{cases} 2x - y + 5z = 0 \\ -x + 7y = 9 \\ 5x + 4y + 3z = -9 \end{cases}$

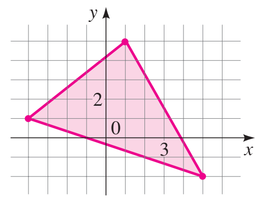
86. $\begin{cases} 3x + 4y - z = 10 \\ x - 4z = 20 \\ 2x + y + 5z = 30 \end{cases}$

87–88 ■ Use the determinant formula for the area of a triangle to find the area of the triangle in the figure.

87.



88.



89–94 ■ Find the partial fraction decomposition of the rational function.

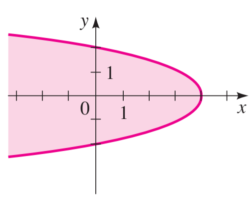
89. $\dfrac{3x + 1}{x^2 - 2x - 15}$ **90.** $\dfrac{8}{x^3 - 4x}$

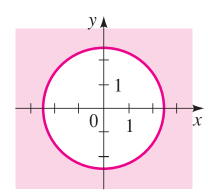
91. $\dfrac{2x - 4}{x(x - 1)^2}$ **92.** $\dfrac{x + 6}{x^3 - 2x^2 + 4x - 8}$

93. $\dfrac{2x - 1}{x^3 + x}$ **94.** $\dfrac{5x^2 - 3x + 10}{x^4 + x^2 - 2}$

95–96 ■ An equation and its graph are given. Find an inequality whose solution is the shaded region.

95. $x + y^2 = 4$ **96.** $x^2 + y^2 = 8$





97–100 ■ Graph the inequality.

97. $3x + y \le 6$

98. $y \ge x^2 - 3$

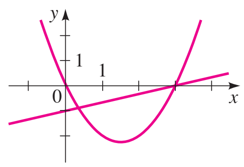
99. $x^2 + y^2 > 9$

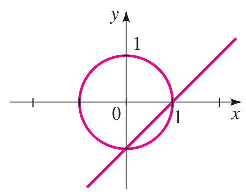
100. $x - y^2 < 4$

101–104 ■ The figure shows the graphs of the equations corresponding to the given inequalities. Shade the solution set of the system of inequalities.

101. $\begin{cases} y \ge x^2 - 3x \\ y \le \frac{1}{3}x - 1 \end{cases}$

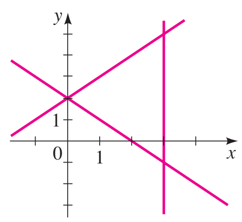
102. $\begin{cases} y \ge x - 1 \\ x^2 + y^2 \le 1 \end{cases}$

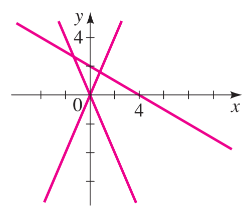




103. $\begin{cases} x + y \ge 2 \\ y - x \le 2 \\ x \le 3 \end{cases}$

104. $\begin{cases} y \ge -2x \\ y \le 2x \\ y \le -\frac{1}{2}x + 2 \end{cases}$





105–108 ■ Graph the solution set of the system of inequalities. Find the coordinates of all vertices, and determine whether the solution set is bounded or unbounded.

105. $\begin{cases} x^2 + y^2 < 9 \\ x + y < 0 \end{cases}$

106. $\begin{cases} y - x^2 \ge 4 \\ y < 20 \end{cases}$

107. $\begin{cases} x \ge 0, \quad y \ge 0 \\ x + 2y \le 12 \\ y \le x + 4 \end{cases}$

108. $\begin{cases} x \ge 4 \\ x + y \ge 24 \\ x \le 2y + 12 \end{cases}$

109–110 ■ Solve for x, y, and z in terms of a, b, and c.

109. $\begin{cases} -x + y + z = a \\ x - y + z = b \\ x + y - z = c \end{cases}$

110. $\begin{cases} ax + by + cz = a - b + c \\ bx + by + cz = c \\ cx + cy + cz = c \end{cases} \qquad (a \ne b, b \ne c, c \ne 0)$

111. For what values of k do the following three lines have a common point of intersection?

$$x + y = 12$$
$$kx - y = 0$$
$$y - x = 2k$$

112. For what value of k does the following system have infinitely many solutions?

$$\begin{cases} kx + y + z = 0 \\ x + 2y + kz = 0 \\ -x + 3z = 0 \end{cases}$$

732 CHAPTER 9 Systems of Equations and Inequalities

9 Test

1–2 ■ A system of equations is given.

(a) Determine whether the system is linear or nonlinear.

(b) Find all solutions of the system.

1. $\begin{cases} x + 3y = 7 \\ 5x + 2y = -4 \end{cases}$ **2.** $\begin{cases} 6x + y^2 = 10 \\ 3x - y = 5 \end{cases}$

3. Use a graphing device to find all solutions of the system correct to two decimal places.

$$\begin{cases} x - 2y = 1 \\ y = x^3 - 2x^2 \end{cases}$$

4. In $2\frac{1}{2}$ h an airplane travels 600 km against the wind. It takes 50 min to travel 300 km with the wind. Find the speed of the wind and the speed of the airplane in still air.

5. Determine whether each matrix is in reduced row-echelon form, row-echelon form, or neither.

(a) $\begin{bmatrix} 1 & 2 & 4 & -6 \\ 0 & 1 & -3 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 3 \end{bmatrix}$

6. Use Gaussian elimination to find the complete solution of the system, or show that no solution exists.

(a) $\begin{cases} x - y + 2z = 0 \\ 2x - 4y + 5z = -5 \\ 2y - 3z = 5 \end{cases}$ (b) $\begin{cases} 2x - 3y + z = 3 \\ x + 2y + 2z = -1 \\ 4x + y + 5z = 4 \end{cases}$

7. Use Gauss-Jordan elimination to find the complete solution of the system.

$$\begin{cases} x + 3y - z = 0 \\ 3x + 4y - 2z = -1 \\ -x + 2y = 1 \end{cases}$$

8. Anne, Barry, and Cathy enter a coffee shop. Anne orders two coffees, one juice, and two donuts, and pays \$6.25. Barry orders one coffee and three donuts, and pays \$3.75. Cathy orders three coffees, one juice, and four donuts, and pays \$9.25. Find the price of coffee, juice, and donuts at this coffee shop.

9. Let

$$A = \begin{bmatrix} 2 & 3 \\ 2 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 4 \\ -1 & 1 \\ 3 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 4 \\ -1 & 1 & 2 \\ 0 & 1 & 3 \end{bmatrix}$$

Carry out the indicated operation, or explain why it cannot be performed.

(a) $A + B$ (b) AB (c) $BA - 3B$ (d) CBA

(e) A^{-1} (f) B^{-1} (g) $\det(B)$ (h) $\det(C)$

10. (a) Write a matrix equation equivalent to the following system.

$$\begin{cases} 4x - 3y = 10 \\ 3x - 2y = 30 \end{cases}$$

(b) Find the inverse of the coefficient matrix, and use it to solve the system.

11. Only one of the following matrices has an inverse. Find the determinant of each matrix, and use the determinants to identify the one that has an inverse. Then find the inverse.

$$A = \begin{bmatrix} 1 & 4 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 4 & 0 \\ 0 & 2 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

12. Solve using Cramer's Rule:

$$\begin{cases} 2x \phantom{{}-y} - z = 14 \\ 3x - y + 5z = 0 \\ 4x + 2y + 3z = -2 \end{cases}$$

13. Find the partial fraction decomposition of the rational function.

(a) $\dfrac{4x - 1}{(x - 1)^2(x + 2)}$ (b) $\dfrac{2x - 3}{x^3 + 3x}$

14. Graph the solution set of the system of inequalities. Label the vertices with their coordinates.

(a) $\begin{cases} 2x + y \le 8 \\ x - y \ge -2 \\ x + 2y \ge 4 \end{cases}$ (b) $\begin{cases} x^2 + y \le 5 \\ y \le 2x + 5 \end{cases}$

Focus on Modeling
Linear Programming

Linear programming is a modeling technique used to determine the optimal allocation of resources in business, the military, and other areas of human endeavor. For example, a manufacturer who makes several different products from the same raw materials can use linear programming to determine how much of each product should be produced to maximize the profit. This modeling technique is probably the most important practical application of systems of linear inequalities. In 1975 Leonid Kantorovich and T. C. Koopmans won the Nobel Prize in economics for their work in the development of this technique.

Although linear programming can be applied to very complex problems with hundreds or even thousands of variables, we consider only a few simple examples to which the graphical methods of Section 9.9 can be applied. (For large numbers of variables, a linear programming method based on matrices is used.) Let's examine a typical problem.

Example 1 Manufacturing for Maximum Profit

A small shoe manufacturer makes two styles of shoes: oxfords and loafers. Two machines are used in the process: a cutting machine and a sewing machine. Each type of shoe requires 15 min per pair on the cutting machine. Oxfords require 10 min of sewing per pair, and loafers require 20 min of sewing per pair. Because the manufacturer can hire only one operator for each machine, each process is available for just 8 hours per day. If the profit is \$15 on each pair of oxfords and \$20 on each pair of loafers, how many pairs of each type should be produced per day for maximum profit?

Because loafers produce more profit per pair, it would seem best to manufacture only loafers. Surprisingly, this does not turn out to be the most profitable solution.

Solution First we organize the given information into a table. To be consistent, let's convert all times to hours.

	Oxfords	Loafers	Time available
Time on cutting machine (h)	$\frac{1}{4}$	$\frac{1}{4}$	8
Time on sewing machine (h)	$\frac{1}{6}$	$\frac{1}{3}$	8
Profit	\$15	\$20	

We describe the model and solve the problem in four steps.

CHOOSING THE VARIABLES To make a mathematical model, we first give names to the variable quantities. For this problem we let

$$x = \text{number of pairs of oxfords made daily}$$

$$y = \text{number of pairs of loafers made daily}$$

FINDING THE OBJECTIVE FUNCTION Our goal is to determine which values for x and y give maximum profit. Since each pair of oxfords generates \$15 profit and each pair of loafers \$20, the total profit is given by

$$P = 15x + 20y$$

This function is called the *objective function*.

GRAPHING THE FEASIBLE REGION The larger x and y are, the greater the profit. But we cannot choose arbitrarily large values for these variables, because of the restrictions, or *constraints*, in the problem. Each restriction is an inequality in the variables.

In this problem the total number of cutting hours needed is $\frac{1}{4}x + \frac{1}{4}y$. Since only 8 hours are available on the cutting machine, we have

$$\tfrac{1}{4}x + \tfrac{1}{4}y \le 8$$

Similarly, by considering the amount of time needed and available on the sewing machine, we get

$$\tfrac{1}{6}x + \tfrac{1}{3}y \le 8$$

We cannot produce a negative number of shoes, so we also have

$$x \ge 0 \qquad \text{and} \qquad y \ge 0$$

Thus, x and y must satisfy the constraints

$$\begin{cases} \frac{1}{4}x + \frac{1}{4}y \le 8 \\ \frac{1}{6}x + \frac{1}{3}y \le 8 \\ x \ge 0 \\ y \ge 0 \end{cases}$$

If we multiply the first inequality by 4 and the second by 6, we obtain the simplified system

$$\begin{cases} x + y \le 32 \\ x + 2y \le 48 \\ x \ge 0 \\ y \ge 0 \end{cases}$$

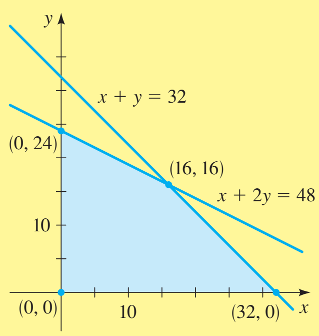


Figure 1

The solution of this system (with vertices labeled) is sketched in Figure 1. The only values that satisfy the restrictions of the problem are the ones that correspond to points of the shaded region in Figure 1. This is called the *feasible region* for the problem.

FINDING MAXIMUM PROFIT As x or y increases, profit increases as well. Thus, it seems reasonable that the maximum profit will occur at a point on one of the outside edges of the feasible region, where it's impossible to increase x or y without going outside the region. In fact, it can be shown that the maximum value occurs at a vertex. This means that we need to check the profit only at the vertices. The largest value of P occurs at the point $(16, 16)$, where $P = \$560$. Thus, the manufacturer should make 16 pairs of oxfords and 16 pairs of loafers, for a maximum daily profit of \$560.

Vertex	$P = 15x + 20y$
$(0, 0)$	0
$(0, 24)$	$15(0) + 20(24) = \$480$
$(16, 16)$	$15(16) + 20(16) = \$560$
$(32, 0)$	$15(32) + 20(0) = \$480$

Maximum profit

Linear programming helps the telephone industry determine the most efficient way to route telephone calls. The computerized routing decisions must be made very rapidly so callers are not kept waiting for connections. Since the database of customers and routes is huge, an extremely fast method for solving linear programming problems is essential. In 1984 the 28-year-old mathematician **Narendra Karmarkar**, working at Bell Labs in Murray Hill, New Jersey, discovered just such a method. His idea is so ingenious and his method so fast that the discovery caused a sensation in the mathematical world. Although mathematical discoveries rarely make the news, this one was reported in *Time*, on December 3, 1984. Today airlines routinely use Karmarkar's technique to minimize costs in scheduling passengers, flight personnel, fuel, baggage, and maintenance workers.

The linear programming problems that we consider all follow the pattern of Example 1. Each problem involves two variables. The problem describes restrictions, called **constraints**, that lead to a system of linear inequalities whose solution is called the **feasible region**. The function we wish to maximize or minimize is called the **objective function**. This function always attains its largest and smallest values at the **vertices** of the feasible region. This modeling technique involves four steps, summarized in the following box.

Guidelines for Linear Programming

1. **Choose the Variables.** Decide what variable quantities in the problem should be named x and y.
2. **Find the Objective Function.** Write an expression for the function we want to maximize or minimize.
3. **Graph the Feasible Region.** Express the constraints as a system of inequalities and graph the solution of this system (the feasible region).
4. **Find the Maximum or Minimum.** Evaluate the objective function at the vertices of the feasible region to determine its maximum or minimum value.

Example 2 A Shipping Problem

A car dealer has warehouses in Millville and Trenton and dealerships in Camden and Atlantic City. Every car sold at the dealerships must be delivered from one of the warehouses. On a certain day the Camden dealers sell 10 cars, and the Atlantic City dealers sell 12. The Millville warehouse has 15 cars available, and the Trenton warehouse has 10. The cost of shipping one car is \$50 from Millville to Camden, \$40 from Millville to Atlantic City, \$60 from Trenton to Camden, and \$55 from Trenton to Atlantic City. How many cars should be moved from each warehouse to each dealership to fill the orders at minimum cost?

Solution Our first step is to organize the given information. Rather than construct a table, we draw a diagram to show the flow of cars from the warehouses to the dealerships (see Figure 2 on the next page). The diagram shows the number of cars available at each warehouse or required at each dealership and the cost of shipping between these locations.

CHOOSING THE VARIABLES The arrows in Figure 2 indicate four possible routes, so the problem seems to involve four variables. But we let

x = number of cars to be shipped from Millville to Camden

y = number of cars to be shipped from Millville to Atlantic City

To fill the orders, we must have

$10 - x$ = number of cars shipped from Trenton to Camden

$12 - y$ = number of cars shipped from Trenton to Atlantic City

So the only variables in the problem are x and y.

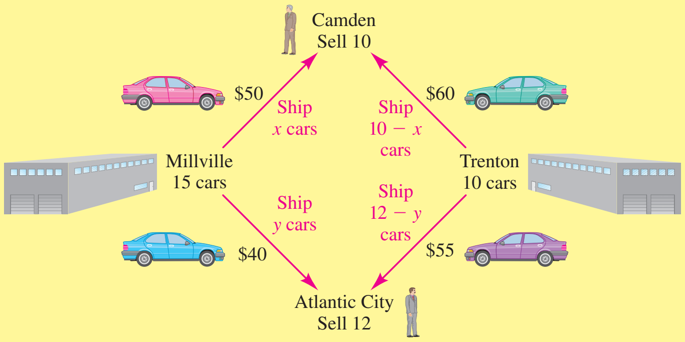


Figure 2

FINDING THE OBJECTIVE FUNCTION The objective of this problem is to minimize cost. From Figure 2 we see that the total cost C of shipping the cars is

$$\begin{aligned} C &= 50x + 40y + 60(10 - x) + 55(12 - y) \\ &= 50x + 40y + 600 - 60x + 660 - 55y \\ &= 1260 - 10x - 15y \end{aligned}$$

This is the objective function.

GRAPHING THE FEASIBLE REGION Now we derive the constraint inequalities that define the feasible region. First, the number of cars shipped on each route can't be negative, so we have

$$\begin{aligned} x &\geq 0 & y &\geq 0 \\ 10 - x &\geq 0 & 12 - y &\geq 0 \end{aligned}$$

Second, the total number of cars shipped from each warehouse can't exceed the number of cars available there, so

$$\begin{aligned} x + y &\leq 15 \\ (10 - x) + (12 - y) &\leq 10 \end{aligned}$$

Simplifying the latter inequality, we get

$$\begin{aligned} 22 - x - y &\leq 10 \\ -x - y &\leq -12 \\ x + y &\geq 12 \end{aligned}$$

The inequalities $10 - x \geq 0$ and $12 - y \geq 0$ can be rewritten as $x \leq 10$ and $y \leq 12$. Thus, the feasible region is described by the constraints

$$\begin{cases} x + y \leq 15 \\ x + y \geq 12 \\ 0 \leq x \leq 10 \\ 0 \leq y \leq 12 \end{cases}$$

The feasible region is graphed in Figure 3.

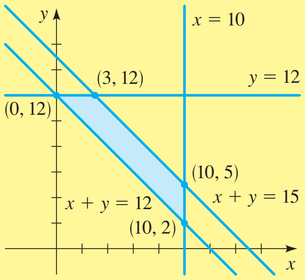


Figure 3

FINDING MINIMUM COST We check the value of the objective function at each vertex of the feasible region.

Vertex	$C = 1260 - 10x - 15y$
$(0, 12)$	$1260 - 10(0) - 15(12) = \$1080$
$(3, 12)$	$1260 - 10(3) - 15(12) = \$1050$
$(10, 5)$	$1260 - 10(10) - 15(5) = \$1085$
$(10, 2)$	$1260 - 10(10) - 15(2) = \$1130$

Minimum cost

The lowest cost is incurred at the point $(3, 12)$. Thus, the dealer should ship

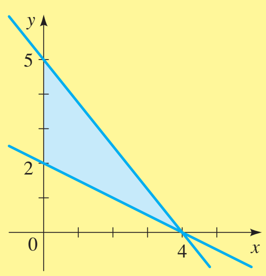
3 cars from Millville to Camden
12 cars from Millville to Atlantic City
7 cars from Trenton to Camden
0 cars from Trenton to Atlantic City ■

In the 1940s mathematicians developed matrix methods for solving linear programming problems that involve more than two variables. These methods were first used by the Allies in World War II to solve supply problems similar to (but, of course, much more complicated than) Example 2. Improving such matrix methods is an active and exciting area of current mathematical research.

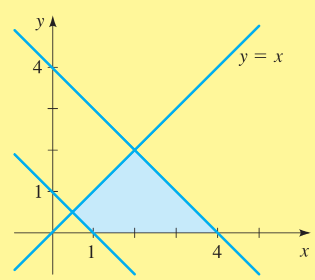
Problems

1–4 ■ Find the maximum and minimum values of the given objective function on the indicated feasible region.

1. $M = 200 - x - y$



2. $N = \frac{1}{2}x + \frac{1}{4}y + 40$



3. $P = 140 - x + 3y$

$$\begin{cases} x \geq 0, \quad y \geq 0 \\ 2x + y \leq 10 \\ 2x + 4y \leq 28 \end{cases}$$

4. $Q = 70x + 82y$

$$\begin{cases} x \geq 0, \quad y \geq 0 \\ x \leq 10, \quad y \leq 20 \\ x + y \geq 5 \\ x + 2y \leq 18 \end{cases}$$

5. Making Furniture A furniture manufacturer makes wooden tables and chairs. The production process involves two basic types of labor: carpentry and finishing. A table requires 2 hours of carpentry and 1 hour of finishing, and a chair requires 3 hours of carpentry and $\frac{1}{2}$ hour of finishing. The profit is $35 per table and $20 per chair. The manufacturer's employees can supply a maximum of 108 hours of carpentry work and 20 hours of finishing work per day. How many tables and chairs should be made each day to maximize profit?

6. **A Housing Development** A housing contractor has subdivided a farm into 100 building lots. He has designed two types of homes for these lots: colonial and ranch style. A colonial requires $30,000 of capital and produces a profit of $4000 when sold. A ranch-style house requires $40,000 of capital and provides an $8000 profit. If he has $3.6 million of capital on hand, how many houses of each type should he build for maximum profit? Will any of the lots be left vacant?

7. **Hauling Fruit** A trucker hauls citrus fruit from Florida to Montreal. Each crate of oranges is 4 ft^3 in volume and weighs 80 lb. Each crate of grapefruit has a volume of 6 ft^3 and weighs 100 lb. Her truck has a maximum capacity of 300 ft^3 and can carry no more than 5600 lb. Moreover, she is not permitted to carry more crates of grapefruit than crates of oranges. If her profit is $2.50 on each crate of oranges and $4 on each crate of grapefruit, how many crates of each fruit should she carry for maximum profit?

8. **Manufacturing Calculators** A manufacturer of calculators produces two models: standard and scientific. Long-term demand for the two models mandates that the company manufacture at least 100 standard and 80 scientific calculators each day. However, because of limitations on production capacity, no more than 200 standard and 170 scientific calculators can be made daily. To satisfy a shipping contract, a total of at least 200 calculators must be shipped every day.
   (a) If the production cost is $5 for a standard calculator and $7 for a scientific one, how many of each model should be produced daily to minimize this cost?
   (b) If each standard calculator results in a $2 loss but each scientific one produces a $5 profit, how many of each model should be made daily to maximize profit?

9. **Shipping Stereos** An electronics discount chain has a sale on a certain brand of stereo. The chain has stores in Santa Monica and El Toro and warehouses in Long Beach and Pasadena. To satisfy rush orders, 15 sets must be shipped from the warehouses to the Santa Monica store, and 19 must be shipped to the El Toro store. The cost of shipping a set is $5 from Long Beach to Santa Monica, $6 from Long Beach to El Toro, $4 from Pasadena to Santa Monica, and $5.50 from Pasadena to El Toro. If the Long Beach warehouse has 24 sets and the Pasadena warehouse has 18 sets in stock, how many sets should be shipped from each warehouse to each store to fill the orders at a minimum shipping cost?

10. **Delivering Plywood** A man owns two building supply stores, one on the east side and one on the west side of a city. Two customers order some $\frac{1}{2}$-inch plywood. Customer A needs 50 sheets and customer B needs 70 sheets. The east-side store has 80 sheets and the west-side store has 45 sheets of this plywood in stock. The east-side store's delivery costs per sheet are $0.50 to customer A and $0.60 to customer B. The west-side store's delivery costs per sheet are $0.40 to A and $0.55 to B. How many sheets should be shipped from each store to each customer to minimize delivery costs?

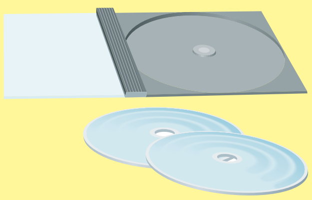
11. **Packaging Nuts** A confectioner sells two types of nut mixtures. The standard-mixture package contains 100 g of cashews and 200 g of peanuts and sells for $1.95. The deluxe-mixture package contains 150 g of cashews and 50 g of peanuts and sells for $2.25. The confectioner has 15 kg of cashews and 20 kg of peanuts available. Based on past sales, he needs to have at least as many standard as deluxe packages available. How many bags of each mixture should he package to maximize his revenue?

12. **Feeding Lab Rabbits** A biologist wishes to feed laboratory rabbits a mixture of two types of foods. Type I contains 8 g of fat, 12 g of carbohydrate, and 2 g of protein per ounce. Type II contains 12 g of fat, 12 g of carbohydrate, and 1 g of protein per ounce.

Type I costs \$0.20 per ounce and type II costs \$0.30 per ounce. The rabbits each receive a daily minimum of 24 g of fat, 36 g of carbohydrate, and 4 g of protein, but get no more than 5 oz of food per day. How many ounces of each food type should be fed to each rabbit daily to satisfy the dietary requirements at minimum cost?

13. Investing in Bonds A woman wishes to invest \$12,000 in three types of bonds: municipal bonds paying 7% interest per year, bank investment certificates paying 8%, and high-risk bonds paying 12%. For tax reasons, she wants the amount invested in municipal bonds to be at least three times the amount invested in bank certificates. To keep her level of risk manageable, she will invest no more than \$2000 in high-risk bonds. How much should she invest in each type of bond to maximize her annual interest yield? [*Hint:* Let x = amount in municipal bonds and y = amount in bank certificates. Then the amount in high-risk bonds will be $12{,}000 - x - y$.]

14. Annual Interest Yield Refer to Problem 13. Suppose the investor decides to increase the maximum invested in high-risk bonds to \$3000 but leaves the other conditions unchanged. By how much will her maximum possible interest yield increase?



15. Business Strategy A small software company publishes computer games and educational and utility software. Their business strategy is to market a total of 36 new programs each year, with at least four of these being games. The number of utility programs published is never more than twice the number of educational programs. On average, the company makes an annual profit of \$5000 on each computer game, \$8000 on each educational program, and \$6000 on each utility program. How many of each type of software should they publish annually for maximum profit?

16. Feasible Region All parts of this problem refer to the following feasible region and objective function.

$$\begin{cases} x \geq 0 \\ x \geq y \\ x + 2y \leq 12 \\ x + y \leq 10 \end{cases}$$

$$P = x + 4y$$

(a) Graph the feasible region.

(b) On your graph from part (a), sketch the graphs of the linear equations obtained by setting P equal to 40, 36, 32, and 28.

(c) If we continue to decrease the value of P, at which vertex of the feasible region will these lines first touch the feasible region?

(d) Verify that the maximum value of P on the feasible region occurs at the vertex you chose in part (c).