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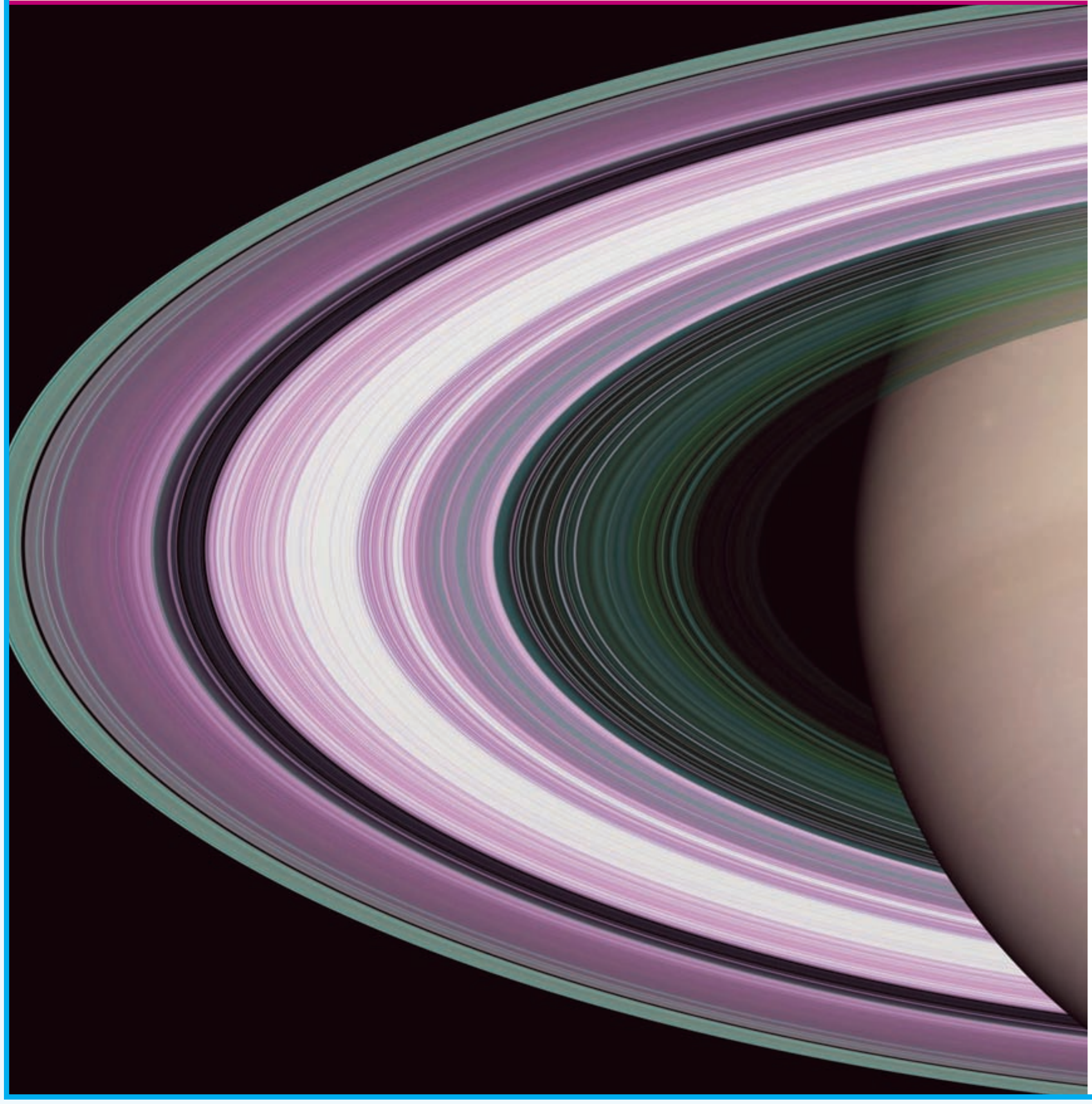
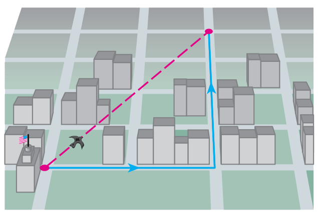
8 Polar Coordinates and Vectors



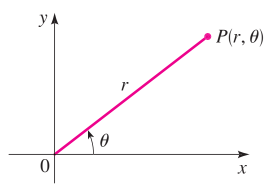


Courtesy of NASA

Chapter Overview

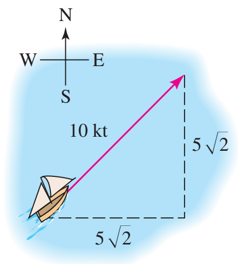
In this chapter we study polar coordinates, a new way of describing the location of points in a plane.

A coordinate system is a method for specifying the location of a point in the plane. We are familiar with rectangular (or Cartesian) coordinates. In rectangular coordinates the location of a point is given by an ordered pair (x, y), which gives the distance of the point to two perpendicular axes. Using rectangular coordinates is like describing a location in a city by saying that it's at the corner of 2nd Street and 4th Avenue. But we might also describe this same location by saying that it's $1\frac{1}{2}$ miles northeast of City Hall. So instead of specifying the location with respect to a grid of streets and avenues, we specify it by giving its distance and direction from a fixed reference point. That is what we do in the polar coordinate system. In polar coordinates the location of a point is given by an ordered pair (r, θ) where r is the distance from the origin (or pole) and θ is the angle from the positive x-axis (see the figure below).



Why do we study different coordinate systems? Because certain curves are more naturally described in one coordinate system rather than the other. In rectangular coordinates we can give simple equations for lines, parabolas, or cubic curves, but the equation of a circle is rather complicated (and it is not a function). In polar coordinates we can give simple equations for circles, ellipses, roses, and figure 8's—curves that are difficult to describe in rectangular coordinates. So, for example, it is more natural to describe a planet's path around the sun in terms of distance from the sun and angle of travel—in other words, in polar coordinates. We will also give polar representations of complex numbers. As you will see, it is easy to multiply complex numbers if they are written in polar form.

In this chapter we also use coordinates to describe directed quantities, or *vectors*. When we talk about temperature, mass, or area, we need only one number. For example, we say the temperature is 70°F. But quantities such as velocity or force are *directed quantities*, because they involve direction as well as magnitude. Thus we say



that a boat is sailing at 10 knots to the northeast. We can also express this graphically by drawing an arrow of length 10 in the direction of travel. The velocity can be completely described by the displacement of the arrow from tail to head, which we express as the vector $\langle 5\sqrt{2}, 5\sqrt{2}\rangle$ (see the figure).

In the *Focus on Modeling* (page 630) we will see how polar coordinates are used to draw a (flat) map of a (spherical) world. In the *Discovery Project* on page 626 we explore how an analysis of the vector forces of wind and current can be used to navigate a sailboat.

8.1 Polar Coordinates

In this section we define polar coordinates, and we learn how polar coordinates are related to rectangular coordinates.

Definition of Polar Coordinates

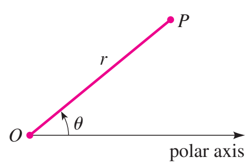


Figure 1

The **polar coordinate system** uses distances and directions to specify the location of a point in the plane. To set up this system, we choose a fixed point O in the plane called the **pole** (or **origin**) and draw from O a ray (half-line) called the **polar axis** as in Figure 1. Then each point P can be assigned polar coordinates $P(r, \theta)$ where

r is the *distance* from O to P

θ is the *angle* between the polar axis and the segment $\overline{OP}$

We use the convention that θ is positive if measured in a counterclockwise direction from the polar axis or negative if measured in a clockwise direction. If r is negative, then $P(r, \theta)$ is defined to be the point that lies $|r|$ units from the pole in the direction opposite to that given by θ (see Figure 2).

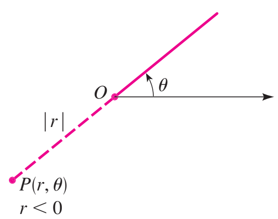


Figure 2

Example 1 Plotting Points in Polar Coordinates

Plot the points whose polar coordinates are given.

(a) $(1, 3\pi/4)$ (b) $(3, -\pi/6)$ (c) $(3, 3\pi)$ (d) $(-4, \pi/4)$

Solution The points are plotted in Figure 3. Note that the point in part (d) lies 4 units from the origin along the angle $5\pi/4$, because the given value of r is negative.

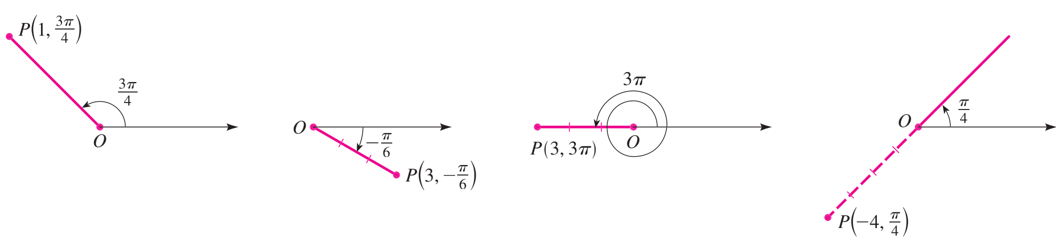


Figure 3

■

SUGGESTED TIME AND EMPHASIS

1 class.
Essential material.

POINT TO STRESS

1. The relationships between rectangular and polar coordinates.
2. Converting equations into different coordinate systems.

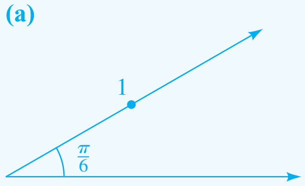
ALTERNATE EXAMPLE 1

Plot the points whose polar coordinates are given:

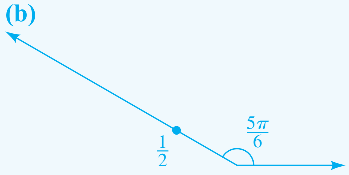
(a) $(1, \pi/6)$
(b) $(1/2, 5\pi/6)$
(c) $(-4, 101\pi/2)$

ANSWER

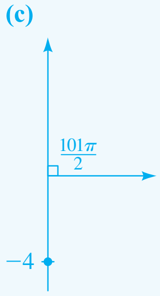
(a)



(b)



(c)



Note that the coordinates (r, θ) and $(-r, \theta + \pi)$ represent the same point, as shown in Figure 4. Moreover, because the angles $\theta + 2n\pi$ (where n is any integer) all have the same terminal side as the angle θ, each point in the plane has infinitely many representations in polar coordinates. In fact, any point $P(r, \theta)$ can also be represented by

$$P(r, \theta + 2n\pi) \qquad \text{and} \qquad P(-r, \theta + (2n+1)\pi)$$

for any integer n.

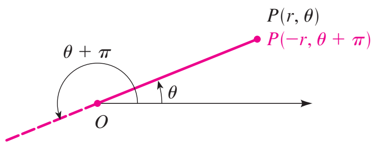


Figure 4

Example 2 Different Polar Coordinates for the Same Point

(a) Graph the point with polar coordinates $P(2, \pi/3)$.

(b) Find two other polar coordinate representations of P with $r > 0$, and two with $r < 0$.

Solution

(a) The graph is shown in Figure 5(a).

(b) Other representations with $r > 0$ are

$$\left(2, \frac{\pi}{3} + 2\pi\right) = \left(2, \frac{7\pi}{3}\right) \qquad \text{Add } 2\pi \text{ to } \theta$$

$$\left(2, \frac{\pi}{3} - 2\pi\right) = \left(2, -\frac{5\pi}{3}\right) \qquad \text{Add } -2\pi \text{ to } \theta$$

Other representations with $r < 0$ are

$$\left(-2, \frac{\pi}{3} + \pi\right) = \left(-2, \frac{4\pi}{3}\right) \qquad \text{Replace } r \text{ by } -r \text{ and add } \pi \text{ to } \theta$$

$$\left(-2, \frac{\pi}{3} - \pi\right) = \left(-2, -\frac{2\pi}{3}\right) \qquad \text{Replace } r \text{ by } -r \text{ and add } -\pi \text{ to } \theta$$

The graphs in Figure 5 explain why these coordinates represent the same point.

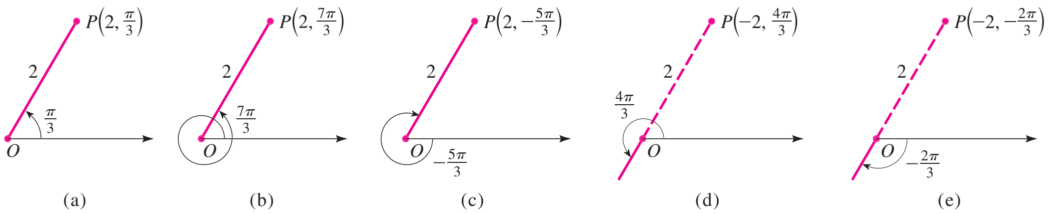


Figure 5

■

IN-CLASS MATERIALS

Begin with an intuitive definition of polar coordinates and then derive the algebraic formulas, noting that the graph of a polar function need not pass the Vertical Line Test.

ALTERNATE EXAMPLE 2

Find rectangular coordinates for the point that has polar coordinates $\left(3, \frac{2\pi}{3}\right)$.

ANSWER

$\left(-\frac{3}{2}, \frac{3\sqrt{3}}{2}\right)$

DRILL QUESTION

Convert the equation $r \sin \theta = 3r \cos \theta + 2$ to polar form.

Answer

$y = 3x + 2$

EXAMPLES

Coordinate conversion:

Rectangular (8,16) is the same as polar $(8\sqrt{5}, 1.107)$.

Rectangular $(5, -5\sqrt{3})$ is the same as polar $\left(10, -\frac{\pi}{3}\right)$.

Polar $\left(4, \frac{13\pi}{6}\right)$ is the same as rectangular $(2\sqrt{3}, 2)$.

Polar (7, 10) is (approximately) the same as rectangular (−5.8735, −3.8081).

ALTERNATE EXAMPLE 3

Find the rectangular coordinates for the point whose polar coordinates are $\left(-4, \frac{5\pi}{2}\right)$.

ANSWER

The rectangular coordinates for the given point are (0, −4)

ALTERNATE EXAMPLE 4

Find two polar coordinates for the point that has rectangular coordinates (−7, −7).

ANSWER

$\left(7\sqrt{2}, -\frac{3\pi}{4}\right), \left(-7\sqrt{2}, -\frac{\pi}{4}\right)$

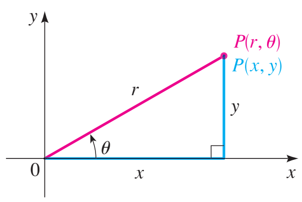


Figure 6

Relationship between Polar and Rectangular Coordinates

Situations often arise in which we need to consider polar and rectangular coordinates simultaneously. The connection between the two systems is illustrated in Figure 6, where the polar axis coincides with the positive x-axis. The formulas in the following box are obtained from the figure using the definitions of the trigonometric functions and the Pythagorean Theorem. (Although we have pictured the case where $r > 0$ and θ is acute, the formulas hold for any angle θ and for any value of r.)

Relationship between Polar and Rectangular Coordinates

1. To change from polar to rectangular coordinates, use the formulas

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

2. To change from rectangular to polar coordinates, use the formulas

$$r^2 = x^2 + y^2 \quad \text{and} \quad \tan\theta = \frac{y}{x} \quad (x \neq 0)$$

Example 3 Converting Polar Coordinates to Rectangular Coordinates

Find rectangular coordinates for the point that has polar coordinates $(4, 2\pi/3)$.

Solution Since $r = 4$ and $\theta = 2\pi/3$, we have

$$x = r\cos\theta = 4\cos\frac{2\pi}{3} = 4\cdot\left(-\frac{1}{2}\right) = -2$$

$$y = r\sin\theta = 4\sin\frac{2\pi}{3} = 4\cdot\frac{\sqrt{3}}{2} = 2\sqrt{3}$$

Thus, the point has rectangular coordinates $(-2, 2\sqrt{3})$. ■

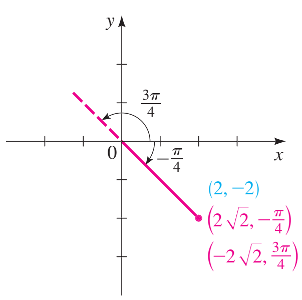


Figure 7

Example 4 Converting Rectangular Coordinates to Polar Coordinates

Find polar coordinates for the point that has rectangular coordinates $(2, -2)$.

Solution Using $x = 2$, $y = -2$, we get

$$r^2 = x^2 + y^2 = 2^2 + (-2)^2 = 8$$

so $r = 2\sqrt{2}$ or $-2\sqrt{2}$. Also

$$\tan\theta = \frac{y}{x} = \frac{-2}{2} = -1$$

so $\theta = 3\pi/4$ or $-\pi/4$. Since the point $(2, -2)$ lies in quadrant IV (see Figure 7), we can represent it in polar coordinates as $(2\sqrt{2}, -\pi/4)$ or $(-2\sqrt{2}, 3\pi/4)$. ■

IN-CLASS MATERIALS

Point out how some equations are simpler to consider in rectangular coordinates ($y = \ln x$ is easier than $r\sin\theta = \ln r + \ln\cos\theta$) but some equations are simpler in polar coordinates ($r = \theta$, the simple spiral, is much easier than $\pm\sqrt{x^2 + y^2} = \tan^{-1}(y/x)$). You can foreshadow Chapter 10 at this point, pointing out that there are curves called rotated ellipses, hyperbolas, and parabolas, that turn out to be very nice when considered as polar equations.

Note that the equations relating polar and rectangular coordinates do not uniquely determine r or θ. When we use these equations to find the polar coordinates of a point, we must be careful that the values we choose for r and θ give us a point in the correct quadrant, as we saw in Example 4.

Polar Equations

In Examples 3 and 4 we converted points from one coordinate system to the other. Now we consider the same problem for equations.

Example 5 Converting an Equation from Rectangular to Polar Coordinates

Express the equation $x^2 = 4y$ in polar coordinates.

Solution We use the formulas $x = r\cos\theta$ and $y = r\sin\theta$.

$$x^2 = 4y \qquad \text{Rectangular equation}$$

$$(r\cos\theta)^2 = 4(r\sin\theta) \qquad \text{Substitute } x = r\cos\theta,\ y = r\sin\theta$$

$$r^2\cos^2\theta = 4r\sin\theta \qquad \text{Expand}$$

$$r = 4\frac{\sin\theta}{\cos^2\theta} \qquad \text{Divide by } r\cos^2\theta$$

$$r = 4\sec\theta\tan\theta \qquad \text{Simplify}$$

■

As Example 5 shows, converting from rectangular to polar coordinates is straightforward—just replace x by $r\cos\theta$ and y by $r\sin\theta$, and then simplify. But converting polar equations to rectangular form often requires more thought.

Example 6 Converting Equations from Polar to Rectangular Coordinates

Express the polar equation in rectangular coordinates. If possible, determine the graph of the equation from its rectangular form.

(a) $r = 5\sec\theta$ (b) $r = 2\sin\theta$ (c) $r = 2 + 2\cos\theta$

Solution

(a) Since $\sec\theta = 1/\cos\theta$, we multiply both sides by $\cos\theta$.

$$r = 5\sec\theta$$

$$r\cos\theta = 5 \qquad \text{Multiply by } \cos\theta$$

$$x = 5 \qquad \text{Substitute } x = r\cos\theta$$

The graph of $x = 5$ is the vertical line in Figure 8.

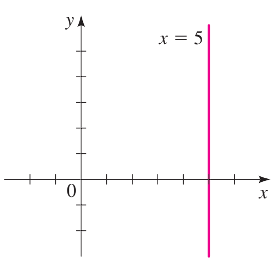


Figure 8

EXAMPLES

Equation conversion:

Rectangular $x^2 + y^2 = 9$ is the same as polar $r = 9$.

Polar $\tan\theta = 1$ is the same as rectangular $y = x$.

ALTERNATE EXAMPLE 5

Convert the equation $x = 1$ to polar form.

ANSWER

$r = \sec(\theta)$

ALTERNATE EXAMPLE 6b

Express the polar equation $r = 4\sin\theta$ in rectangular coordinates.

ANSWER

$x^2 + (y - 2)^2 = 4$

ALTERNATE EXAMPLE 6c

Convert the equation $r = 5 + 4\cos\theta$ to rectangular coordinates.

ANSWER

$(x^2 + y^2 - 4x)^2 = 25(x^2 + y^2)$

SAMPLE QUESTION

Text Question

Where do the conversion equations $x = r\cos\theta$, $y = r\sin\theta$ come from?

Answer

Several correct answers are possible. Anything addressing the definitions of sine and cosine, for example, should be given credit.

IN-CLASS MATERIALS

Do several examples of converting Cartesian equations into polar equations, such as $y^2 = 4x$ to $r = 4\csc\theta\cot\theta$, and of converting polar equations into Cartesian equations (the result of which are sometimes implicit equations) such as $r = 2\sec\theta$ to $x = 2$ and $r = 2(1 + \cos\theta)$ to $(x^2 + y^2 - 2x)^2 = 4(x^2 + y^2)$.

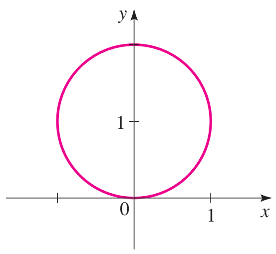


Figure 9

(b) We multiply both sides of the equation by r, because then we can use the formulas $r^2 = x^2 + y^2$ and $r \sin\theta = y$.

$$r^2 = 2r\sin\theta \qquad \text{Multiply by } r$$
$$x^2 + y^2 = 2y \qquad r^2 = x^2 + y^2 \text{ and } r\sin\theta = y$$
$$x^2 + y^2 - 2y = 0 \qquad \text{Subtract } 2y$$
$$x^2 + (y-1)^2 = 1 \qquad \text{Complete the square in } y$$

This is the equation of a circle of radius 1 centered at the point $(0, 1)$. It is graphed in Figure 9.

(c) We first multiply both sides of the equation by r:

$$r^2 = 2r + 2r\cos\theta$$

Using $r^2 = x^2 + y^2$ and $x = r\cos\theta$, we can convert two of the three terms in the equation into rectangular coordinates, but eliminating the remaining r requires more work:

$$x^2 + y^2 = 2r + 2x \qquad r^2 = x^2 + y^2 \text{ and } r\cos\theta = x$$
$$x^2 + y^2 - 2x = 2r \qquad \text{Subtract } 2x$$
$$(x^2 + y^2 - 2x)^2 = 4r^2 \qquad \text{Square both sides}$$
$$(x^2 + y^2 - 2x)^2 = 4(x^2 + y^2) \qquad r^2 = x^2 + y^2$$

In this case, the rectangular equation looks more complicated than the polar equation. Although we cannot easily determine the graph of the equation from its rectangular form, we will see in the next section how to graph it using the polar equation. ■

8.1 Exercises

1–6 ■ Plot the point that has the given polar coordinates.

1. $(4, \pi/4)$ **2.** $(1, 0)$ **3.** $(6, -7\pi/6)$

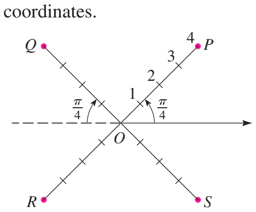
4. $(3, -2\pi/3)$ **5.** $(-2, 4\pi/3)$ **6.** $(-5, -17\pi/6)$

7–12 ■ Plot the point that has the given polar coordinates. Then give two other polar coordinate representations of the point, one with $r < 0$ and the other with $r > 0$.

7. $(3, \pi/2)$ **8.** $(2, 3\pi/4)$ **9.** $(-1, 7\pi/6)$

10. $(-2, -\pi/3)$ **11.** $(-5, 0)$ **12.** $(3, 1)$

13–20 ■ Determine which point in the figure, P, Q, R, or S, has the given polar coordinates.



13. $(4, 3\pi/4)$ **14.** $(4, -3\pi/4)$

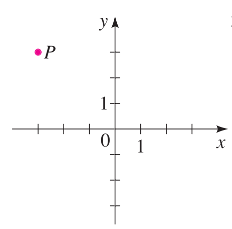
15. $(-4, -\pi/4)$ **16.** $(-4, 13\pi/4)$

17. $(4, -23\pi/4)$ **18.** $(-4, 23\pi/4)$

19. $(-4, 101\pi/4)$ **20.** $(4, 103\pi/4)$

21–22 ■ A point is graphed in rectangular form. Find polar coordinates for the point, with $r > 0$ and $0 < \theta < 2\pi$.

21.



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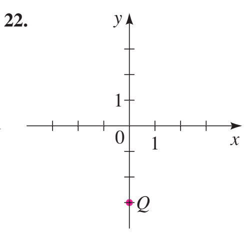
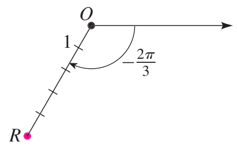


23–24 ■ A point is graphed in polar form. Find its rectangular coordinates.

23.



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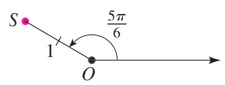


25–32 ■ Find the rectangular coordinates for the point whose polar coordinates are given.

25. $(4, \pi/6)$ **26.** $(6, 2\pi/3)$

27. $(\sqrt{2}, -\pi/4)$ **28.** $(-1, 5\pi/2)$

29. $(5, 5\pi)$ **30.** $(0, 13\pi)$

31. $(6\sqrt{2}, 11\pi/6)$ **32.** $(\sqrt{3}, -5\pi/3)$

33–40 ■ Convert the rectangular coordinates to polar coordinates with $r > 0$ and $0 \le \theta < 2\pi$.

33. $(-1, 1)$ **34.** $(3\sqrt{3}, -3)$

35. $(\sqrt{8}, \sqrt{8})$ **36.** $(-\sqrt{6}, -\sqrt{2})$

37. $(3, 4)$ **38.** $(1, -2)$

39. $(-6, 0)$ **40.** $(0, -\sqrt{3})$

41–46 ■ Convert the equation to polar form.

41. $x = y$ **42.** $x^2 + y^2 = 9$

43. $y = x^2$ **44.** $y = 5$

45. $x = 4$ **46.** $x^2 - y^2 = 1$

47–60 ■ Convert the polar equation to rectangular coordinates.

47. $r = 7$ **48.** $\theta = \pi$

49. $r\cos\theta = 6$ **50.** $r = 6\cos\theta$

51. $r^2 = \tan\theta$ **52.** $r^2 = \sin 2\theta$

53. $r = \dfrac{1}{\sin\theta - \cos\theta}$ **54.** $r = \dfrac{1}{1 + \sin\theta}$

55. $r = 1 + \cos\theta$ **56.** $r = \dfrac{4}{1 + 2\sin\theta}$

57. $r = 2\sec\theta$ **58.** $r = 2 - \cos\theta$

59. $\sec\theta = 2$ **60.** $\cos 2\theta = 1$

Discovery • Discussion

61. The Distance Formula in Polar Coordinates

(a) Use the Law of Cosines to prove that the distance between the polar points (r_1, θ_1) and (r_2, θ_2) is

$$d = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_2 - \theta_1)}$$

(b) Find the distance between the points whose polar coordinates are $(3, 3\pi/4)$ and $(1, 7\pi/6)$, using the formula from part (a).

(c) Now convert the points in part (b) to rectangular coordinates. Find the distance between them using the usual Distance Formula. Do you get the same answer?

8.2 Graphs of Polar Equations

The **graph of a polar equation** $r = f(\theta)$ consists of all points P that have at least one polar representation (r, θ) whose coordinates satisfy the equation. Many curves that arise in mathematics and its applications are more easily and naturally represented by polar equations rather than rectangular equations.

A rectangular grid is helpful for plotting points in rectangular coordinates (see Figure 1(a) on the next page). To plot points in polar coordinates, it is conven-

SUGGESTED TIME AND EMPHASIS

1 class.
Essential material.

POINTS TO STRESS

1. Graphs in polar coordinates.
2. Tests for symmetry.
3. Using graphing devices to obtain polar graphs.

DRILL QUESTION

What is the polar equation of a circle with radius 3 centered at the origin?

Answer

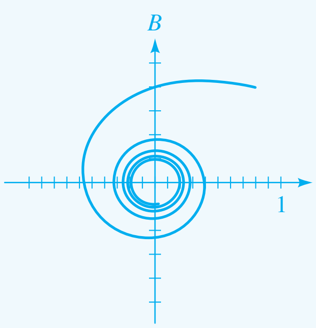
$r = 3$

ALTERNATE EXAMPLE 1

Sketch the graph of the equation

$r = \dfrac{1}{\sqrt{\theta}}$.

ANSWER



ient to use a grid consisting of circles centered at the pole and rays emanating from the pole, as in Figure 1(b). We will use such grids to help us sketch polar graphs.

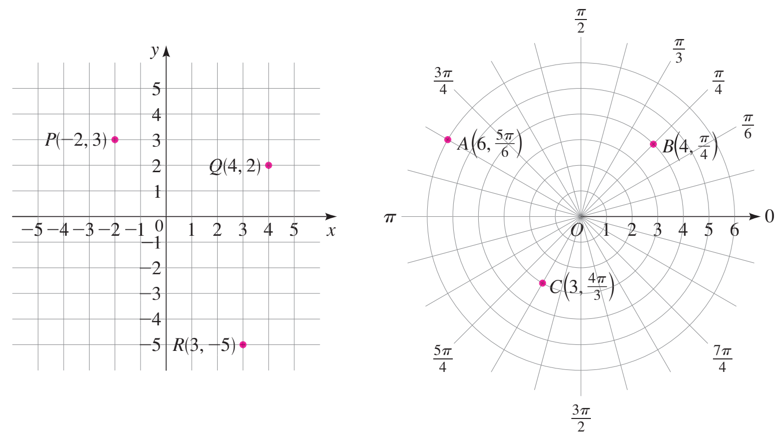


Figure 1 (a) Grid for rectangular coordinates (b) Grid for polar coordinates

In Examples 1 and 2 we see that circles centered at the origin and lines that pass through the origin have particularly simple equations in polar coordinates.

Example 1 Sketching the Graph of a Polar Equation

Sketch the graph of the equation $r = 3$ and express the equation in rectangular coordinates.

Solution The graph consists of all points whose r-coordinate is 3, that is, all points that are 3 units away from the origin. So the graph is a circle of radius 3 centered at the origin, as shown in Figure 2.

Squaring both sides of the equation, we get

$$r^2 = 3^2 \qquad \text{Square both sides}$$

$$x^2 + y^2 = 9 \qquad \text{Substitute } r^2 = x^2 + y^2$$

So the equivalent equation in rectangular coordinates is $x^2 + y^2 = 9$. ■

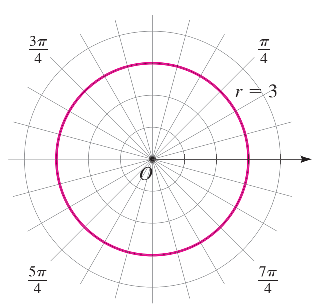


Figure 2

In general, the graph of the equation $r = a$ is a circle of radius $|a|$ centered at the origin. Squaring both sides of this equation, we see that the equivalent equation in rectangular coordinates is $x^2 + y^2 = a^2$.

Example 2 Sketching the Graph of a Polar Equation

Sketch the graph of the equation $\theta = \pi/3$ and express the equation in rectangular coordinates.

Solution The graph consists of all points whose θ-coordinate is $\pi/3$. This is the straight line that passes through the origin and makes an angle of $\pi/3$ with the polar

IN-CLASS MATERIALS

After showing students how to graph polar functions on a calculator, give them a chance to experiment and try to come up with interesting-looking polar graphs. Make sure they know that if their graphs have a lot of cusps, it could be that their Δt is set too large.

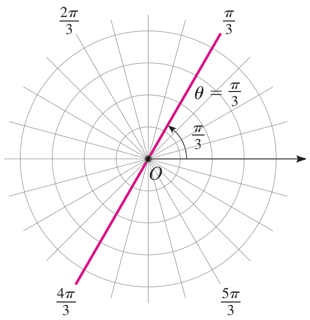


Figure 3

axis (see Figure 3). Note that the points $(r, \pi/3)$ on the line with $r > 0$ lie in quadrant I, whereas those with $r < 0$ lie in quadrant III. If the point (x, y) lies on this line, then

$$\frac{y}{x} = \tan\theta = \tan\frac{\pi}{3} = \sqrt{3}$$

Thus, the rectangular equation of this line is $y = \sqrt{3}\,x$. ■

To sketch a polar curve whose graph isn't as obvious as the ones in the preceding examples, we plot points calculated for sufficiently many values of θ and then join them in a continuous curve. (This is what we did when we first learned to graph functions in rectangular coordinates.)

Example 3 Sketching the Graph of a Polar Equation

Sketch the graph of the polar equation $r = 2\sin\theta$.

Solution We first use the equation to determine the polar coordinates of several points on the curve. The results are shown in the following table.

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$	π
$r = 2\sin\theta$	0	1	$\sqrt{2}$	$\sqrt{3}$	2	$\sqrt{3}$	$\sqrt{2}$	1	0

We plot these points in Figure 4 and then join them to sketch the curve. The graph appears to be a circle. We have used values of θ only between 0 and π, since the same points (this time expressed with negative r-coordinates) would be obtained if we allowed θ to range from π to 2π.

The polar equation $r = 2\sin\theta$ in rectangular coordinates is

$$x^2 + (y - 1)^2 = 1$$

(See Section 8.1, Example 6(b)). From the rectangular form of the equation we see that the graph is a circle of radius 1 centered at $(0, 1)$.

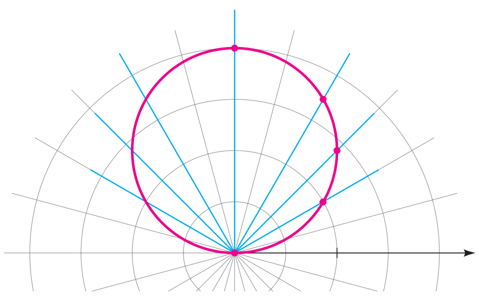


Figure 4
$r = 2\sin\theta$ ■

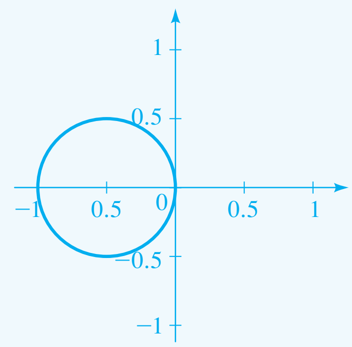
In general, the graphs of equations of the form

$$r = 2a\sin\theta \qquad \text{and} \qquad r = 2a\cos\theta$$

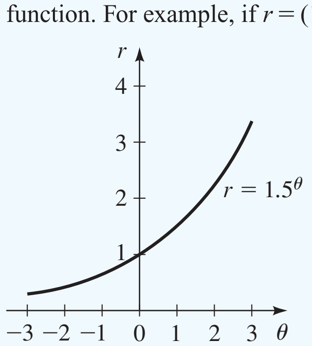
are circles with radius $|a|$ centered at the points with polar coordinates $(a, \pi/2)$ and $(a, 0)$, respectively.

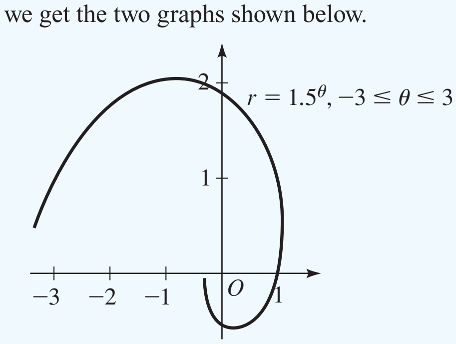
ALTERNATE EXAMPLE 3
Sketch the graph of the polar equation $r = -\cos\theta$ by plotting points as done in the text.

ANSWER



EXAMPLE
Do a problem where $r = f(\theta)$ is graphed both as a function in rectangular coordinates and as a polar function. For example, if $r = (1.5)^\theta$ we get the two graphs shown below.

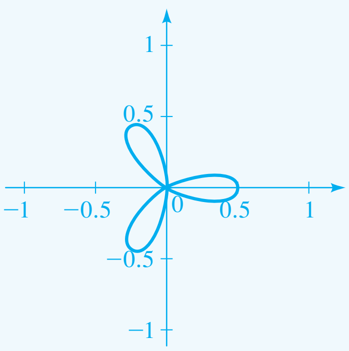




ALTERNATE EXAMPLE 4

Sketch the graph of $r = \dfrac{\cos 3\theta}{2}$.

ANSWER



Example 4 Sketching the Graph of a Polar Equation

Sketch the graph of $r = 2 + 2\cos\theta$.

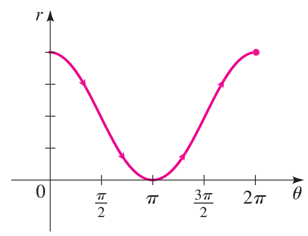


Figure 5
$r = 2 + 2\cos\theta$

Solution Instead of plotting points as in Example 3, we first sketch the graph of $r = 2 + 2\cos\theta$ in *rectangular* coordinates in Figure 5. We can think of this graph as a table of values that enables us to read at a glance the values of r that correspond to increasing values of θ. For instance, we see that as θ increases from 0 to $\pi/2$, r (the distance from O) decreases from 4 to 2, so we sketch the corresponding part of the polar graph in Figure 6(a). As θ increases from $\pi/2$ to π, Figure 5 shows that r decreases from 2 to 0, so we sketch the next part of the graph as in Figure 6(b). As θ increases from π to $3\pi/2$, r increases from 0 to 2, as shown in part (c). Finally, as θ increases from $3\pi/2$ to 2π, r increases from 2 to 4, as shown in part (d). If we let θ increase beyond 2π or decrease beyond 0, we would simply retrace our path. Combining the portions of the graph from parts (a) through (d) of Figure 6, we sketch the complete graph in part (e).

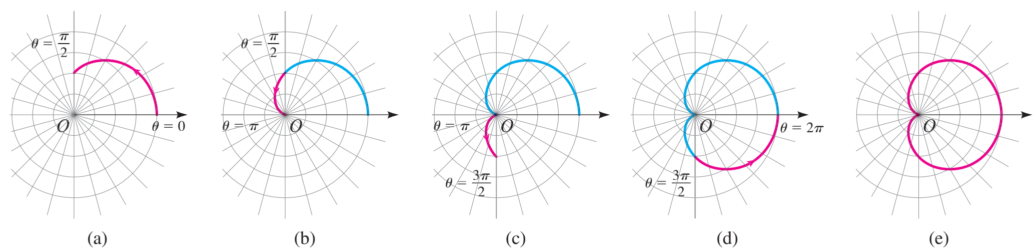


Figure 6 Steps in sketching $r = 2 + 2\cos\theta$ ■

The polar equation $r = 2 + 2\cos\theta$ in rectangular coordinates is

$$(x^2 + y^2 - 2x)^2 = 4(x^2 + y^2)$$

(See Section 8.1, Example 6(c)). The simpler form of the polar equation shows that it is more natural to describe cardioids using polar coordinates.

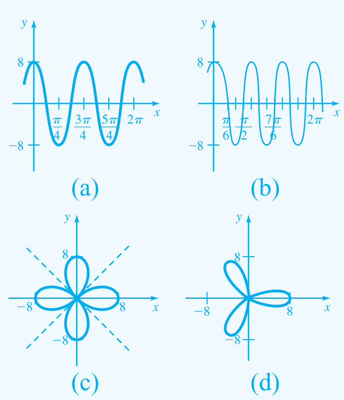
The curve in Figure 6 is called a **cardioid** because it is heart-shaped. In general, the graph of any equation of the form

$$r = a(1 \pm \cos\theta) \qquad \text{or} \qquad r = a(1 \pm \sin\theta)$$

is a cardioid.

ALTERNATE EXAMPLE 5

One of the figures below represents the graph of the curve $r = 8\cos 2\theta$.



ANSWER
(c)

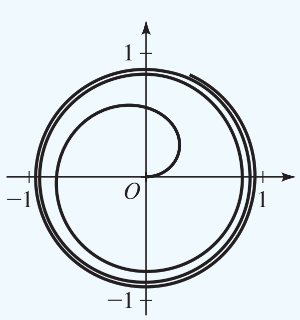
Example 5 Sketching the Graph of a Polar Equation

Sketch the curve $r = \cos 2\theta$.

Solution As in Example 4, we first sketch the graph of $r = \cos 2\theta$ in *rectangular* coordinates, as shown in Figure 7. As θ increases from 0 to $\pi/4$, Figure 7 shows that r decreases from 1 to 0, and so we draw the corresponding portion of the polar curve in Figure 8 (indicated by ①). As θ increases from $\pi/4$ to $\pi/2$, the value of r goes from 0 to -1. This means that the distance from the origin increases from 0 to 1, but instead of being in quadrant I, this portion of the polar curve (indicated by ②) lies on the opposite side of the origin in quadrant III. The remainder of the curve is drawn in a similar fashion, with the arrows and numbers indicating the order in

IN-CLASS MATERIALS

An interesting graph to look at is $r = \cos\dfrac{\theta}{2}$.



Have the class try to figure out if, as $\theta \to \infty$, the curve will get infinitely close to the curve $r = 1$, and if so, why. If this curve is combined with the circle $r = 1$, the resulting set of points is what topologists call "connected, but not path-connected." The set of points is not path-connected because there is no path from the origin that touches the outer circle. It is called "connected" because (to simplify things somewhat) there is no curve that separates the two components without touching either."

which the portions are traced out. The resulting curve has four petals and is called a **four-leaved rose**.

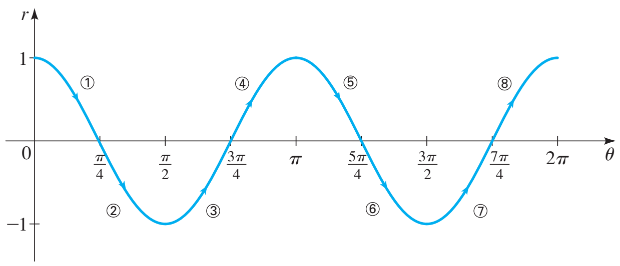


Figure 7
Graph of $r = \cos 2\theta$ sketched in rectangular coordinates

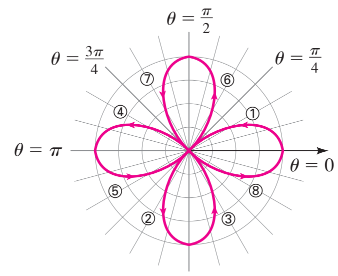


Figure 8
Four-leaved rose $r = \cos 2\theta$ sketched in polar coordinates ■

In general, the graph of an equation of the form

$$r = a \cos n\theta \qquad \text{or} \qquad r = a \sin n\theta$$

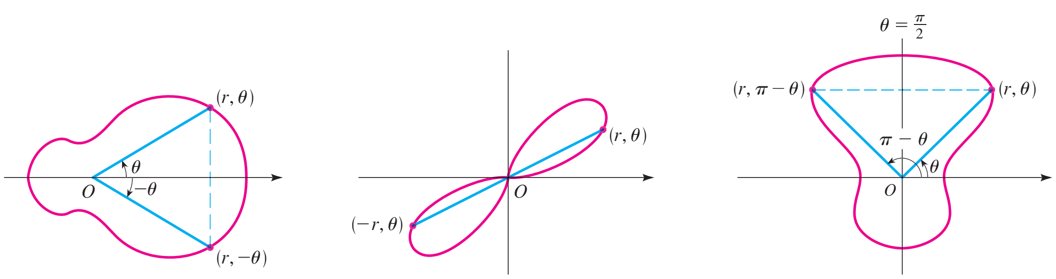
is an ***n*-leaved rose** if n is odd or a $2n$-leaved rose if n is even (as in Example 5).

Symmetry

When graphing a polar equation, it's often helpful to take advantage of symmetry. We list three tests for symmetry; Figure 9 shows why these tests work.

Tests for Symmetry

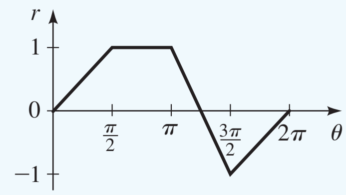
1. If a polar equation is unchanged when we replace θ by $-\theta$, then the graph is symmetric about the polar axis (Figure 9(a)).
2. If the equation is unchanged when we replace r by $-r$, then the graph is symmetric about the pole (Figure 9(b)).
3. If the equation is unchanged when we replace θ by $\pi - \theta$, the graph is symmetric about the vertical line $\theta = \pi/2$ (the y-axis) (Figure 9(c)).



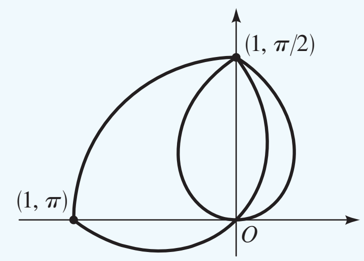
(a) Symmetry about the polar axis (b) Symmetry about the pole (c) Symmetry about the line $\theta = \frac{\pi}{2}$

Figure 9

EXAMPLE
Sketch a graph of the polar curve $r = f(\theta)$ where $f(\theta)$ is the function whose representation in rectangular coordinates is given below.

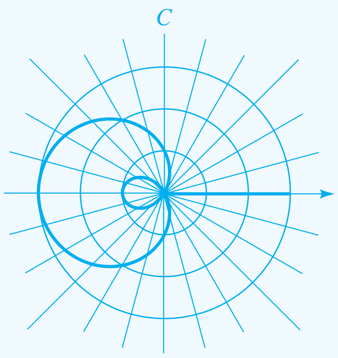


ANSWER



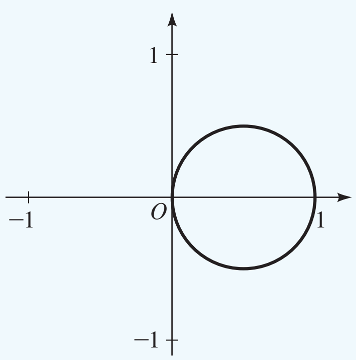
ALTERNATE EXAMPLE 6
Sketch the graph of the equation $r = 1 - 2\cos\theta$.

ANSWER



SAMPLE QUESTION
Text Question
Sketch the graph of $r = \cos\theta$.

Answer



The graphs in Figures 2, 6(e), and 8 are symmetric about the polar axis. The graph in Figure 8 is also symmetric about the pole. Figures 4 and 8 show graphs that are symmetric about $\theta = \pi/2$. Note that the four-leaved rose in Figure 8 meets all three tests for symmetry.

In rectangular coordinates, the zeros of the function $y = f(x)$ correspond to the x-intercepts of the graph. In polar coordinates, the zeros of the function $r = f(\theta)$ are the angles θ at which the curve crosses the pole. The zeros help us sketch the graph, as illustrated in the next example.

Example 6 Using Symmetry to Sketch a Polar Graph

Sketch the graph of the equation $r = 1 + 2\cos\theta$.

Solution We use the following as aids in sketching the graph.

- **Symmetry** Since the equation is unchanged when θ is replaced by $-\theta$, the graph is symmetric about the polar axis.
- **Zeros** To find the zeros, we solve

$$0 = 1 + 2\cos\theta$$
$$\cos\theta = -\frac{1}{2}$$
$$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$$

- **Table of values** As in Example 4, we sketch the graph of $r = 1 + 2\cos\theta$ in *rectangular* coordinates to serve as a table of values (Figure 10).

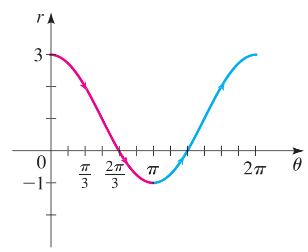


Figure 10

Now we sketch the polar graph of $r = 1 + 2\cos\theta$ from $\theta = 0$ to $\theta = \pi$, and then use symmetry to complete the graph in Figure 11. ■

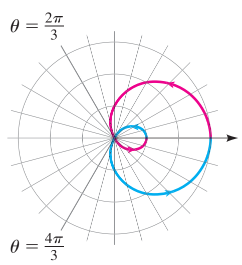


Figure 11
$r = 1 + 2\cos\theta$

The curve in Figure 11 is called a **limaçon**, after the Middle French word for snail. In general, the graph of an equation of the form

$$r = a \pm b\cos\theta \quad \text{or} \quad r = a \pm b\sin\theta$$

is a limaçon. The shape of the limaçon depends on the relative size of a and b (see the table on page 594).

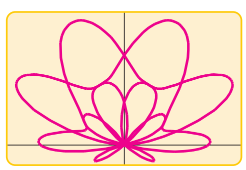


Figure 12
$r = \sin\theta + \sin^3(5\theta/2)$

Graphing Polar Equations with Graphing Devices

Although it's useful to be able to sketch simple polar graphs by hand, we need a graphing calculator or computer when the graph is as complicated as the one in Figure 12. Fortunately, most graphing calculators are capable of graphing polar equations directly.

Example 7 Drawing the Graph of a Polar Equation

Graph the equation $r = \cos(2\theta/3)$.

Solution We need to determine the domain for θ. So we ask ourselves: How many complete rotations are required before the graph starts to repeat itself? The graph repeats itself when the same value of r is obtained at θ and $\theta + 2n\pi$. Thus, we need to find an integer n, so that

$$\cos\frac{2(\theta + 2n\pi)}{3} = \cos\frac{2\theta}{3}$$

For this equality to hold, $4n\pi/3$ must be a multiple of 2π, and this first happens when $n = 3$. Therefore, we obtain the entire graph if we choose values of θ between $\theta = 0$ and $\theta = 0 + 2(3)\pi = 6\pi$. The graph is shown in Figure 13.

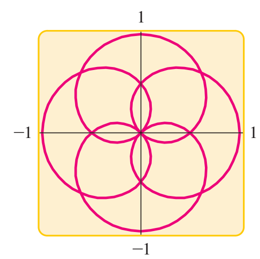


Figure 13
$r = \cos(2\theta/3)$

■

Example 8 A Family of Polar Equations

Graph the family of polar equations $r = 1 + c \sin\theta$ for $c = 3, 2.5, 2, 1.5, 1$. How does the shape of the graph change as c changes?

Solution Figure 14 shows computer-drawn graphs for the given values of c. For $c > 1$, the graph has an inner loop; the loop decreases in size as c decreases. When $c = 1$, the loop disappears and the graph becomes a cardioid (see Example 4).

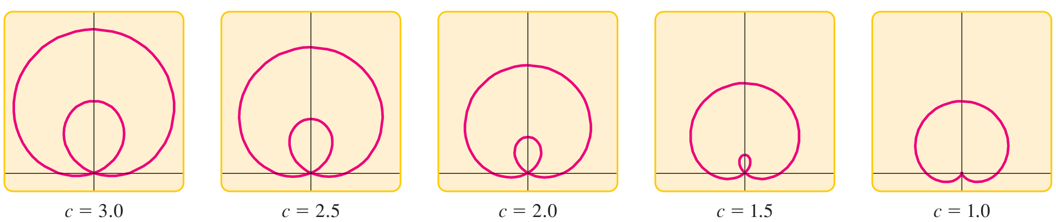


Figure 14 A family of limaçons $r = 1 + c \sin\theta$ in the viewing rectangle $[-2.5, 2.5]$ by $[-0.5, 4.5]$ ■

The following box gives a summary of some of the basic polar graphs used in calculus.

ALTERNATE EXAMPLE 7

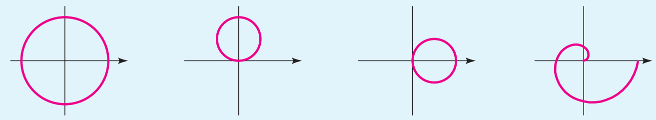
For the graph of the equation $r = \cos\dfrac{\theta}{2}$, determine the number of complete rotations required before the graph starts to repeat itself.

ANSWER

2

Some Common Polar Curves

Circles and Spiral



$r = a$ circle | $r = a\sin\theta$ circle | $r = a\cos\theta$ circle | $r = a\theta$ spiral

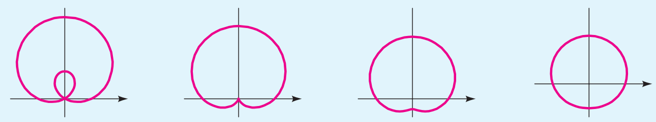
Limaçons

$r = a \pm b\sin\theta$

$r = a \pm b\cos\theta$

$(a > 0, b > 0)$

Orientation depends on the trigonometric function (sine or cosine) and the sign of b.



$a < b$ limaçon with inner loop | $a = b$ cardioid | $a > b$ dimpled limaçon | $a \geq 2b$ convex limaçon

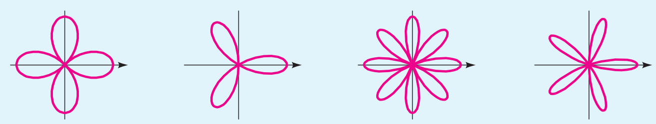
Roses

$r = a\sin n\theta$

$r = a\cos n\theta$

n-leaved if n is odd

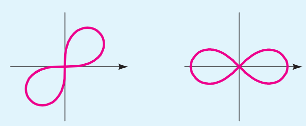
$2n$-leaved if n is even



$r = a\cos 2\theta$ 4-leaved rose | $r = a\cos 3\theta$ 3-leaved rose | $r = a\cos 4\theta$ 8-leaved rose | $r = a\cos 5\theta$ 5-leaved rose

Lemniscates

Figure-eight-shaped curves



$r^2 = a^2\sin 2\theta$ lemniscate | $r^2 = a^2\cos 2\theta$ lemniscate

8.2 Exercises

1–6 ■ Match the polar equation with the graphs labeled I–VI. Use the table above to help you.

1. $r = 3\cos\theta$

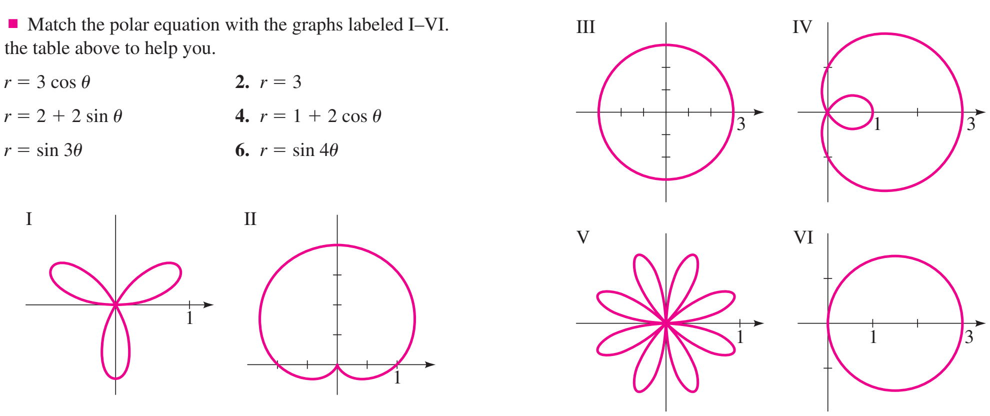
2. $r = 3$

3. $r = 2 + 2\sin\theta$

4. $r = 1 + 2\cos\theta$

5. $r = \sin 3\theta$

6. $r = \sin 4\theta$



7–14 ■ Test the polar equation for symmetry with respect to the polar axis, the pole, and the line $\theta = \pi/2$.

7. $r = 2 - \sin\theta$

8. $r = 4 + 8\cos\theta$

9. $r = 3\sec\theta$

10. $r = 5\cos\theta\ \csc\theta$

11. $r = \dfrac{4}{3 - 2\sin\theta}$

12. $r = \dfrac{5}{1 + 3\cos\theta}$

13. $r^2 = 4\cos 2\theta$

14. $r^2 = 9\sin\theta$

15–36 ■ Sketch the graph of the polar equation.

15. $r = 2$

16. $r = -1$

17. $\theta = -\pi/2$

18. $\theta = 5\pi/6$

19. $r = 6\sin\theta$

20. $r = \cos\theta$

21. $r = -2\cos\theta$

22. $r = 2\sin\theta + 2\cos\theta$

23. $r = 2 - 2\cos\theta$

24. $r = 1 + \sin\theta$

25. $r = -3(1 + \sin\theta)$

26. $r = \cos\theta - 1$

27. $r = \theta, \quad \theta \geq 0$ (spiral)

28. $r\theta = 1, \quad \theta > 0$ (reciprocal spiral)

29. $r = \sin 2\theta$ (four-leaved rose)

30. $r = 2\cos 3\theta$ (three-leaved rose)

31. $r^2 = \cos 2\theta$ (lemniscate)

32. $r^2 = 4\sin 2\theta$ (lemniscate)

33. $r = 2 + \sin\theta$ (limaçon)

34. $r = 1 - 2\cos\theta$ (limaçon)

35. $r = 2 + \sec\theta$ (conchoid)

36. $r = \sin\theta\tan\theta$ (cissoid)

37–40 ■ Use a graphing device to graph the polar equation. Choose the domain of θ to make sure you produce the entire graph.

37. $r = \cos(\theta/2)$

38. $r = \sin(8\theta/5)$

39. $r = 1 + 2\sin(\theta/2)$ (nephroid)

40. $r = \sqrt{1 - 0.8\sin^2\theta}$ (hippopede)

41. Graph the family of polar equations $r = 1 + \sin n\theta$ for $n = 1, 2, 3, 4$, and 5. How is the number of loops related to n?

42. Graph the family of polar equations $r = 1 + c\sin 2\theta$ for $c = 0.3, 0.6, 1, 1.5$, and 2. How does the graph change as c increases?

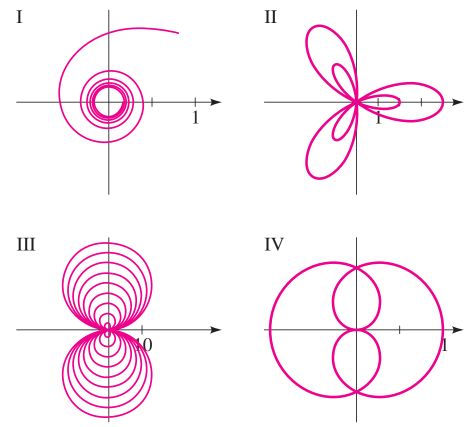
43–46 ■ Match the polar equation with the graphs labeled I–IV. Give reasons for your answers.

43. $r = \sin(\theta/2)$

44. $r = 1/\sqrt{\theta}$

45. $r = \theta\sin\theta$

46. $r = 1 + 3\cos(3\theta)$



47–50 ■ Sketch a graph of the rectangular equation. [*Hint:* First convert the equation to polar coordinates.]

47. $(x^2 + y^2)^3 = 4x^2y^2$

48. $(x^2 + y^2)^3 = (x^2 - y^2)^2$

49. $(x^2 + y^2)^2 = x^2 - y^2$

50. $x^2 + y^2 = (x^2 + y^2 - x)^2$

51. Show that the graph of $r = a\cos\theta + b\sin\theta$ is a circle, and find its center and radius.

52. **(a)** Graph the polar equation $r = \tan\theta\sec\theta$ in the viewing rectangle $[-3, 3]$ by $[-1, 9]$.

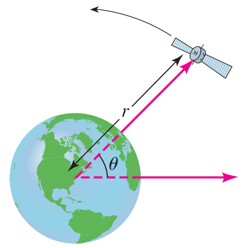
(b) Note that your graph in part (a) looks like a parabola (see Section 2.5). Confirm this by converting the equation to rectangular coordinates.

Applications

53. Orbit of a Satellite Scientists and engineers often use polar equations to model the motion of satellites in earth orbit. Let's consider a satellite whose orbit is modeled by the equation $r = 22500/(4 - \cos\theta)$, where r is the distance in miles between the satellite and the center of the earth and θ is the angle shown in the figure on the next page.

(a) On the same viewing screen, graph the circle $r = 3960$ (to represent the earth, which we will assume to be a sphere of radius 3960 mi) and the polar equation of the satellite's orbit. Describe the motion of the satellite as θ increases from 0 to 2π.

(b) For what angle θ is the satellite closest to the earth? Find the height of the satellite above the earth's surface for this value of θ.



54. An Unstable Orbit The orbit described in Exercise 53 is stable because the satellite traverses the same path over and over as θ increases. Suppose that a meteor strikes the satellite and changes its orbit to

$$r = \frac{22500\left(1 - \dfrac{\theta}{40}\right)}{4 - \cos\theta}$$

(a) On the same viewing screen, graph the circle $r = 3960$ and the new orbit equation, with θ increasing from 0 to 3π. Describe the new motion of the satellite.

(b) Use the TRACE feature on your graphing calculator to find the value of θ at the moment the satellite crashes into the earth.

Discovery • Discussion

55. A Transformation of Polar Graphs How are the graphs of $r = 1 + \sin(\theta - \pi/6)$ and $r = 1 + \sin(\theta - \pi/3)$ related to the graph of $r = 1 + \sin\theta$? In general, how is the graph of $r = f(\theta - \alpha)$ related to the graph of $r = f(\theta)$?

56. Choosing a Convenient Coordinate System Compare the polar equation of the circle $r = 2$ with its equation in rectangular coordinates. In which coordinate system is the equation simpler? Do the same for the equation of the four-leaved rose $r = \sin 2\theta$. Which coordinate system would you choose to study these curves?

57. Choosing a Convenient Coordinate System Compare the rectangular equation of the line $y = 2$ with its polar equation. In which coordinate system is the equation simpler? Which coordinate system would you choose to study lines?

SUGGESTED TIME AND EMPHASIS

$\frac{1}{2}$–1 class.

Recommended material.

8.3 Polar Form of Complex Numbers; DeMoivre's Theorem

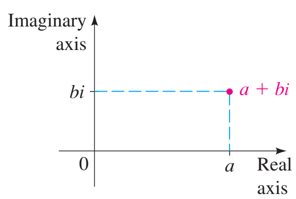


Figure 1

In this section we represent complex numbers in polar (or trigonometric) form. This enables us to find the nth roots of complex numbers. To describe the polar form of complex numbers, we must first learn to work with complex numbers graphically.

Graphing Complex Numbers

To graph real numbers or sets of real numbers, we have been using the number line, which has just one dimension. Complex numbers, however, have two components: a real part and an imaginary part. This suggests that we need two axes to graph complex numbers: one for the real part and one for the imaginary part. We call these the **real axis** and the **imaginary axis**, respectively. The plane determined by these two axes is called the **complex plane**. To graph the complex number $a + bi$, we plot the ordered pair of numbers (a, b) in this plane, as indicated in Figure 1.

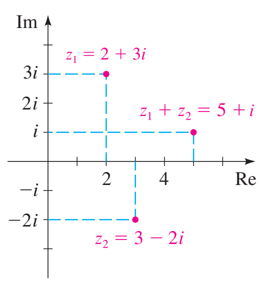
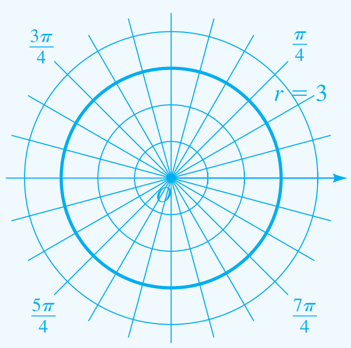


Figure 2

ALTERNATE EXAMPLE 1
Graph the complex numbers $z_1 = 3 - 4i$, $z_2 = -2i + 1$.

ANSWER



Example 1 Graphing Complex Numbers

Graph the complex numbers $z_1 = 2 + 3i$, $z_2 = 3 - 2i$, and $z_1 + z_2$.

Solution We have $z_1 + z_2 = (2 + 3i) + (3 - 2i) = 5 + i$. The graph is shown in Figure 2. ■

POINTS TO STRESS

1. Three representations of a complex number: as a point, as a number $a + bi$, and in trigonometric form.
2. Multiplication and division of complex numbers in trigonometric form.
3. DeMoivre's Theorem.

Example 2 Graphing Sets of Complex Numbers

Graph each set of complex numbers.

(a) $S = \{a + bi \mid a \geq 0\}$ (b) $T = \{a + bi \mid a < 1, b \geq 0\}$

Solution

(a) S is the set of complex numbers whose real part is nonnegative. The graph is shown in Figure 3(a).

(b) T is the set of complex numbers for which the real part is less than 1 and the imaginary part is nonnegative. The graph is shown in Figure 3(b).

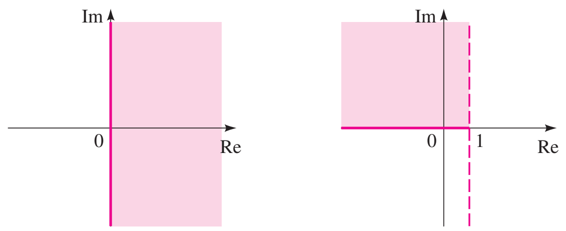


Figure 3 (a) (b) ■

Recall that the absolute value of a real number can be thought of as its distance from the origin on the real number line (see Section 1.1). We define absolute value for complex numbers in a similar fashion. Using the Pythagorean Theorem, we can see from Figure 4 that the distance between $a + bi$ and the origin in the complex plane is $\sqrt{a^2 + b^2}$. This leads to the following definition.

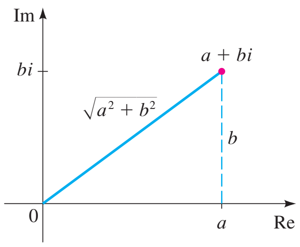


Figure 4

> The **modulus** (or **absolute value**) of the complex number $z = a + bi$ is
>
> $$|z| = \sqrt{a^2 + b^2}$$

Example 3 Calculating the Modulus

Find the moduli of the complex numbers $3 + 4i$ and $8 - 5i$.

The plural of *modulus* is *moduli*.

Solution

$$|3 + 4i| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

$$|8 - 5i| = \sqrt{8^2 + (-5)^2} = \sqrt{89}$$ ■

Example 4 Absolute Value of Complex Numbers

Graph each set of complex numbers.

(a) $C = \{z \mid |z| = 1\}$ (b) $D = \{z \mid |z| \leq 1\}$

Solution

(a) C is the set of complex numbers whose distance from the origin is 1. Thus, C is a circle of radius 1 with center at the origin, as shown in Figure 5.

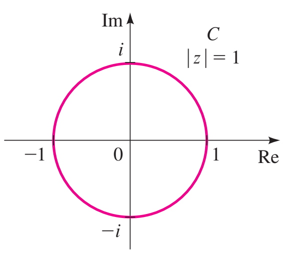
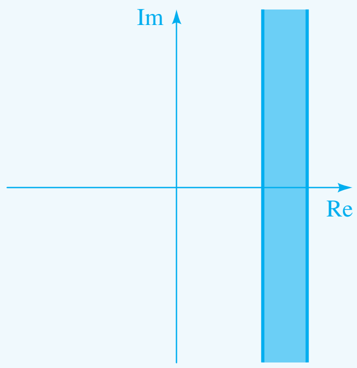


Figure 5

ALTERNATE EXAMPLE 2

Graph the set of complex numbers $S = \{a + bi \mid 2 \leq a \leq 3\}$.

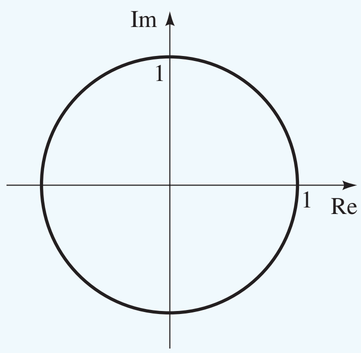


SAMPLE QUESTION

Text Question

Graph the following set of complex numbers: $\{z \mid |z| = 1\}$

Answer



ALTERNATE EXAMPLE 3

Calculate the moduli of the complex numbers $-7 + 24i$ and $40 + 9i$.

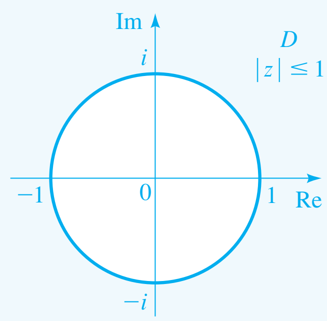
ANSWER

25, 41

ALTERNATE EXAMPLE 4

Graph the set of complex numbers $E = \{z \mid |z| \geq 1\}$.

ANSWER



EXAMPLES

Switching between forms:

- $4\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) = -2\sqrt{2} - 2\sqrt{2}i$
- $4(\cos 1 + i\sin 1) \approx 2.161 + 3.366i$
- $3(\cos 20° + i\sin 20°) \approx 2.81907786 + 1.02606043i$
- $(4 - 3i)^5 = -3116 + 237i$

DRILL QUESTION
Compute $(\sqrt{3} - i)^{10}$.

Answer
$512 + 512\sqrt{3}i$

ALTERNATE EXAMPLE 5
Write the complex number $-1 - i$ in trigonometric form.

ANSWER
$\sqrt{2}\left(\cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right)\right)$

ALTERNATE EXAMPLE 5b
Let $z_1 = 4\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$ and $z_2 = 9\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$.
Find $\frac{z_1}{z_2}$.

ANSWER
$\frac{z_1}{z_2} = \frac{4}{9}\left(\cos\left(\frac{\pi}{12}\right) - i\sin\left(\frac{\pi}{12}\right)\right)$

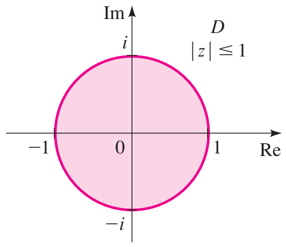


Figure 6

(b) D is the set of complex numbers whose distance from the origin is less than or equal to 1. Thus, D is the disk that consists of all complex numbers on and inside the circle C of part (a), as shown in Figure 6. ■

Polar Form of Complex Numbers

Let $z = a + bi$ be a complex number, and in the complex plane let's draw the line segment joining the origin to the point $a + bi$ (see Figure 7). The length of this line segment is $r = |z| = \sqrt{a^2 + b^2}$. If θ is an angle in standard position whose terminal side coincides with this line segment, then by the definitions of sine and cosine (see Section 6.2)

$$a = r\cos\theta \qquad \text{and} \qquad b = r\sin\theta$$

so $z = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$. We have shown the following.

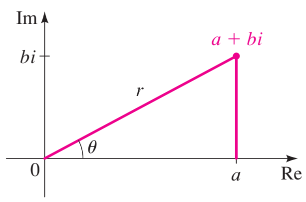


Figure 7

Polar Form of Complex Numbers

A complex number $z = a + bi$ has the **polar form** (or **trigonometric form**)

$$z = r(\cos\theta + i\sin\theta)$$

where $r = |z| = \sqrt{a^2 + b^2}$ and $\tan\theta = b/a$. The number r is the **modulus** of z, and θ is an **argument** of z.

The argument of z is not unique, but any two arguments of z differ by a multiple of 2π.

Example 5 Writing Complex Numbers in Polar Form

Write each complex number in trigonometric form.

(a) $1 + i$ (b) $-1 + \sqrt{3}i$ (c) $-4\sqrt{3} - 4i$ (d) $3 + 4i$

Solution These complex numbers are graphed in Figure 8, which helps us find their arguments.

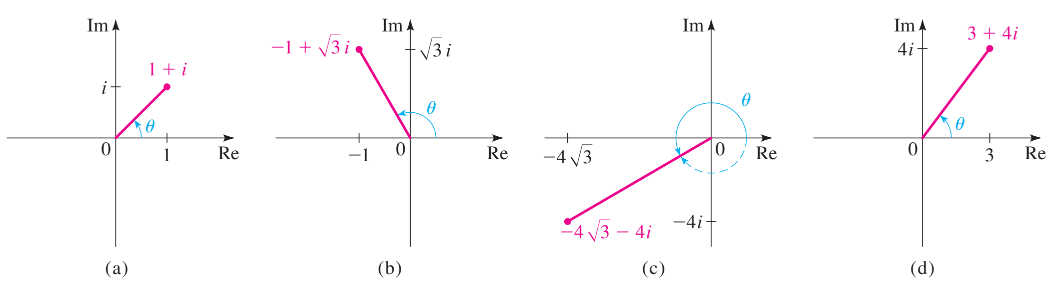


Figure 8

IN-CLASS MATERIALS

Verify, using the multiplication of complex numbers in trigonometric form, that $i^2 = -1$. This leads to another way to view complex numbers, one which is more intuitively justifiable for some than allowing negative square roots. We define a "complex number" as a point in the plane where we want to define addition in a certain way (that will parallel vector addition defined in the next section) and multiplication of quantities of the same magnitude to correspond to a rotation. The idea that the point (0, 1) times itself is (−1, 0) is then a logical consequence of this definition of multiplication. Then, if we define the point (1, 0) to be "1" and the point (0, 1) to be "i" we obtain $i^2 = -1$. This is not violating the laws of conventional mathematics, it is just a result of our definitions of operations on these point quantities. In the real world, it turns out there are quantities (such as waveforms, impedance of an AC circuit, etc.) that are most usefully represented in this way.

$\tan\theta = \frac{1}{1} = 1$
$\theta = \frac{\pi}{4}$

(a) An argument is $\theta = \pi/4$ and $r = \sqrt{1+1} = \sqrt{2}$. Thus

$$1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$\tan\theta = \frac{\sqrt{3}}{-1} = -\sqrt{3}$
$\theta = \frac{2\pi}{3}$

(b) An argument is $\theta = 2\pi/3$ and $r = \sqrt{1+3} = 2$. Thus

$$-1 + \sqrt{3}i = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$

$\tan\theta = \frac{-4}{-4\sqrt{3}} = \frac{1}{\sqrt{3}}$
$\theta = \frac{7\pi}{6}$

(c) An argument is $\theta = 7\pi/6$ (or we could use $\theta = -5\pi/6$), and $r = \sqrt{48+16} = 8$. Thus

$$-4\sqrt{3} - 4i = 8\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right)$$

$\tan\theta = \frac{4}{3}$
$\theta = \tan^{-1}\frac{4}{3}$

(d) An argument is $\theta = \tan^{-1}\frac{4}{3}$ and $r = \sqrt{3^2 + 4^2} = 5$. So

$$3 + 4i = 5\left[\cos\left(\tan^{-1}\tfrac{4}{3}\right) + i\sin\left(\tan^{-1}\tfrac{4}{3}\right)\right]$$ ■

The addition formulas for sine and cosine that we discussed in Section 7.2 greatly simplify the multiplication and division of complex numbers in polar form. The following theorem shows how.

Multiplication and Division of Complex Numbers

If the two complex numbers z_1 and z_2 have the polar forms

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \qquad \text{and} \qquad z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

then

$$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] \qquad \text{Multiplication}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)] \qquad (z_2 \neq 0) \qquad \text{Division}$$

This theorem says:

To multiply two complex numbers, multiply the moduli and add the arguments.
To divide two complex numbers, divide the moduli and subtract the arguments.

■ **Proof** To prove the multiplication formula, we simply multiply the two complex numbers.

$$\begin{aligned} z_1 z_2 &= r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2[\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)] \\ &= r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] \end{aligned}$$

In the last step we used the addition formulas for sine and cosine. ■

The proof of the division formula is left as an exercise.

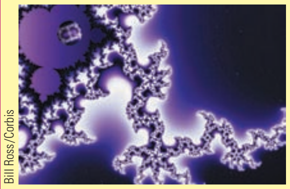
ALTERNATE EXAMPLE 6

Let $z_1 = 7\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$ and $z_2 = 8\left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)$. Find z_1z_2.

ANSWER

$$z_1z_2 = 56\left(\cos\left(\frac{11\pi}{30}\right) + i\sin\left(\frac{11\pi}{30}\right)\right)$$

Mathematics in the Modern World



Bill Ross/Corbis

Fractals

Many of the things we model in this book have regular predictable shapes. But recent advances in mathematics have made it possible to model such seemingly random or even chaotic shapes as those of a cloud, a flickering flame, a mountain, or a jagged coastline. The basic tools in this type of modeling are the fractals invented by the mathematician Benoit Mandelbrot. A *fractal* is a geometric shape built up from a simple basic shape by scaling and repeating the shape indefinitely according to a given rule. Fractals have infinite detail; this means the closer you look, the more you see. They are also *self-similar*; that is, zooming in on a portion of the fractal yields the same detail as the original shape. Because of their beautiful shapes, fractals are used by movie makers to create fictional landscapes and exotic backgrounds.

Although a fractal is a complex shape, it is produced according to very simple rules (see page 605). This property of fractals is exploited in a process of storing pictures on a computer called *fractal image compression*. In this process a picture is stored as a simple basic shape and a rule; repeating the shape according to the rule produces the original picture. This is an extremely efficient method of storage; that's how thousands of color pictures can be put on a single compact disc.

Example 6 Multiplying and Dividing Complex Numbers

Let

$$z_1 = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \quad\text{and}\quad z_2 = 5\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

Find (a) z_1z_2 and (b) z_1/z_2.

Solution

(a) By the multiplication formula

$$z_1z_2 = (2)(5)\left[\cos\left(\frac{\pi}{4} + \frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{4} + \frac{\pi}{3}\right)\right]$$
$$= 10\left(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right)$$

To approximate the answer, we use a calculator in radian mode and get

$$z_1z_2 \approx 10(-0.2588 + 0.9659i) = -2.588 + 9.659i$$

(b) By the division formula

$$\frac{z_1}{z_2} = \frac{2}{5}\left[\cos\left(\frac{\pi}{4} - \frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{4} - \frac{\pi}{3}\right)\right]$$
$$= \frac{2}{5}\left[\cos\left(-\frac{\pi}{12}\right) + i\sin\left(-\frac{\pi}{12}\right)\right]$$
$$= \frac{2}{5}\left(\cos\frac{\pi}{12} - i\sin\frac{\pi}{12}\right)$$

Using a calculator in radian mode, we get the approximate answer:

$$\frac{z_1}{z_2} \approx \tfrac{2}{5}(0.9659 - 0.2588i) = 0.3864 - 0.1035i$$ ■

DeMoivre's Theorem

Repeated use of the multiplication formula gives the following useful formula for raising a complex number to a power n for any positive integer n.

DeMoivre's Theorem

If $z = r(\cos\theta + i\sin\theta)$, then for any integer n

$$z^n = r^n(\cos n\theta + i\sin n\theta)$$

This theorem says: *To take the nth power of a complex number, we take the nth power of the modulus and multiply the argument by n.*

■ **Proof** By the multiplication formula

$$z^2 = zz = r^2[\cos(\theta + \theta) + i\sin(\theta + \theta)]$$
$$= r^2(\cos 2\theta + i\sin 2\theta)$$

Now we multiply z^2 by z to get

$$z^3 = z^2 z = r^3[\cos(2\theta + \theta) + i\sin(2\theta + \theta)]$$
$$= r^3(\cos 3\theta + i \sin 3\theta)$$

Repeating this argument, we see that for any positive integer n

$$z^n = r^n(\cos n\theta + i\sin n\theta)$$

A similar argument using the division formula shows that this also holds for negative integers. ■

Example 7 **Finding a Power Using DeMoivre's Theorem**

Find $(\frac{1}{2} + \frac{1}{2}i)^{10}$.

Solution Since $\frac{1}{2} + \frac{1}{2}i = \frac{1}{2}(1 + i)$, it follows from Example 5(a) that

$$\frac{1}{2} + \frac{1}{2}i = \frac{\sqrt{2}}{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

So by DeMoivre's Theorem,

$$\left(\frac{1}{2} + \frac{1}{2}i\right)^{10} = \left(\frac{\sqrt{2}}{2}\right)^{10}\left(\cos\frac{10\pi}{4} + i\sin\frac{10\pi}{4}\right)$$
$$= \frac{2^5}{2^{10}}\left(\cos\frac{5\pi}{2} + i\sin\frac{5\pi}{2}\right) = \frac{1}{32}i$$ ■

ALTERNATE EXAMPLE 7

Find $\left(\frac{1}{2} + \frac{1}{2}i\right)^{12}$.

ANSWER

$-\frac{1}{64}$

*n*th Roots of Complex Numbers

An ***n*th root** of a complex number z is any complex number w such that $w^n = z$. DeMoivre's Theorem gives us a method for calculating the nth roots of any complex number.

*n*th Roots of Complex Numbers

If $z = r(\cos\theta + i\sin\theta)$ and n is a positive integer, then z has the n distinct nth roots

$$w_k = r^{1/n}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right]$$

for $k = 0, 1, 2, \ldots, n - 1$.

■ **Proof** To find the nth roots of z, we need to find a complex number w such that

$$w^n = z$$

Let's write z in polar form:

$$z = r(\cos\theta + i\sin\theta)$$

One nth root of z is

$$w = r^{1/n}\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right)$$

since by DeMoivre's Theorem, $w^n = z$. But the argument θ of z can be replaced by $\theta + 2k\pi$ for any integer k. Since this expression gives a different value of w for $k = 0, 1, 2, \dots, n - 1$, we have proved the formula in the theorem. ■

The following observations help us use the preceding formula.

1. The modulus of each nth root is $r^{1/n}$.
2. The argument of the first root is θ/n.
3. We repeatedly add $2\pi/n$ to get the argument of each successive root.

These observations show that, when graphed, the nth roots of z are spaced equally on the circle of radius $r^{1/n}$.

ALTERNATE EXAMPLE 8
Find the six sixth roots of $z = -729$.

ANSWER
$\frac{3\sqrt{3}}{2} + \frac{3}{2}i, 3i, -\frac{3\sqrt{3}}{2} + \frac{3}{2}i,$
$-\frac{3\sqrt{3}}{2} - \frac{3}{2}i, -3i, \frac{3\sqrt{3}}{2} - \frac{3}{2}i$

Example 8 Finding Roots of a Complex Number

Find the six sixth roots of $z = -64$, and graph these roots in the complex plane.

Solution In polar form, $z = 64(\cos \pi + i \sin \pi)$. Applying the formula for nth roots with $n = 6$, we get

$$w_k = 64^{1/6}\left[\cos\left(\frac{\pi + 2k\pi}{6}\right) + i\sin\left(\frac{\pi + 2k\pi}{6}\right)\right]$$

for $k = 0, 1, 2, 3, 4, 5$. Using $64^{1/6} = 2$, we find that the six sixth roots of -64 are

We add $2\pi/6 = \pi/3$ to each argument to get the argument of the next root.

$$w_0 = 64^{1/6}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = \sqrt{3} + i$$

$$w_1 = 64^{1/6}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = 2i$$

$$w_2 = 64^{1/6}\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) = -\sqrt{3} + i$$

$$w_3 = 64^{1/6}\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right) = -\sqrt{3} - i$$

$$w_4 = 64^{1/6}\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right) = -2i$$

$$w_5 = 64^{1/6}\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right) = \sqrt{3} - i$$

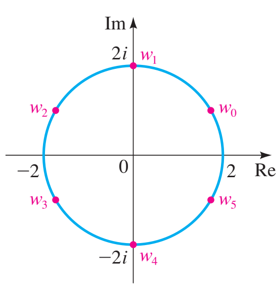


Figure 9
The six sixth roots of $z = -64$

All these points lie on a circle of radius 2, as shown in Figure 9. ■

When finding roots of complex numbers, we sometimes write the argument θ of the complex number in degrees. In this case, the nth roots are obtained from the formula

$$w_k = r^{1/n}\left[\cos\left(\frac{\theta + 360°k}{n}\right) + i\sin\left(\frac{\theta + 360°k}{n}\right)\right]$$

for $k = 0, 1, 2, \dots, n - 1$.

IN-CLASS MATERIALS

Notice that we now have three distinct ways of representing the same quantity. Each has advantages and disadvantages. For example, it is cumbersome to compute

$$3\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) + 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = \sqrt{19}\,(\cos 1.162 + i \sin 1.162)$$

but not too hard to compute

$$3i + (\sqrt{3} + i) = \sqrt{3} + 4i$$

Similarly, it is annoying to divide

$$\frac{3 - 6i}{4 + 2i} = \frac{3}{2}i$$

but it is not hard to divide

$$\frac{8(\cos 4 + i \sin 4)}{4[\cos(-1) + i\sin(-1)]} = 2(\cos 5 + i \sin 5)$$

Example 9 Finding Cube Roots of a Complex Number

Find the three cube roots of $z = 2 + 2i$, and graph these roots in the complex plane.

Solution First we write z in polar form using degrees. We have $r = \sqrt{2^2 + 2^2} = 2\sqrt{2}$ and $\theta = 45°$. Thus

$$z = 2\sqrt{2}(\cos 45° + i \sin 45°)$$

Applying the formula for nth roots (in degrees) with $n = 3$, we find the cube roots of z are of the form

$$w_k = (2\sqrt{2})^{1/3}\left[\cos\left(\frac{45° + 360°k}{3}\right) + i \sin\left(\frac{45° + 360°k}{3}\right)\right]$$

$(2\sqrt{2})^{1/3} = (2^{3/2})^{1/3} = 2^{1/2} = \sqrt{2}$

We add $360°/3 = 120°$ to each argument to get the argument of the next root.

where $k = 0, 1, 2$. Thus, the three cube roots are

$$w_0 = \sqrt{2}(\cos 15° + i \sin 15°) \approx 1.366 + 0.366i$$

$$w_1 = \sqrt{2}(\cos 135° + i \sin 135°) = -1 + i$$

$$w_2 = \sqrt{2}(\cos 255° + i \sin 255°) \approx -0.366 - 1.366i$$

The three cube roots of z are graphed in Figure 10. These roots are spaced equally on a circle of radius $\sqrt{2}$. ■

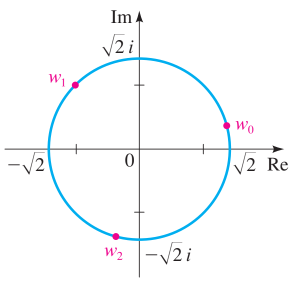


Figure 10
The three cube roots of $z = 2 + 2i$

Example 10 Solving an Equation Using the *n*th Roots Formula

Solve the equation $z^6 + 64 = 0$.

Solution This equation can be written as $z^6 = -64$. Thus, the solutions are the sixth roots of -64, which we found in Example 8. ■

8.3 Exercises

1–8 ■ Graph the complex number and find its modulus.

1. $4i$ **2.** $-3i$

3. -2 **4.** 6

5. $5 + 2i$ **6.** $7 - 3i$

7. $\sqrt{3} + i$ **8.** $-1 - \frac{\sqrt{3}}{3}i$

9. $\frac{3 + 4i}{5}$ **10.** $\frac{-\sqrt{2} + i\sqrt{2}}{2}$

11–12 ■ Sketch the complex number z, and also sketch $2z$, $-z$, and $\frac{1}{2}z$ on the same complex plane.

11. $z = 1 + i$ **12.** $z = -1 + i\sqrt{3}$

13–14 ■ Sketch the complex number z and its complex conjugate $\bar{z}$ on the same complex plane.

13. $z = 8 + 2i$ **14.** $z = -5 + 6i$

15–16 ■ Sketch z_1, z_2, $z_1 + z_2$, and $z_1 z_2$ on the same complex plane.

15. $z_1 = 2 - i, \quad z_2 = 2 + i$

16. $z_1 = -1 + i, \quad z_2 = 2 - 3i$

17–24 ■ Sketch the set in the complex plane.

17. $\{z = a + bi \mid a \le 0, b \ge 0\}$

18. $\{z = a + bi \mid a > 1, b > 1\}$

19. $\{z \mid |z| = 3\}$ **20.** $\{z \mid |z| \ge 1\}$

21. $\{z \mid |z| < 2\}$ **22.** $\{z \mid 2 \le |z| \le 5\}$

23. $\{z = a + bi \mid a + b < 2\}$

24. $\{z = a + bi \mid a \ge b\}$

ALTERNATE EXAMPLE 9
Find the three cube roots of $8i$.

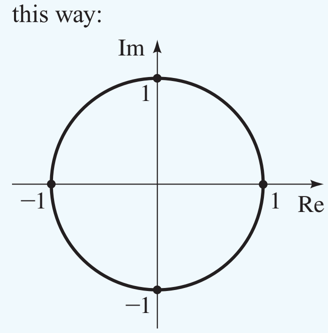
ANSWER
$-2i, -\sqrt{3} + i, \sqrt{3} + i$

ALTERNATE EXAMPLE 10
Solve the equation $z^6 + 729 = 0$.

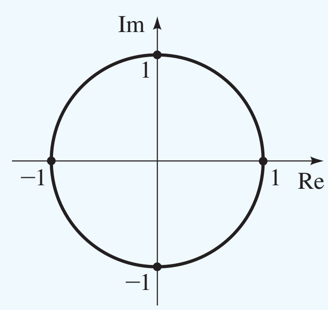
ANSWER
$\frac{3\sqrt{3}}{2} + \frac{3}{2}i, 3i, -\frac{3\sqrt{3}}{2} + \frac{3}{2}i, -\frac{3\sqrt{3}}{2} - \frac{3}{2}i, -3i, \frac{3\sqrt{3}}{2} - \frac{3}{2}i$

IN-CLASS MATERIALS

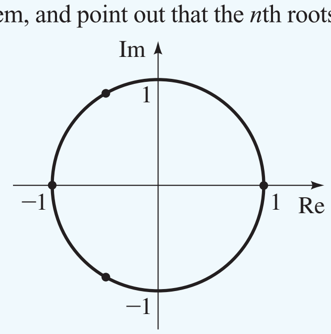
Ask the class, "What are the square roots of 1?" Elicit the answers ± 1, and draw them this way:



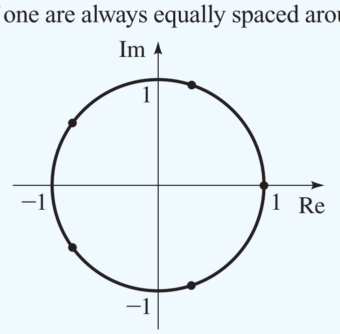
Ask about the fourth roots of 1, obtaining ± 1 and $\pm i$, and draw them as well:



Now ask about the third roots of 1 and fifth roots of 1. You will probably have to do these as an example on the board. Draw them, and point out that the nth roots of one are always equally spaced around the unit circle.



Third roots of 1



Fifth roots of 1

25–48 ■ Write the complex number in polar form with argument θ between 0 and 2π.

25. $1 + i$ **26.** $1 + \sqrt{3}\,i$ **27.** $\sqrt{2} - \sqrt{2}\,i$

28. $1 - i$ **29.** $2\sqrt{3} - 2i$ **30.** $-1 + i$

31. $-3i$ **32.** $-3 - 3\sqrt{3}\,i$ **33.** $5 + 5i$

34. 4 **35.** $4\sqrt{3} - 4i$ **36.** $8i$

37. -20 **38.** $\sqrt{3} + i$ **39.** $3 + 4i$

40. $i(2 - 2i)$ **41.** $3i(1 + i)$ **42.** $2(1 - i)$

43. $4(\sqrt{3} + i)$ **44.** $-3 - 3i$ **45.** $2 + i$

46. $3 + \sqrt{3}\,i$ **47.** $\sqrt{2} + \sqrt{2}\,i$ **48.** $-\pi i$

49–56 ■ Find the product z_1z_2 and the quotient z_1/z_2. Express your answer in polar form.

49. $z_1 = \cos\pi + i\sin\pi, \quad z_2 = \cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}$

50. $z_1 = \cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4}, \quad z_2 = \cos\dfrac{3\pi}{4} + i\sin\dfrac{3\pi}{4}$

51. $z_1 = 3\left(\cos\dfrac{\pi}{6} + i\sin\dfrac{\pi}{6}\right), \quad z_2 = 5\left(\cos\dfrac{4\pi}{3} + i\sin\dfrac{4\pi}{3}\right)$

52. $z_1 = 7\left(\cos\dfrac{9\pi}{8} + i\sin\dfrac{9\pi}{8}\right), \quad z_2 = 2\left(\cos\dfrac{\pi}{8} + i\sin\dfrac{\pi}{8}\right)$

53. $z_1 = 4(\cos 120° + i\sin 120°)$,
$z_2 = 2(\cos 30° + i\sin 30°)$

54. $z_1 = \sqrt{2}(\cos 75° + i\sin 75°)$,
$z_2 = 3\sqrt{2}(\cos 60° + i\sin 60°)$

55. $z_1 = 4(\cos 200° + i\sin 200°)$,
$z_2 = 25(\cos 150° + i\sin 150°)$

56. $z_1 = \frac{4}{5}(\cos 25° + i\sin 25°)$,
$z_2 = \frac{1}{5}(\cos 155° + i\sin 155°)$

57–64 ■ Write z_1 and z_2 in polar form, and then find the product z_1z_2 and the quotients z_1/z_2 and $1/z_1$.

57. $z_1 = \sqrt{3} + i, \quad z_2 = 1 + \sqrt{3}\,i$

58. $z_1 = \sqrt{2} - \sqrt{2}\,i, \quad z_2 = 1 - i$

59. $z_1 = 2\sqrt{3} - 2i, \quad z_2 = -1 + i$

60. $z_1 = -\sqrt{2}\,i, \quad z_2 = -3 - 3\sqrt{3}\,i$

61. $z_1 = 5 + 5i, \quad z_2 = 4$ **62.** $z_1 = 4\sqrt{3} - 4i, \quad z_2 = 8i$

63. $z_1 = -20, \quad z_2 = \sqrt{3} + i$ **64.** $z_1 = 3 + 4i, \quad z_2 = 2 - 2i$

65–76 ■ Find the indicated power using DeMoivre's Theorem.

65. $(1 + i)^{20}$ **66.** $(1 - \sqrt{3}\,i)^5$

67. $(2\sqrt{3} + 2i)^5$ **68.** $(1 - i)^8$

69. $\left(\dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{2}}{2}i\right)^{12}$ **70.** $(\sqrt{3} - i)^{-10}$

71. $(2 - 2i)^8$ **72.** $\left(-\dfrac{1}{2} - \dfrac{\sqrt{3}}{2}i\right)^{15}$

73. $(-1 - i)^7$ **74.** $(3 + \sqrt{3}\,i)^4$

75. $(2\sqrt{3} + 2i)^{-5}$ **76.** $(1 - i)^{-8}$

77–86 ■ Find the indicated roots, and graph the roots in the complex plane.

77. The square roots of $4\sqrt{3} + 4i$

78. The cube roots of $4\sqrt{3} + 4i$

79. The fourth roots of $-81i$ **80.** The fifth roots of 32

81. The eighth roots of 1 **82.** The cube roots of $1 + i$

83. The cube roots of i **84.** The fifth roots of i

85. The fourth roots of -1

86. The fifth roots of $-16 - 16\sqrt{3}i$

87–92 ■ Solve the equation.

87. $z^4 + 1 = 0$ **88.** $z^8 - i = 0$

89. $z^3 - 4\sqrt{3} - 4i = 0$ **90.** $z^6 - 1 = 0$

91. $z^3 + 1 = -i$ **92.** $z^3 - 1 = 0$

93. **(a)** Let $w = \cos\dfrac{2\pi}{n} + i\sin\dfrac{2\pi}{n}$ where n is a positive integer. Show that $1, w, w^2, w^3, \ldots, w^{n-1}$ are the n distinct nth roots of 1.

(b) If $z \neq 0$ is any complex number and $s^n = z$, show that the n distinct nth roots of z are

$$s,\ sw,\ sw^2,\ sw^3,\ \ldots,\ sw^{n-1}$$

Discovery • Discussion

94. Sums of Roots of Unity Find the exact values of all three cube roots of 1 (see Exercise 93) and then add them. Do the same for the fourth, fifth, sixth, and eighth roots of 1. What do you think is the sum of the nth roots of 1, for any n?

95. Products of Roots of Unity Find the product of the three cube roots of 1 (see Exercise 93). Do the same for the fourth, fifth, sixth, and eighth roots of 1. What do you think is the product of the nth roots of 1, for any n?

96. Complex Coefficients and the Quadratic Formula The quadratic formula works whether the coefficients of the equation are real or complex. Solve these equations using the quadratic formula, and, if necessary, DeMoivre's Theorem.

(a) $z^2 + (1 + i)z + i = 0$

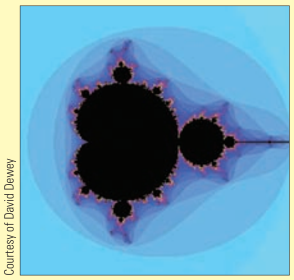
(b) $z^2 - iz + 1 = 0$

(c) $z^2 - (2 - i)z - \frac{1}{4}i = 0$

DISCOVERY PROJECT

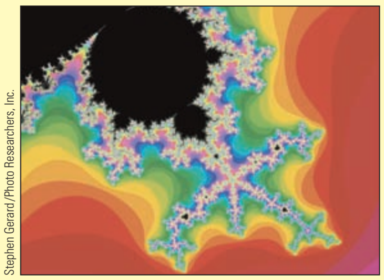
Fractals

Fractals are geometric objects that exhibit more and more detail the more we magnify them (see *Mathematics in the Modern World* on page 600). Many fractals can be described by iterating functions of complex numbers. The most famous such fractal is illustrated in Figures 1 and 2. It is called the *Mandelbrot set*, named after Benoit Mandelbrot, the mathematician who discovered it in the 1950s.



Courtesy of David Dewey

Figure 1
The Mandelbrot set



Stephen Gerard/Photo Researchers, Inc.

Figure 2
Detail from the Mandelbrot set

Here is how the Mandelbrot set is defined. Choose a complex number c, and define the complex quadratic function

$$f(z) = z^2 + c$$

Starting with $z_0 = 0$, we form the iterates of f as follows:

$$\begin{aligned} z_1 &= f(0) = c \\ z_2 &= f(f(0)) = f(c) = c^2 + c \\ z_3 &= f(f(f(0))) = f(c^2 + c) = (c^2 + c)^2 + c \\ &\vdots \end{aligned}$$

See page 597 for the definition of *modulus* (plural *moduli*).

As we continue calculating the iterates, one of two things will happen, depending on the value of c. Either the iterates $z_0, z_1, z_2, z_3, \ldots$ form a bounded set (that is, the moduli of the iterates are all less than some fixed number K), or else they eventually grow larger and larger without bound. The calculations in the table on page 606 show that for $c = 0.1 + 0.2i$, the iterates eventually stabilize at about $0.05 + 0.22i$, whereas for $c = 1 - i$, the iterates quickly become so large that a calculator can't handle them.

You can use your calculator to find the iterates, just like in the Discovery Project on page 233. With the TI-83, first put the calculator into `a+bi` mode. Then press the `Y=` key and enter the function `Y1=X²+C`. Now if $c = 1 + i$, for instance, enter the following commands:

```
1-i→C
0→X
Y1→X
```

Press the `ENTER` key repeatedly to get the list of iterates. (With this value of c, you should end up with the values in the right-hand column of the table.)

$f(z) = z^2 + 0.1 + 0.2i$	$f(z) = z^2 + 1 - i$
$z_1 = f(z_0) = .1 + .2i$	$z_1 = f(z_0) = 1 - i$
$z_2 = f(z_1) = .07 + .24i$	$z_2 = f(z_1) = 1 - 3i$
$z_3 = f(z_2) = .047 + .234i$	$z_3 = f(z_2) = -7 - 7i$
$z_4 = f(z_3) = .048 + .222i$	$z_4 = f(z_3) = 1 + 97i$
$z_5 = f(z_4) = .053 + .221i$	$z_5 = f(z_4) = -9407 + 193i$
$z_6 = f(z_5) = .054 + .223i$	$z_6 = f(z_5) = 88454401 - 3631103i$
$z_7 = f(z_6) = .053 + .224i$	$z_7 = f(z_6) = 7.8 \times 10^{15} - 6.4 \times 10^{14}i$

The **Mandelbrot set** consists of those complex numbers c for which the iterates of $f(z) = z^2 + c$ are bounded. (In fact, for this function it turns out that if the iterates are bounded, the moduli of all the iterates will be less than $K = 2$.) The numbers c that belong to the Mandelbrot set can be graphed in the complex plane. The result is the black part in Figure 1. The points not in the Mandelbrot set are assigned colors depending on how quickly the iterates become unbounded.

Figure 3

The TI-83 program below draws a rough graph of the Mandelbrot set. The program takes a long time to finish, even though it performs only 10 iterations for each c. For some values of c, you actually have to do many more iterations to tell whether the iterates are unbounded. (See, for instance, Problem 1(f) below.) That's why the program produces only a rough graph. But the calculator output in Figure 3 is actually a good approximation.

```
PROGRAM:MANDLBRT
:ClrDraw
:AxesOff
:(Xmax-Xmin)/94→H
:(Ymax-Ymin)/62→V
:For(I,0,93)
:For(J,0,61)
:Xmin+I*H→X
:Ymin+J*V→Y
:X+Yi→C
:0→Z
:For(N,1,10)
:If abs(Z)≤2
:Z²+C→Z
:End
:If abs(Z)≤2
:Pt-On(real(C),imag(C))
:DispGraph
:End
:End
:StorePic 1
```

Use the viewing rectangle $[-2, 1]$ by $[-1, 1]$ and make sure the calculator is in "`a+bi`" mode

`H` is the horizontal width of one pixel

`V` is the vertical height of one pixel

These two "`For`" loops find the complex number associated with each pixel on the screen

This "`For`" loop calculates 10 iterates, but stops iterating if `Z` has modulus larger than 2

If the iterates have modulus less than or equal to 2, the point `C` is plotted

This stores the final image under "1" so that it can be recalled later

1. Use your calculator as described in the margin on page 606 to decide whether the complex number c is in the Mandelbrot set. (For part (f), calculate at least 60 iterates.)

(a) $c = 1$ (b) $c = -1$

(c) $c = -0.7 + 0.15i$ (d) $c = 0.5 + 0.5i$

(e) $c = i$ (f) $c = -1.0404 + 0.2509i$

2. Use the `MANDLBRT` program with a smaller viewing rectangle to zoom in on a portion of the Mandelbrot set near its edge. (Store the final image in a different location if you want to keep the complete Mandelbrot picture in "1.") Do you see more detail?

3. (a) Write a calculator program that takes as input a complex number c, iterates the function $f(z) = z^2 + c$ a hundred times, and then gives the following output:

- "`UNBOUNDED AT` N", if z_N is the first iterate whose modulus is greater than 2
- "`BOUNDED`" if each iterate from z_1 to z_{100} has modulus less than or equal to 2

In the first case, the number c is not in the Mandelbrot set, and the index N tells us how "quickly" the iterates become unbounded. In the second case, it is likely that c is in the Mandelbrot set.

(b) Use your program to test each of the numbers in Problem 1.

(c) Choose other complex numbers and use your program to test them.

8.4 Vectors

SUGGESTED TIME AND EMPHASIS
1 class.
Essential material.

In applications of mathematics, certain quantities are determined completely by their magnitude—for example, length, mass, area, temperature, and energy. We speak of a length of 5 m or a mass of 3 kg; only one number is needed to describe each of these quantities. Such a quantity is called a **scalar**.

On the other hand, to describe the displacement of an object, two numbers are required: the *magnitude* and the *direction* of the displacement. To describe the velocity of a moving object, we must specify both the *speed* and the *direction* of travel. Quantities such as displacement, velocity, acceleration, and force that involve magnitude as well as direction are called *directed quantities*. One way to represent such quantities mathematically is through the use of *vectors*.

Geometric Description of Vectors

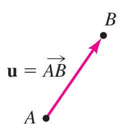


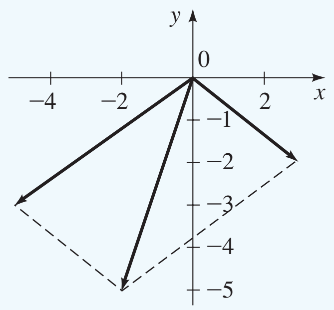
Figure 1

A **vector** in the plane is a line segment with an assigned direction. We sketch a vector as shown in Figure 1 with an arrow to specify the direction. We denote this vector by $\overrightarrow{AB}$. Point A is the **initial point**, and B is the **terminal point** of the vector

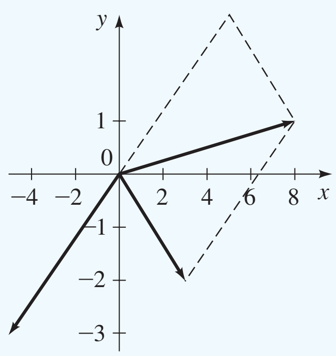
POINTS TO STRESS
1. The difference between a vector and a scalar.
2. Translational invariance of vectors, including the concept of standard position.
3. The geometric interpretation of vector addition and scalar multiplication.
4. The magnitude of a vector, its direction, and the resolution of a vector into its components.

EXAMPLES

Geometric and algebraic addition and subtraction:



$\langle 3, -2\rangle - \langle -5, -3\rangle = \langle -2, -5\rangle$



$\langle 3, -2\rangle - \langle -5, -3\rangle = \langle 8, 1\rangle$

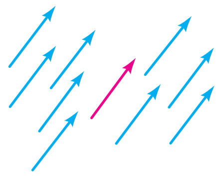


Figure 2

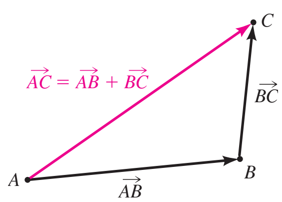


Figure 3

$\overrightarrow{AB}$. The length of the line segment AB is called the **magnitude** or **length** of the vector and is denoted by $|\overrightarrow{AB}|$. We use boldface letters to denote vectors. Thus, we write $\mathbf{u} = \overrightarrow{AB}$.

Two vectors are considered **equal** if they have equal magnitude and the same direction. Thus, all the vectors in Figure 2 are equal. This definition of equality makes sense if we think of a vector as representing a displacement. Two such displacements are the same if they have equal magnitudes and the same direction. So the vectors in Figure 2 can be thought of as the *same* displacement applied to objects in different locations in the plane.

If the displacement $\mathbf{u} = \overrightarrow{AB}$ is followed by the displacement $\mathbf{v} = \overrightarrow{BC}$, then the resulting displacement is $\overrightarrow{AC}$ as shown in Figure 3. In other words, the single displacement represented by the vector $\overrightarrow{AC}$ has the same effect as the other two displacements together. We call the vector $\overrightarrow{AC}$ the **sum** of the vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ and we write $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$. (The **zero vector**, denoted by $\mathbf{0}$, represents no displacement.) Thus, to find the sum of any two vectors $\mathbf{u}$ and $\mathbf{v}$, we sketch vectors equal to $\mathbf{u}$ and $\mathbf{v}$ with the initial point of one at the terminal point of the other (see Figure 4(a)). If we draw $\mathbf{u}$ and $\mathbf{v}$ starting at the same point, then $\mathbf{u} + \mathbf{v}$ is the vector that is the diagonal of the parallelogram formed by $\mathbf{u}$ and $\mathbf{v}$, as shown in Figure 4(b).

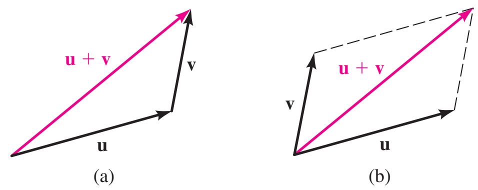


Figure 4
Addition of vectors

If a is a real number and $\mathbf{v}$ is a vector, we define a new vector $a\mathbf{v}$ as follows: The vector $a\mathbf{v}$ has magnitude $|a||\mathbf{v}|$ and has the same direction as $\mathbf{v}$ if $a > 0$, or the opposite direction if $a < 0$. If $a = 0$, then $a\mathbf{v} = \mathbf{0}$, the zero vector. This process is called **multiplication of a vector by a scalar**. Multiplying a vector by a scalar has the effect of stretching or shrinking the vector. Figure 5 shows graphs of the vector $a\mathbf{v}$ for different values of a. We write the vector $(-1)\mathbf{v}$ as $-\mathbf{v}$. Thus, $-\mathbf{v}$ is the vector with the same length as $\mathbf{v}$ but with the opposite direction.

The **difference** of two vectors $\mathbf{u}$ and $\mathbf{v}$ is defined by $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$. Figure 6 shows that the vector $\mathbf{u} - \mathbf{v}$ is the other diagonal of the parallelogram formed by $\mathbf{u}$ and $\mathbf{v}$.

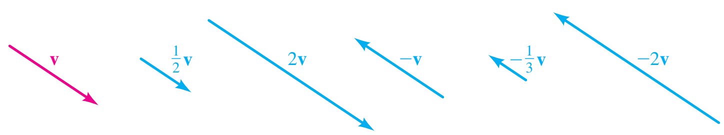


Figure 5
Multiplication of a vector by a scalar

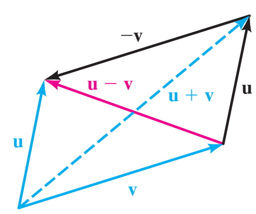


Figure 6
Subtraction of vectors

Vectors in the Coordinate Plane

So far we've discussed vectors geometrically. By placing a vector in a coordinate plane, we can describe it analytically (that is, by using components). In Figure 7(a), to go from the initial point of the vector **v** to the terminal point, we move a units to the right and b units upward. We represent **v** as an ordered pair of real numbers.

Note the distinction between the *vector* $\langle a, b\rangle$ and the *point* (a, b).

$$\mathbf{v} = \langle a, b\rangle$$

where a is the **horizontal component** of **v** and b is the **vertical component** of **v**. Remember that a vector represents a magnitude and a direction, not a particular arrow in the plane. Thus, the vector $\langle a, b\rangle$ has many different representations, depending on its initial point (see Figure 7(b)).

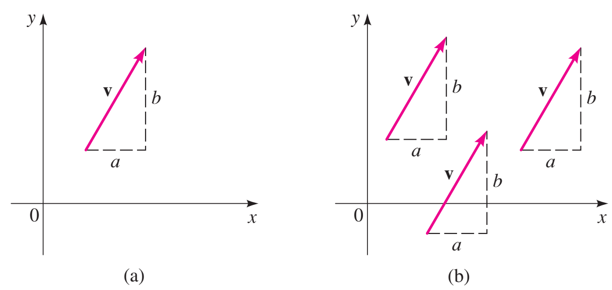


Figure 7

Using Figure 8, the relationship between a geometric representation of a vector and the analytic one can be stated as follows.

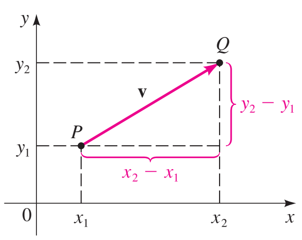


Figure 8

Component Form of a Vector

If a vector **v** is represented in the plane with initial point $P(x_1, y_1)$ and terminal point $Q(x_2, y_2)$, then

$$\mathbf{v} = \langle x_2 - x_1, y_2 - y_1\rangle$$

Example 1 Describing Vectors in Component Form

(a) Find the component form of the vector **u** with initial point $(-2, 5)$ and terminal point $(3, 7)$.

(b) If the vector $\mathbf{v} = \langle 3, 7\rangle$ is sketched with initial point $(2, 4)$, what is its terminal point?

(c) Sketch representations of the vector $\mathbf{w} = \langle 2, 3\rangle$ with initial points at $(0, 0)$, $(2, 2)$, $(-2, -1)$, and $(1, 4)$.

Solution

(a) The desired vector is

$$\mathbf{u} = \langle 3 - (-2), 7 - 5\rangle = \langle 5, 2\rangle$$

ALTERNATE EXAMPLE 1a

Find the vector **u** with initial point $(-5, 4)$ and terminal point $(4, 8)$.

ANSWER

$\begin{pmatrix} 9 \\ 4 \end{pmatrix}$

ALTERNATE EXAMPLE 1b

If the vector $\mathbf{v} = \langle 4, 7\rangle$ is sketched with initial point $(4, 5)$, what is its terminal point?

ANSWER

$(8, 12)$

SAMPLE QUESTIONS

Text Question

What are the components of the vector from (3, 3) to (4, 4)?

Answer

$\langle 1, 1\rangle$

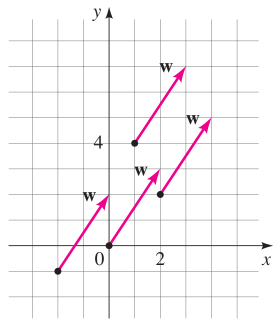


Figure 9

(b) Let the terminal point of **v** be (x, y). Then

$$\langle x - 2, y - 4 \rangle = \langle 3, 7 \rangle$$

So $x - 2 = 3$ and $y - 4 = 7$, or $x = 5$ and $y = 11$. The terminal point is $(5, 11)$
.

(c) Representations of the vector **w** are sketched in Figure 9. ■

We now give analytic definitions of the various operations on vectors that we have described geometrically. Let's start with equality of vectors. We've said that two vectors are equal if they have equal magnitude and the same direction. For the vectors $\mathbf{u} = \langle a_1, b_1 \rangle$ and $\mathbf{v} = \langle a_2, b_2 \rangle$, this means that $a_1 = a_2$ and $b_1 = b_2$. In other words, two vectors are **equal** if and only if their corresponding components are equal. Thus, all the arrows in Figure 7(b) represent the same vector, as do all the arrows in Figure 9.

Applying the Pythagorean Theorem to the triangle in Figure 10, we obtain the following formula for the magnitude of a vector.

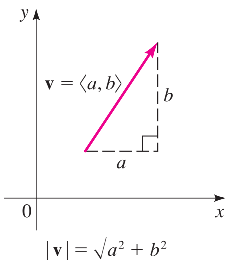


Figure 10

Magnitude of a Vector

The **magnitude** or **length** of a vector $\mathbf{v} = \langle a, b \rangle$ is

$$|\mathbf{v}| = \sqrt{a^2 + b^2}$$

Example 2 Magnitudes of Vectors

Find the magnitude of each vector.

(a) $\mathbf{u} = \langle 2, -3 \rangle$ (b) $\mathbf{v} = \langle 5, 0 \rangle$ (c) $\mathbf{w} = \langle \frac{3}{5}, \frac{4}{5} \rangle$

Solution

(a) $|\mathbf{u}| = \sqrt{2^2 + (-3)^2} = \sqrt{13}$

(b) $|\mathbf{v}| = \sqrt{5^2 + 0^2} = \sqrt{25} = 5$

(c) $|\mathbf{w}| = \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = \sqrt{\frac{9}{25} + \frac{16}{25}} = 1$ ■

The following definitions of addition, subtraction, and scalar multiplication of vectors correspond to the geometric descriptions given earlier. Figure 11 shows how the analytic definition of addition corresponds to the geometric one.

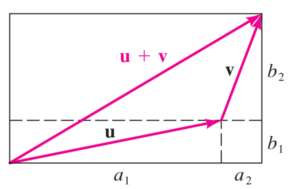


Figure 11

Algebraic Operations on Vectors

If $\mathbf{u} = \langle a_1, b_1 \rangle$ and $\mathbf{v} = \langle a_2, b_2 \rangle$, then

$$\mathbf{u} + \mathbf{v} = \langle a_1 + a_2, b_1 + b_2 \rangle$$

$$\mathbf{u} - \mathbf{v} = \langle a_1 - a_2, b_1 - b_2 \rangle$$

$$c\mathbf{u} = \langle ca_1, cb_1 \rangle, \qquad c \in \mathbb{R}$$

DRILL QUESTION

What are the magnitude and direction of the vector $3i + 3j$?

Answer

Magnitude $3\sqrt{2}$, direction $\frac{\pi}{4}$

ALTERNATE EXAMPLE 2ab

Find the magnitude of the vector $\mathbf{u} = \langle 4, 0 \rangle$.

ANSWER

4

ALTERNATE EXAMPLE 2c

Find the magnitude of the vector $\mathbf{u} = \left\langle \frac{12}{13}, \frac{5}{13} \right\rangle$.

ANSWER

1

IN-CLASS MATERIALS

Ask students to find r and s such that $\langle 3, 2 \rangle = r\langle 1, 1 \rangle + s\langle 0, 4 \rangle$. Repeat for $\langle 3, 2 \rangle = r\langle 1, 1 \rangle - s\langle 0, 4 \rangle$. Ask students if there is a vector $\langle x, y \rangle$ such that we cannot find r and s with $\langle x, y \rangle = r\langle 1, 1 \rangle + s\langle 0, 4 \rangle$. Note that this kind of question becomes very important in linear algebra.

Example 3 Operations with Vectors

If $\mathbf{u} = \langle 2, -3 \rangle$ and $\mathbf{v} = \langle -1, 2 \rangle$, find $\mathbf{u} + \mathbf{v}$, $\mathbf{u} - \mathbf{v}$, $2\mathbf{u}$, $-3\mathbf{v}$, and $2\mathbf{u} + 3\mathbf{v}$.

Solution By the definitions of the vector operations, we have

$$\mathbf{u} + \mathbf{v} = \langle 2, -3 \rangle + \langle -1, 2 \rangle = \langle 1, -1 \rangle$$

$$\mathbf{u} - \mathbf{v} = \langle 2, -3 \rangle - \langle -1, 2 \rangle = \langle 3, -5 \rangle$$

$$2\mathbf{u} = 2\langle 2, -3 \rangle = \langle 4, -6 \rangle$$

$$-3\mathbf{v} = -3\langle -1, 2 \rangle = \langle 3, -6 \rangle$$

$$2\mathbf{u} + 3\mathbf{v} = 2\langle 2, -3 \rangle + 3\langle -1, 2 \rangle = \langle 4, -6 \rangle + \langle -3, 6 \rangle = \langle 1, 0 \rangle$$

■

The following properties for vector operations can be easily proved from the definitions. The **zero vector** is the vector $\mathbf{0} = \langle 0, 0 \rangle$. It plays the same role for addition of vectors as the number 0 does for addition of real numbers.

Properties of Vectors

Vector addition	Multiplication by a scalar
$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$	$c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$	$(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
$\mathbf{u} + \mathbf{0} = \mathbf{u}$	$(cd)\mathbf{u} = c(d\mathbf{u}) = d(c\mathbf{u})$
$\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$	$1\mathbf{u} = \mathbf{u}$
Length of a vector	$0\mathbf{u} = \mathbf{0}$
$\lvert c\mathbf{u} \rvert = \lvert c \rvert \lvert \mathbf{u} \rvert$	$c\mathbf{0} = \mathbf{0}$

A vector of length 1 is called a **unit vector**. For instance, in Example 2(c), the vector $\mathbf{w} = \langle \frac{3}{5}, \frac{4}{5} \rangle$ is a unit vector. Two useful unit vectors are **i** and **j**, defined by

$$\mathbf{i} = \langle 1, 0 \rangle \qquad \mathbf{j} = \langle 0, 1 \rangle$$

These vectors are special because any vector can be expressed in terms of them.

Vectors in Terms of i and j

The vector $\mathbf{v} = \langle a, b \rangle$ can be expressed in terms of **i** and **j** by

$$\mathbf{v} = \langle a, b \rangle = a\mathbf{i} + b\mathbf{j}$$

Example 4 Vectors in Terms of i and j

(a) Write the vector $\mathbf{u} = \langle 5, -8 \rangle$ in terms of **i** and **j**.

(b) If $\mathbf{u} = 3\mathbf{i} + 2\mathbf{j}$ and $\mathbf{v} = -\mathbf{i} + 6\mathbf{j}$, write $2\mathbf{u} + 5\mathbf{v}$ in terms of **i** and **j**.

Solution

(a) $\mathbf{u} = 5\mathbf{i} + (-8)\mathbf{j} = 5\mathbf{i} - 8\mathbf{j}$

ALTERNATE EXAMPLE 3
If $\mathbf{u} = \langle 4, 5 \rangle$ and $\mathbf{v} = \langle -3, 4 \rangle$, find $2\mathbf{u}$, $-3\mathbf{v}$, and $3\mathbf{u} - 2\mathbf{v}$.

ANSWER
$\langle 8, 10 \rangle$, $\langle 9, -12 \rangle$, $\langle 18, -23 \rangle$

IN-CLASS MATERIALS

Go over the properties of vectors from both algebraic and geometric perspectives.

ALTERNATE EXAMPLE 4
If $\mathbf{u} = \langle 5, 3 \rangle$ and $\mathbf{v} = \langle -4, 6 \rangle$, write $2\mathbf{u} + 5\mathbf{v}$ in terms of **i** and **j**.

ANSWER
$-10\mathbf{i} + 36\mathbf{j}$

IN-CLASS MATERIALS

Assume $\mathbf{v}_1$ and $\mathbf{v}_2$ are in standard position. Demonstrate that the vector $\mathbf{v}_3 = \mathbf{v}_1 - \mathbf{v}_2$ can be viewed as the vector with initial point $\mathbf{v}_2$ and terminal point $\mathbf{v}_1$. Draw this picture in $\mathbb{R}^2$ and (if you can) in $\mathbb{R}^3$.

ALTERNATE EXAMPLE 5a
A vector **v** has length 4 and direction $\frac{\pi}{6}$. Write **v** in terms of **i** and **j** by finding its horizontal and vertical components.

ANSWER
$2\sqrt{3}\mathbf{i} + 2\mathbf{j}$

(b) The properties of addition and scalar multiplication of vectors show that we can manipulate vectors in the same way as algebraic expressions. Thus

$$\begin{aligned}2\mathbf{u} + 5\mathbf{v} &= 2(3\mathbf{i} + 2\mathbf{j}) + 5(-\mathbf{i} + 6\mathbf{j}) \\ &= (6\mathbf{i} + 4\mathbf{j}) + (-5\mathbf{i} + 30\mathbf{j}) \\ &= \mathbf{i} + 34\mathbf{j}\end{aligned}$$

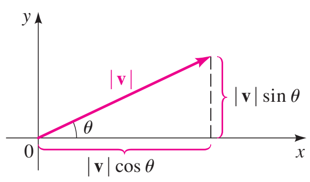


Figure 12

Let **v** be a vector in the plane with its initial point at the origin. The **direction** of **v** is θ, the smallest positive angle in standard position formed by the positive x-axis and **v** (see Figure 12). If we know the magnitude and direction of a vector, then Figure 12 shows that we can find the horizontal and vertical components of the vector.

Horizontal and Vertical Components of a Vector

Let **v** be a vector with magnitude $|\mathbf{v}|$ and direction θ.
Then $\mathbf{v} = \langle a, b\rangle = a\mathbf{i} + b\mathbf{j}$, where

$$a = |\mathbf{v}|\cos\theta \qquad \text{and} \qquad b = |\mathbf{v}|\sin\theta$$

Thus, we can express **v** as

$$\mathbf{v} = |\mathbf{v}|\cos\theta\,\mathbf{i} + |\mathbf{v}|\sin\theta\,\mathbf{j}$$

Example 5 Components and Direction of a Vector

(a) A vector **v** has length 8 and direction $\pi/3$. Find the horizontal and vertical components, and write **v** in terms of **i** and **j**.

(b) Find the direction of the vector $\mathbf{u} = -\sqrt{3}\mathbf{i} + \mathbf{j}$.

Solution

(a) We have $\mathbf{v} = \langle a, b\rangle$, where the components are given by

$$a = 8\cos\frac{\pi}{3} = 4 \qquad \text{and} \qquad b = 8\sin\frac{\pi}{3} = 4\sqrt{3}$$

Thus, $\mathbf{v} = \langle 4, 4\sqrt{3}\rangle = 4\mathbf{i} + 4\sqrt{3}\mathbf{j}$.

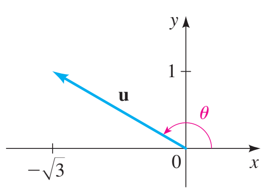


Figure 13

(b) From Figure 13 we see that the direction θ has the property that

$$\tan\theta = \frac{1}{-\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

Thus, the reference angle for θ is $\pi/6$. Since the terminal point of the vector **u** is in quadrant II, it follows that $\theta = 5\pi/6$. ■

Using Vectors to Model Velocity and Force

The use of bearings (such as N 30° E) to describe directions is explained on page 511 in Section 6.5.

The **velocity** of a moving object is modeled by a vector whose direction is the direction of motion and whose magnitude is the speed. Figure 14 shows some vectors **u**, representing the velocity of wind flowing in the direction N 30° E, and a vector **v**, representing the velocity of an airplane flying through this wind at the point P. It's obvious from our experience that wind affects both the speed and the direction of an airplane. Figure 15 indicates that the true velocity of the plane (relative to the ground) is given by the vector $\mathbf{w} = \mathbf{u} + \mathbf{v}$.

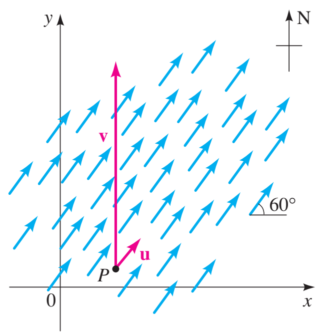


Figure 14

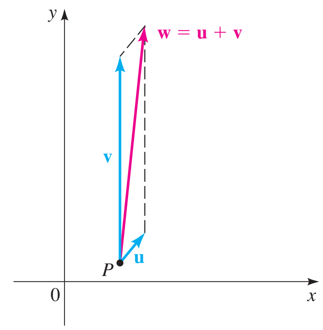


Figure 15

Example 6 The True Speed and Direction of an Airplane

An airplane heads due north at 300 mi/h. It experiences a 40 mi/h crosswind flowing in the direction N 30° E, as shown in Figure 14.

(a) Express the velocity $\mathbf{v}$ of the airplane relative to the air, and the velocity $\mathbf{u}$ of the wind, in component form.

(b) Find the true velocity of the airplane as a vector.

(c) Find the true speed and direction of the airplane.

Solution

(a) The velocity of the airplane relative to the air is $\mathbf{v} = 0\mathbf{i} + 300\mathbf{j} = 300\mathbf{j}$.
By the formulas for the components of a vector, we find that the velocity of the wind is

$$\begin{aligned}\mathbf{u} &= (40\cos 60°)\mathbf{i} + (40\sin 60°)\mathbf{j}\\ &= 20\mathbf{i} + 20\sqrt{3}\,\mathbf{j}\\ &\approx 20\mathbf{i} + 34.64\,\mathbf{j}\end{aligned}$$

(b) The true velocity of the airplane is given by the vector $\mathbf{w} = \mathbf{u} + \mathbf{v}$.

$$\begin{aligned}\mathbf{w} = \mathbf{u} + \mathbf{v} &= (20\mathbf{i} + 20\sqrt{3}\,\mathbf{j}) + (300\mathbf{j})\\ &= 20\mathbf{i} + (20\sqrt{3} + 300)\mathbf{j}\\ &\approx 20\mathbf{i} + 334.64\,\mathbf{j}\end{aligned}$$

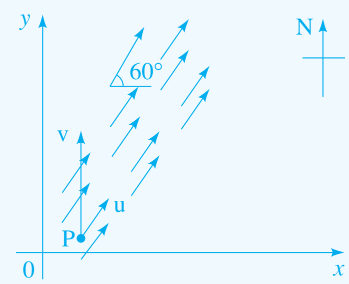
(c) The true speed of the airplane is given by the magnitude of $\mathbf{w}$.

$$|\mathbf{w}| \approx \sqrt{(20)^2 + (334.64)^2} \approx 335.2 \text{ mi/h}$$

The direction of the airplane is the direction θ of the vector $\mathbf{w}$. The angle θ has the property that $\tan\theta \approx 334.64/20 = 16.732$ and so $\theta \approx 86.6°$. Thus, the airplane is heading in the direction N 3.4° E. ■

ALTERNATE EXAMPLE 6

An airplane heads due north at 280 mi/h. It experiences a 38 mi/h crosswind flowing in the direction N 60° E, as shown in the figure below. Find the true velocity of the airplane as a vector expressed in terms of **i** and **j**. Find the true speed of the airplane.

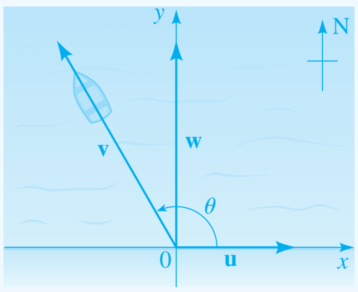


ANSWER

$19\mathbf{i} + 312.91\mathbf{j}$, 313.5

ALTERNATE EXAMPLE 7

A woman launches a boat from one shore of a straight river and wants to land at the point directly on the opposite shore. If the speed of the boat (in still water) is 16 mi/h and the river is flowing east at the rate of 8 mi/h, in what direction (you should find the angle θ) should she head the boat in order to arrive at the desired landing point?



ANSWER
120°

Example 7 Calculating a Heading

A woman launches a boat from one shore of a straight river and wants to land at the point directly on the opposite shore. If the speed of the boat (relative to the water) is 10 mi/h and the river is flowing east at the rate of 5 mi/h, in what direction should she head the boat in order to arrive at the desired landing point?

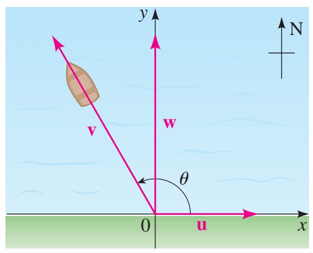


Figure 16

Solution We choose a coordinate system with the origin at the initial position of the boat as shown in Figure 16. Let **u** and **v** represent the velocities of the river and the boat, respectively. Clearly, $\mathbf{u} = 5\mathbf{i}$ and, since the speed of the boat is 10 mi/h, we have $|\mathbf{v}| = 10$, so

$$\mathbf{v} = (10\cos\theta)\mathbf{i} + (10\sin\theta)\mathbf{j}$$

where the angle θ is as shown in Figure 16. The true course of the boat is given by the vector $\mathbf{w} = \mathbf{u} + \mathbf{v}$. We have

$$\begin{aligned} \mathbf{w} = \mathbf{u} + \mathbf{v} &= 5\mathbf{i} + (10\cos\theta)\mathbf{i} + (10\sin\theta)\mathbf{j} \\ &= (5 + 10\cos\theta)\mathbf{i} + (10\sin\theta)\mathbf{j} \end{aligned}$$

Since the woman wants to land at a point directly across the river, her direction should have horizontal component 0. In other words, she should choose θ in such a way that

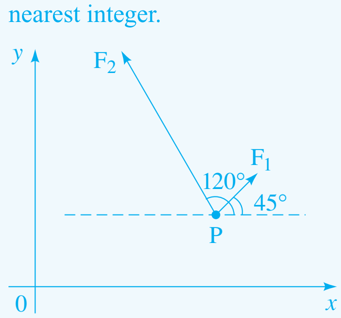
$$\begin{aligned} 5 + 10\cos\theta &= 0 \\ \cos\theta &= -\tfrac{1}{2} \\ \theta &= 120^\circ \end{aligned}$$

Thus, she should head the boat in the direction $\theta = 120^\circ$ (or N 30° W). ■

Force is also represented by a vector. Intuitively, we can think of force as describing a push or a pull on an object, for example, a horizontal push of a book across a table or the downward pull of the earth's gravity on a ball. Force is measured in pounds (or in newtons, in the metric system). For instance, a man weighing 200 lb exerts a force of 200 lb downward on the ground. If several forces are acting on an object, the **resultant force** experienced by the object is the vector sum of these forces.

ALTERNATE EXAMPLE 8

Two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ with magnitudes 6 and 24 lb, respectively, act on an object at a point P as shown in the figure below. Find the resultant force acting at P as a vector expressed in terms of **i** and **j**. Round the horizontal and vertical components of the vector to the nearest integer.



ANSWER
$-8\mathbf{i} + 25\mathbf{j}$

Example 8 Resultant Force

Two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ with magnitudes 10 and 20 lb, respectively, act on an object at a point P as shown in Figure 17. Find the resultant force acting at P.

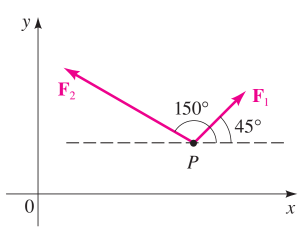


Figure 17

Solution We write $\mathbf{F}_1$ and $\mathbf{F}_2$ in component form:

$$\mathbf{F}_1 = (10\cos 45^\circ)\mathbf{i} + (10\sin 45^\circ)\mathbf{j} = 10\frac{\sqrt{2}}{2}\mathbf{i} + 10\frac{\sqrt{2}}{2}\mathbf{j} = 5\sqrt{2}\mathbf{i} + 5\sqrt{2}\mathbf{j}$$

$$\begin{aligned} \mathbf{F}_2 &= (20\cos 150^\circ)\mathbf{i} + (20\sin 150^\circ)\mathbf{j} = -20\frac{\sqrt{3}}{2}\mathbf{i} + 20\left(\frac{1}{2}\right)\mathbf{j} \\ &= -10\sqrt{3}\mathbf{i} + 10\mathbf{j} \end{aligned}$$

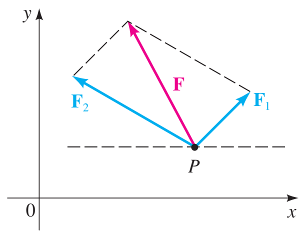


Figure 18

So the resultant force **F** is

$$\begin{aligned}\mathbf{F} &= \mathbf{F}_1 + \mathbf{F}_2 \\ &= (5\sqrt{2}\mathbf{i} + 5\sqrt{2}\mathbf{j}) + (-10\sqrt{3}\mathbf{i} + 10\mathbf{j}) \\ &= (5\sqrt{2} - 10\sqrt{3})\mathbf{i} + (5\sqrt{2} + 10)\mathbf{j} \\ &\approx -10\mathbf{i} + 17\mathbf{j}\end{aligned}$$

The resultant force **F** is shown in Figure 18. ■

8.4 Exercises

1–6 ■ Sketch the vector indicated. (The vectors **u** and **v** are shown in the figure.)

1. 2**u**

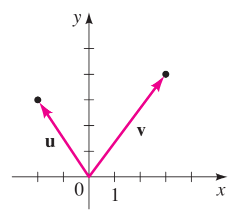
2. −**v**

3. **u** + **v**

4. **u** − **v**

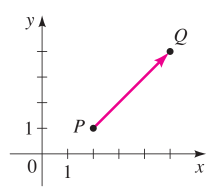
5. **v** − 2**u**

6. 2**u** + **v**

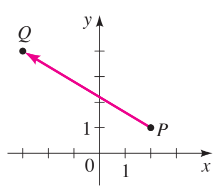


7–16 ■ Express the vector with initial point *P* and terminal point *Q* in component form.

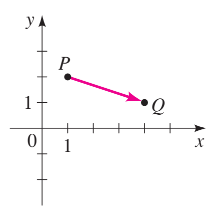
7.



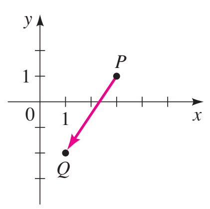
8.



9.



10.



11. $P(3, 2)$, $Q(8, 9)$

12. $P(1, 1)$, $Q(9, 9)$

13. $P(5, 3)$, $Q(1, 0)$

14. $P(-1, 3)$, $Q(-6, -1)$

15. $P(-1, -1)$, $Q(-1, 1)$

16. $P(-8, -6)$, $Q(-1, -1)$

17–22 ■ Find 2**u**, −3**v**, **u** + **v**, and 3**u** − 4**v** for the given vectors **u** and **v**.

17. $\mathbf{u} = \langle 2, 7\rangle$, $\mathbf{v} = \langle 3, 1\rangle$

18. $\mathbf{u} = \langle -2, 5\rangle$, $\mathbf{v} = \langle 2, -8\rangle$

19. $\mathbf{u} = \langle 0, -1\rangle$, $\mathbf{v} = \langle -2, 0\rangle$

20. $\mathbf{u} = \mathbf{i}$, $\mathbf{v} = -2\mathbf{j}$

21. $\mathbf{u} = 2\mathbf{i}$, $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j}$

22. $\mathbf{u} = \mathbf{i} + \mathbf{j}$, $\mathbf{v} = \mathbf{i} - \mathbf{j}$

23–26 ■ Find $|\mathbf{u}|$, $|\mathbf{v}|$, $|2\mathbf{u}|$, $|\frac{1}{2}\mathbf{v}|$, $|\mathbf{u} + \mathbf{v}|$, $|\mathbf{u} - \mathbf{v}|$, and $|\mathbf{u}| - |\mathbf{v}|$.

23. $\mathbf{u} = 2\mathbf{i} + \mathbf{j}$, $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j}$

24. $\mathbf{u} = -2\mathbf{i} + 3\mathbf{j}$, $\mathbf{v} = \mathbf{i} - 2\mathbf{j}$

25. $\mathbf{u} = \langle 10, -1\rangle$, $\mathbf{v} = \langle -2, -2\rangle$

26. $\mathbf{u} = \langle -6, 6\rangle$, $\mathbf{v} = \langle -2, -1\rangle$

27–32 ■ Find the horizontal and vertical components of the vector with given length and direction, and write the vector in terms of the vectors **i** and **j**.

27. $|\mathbf{v}| = 40$, $\theta = 30°$

28. $|\mathbf{v}| = 50$, $\theta = 120°$

29. $|\mathbf{v}| = 1$, $\theta = 225°$

30. $|\mathbf{v}| = 800$, $\theta = 125°$

31. $|\mathbf{v}| = 4$, $\theta = 10°$

32. $|\mathbf{v}| = \sqrt{3}$, $\theta = 300°$

33–38 ■ Find the magnitude and direction (in degrees) of the vector.

33. $\mathbf{v} = \langle 3, 4\rangle$

34. $\mathbf{v} = \left\langle -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right\rangle$

35. $\mathbf{v} = \langle -12, 5\rangle$

36. $\mathbf{v} = \langle 40, 9\rangle$

37. $\mathbf{v} = \mathbf{i} + \sqrt{3}\mathbf{j}$

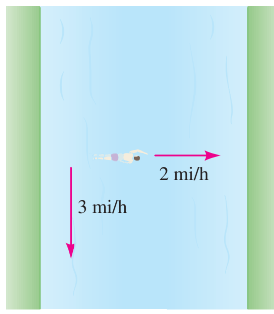
38. $\mathbf{v} = \mathbf{i} + \mathbf{j}$

Applications

39. Components of a Force A man pushes a lawn mower with a force of 30 lb exerted at an angle of 30° to the ground. Find the horizontal and vertical components of the force.

40. Components of a Velocity A jet is flying in a direction N 20° E with a speed of 500 mi/h. Find the north and east components of the velocity.

41. Velocity A river flows due south at 3 mi/h. A swimmer attempting to cross the river heads due east swimming at 2 mi/h relative to the water. Find the true velocity of the swimmer as a vector.



42. Velocity A migrating salmon heads in the direction N 45° E, swimming at 5 mi/h relative to the water. The prevailing ocean currents flow due east at 3 mi/h. Find the true velocity of the fish as a vector.

43. True Velocity of a Jet A pilot heads his jet due east. The jet has a speed of 425 mi/h relative to the air. The wind is blowing due north with a speed of 40 mi/h.

(a) Express the velocity of the wind as a vector in component form.

(b) Express the velocity of the jet relative to the air as a vector in component form.

(c) Find the true velocity of the jet as a vector.

(d) Find the true speed and direction of the jet.

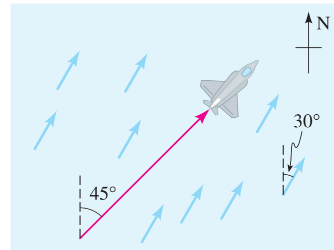
44. True Velocity of a Jet A jet is flying through a wind that is blowing with a speed of 55 mi/h in the direction N 30° E (see the figure). The jet has a speed of 765 mi/h relative to the air, and the pilot heads the jet in the direction N 45° E.

(a) Express the velocity of the wind as a vector in component form.

(b) Express the velocity of the jet relative to the air as a vector in component form.

(c) Find the true velocity of the jet as a vector.

(d) Find the true speed and direction of the jet.



45. True Velocity of a Jet Find the true speed and direction of the jet in Exercise 44 if the pilot heads the plane in the direction N 30° W.

46. True Velocity of a Jet In what direction should the pilot in Exercise 44 head the plane for the true course to be due north?

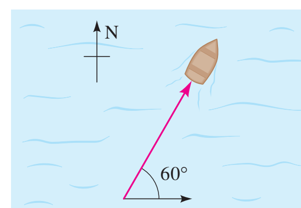
47. Velocity of a Boat A straight river flows east at a speed of 10 mi/h. A boater starts at the south shore of the river and heads in a direction 60° from the shore (see the figure). The motorboat has a speed of 20 mi/h relative to the water.

(a) Express the velocity of the river as a vector in component form.

(b) Express the velocity of the motorboat relative to the water as a vector in component form.

(c) Find the true velocity of the motorboat.

(d) Find the true speed and direction of the motorboat.



48. Velocity of a Boat The boater in Exercise 47 wants to arrive at a point on the north shore of the river directly opposite the starting point. In what direction should the boat be headed?

49. Velocity of a Boat A boat heads in the direction N 72° E. The speed of the boat relative to the water is 24 mi/h. The water is flowing directly south. It is observed that the true direction of the boat is directly east.

(a) Express the velocity of the boat relative to the water as a vector in component form.

(b) Find the speed of the water and the true speed of the boat.

50. Velocity A woman walks due west on the deck of an ocean liner at 2 mi/h. The ocean liner is moving due north at a speed of 25 mi/h. Find the speed and direction of the woman relative to the surface of the water.

51–56 ■ Equilibrium of Forces The forces $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_n$ acting at the same point P are said to be in equilibrium if the resultant force is zero, that is, if $\mathbf{F}_1 + \mathbf{F}_2 + \cdots + \mathbf{F}_n = 0$. Find **(a)** the resultant forces acting at P, and **(b)** the additional force required (if any) for the forces to be in equilibrium.

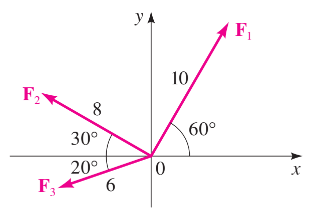
51. $\mathbf{F}_1 = \langle 2, 5 \rangle, \quad \mathbf{F}_2 = \langle 3, -8 \rangle$

52. $\mathbf{F}_1 = \langle 3, -7 \rangle, \quad \mathbf{F}_2 = \langle 4, -2 \rangle, \quad \mathbf{F}_3 = \langle -7, 9 \rangle$

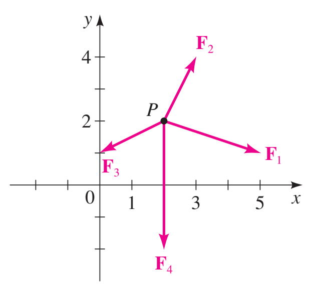
53. $\mathbf{F}_1 = 4\mathbf{i} - \mathbf{j}, \quad \mathbf{F}_2 = 3\mathbf{i} - 7\mathbf{j}, \quad \mathbf{F}_3 = -8\mathbf{i} + 3\mathbf{j}, \quad \mathbf{F}_4 = \mathbf{i} + \mathbf{j}$

54. $\mathbf{F}_1 = \mathbf{i} - \mathbf{j}, \quad \mathbf{F}_2 = \mathbf{i} + \mathbf{j}, \quad \mathbf{F}_3 = -2\mathbf{i} + \mathbf{j}$

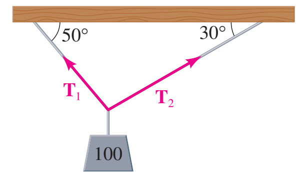
55.



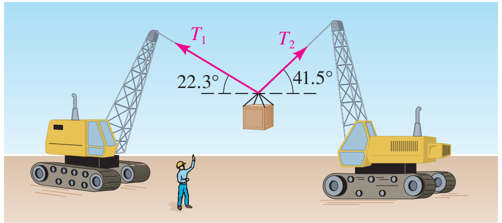
56.



57. Equilibrium of Tensions A 100-lb weight hangs from a string as shown in the figure. Find the tensions $\mathbf{T}_1$ and $\mathbf{T}_2$ in the string.

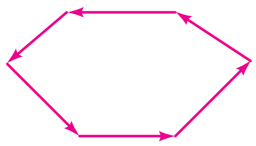


58. Equilibrium of Tensions The cranes in the figure are lifting an object that weighs 18,278 lb. Find the tensions $\mathbf{T}_1$ and $\mathbf{T}_2$.



Discovery • Discussion

59. Vectors That Form a Polygon Suppose that n vectors can be placed head to tail in the plane so that they form a polygon. (The figure shows the case of a hexagon.) Explain why the sum of these vectors is $\mathbf{0}$.



8.5 The Dot Product

In this section we define an operation on vectors called the dot product. This concept is especially useful in calculus and in applications of vectors to physics and engineering.

The Dot Product of Vectors

We begin by defining the dot product of two vectors.

SUGGESTED TIME AND EMPHASIS

1 class.
Essential material.

POINTS TO STRESS

1. The dot product and its properties.
2. Definition of orthogonality.
3. Length of a vector.
4. Projections and their applications to computer science.

Definition of the Dot Product

If $\mathbf{u} = \langle a_1, b_1 \rangle$ and $\mathbf{v} = \langle a_2, b_2 \rangle$ are vectors, then their **dot product,** denoted by $\mathbf{u} \cdot \mathbf{v}$, is defined by

$$\mathbf{u} \cdot \mathbf{v} = a_1 a_2 + b_1 b_2$$

Thus, to find the dot product of **u** and **v** we multiply corresponding components and add. The dot product is *not* a vector; it is a real number, or scalar.

ALTERNATE EXAMPLE 1
Compute $\mathbf{a} \cdot \mathbf{b}$.

(a) $\mathbf{a} = \langle 3, -5 \rangle$ $\mathbf{b} = \langle 1, 4 \rangle$

(b) $\mathbf{a} = \langle 6\mathbf{i} + 2\mathbf{j} \rangle$ $\mathbf{b} = \langle -\mathbf{i} - 2\mathbf{j} \rangle$

ANSWERS
(a) -17 **(b)** -10

ALTERNATE EXAMPLE 1b
Calculate the dot product of **u** and **v**.
$\mathbf{u} = \langle 3, -4 \rangle$ and $\mathbf{v} = \langle 7, 8 \rangle$

ANSWER
-11

Example 1 Calculating Dot Products

(a) If $\mathbf{u} = \langle 3, -2 \rangle$ and $\mathbf{v} = \langle 4, 5 \rangle$ then

$$\mathbf{u} \cdot \mathbf{v} = (3)(4) + (-2)(5) = 2$$

(b) If $\mathbf{u} = 2\mathbf{i} + \mathbf{j}$ and $\mathbf{v} = 5\mathbf{i} - 6\mathbf{j}$, then

$$\mathbf{u} \cdot \mathbf{v} = (2)(5) + (1)(-6) = 4$$ ■

The proofs of the following properties of the dot product follow easily from the definition.

Properties of the Dot Product

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $(a\mathbf{u}) \cdot \mathbf{v} = a(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (a\mathbf{v})$
3. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$
4. $|\mathbf{u}|^2 = \mathbf{u} \cdot \mathbf{u}$

■ **Proof** We prove only the last property. The proofs of the others are left as exercises. Let $\mathbf{u} = \langle a, b \rangle$. Then

$$\mathbf{u} \cdot \mathbf{u} = \langle a, b \rangle \cdot \langle a, b \rangle = a^2 + b^2 = |\mathbf{u}|^2$$ ■

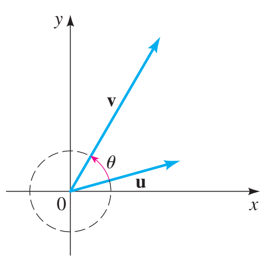


Figure 1

Let **u** and **v** be vectors and sketch them with initial points at the origin. We define the **angle** θ **between u and v** to be the smaller of the angles formed by these representations of **u** and **v** (see Figure 1). Thus, $0 \le \theta \le \pi$. The next theorem relates the angle between two vectors to their dot product.

The Dot Product Theorem

If θ is the angle between two nonzero vectors **u** and **v**, then

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos \theta$$

IN-CLASS MATERIALS

Demonstrate the proper formation of statements involving dot products. For example, the statement $c\,(\mathbf{a} \cdot \mathbf{b})$ makes sense, while the statements $\mathbf{d} \cdot (\mathbf{a} \cdot \mathbf{b})$ and $c \cdot \mathbf{a}$ do not.

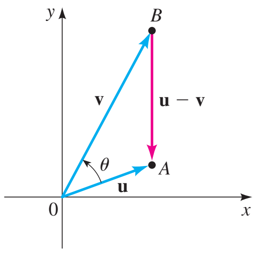


Figure 2

■ **Proof** The proof is a nice application of the Law of Cosines. Applying the Law of Cosines to triangle *AOB* in Figure 2 gives

$$|\mathbf{u}-\mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}||\mathbf{v}|\cos\theta$$

Using the properties of the dot product, we write the left-hand side as follows:

$$\begin{aligned}|\mathbf{u}-\mathbf{v}|^2 &= (\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v})\\ &= \mathbf{u}\cdot\mathbf{u} - \mathbf{u}\cdot\mathbf{v} - \mathbf{v}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v}\\ &= |\mathbf{u}|^2 - 2(\mathbf{u}\cdot\mathbf{v}) + |\mathbf{v}|^2\end{aligned}$$

Equating the right-hand sides of the displayed equations, we get

$$\begin{aligned}|\mathbf{u}|^2 - 2(\mathbf{u}\cdot\mathbf{v}) + |\mathbf{v}|^2 &= |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}||\mathbf{v}|\cos\theta\\ -2(\mathbf{u}\cdot\mathbf{v}) &= -2|\mathbf{u}||\mathbf{v}|\cos\theta\\ \mathbf{u}\cdot\mathbf{v} &= |\mathbf{u}||\mathbf{v}|\cos\theta\end{aligned}$$

This proves the theorem. ■

The Dot Product Theorem is useful because it allows us to find the angle between two vectors if we know the components of the vectors. The angle is obtained simply by solving the equation in the Dot Product Theorem for $\cos\theta$. We state this important result explicitly.

Angle between Two Vectors

If θ is the angle between two nonzero vectors **u** and **v**, then

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$$

Example 2 Finding the Angle between Two Vectors

Find the angle between the vectors $\mathbf{u} = \langle 2, 5\rangle$ and $\mathbf{v} = \langle 4, -3\rangle$.

Solution By the formula for the angle between two vectors, we have

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \frac{(2)(4)+(5)(-3)}{\sqrt{4+25}\sqrt{16+9}} = \frac{-7}{5\sqrt{29}}$$

Thus, the angle between **u** and **v** is

$$\theta = \cos^{-1}\left(\frac{-7}{5\sqrt{29}}\right) \approx 105.1°$$ ■

Two nonzero vectors **u** and **v** are called **perpendicular**, or **orthogonal**, if the angle between them is $\pi/2$. The following theorem shows that we can determine if two vectors are perpendicular by finding their dot product.

Orthogonal Vectors

Two nonzero vectors **u** and **v** are perpendicular if and only if $\mathbf{u}\cdot\mathbf{v} = 0$.

ALTERNATE EXAMPLE 2
Calculate the dot product of **u** and **v**.
$\mathbf{u} = 4\mathbf{i} + \mathbf{j}$ and $\mathbf{v} = 6\mathbf{i} - 5\mathbf{j}$

ANSWER
19

IN-CLASS MATERIALS

Demonstrate the distributive property in general. Note that while the dot product is commutative and distributive, the associative property makes no sense, as it is not possible to take the dot product of three vectors.

Proof If **u** and **v** are perpendicular, then the angle between them is $\pi/2$ and so

$$\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\frac{\pi}{2} = 0$$

Conversely, if $\mathbf{u}\cdot\mathbf{v} = 0$, then

$$|\mathbf{u}||\mathbf{v}|\cos\theta = 0$$

Since **u** and **v** are nonzero vectors, we conclude that $\cos\theta = 0$, and so $\theta = \pi/2$. Thus, **u** and **v** are orthogonal. ■

ALTERNATE EXAMPLE 3
Determine whether the pair of vectors are perpendicular.

(a) $\mathbf{a} = \langle 1, 6\rangle$ $\mathbf{b} = \langle 12, -2\rangle$
(b) $\mathbf{a} = \langle 3, 4\rangle$ $\mathbf{b} = \langle -3, 4\rangle$

ANSWERS
(a) $\mathbf{a}\cdot\mathbf{b} = 0$, so they are perpendicular.

(b) $\mathbf{a}\cdot\mathbf{b} = 7 \neq 0$, so they are not perpendicular.

Example 3 Checking Vectors for Perpendicularity

Determine whether the vectors in each pair are perpendicular.

(a) $\mathbf{u} = \langle 3, 5\rangle$ and $\mathbf{v} = \langle 2, -8\rangle$ (b) $\mathbf{u} = \langle 2, 1\rangle$ and $\mathbf{v} = \langle -1, 2\rangle$

Solution

(a) $\mathbf{u}\cdot\mathbf{v} = (3)(2) + (5)(-8) = -34 \neq 0$, so **u** and **v** are not perpendicular.

(b) $\mathbf{u}\cdot\mathbf{v} = (2)(-1) + (1)(2) = 0$, so **u** and **v** are perpendicular. ■

The Component of u Along v

Note that the component of **u** along **v** is a scalar, not a vector.

The **component of u along v** (or the **component of u in the direction of v**) is defined to be

$$|\mathbf{u}|\cos\theta$$

where θ is the angle between **u** and **v**. Figure 3 gives a geometric interpretation of this concept. Intuitively, the component of **u** along **v** is the magnitude of the portion of **u** that points in the direction of **v**. Notice that the component of **u** along **v** is negative if $\pi/2 < \theta \leq \pi$.

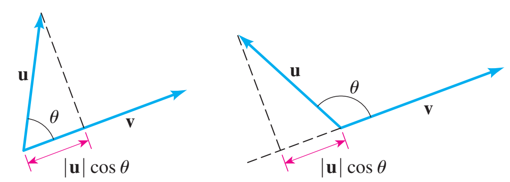


Figure 3

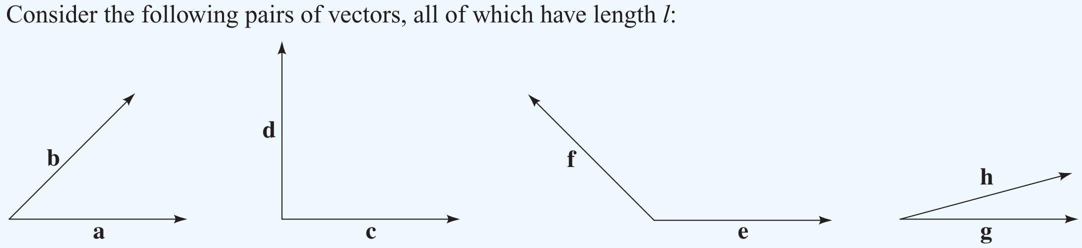
When analyzing forces in physics and engineering, it's often helpful to express a vector as a sum of two vectors lying in perpendicular directions. For example, suppose a car is parked on an inclined driveway as in Figure 4. The weight of the car is a vector **w** that points directly downward. We can write

$$\mathbf{w} = \mathbf{u} + \mathbf{v}$$

where **u** is parallel to the driveway and **v** is perpendicular to the driveway. The vector **u** is the force that tends to roll the car down the driveway, and **v** is the force experienced

DRILL QUESTION

Consider the following pairs of vectors, all of which have length l:



Put the following quantities in order, from smallest to largest: $\mathbf{a}\cdot\mathbf{b}$ $\mathbf{c}\cdot\mathbf{d}$ $\mathbf{e}\cdot\mathbf{f}$ $\mathbf{g}\cdot\mathbf{h}$

Answer

$\mathbf{e}\cdot\mathbf{f}$, $\mathbf{c}\cdot\mathbf{d}$, $\mathbf{a}\cdot\mathbf{b}$, $\mathbf{g}\cdot\mathbf{h}$

by the surface of the driveway. The magnitudes of these forces are the components of **w** along **u** and **v**, respectively.

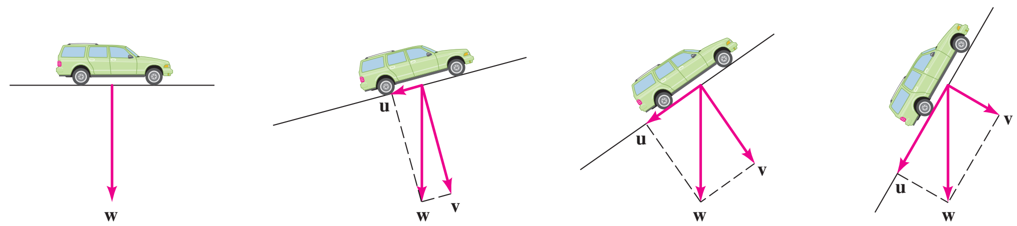


Figure 4

Example 4 Resolving a Force into Components

A car weighing 3000 lb is parked on a driveway that is inclined 15° to the horizontal, as shown in Figure 5.

(a) Find the magnitude of the force required to prevent the car from rolling down the driveway.

(b) Find the magnitude of the force experienced by the driveway due to the weight of the car.

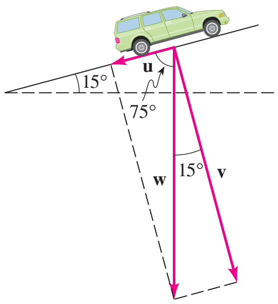


Figure 5

Solution The car exerts a force **w** of 3000 lb directly downward. We resolve **w** into the sum of two vectors **u** and **v**, one parallel to the surface of the driveway and the other perpendicular to it, as shown in Figure 5.

(a) The magnitude of the part of the force **w** that causes the car to roll down the driveway is

$$|\mathbf{u}| = \text{component of } \mathbf{w} \text{ along } \mathbf{u} = 3000 \cos 75° \approx 776$$

Thus, the force needed to prevent the car from rolling down the driveway is about 776 lb.

(b) The magnitude of the force exerted by the car on the driveway is

$$|\mathbf{v}| = \text{component of } \mathbf{w} \text{ along } \mathbf{v} = 3000 \cos 15° \approx 2898$$

The force experienced by the driveway is about 2898 lb. ■

The component of **u** along **v** can be computed using dot products:

$$|\mathbf{u}|\cos\theta = \frac{|\mathbf{v}||\mathbf{u}|\cos\theta}{|\mathbf{v}|} = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|}$$

We have shown the following.

Calculating Components

The component of **u** along **v** is $\dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|}$.

SAMPLE QUESTION
Text Question

Compute $\langle 3, 4\rangle \cdot \langle -1, 2\rangle$.

Answer

5

ALTERNATE EXAMPLE 4

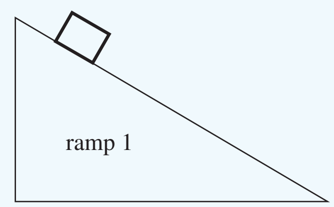
Find the angle between the vectors $\mathbf{u} = \langle 7, 5\rangle$ and $\mathbf{v} = \langle 40, -9\rangle$. Please round the answer to the nearest tenth of a degree.

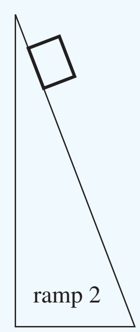
ANSWER

48.2°

IN-CLASS MATERIALS

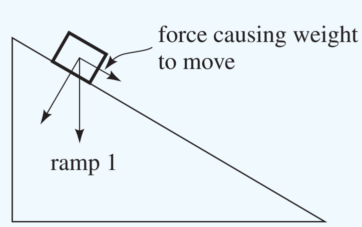
This is a nice, direct application of vector projections. It is clear that a weight will slide more quickly down ramp 2 than down ramp 1:

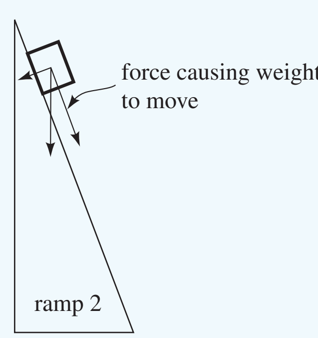




Gravity is the same in both cases, yet there is a definite difference in speed. The reason behind this is interesting. Gravity is doing two things at once: It is letting the weight slide down, and it is also preventing the weight from floating off the ramp and flying into outer space. We can draw a "free body diagram" that shows how the gravity available to let the weight slide down is affected by the angle of the ramp.

A block slides faster on a steeper slope because the projection of the gravitational force in the direction of the slope is larger. There is more force pushing the block down the slope, and less of a force holding it to the surface of the slope.





ALTERNATE EXAMPLE 5
Determine whether the vectors in the pair $\mathbf{u} = \langle 3, 2 \rangle$ and $\mathbf{v} = \langle 4, -8 \rangle$ are perpendicular.

ANSWER
No

EXAMPLES

- $\langle 5, 6, 2 \rangle \cdot \langle -3, 1, 0 \rangle = -9$
- Two vectors that are orthogonal: $\mathbf{a} = \langle 5, 6, 2 \rangle$, $\mathbf{b} = \left\langle 1, -1, \frac{1}{2} \right\rangle$
- Projections: If $\mathbf{a} = \langle 2, 1, -1 \rangle$ and $\mathbf{b} = \langle 3, 2, 7 \rangle$, then $\text{proj}_{\mathbf{a}}\,\mathbf{b} = \left\langle \frac{1}{3}, \frac{1}{6}, \frac{1}{6} \right\rangle$, $\text{proj}_{\mathbf{b}}\,\mathbf{a} = \left\langle \frac{3}{62}, \frac{1}{31}, -\frac{7}{62} \right\rangle$, and $|\mathbf{a}| = \sqrt{6}$.

Example 5 Finding Components

Let $\mathbf{u} = \langle 1, 4 \rangle$ and $\mathbf{v} = \langle -2, 1 \rangle$. Find the component of $\mathbf{u}$ along $\mathbf{v}$.

Solution We have

$$\text{component of } \mathbf{u} \text{ along } \mathbf{v} = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|} = \frac{(1)(-2) + (4)(1)}{\sqrt{4+1}} = \frac{2}{\sqrt{5}}$$

■

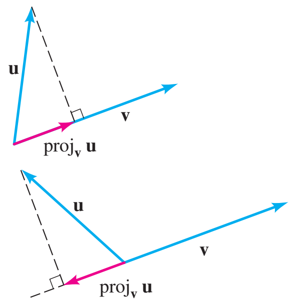


Figure 6

The Projection of u onto v

Figure 6 shows representations of the vectors $\mathbf{u}$ and $\mathbf{v}$. The projection of $\mathbf{u}$ onto $\mathbf{v}$, denoted by $\text{proj}_{\mathbf{v}}\,\mathbf{u}$, is the vector whose *direction* is the same as $\mathbf{v}$ and whose *length* is the component of $\mathbf{u}$ along $\mathbf{v}$. To find an expression for $\text{proj}_{\mathbf{v}}\,\mathbf{u}$, we first find a unit vector in the direction of $\mathbf{v}$ and then multiply it by the component of $\mathbf{u}$ along $\mathbf{v}$.

$$\text{proj}_{\mathbf{v}}\,\mathbf{u} = (\text{component of } \mathbf{u} \text{ along } \mathbf{v})(\text{unit vector in direction of } \mathbf{v})$$

$$= \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|}\right)\frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v}$$

We often need to **resolve** a vector $\mathbf{u}$ into the sum of two vectors, one parallel to $\mathbf{v}$ and one orthogonal to $\mathbf{v}$. That is, we want to write $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$ where $\mathbf{u}_1$ is parallel to $\mathbf{v}$ and $\mathbf{u}_2$ is orthogonal to $\mathbf{v}$. In this case, $\mathbf{u}_1 = \text{proj}_{\mathbf{v}}\,\mathbf{u}$ and $\mathbf{u}_2 = \mathbf{u} - \text{proj}_{\mathbf{v}}\,\mathbf{u}$ (see Exercise 37).

Calculating Projections

The **projection of u onto v** is the vector $\text{proj}_{\mathbf{v}}\,\mathbf{u}$ given by

$$\text{proj}_{\mathbf{v}}\,\mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v}$$

If the vector $\mathbf{u}$ is **resolved** into $\mathbf{u}_1$ and $\mathbf{u}_2$, where $\mathbf{u}_1$ is parallel to $\mathbf{v}$ and $\mathbf{u}_2$ is orthogonal to $\mathbf{v}$, then

$$\mathbf{u}_1 = \text{proj}_{\mathbf{v}}\,\mathbf{u} \qquad \text{and} \qquad \mathbf{u}_2 = \mathbf{u} - \text{proj}_{\mathbf{v}}\,\mathbf{u}$$

Example 6 Resolving a Vector into Orthogonal Vectors

Let $\mathbf{u} = \langle -2, 9 \rangle$ and $\mathbf{v} = \langle -1, 2 \rangle$.

(a) Find $\text{proj}_{\mathbf{v}}\,\mathbf{u}$.

(b) Resolve $\mathbf{u}$ into $\mathbf{u}_1$ and $\mathbf{u}_2$, where $\mathbf{u}_1$ is parallel to $\mathbf{v}$ and $\mathbf{u}_2$ is orthogonal to $\mathbf{v}$.

Solution

(a) By the formula for the projection of one vector onto another we have

$$\text{proj}_{\mathbf{v}}\,\mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v} \qquad \text{Formula for projection}$$

$$= \left(\frac{\langle -2, 9 \rangle \cdot \langle -1, 2 \rangle}{(-1)^2 + 2^2}\right)\langle -1, 2 \rangle \qquad \text{Definition of } \mathbf{u} \text{ and } \mathbf{v}$$

$$= 4\langle -1, 2 \rangle = \langle -4, 8 \rangle$$

IN-CLASS MATERIALS

Assume that $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$ and $\mathbf{a} \neq 0$. Pose the question, "Is it necessarily true that $\mathbf{b} = \mathbf{c}$?" When you've convinced the students (perhaps by example) that the answer is "no," the next logical question to ask is, "What *can* we say about $\mathbf{b}$ and $\mathbf{c}$?" It can be shown that $\mathbf{b}$ and $\mathbf{c}$ have the same projection onto $\mathbf{a}$, since $\mathbf{a} \perp (\mathbf{b} - \mathbf{c})$.

(b) By the formula in the preceding box we have $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$, where

$$\mathbf{u}_1 = \text{proj}_{\mathbf{v}}\,\mathbf{u} = \langle -4, 8 \rangle \qquad \text{From part (a)}$$

$$\mathbf{u}_2 = \mathbf{u} - \text{proj}_{\mathbf{v}}\,\mathbf{u} = \langle -2, 9 \rangle - \langle -4, 8 \rangle = \langle 2, 1 \rangle$$

■

Work

One use of the dot product occurs in calculating work. In everyday use, the term *work* means the total amount of effort required to perform a task. In physics, *work* has a technical meaning that conforms to this intuitive meaning. If a constant force of magnitude F moves an object through a distance d along a straight line, then the **work** done is

$$W = Fd \qquad \text{or} \qquad \text{work} = \text{force} \times \text{distance}$$

If F is measured in pounds and d in feet, then the unit of work is a foot-pound (ft-lb). For example, how much work is done in lifting a 20-lb weight 6 ft off the ground? Since a force of 20 lb is required to lift this weight and since the weight moves through a distance of 6 ft, the amount of work done is

$$W = Fd = (20)(6) = 120 \text{ ft-lb}$$

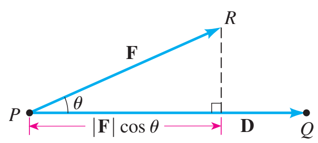


Figure 7

This formula applies only when the force is directed along the direction of motion. In the general case, if the force $\mathbf{F}$ moves an object from P to Q, as in Figure 7, then only the component of the force in the direction of $\mathbf{D} = \overrightarrow{PQ}$ affects the object. Thus, the effective magnitude of the force on the object is

$$\text{component of } \mathbf{F} \text{ along } \mathbf{D} = |\mathbf{F}| \cos\theta$$

So, the work done is

$$W = \text{force} \times \text{distance} = (|\mathbf{F}| \cos\theta)|\mathbf{D}| = |\mathbf{F}||\mathbf{D}| \cos\theta = \mathbf{F} \cdot \mathbf{D}$$

We have derived the following simple formula for calculating work.

Work

The **work** W done by a force $\mathbf{F}$ in moving along a vector $\mathbf{D}$ is

$$W = \mathbf{F} \cdot \mathbf{D}$$

Example 7 Calculating Work

A force is given by the vector $\mathbf{F} = \langle 2, 3 \rangle$ and moves an object from the point $(1, 3)$ to the point $(5, 9)$. Find the work done.

Solution The displacement vector is

$$\mathbf{D} = \langle 5 - 1, 9 - 3 \rangle = \langle 4, 6 \rangle$$

So the work done is

$$W = \mathbf{F} \cdot \mathbf{D} = \langle 2, 3 \rangle \cdot \langle 4, 6 \rangle = 26$$

If the unit of force is pounds and the distance is measured in feet, then the work done is 26 ft-lb. ■

ALTERNATE EXAMPLE 7
A force is given by the vector $\mathbf{F} = \langle 7, 6 \rangle$ and moves an object from the point (2, 1) to the point (5, 6). Find the work done.

ANSWER
51

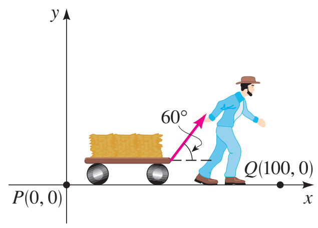


Figure 8

Example 8 Calculating Work

A man pulls a wagon horizontally by exerting a force of 20 lb on the handle. If the handle makes an angle of 60° with the horizontal, find the work done in moving the wagon 100 ft.

Solution We choose a coordinate system with the origin at the initial position of the wagon (see Figure 8). That is, the wagon moves from the point $P(0, 0)$ to the point $Q(100, 0)$. The vector that represents this displacement is

$$\mathbf{D} = 100\mathbf{i}$$

The force on the handle can be written in terms of components (see Section 8.4) as

$$\mathbf{F} = (20\cos 60°)\mathbf{i} + (20\sin 60°)\mathbf{j} = 10\mathbf{i} + 10\sqrt{3}\mathbf{j}$$

Thus, the work done is

$$W = \mathbf{F}\cdot\mathbf{D} = (10\mathbf{i} + 10\sqrt{3}\mathbf{j})\cdot(100\mathbf{i}) = 1000 \text{ ft-lb}$$

■

8.5 Exercises

1–8 ■ Find **(a)** $\mathbf{u}\cdot\mathbf{v}$ and **(b)** the angle between $\mathbf{u}$ and $\mathbf{v}$ to the nearest degree.

1. $\mathbf{u} = \langle 2, 0\rangle$, $\mathbf{v} = \langle 1, 1\rangle$

2. $\mathbf{u} = \mathbf{i} + \sqrt{3}\mathbf{j}$, $\mathbf{v} = -\sqrt{3}\mathbf{i} + \mathbf{j}$

3. $\mathbf{u} = \langle 2, 7\rangle$, $\mathbf{v} = \langle 3, 1\rangle$

4. $\mathbf{u} = \langle -6, 6\rangle$, $\mathbf{v} = \langle 1, -1\rangle$

5. $\mathbf{u} = \langle 3, -2\rangle$, $\mathbf{v} = \langle 1, 2\rangle$

6. $\mathbf{u} = 2\mathbf{i} + \mathbf{j}$, $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j}$

7. $\mathbf{u} = -5\mathbf{j}$, $\mathbf{v} = -\mathbf{i} - \sqrt{3}\mathbf{j}$

8. $\mathbf{u} = \mathbf{i} + \mathbf{j}$, $\mathbf{v} = \mathbf{i} - \mathbf{j}$

9–14 ■ Determine whether the given vectors are orthogonal.

9. $\mathbf{u} = \langle 6, 4\rangle$, $\mathbf{v} = \langle -2, 3\rangle$

10. $\mathbf{u} = \langle 0, -5\rangle$, $\mathbf{v} = \langle 4, 0\rangle$

11. $\mathbf{u} = \langle -2, 6\rangle$, $\mathbf{v} = \langle 4, 2\rangle$

12. $\mathbf{u} = 2\mathbf{i}$, $\mathbf{v} = -7\mathbf{j}$

13. $\mathbf{u} = 2\mathbf{i} - 8\mathbf{j}$, $\mathbf{v} = -12\mathbf{i} - 3\mathbf{j}$

14. $\mathbf{u} = 4\mathbf{i}$, $\mathbf{v} = -\mathbf{i} + 3\mathbf{j}$

15–18 ■ Find the indicated quantity, assuming $\mathbf{u} = 2\mathbf{i} + \mathbf{j}$, $\mathbf{v} = \mathbf{i} - 3\mathbf{j}$, and $\mathbf{w} = 3\mathbf{i} + 4\mathbf{j}$.

15. $\mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{w}$

16. $\mathbf{u}\cdot(\mathbf{v} + \mathbf{w})$

17. $(\mathbf{u} + \mathbf{v})\cdot(\mathbf{u} - \mathbf{v})$

18. $(\mathbf{u}\cdot\mathbf{v})(\mathbf{u}\cdot\mathbf{w})$

19–22 ■ Find the component of $\mathbf{u}$ along $\mathbf{v}$.

19. $\mathbf{u} = \langle 4, 6\rangle$, $\mathbf{v} = \langle 3, -4\rangle$

20. $\mathbf{u} = \langle -3, 5\rangle$, $\mathbf{v} = \langle 1/\sqrt{2}, 1/\sqrt{2}\rangle$

21. $\mathbf{u} = 7\mathbf{i} - 24\mathbf{j}$, $\mathbf{v} = \mathbf{j}$

22. $\mathbf{u} = 7\mathbf{i}$, $\mathbf{v} = 8\mathbf{i} + 6\mathbf{j}$

23–28 ■ **(a)** Calculate $\text{proj}_{\mathbf{v}}\,\mathbf{u}$. **(b)** Resolve $\mathbf{u}$ into $\mathbf{u}_1$ and $\mathbf{u}_2$, where $\mathbf{u}_1$ is parallel to $\mathbf{v}$ and $\mathbf{u}_2$ is orthogonal to $\mathbf{v}$.

23. $\mathbf{u} = \langle -2, 4\rangle$, $\mathbf{v} = \langle 1, 1\rangle$

24. $\mathbf{u} = \langle 7, -4\rangle$, $\mathbf{v} = \langle 2, 1\rangle$

25. $\mathbf{u} = \langle 1, 2\rangle$, $\mathbf{v} = \langle 1, -3\rangle$

26. $\mathbf{u} = \langle 11, 3\rangle$, $\mathbf{v} = \langle -3, -2\rangle$

27. $\mathbf{u} = \langle 2, 9\rangle$, $\mathbf{v} = \langle -3, 4\rangle$

28. $\mathbf{u} = \langle 1, 1\rangle$, $\mathbf{v} = \langle 2, -1\rangle$

29–32 ■ Find the work done by the force $\mathbf{F}$ in moving an object from P to Q.

29. $\mathbf{F} = 4\mathbf{i} - 5\mathbf{j}$; $P(0, 0)$, $Q(3, 8)$

30. $\mathbf{F} = 400\mathbf{i} + 50\mathbf{j}$; $P(-1, 1)$, $Q(200, 1)$

31. $\mathbf{F} = 10\mathbf{i} + 3\mathbf{j}$; $P(2, 3)$, $Q(6, -2)$

32. $\mathbf{F} = -4\mathbf{i} + 20\mathbf{j}$; $P(0, 10)$, $Q(5, 25)$

ALTERNATE EXAMPLE 8

A man pulls a wagon horizontally by exerting a force of 30 lb on the handle. If the handle makes an angle of 70° with the horizontal, find the work done in moving the wagon 200 ft.

ANSWER

$\mathbf{F} = (30\cos 70°)\mathbf{i} + (30\sin 70°)\mathbf{j}$

$\mathbf{D} = 200\mathbf{i}$

$\mathbf{F}\cdot\mathbf{D} = 205.2$ ft-lb

33–36 ■ Let **u**, **v**, and **w** be vectors and let a be a scalar. Prove the given property.

33. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

34. $(a\mathbf{u}) \cdot \mathbf{v} = a(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (a\mathbf{v})$

35. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$

36. $(\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) = |\mathbf{u}|^2 - |\mathbf{v}|^2$

37. Show that the vectors $\text{proj}_{\mathbf{v}}\, \mathbf{u}$ and $\mathbf{u} - \text{proj}_{\mathbf{v}}\, \mathbf{u}$ are orthogonal.

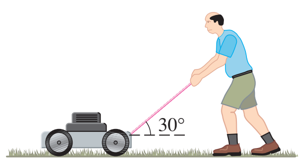
38. Evaluate $\mathbf{v} \cdot \text{proj}_{\mathbf{v}}\, \mathbf{u}$.

Applications

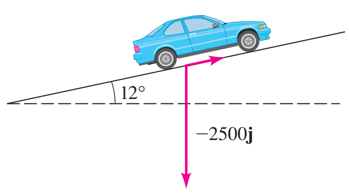
39. Work The force $\mathbf{F} = 4\mathbf{i} - 7\mathbf{j}$ moves an object 4 ft along the x-axis in the positive direction. Find the work done if the unit of force is the pound.

40. Work A constant force $\mathbf{F} = \langle 2, 8 \rangle$ moves an object along a straight line from the point $(2, 5)$ to the point $(11, 13)$. Find the work done if the distance is measured in feet and the force is measured in pounds.

41. Work A lawn mower is pushed a distance of 200 ft along a horizontal path by a constant force of 50 lb. The handle of the lawn mower is held at an angle of 30° from the horizontal (see the figure). Find the work done.



42. Work A car drives 500 ft on a road that is inclined 12° to the horizontal, as shown in the figure. The car weighs 2500 lb. Thus, gravity acts straight down on the car with a constant force $\mathbf{F} = -2500\mathbf{j}$. Find the work done by the car in overcoming gravity.



43. Force A car is on a driveway that is inclined 25° to the horizontal. If the car weighs 2755 lb, find the force required to keep it from rolling down the driveway.

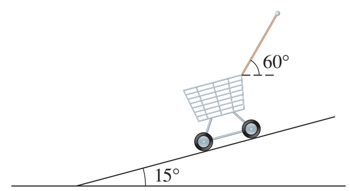
44. Force A car is on a driveway that is inclined 10° to the horizontal. A force of 490 lb is required to keep the car from rolling down the driveway.

(a) Find the weight of the car.

(b) Find the force the car exerts against the driveway.

45. Force A package that weighs 200 lb is placed on an inclined plane. If a force of 80 lb is just sufficient to keep the package from sliding, find the angle of inclination of the plane. (Ignore the effects of friction.)

46. Force A cart weighing 40 lb is placed on a ramp inclined at 15° to the horizontal. The cart is held in place by a rope inclined at 60° to the horizontal, as shown in the figure. Find the force that the rope must exert on the cart to keep it from rolling down the ramp.



Discovery • Discussion

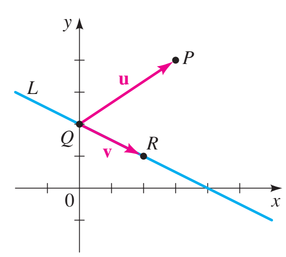
47. Distance from a Point to a Line Let L be the line $2x + 4y = 8$ and let P be the point $(3, 4)$.

(a) Show that the points $Q(0, 2)$ and $R(2, 1)$ lie on L.

(b) Let $\mathbf{u} = \overrightarrow{QP}$ and $\mathbf{v} = \overrightarrow{QR}$, as shown in the figure. Find $\mathbf{w} = \text{proj}_{\mathbf{v}}\, \mathbf{u}$.

(c) Sketch a graph that explains why $|\mathbf{u} - \mathbf{w}|$ is the distance from P to L. Find this distance.

(d) Write a short paragraph describing the steps you would take to find the distance from a given point to a given line.



DISCOVERY PROJECT

J. L. Amos/SuperStock

Sailing Against the Wind

Sailors depend on the wind to propel their boats. But what if the wind is blowing in a direction opposite to that in which they want to travel? Although it is obviously impossible to sail directly against the wind, it *is* possible to sail at an angle *into* the wind. Then by *tacking*, that is, zig-zagging on alternate sides of the wind direction, a sailor can make headway against the wind (see Figure 1).

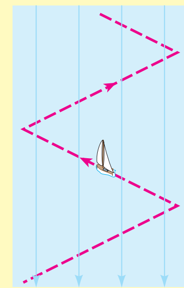


Figure 1
Tacking

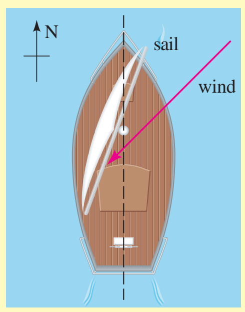


Figure 2

How should the sail be aligned to propel the boat in the desired direction into the wind? This question can be answered by modeling the wind as a vector and studying its components along the keel and the sail.

For example, suppose a sailboat headed due north has its sail inclined in the direction N 20° E. The wind is blowing into the sail in the direction S 45° W with a force of magnitude F (see Figure 2).

1. Show that the effective force of the wind on the sail is $F \sin 25°$. You can do this by finding the components of the wind parallel to the sail and perpendicular to the sail. The component parallel to the sail slips by and does not propel the boat. Only the perpendicular component pushes against the sail.

2. If the keel of the boat is aligned due north, what fraction of the force F actually drives the boat forward? Only the component of the force found in Problem 1 that is parallel to the keel drives the boat forward.

(In real life, other factors, including the aerodynamic properties of the sail, influence the speed of the sailboat.)

3. If a boat heading due north has its sail inclined in the direction N α° E, and the wind is blowing with force F in the direction S β° W where $0 < \alpha < \beta < 180$, find a formula for the magnitude of the force that actually drives the boat forward.

8 Review

Concept Check

1. Describe how polar coordinates represent the position of a point in the plane.

2. (a) What equations do you use to change from polar to rectangular coordinates?

(b) What equations do you use to change from rectangular to polar coordinates?

3. How do you sketch the graph of a polar equation $r = f(\theta)$?

4. What type of curve has a polar equation of the given form?

(a) $r = a\cos\theta$ or $r = a\sin\theta$

(b) $r = a(1 \pm \cos\theta)$ or $r = a(1 \pm \sin\theta)$

(c) $r = a \pm b\cos\theta$ or $r = a \pm b\sin\theta$

(d) $r = a\cos n\theta$ or $r = a\sin n\theta$

5. How do you graph a complex number z? What is the polar form of a complex number z? What is the modulus of z? What is the argument of z?

6. (a) How do you multiply two complex numbers if they are given in polar form?

(b) How do you divide two such numbers?

7. (a) State DeMoivre's Theorem.

(b) How do you find the nth roots of a complex number?

8. (a) What is the difference between a scalar and a vector?

(b) Draw a diagram to show how to add two vectors.

(c) Draw a diagram to show how to subtract two vectors.

(d) Draw a diagram to show how to multiply a vector by the scalars $2, \frac{1}{2}, -2$, and $-\frac{1}{2}$.

9. If $\mathbf{u} = \langle a_1, b_1\rangle$, $\mathbf{v} = \langle a_2, b_2\rangle$ and c is a scalar, write expressions for $\mathbf{u} + \mathbf{v}$, $\mathbf{u} - \mathbf{v}$, $c\mathbf{u}$, and $|\mathbf{u}|$.

10. (a) If $\mathbf{v} = \langle a, b\rangle$, write $\mathbf{v}$ in terms of $\mathbf{i}$ and $\mathbf{j}$.

(b) Write the components of $\mathbf{v}$ in terms of the magnitude and direction of $\mathbf{v}$.

11. If $\mathbf{u} = \langle a_1, b_1\rangle$ and $\mathbf{v} = \langle a_2, b_2\rangle$, what is the dot product $\mathbf{u} \cdot \mathbf{v}$?

12. (a) How do you use the dot product to find the angle between two vectors?

(b) How do you use the dot product to determine whether two vectors are perpendicular?

13. What is the component of $\mathbf{u}$ along $\mathbf{v}$, and how do you calculate it?

14. What is the projection of $\mathbf{u}$ onto $\mathbf{v}$, and how do you calculate it?

15. How much work is done by the force $\mathbf{F}$ in moving an object along a displacement $\mathbf{D}$?

Exercises

1–6 ■ A point $P(r, \theta)$ is given in polar coordinates. **(a)** Plot the point P. **(b)** Find rectangular coordinates for P.

1. $\left(12, \frac{\pi}{6}\right)$ **2.** $\left(8, -\frac{3\pi}{4}\right)$

3. $\left(-3, \frac{7\pi}{4}\right)$ **4.** $\left(-\sqrt{3}, \frac{2\pi}{3}\right)$

5. $\left(4\sqrt{3}, -\frac{5\pi}{3}\right)$ **6.** $\left(-6\sqrt{2}, -\frac{\pi}{4}\right)$

7–12 ■ A point $P(x, y)$ is given in rectangular coordinates.
(a) Plot the point P.
(b) Find polar coordinates for P with $r \geq 0$.
(c) Find polar coordinates for P with $r \leq 0$.

7. $(8, 8)$ **8.** $(-\sqrt{2}, \sqrt{6})$

9. $(-6\sqrt{2}, -6\sqrt{2})$ **10.** $(3\sqrt{3}, 3)$

11. $(-3, \sqrt{3})$ **12.** $(4, -4)$

13–16 ■ **(a)** Convert the equation to polar coordinates and simplify. **(b)** Graph the equation. [*Hint:* Use the form of the equation that you find easier to graph.]

13. $x + y = 4$ **14.** $xy = 1$

15. $x^2 + y^2 = 4x + 4y$ **16.** $(x^2 + y^2)^2 = 2xy$

17–24 ■ **(a)** Sketch the graph of the polar equation.
(b) Express the equation in rectangular coordinates.

17. $r = 3 + 3\cos\theta$ **18.** $r = 3\sin\theta$

19. $r = 2\sin 2\theta$ **20.** $r = 4\cos 3\theta$

21. $r^2 = \sec 2\theta$ **22.** $r^2 = 4\sin 2\theta$

23. $r = \sin\theta + \cos\theta$ **24.** $r = \dfrac{4}{2 + \cos\theta}$

25–28 ■ Use a graphing device to graph the polar equation. Choose the domain of θ to make sure you produce the entire graph.

25. $r = \cos(\theta/3)$

26. $r = \sin(9\theta/4)$

27. $r = 1 + 4\cos(\theta/3)$

28. $r = \theta \sin\theta, \quad -6\pi \le \theta \le 6\pi$

29–34 ■ A complex number is given.

(a) Graph the complex number in the complex plane.

(b) Find the modulus and argument.

(c) Write the number in polar form.

29. $4 + 4i$

30. $-10i$

31. $5 + 3i$

32. $1 + \sqrt{3}i$

33. $-1 + i$

34. -20

35–38 ■ Use DeMoivre's Theorem to find the indicated power.

35. $(1 - \sqrt{3}i)^4$

36. $(1 + i)^8$

37. $(\sqrt{3} + i)^{-4}$

38. $\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^{20}$

39–42 ■ Find the indicated roots.

39. The square roots of $-16i$

40. The cube roots of $4 + 4\sqrt{3}i$

41. The sixth roots of 1

42. The eighth roots of i

43–44 ■ Find $|\mathbf{u}|$, $\mathbf{u} + \mathbf{v}$, $\mathbf{u} - \mathbf{v}$, $2\mathbf{u}$, and $3\mathbf{u} - 2\mathbf{v}$.

43. $\mathbf{u} = \langle -2, 3\rangle$, $\mathbf{v} = \langle 8, 1\rangle$

44. $\mathbf{u} = 2\mathbf{i} + \mathbf{j}$, $\mathbf{v} = \mathbf{i} - 2\mathbf{j}$

45. Find the vector $\mathbf{u}$ with initial point $P(0, 3)$ and terminal point $Q(3, -1)$.

46. Find the vector $\mathbf{u}$ having length $|\mathbf{u}| = 20$ and direction $\theta = 60°$.

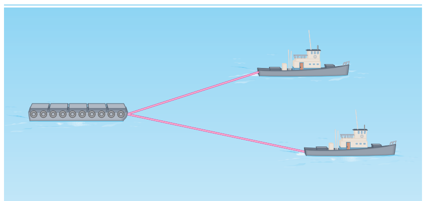
47. If the vector $5\mathbf{i} - 8\mathbf{j}$ is placed in the plane with its initial point at $P(5, 6)$, find its terminal point.

48. Find the direction of the vector $2\mathbf{i} - 5\mathbf{j}$.

49. Two tugboats are pulling a barge, as shown. One pulls with a force of 2.0×10^4 lb in the direction N 50° E and the other with a force of 3.4×10^4 lb in the direction S 75° E.

(a) Find the resultant force on the barge as a vector.

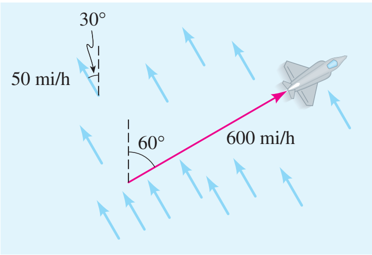
(b) Find the magnitude and direction of the resultant force.



50. An airplane heads N 60° E at a speed of 600 mi/h relative to the air. A wind begins to blow in the direction N 30° W at 50 mi/h.

(a) Find the velocity of the airplane as a vector.

(b) Find the true speed and direction of the airplane.



51–54 ■ Find $|\mathbf{u}|$, $\mathbf{u} \cdot \mathbf{u}$, and $\mathbf{u} \cdot \mathbf{v}$.

51. $\mathbf{u} = \langle 4, -3\rangle, \quad \mathbf{v} = \langle 9, -8\rangle$

52. $\mathbf{u} = \langle 5, 12\rangle, \quad \mathbf{v} = \langle 10, -4\rangle$

53. $\mathbf{u} = -2\mathbf{i} + 2\mathbf{j}, \quad \mathbf{v} = \mathbf{i} + \mathbf{j}$

54. $\mathbf{u} = 10\mathbf{j}, \quad \mathbf{v} = 5\mathbf{i} - 3\mathbf{j}$

55–58 ■ Are $\mathbf{u}$ and $\mathbf{v}$ orthogonal? If not, find the angle between them.

55. $\mathbf{u} = \langle -4, 2\rangle, \quad \mathbf{v} = \langle 3, 6\rangle$

56. $\mathbf{u} = \langle 5, 3\rangle, \quad \mathbf{v} = \langle -2, 6\rangle$

57. $\mathbf{u} = 2\mathbf{i} + \mathbf{j}, \quad \mathbf{v} = \mathbf{i} + 3\mathbf{j}$

58. $\mathbf{u} = \mathbf{i} - \mathbf{j}, \quad \mathbf{v} = \mathbf{i} + \mathbf{j}$

59–60 ■ The vectors $\mathbf{u}$ and $\mathbf{v}$ are given.

(a) Find the component of $\mathbf{u}$ along $\mathbf{v}$.

(b) Find $\text{proj}_{\mathbf{v}}\,\mathbf{u}$.

(c) Resolve $\mathbf{u}$ into the vectors $\mathbf{u}_1$ and $\mathbf{u}_2$, where $\mathbf{u}_1$ is parallel to $\mathbf{v}$ and $\mathbf{u}_2$ is perpendicular to $\mathbf{v}$.

59. $\mathbf{u} = \langle 3, 1\rangle, \quad \mathbf{v} = \langle 6, -1\rangle$

60. $\mathbf{u} = \langle -8, 6\rangle, \quad \mathbf{v} = \langle 20, 20\rangle$

61. Find the work done by the force $\mathbf{F} = 2\mathbf{i} + 9\mathbf{j}$ in moving an object from the point $(1, 1)$ to the point $(7, -1)$.

62. A force $\mathbf{F}$ with magnitude 250 lb moves an object in the direction of the vector $\mathbf{D}$ a distance of 20 ft. If the work done is 3800 ft-lb, find the angle between $\mathbf{F}$ and $\mathbf{D}$.

8 Test

1. (a) Convert the point whose polar coordinates are $(8, 5\pi/4)$ to rectangular coordinates.
(b) Find two polar coordinate representations for the rectangular coordinate point $(-6, 2\sqrt{3})$, one with $r > 0$ and one with $r < 0$, and both with $0 \le \theta < 2\pi$.

2. (a) Graph the polar equation $r = 8\cos\theta$. What type of curve is this?
(b) Convert the equation to rectangular coordinates.

3. Let $z = 1 + \sqrt{3}i$.
(a) Graph z in the complex plane.
(b) Write z in polar form.
(c) Find the complex number z^9.

4. Let $z_1 = 4\left(\cos\dfrac{7\pi}{12} + i\sin\dfrac{7\pi}{12}\right)$ and $z_2 = 2\left(\cos\dfrac{5\pi}{12} + i\sin\dfrac{5\pi}{12}\right)$.

Find z_1z_2 and $\dfrac{z_1}{z_2}$.

5. Find the cube roots of $27i$, and sketch these roots in the complex plane.

6. Let **u** be the vector with initial point $P(3, -1)$ and terminal point $Q(-3, 9)$.
(a) Express **u** in terms of **i** and **j**.
(b) Find the length of **u**.

7. Let $\mathbf{u} = \langle 1, 3\rangle$ and $\mathbf{v} = \langle -6, 2\rangle$.
(a) Find $\mathbf{u} - 3\mathbf{v}$.
(b) Find $|\mathbf{u} + \mathbf{v}|$.
(c) Find $\mathbf{u} \cdot \mathbf{v}$.
(d) Are **u** and **v** perpendicular?

8. Let $\mathbf{u} = \langle -4\sqrt{3}, 4\rangle$.
(a) Graph **u** with initial point $(0, 0)$.
(b) Find the length and direction of **u**.

9. A river is flowing due east at 8 mi/h. A man heads his motorboat in a direction N 30° E in the river. The speed of the motorboat relative to the water is 12 mi/h.
(a) Express the true velocity of the motorboat as a vector.
(b) Find the true speed and direction of the motorboat.

10. Let $\mathbf{u} = 3\mathbf{i} + 2\mathbf{j}$ and $\mathbf{v} = 5\mathbf{i} - \mathbf{j}$.
(a) Find the angle between **u** and **v**.
(b) Find the component of **u** along **v**.
(c) Find $\text{proj}_{\mathbf{v}}\,\mathbf{u}$.

11. Find the work done by the force $\mathbf{F} = 3\mathbf{i} - 5\mathbf{j}$ in moving an object from the point $(2, 2)$ to the point $(7, -13)$.

Focus on Modeling

Mapping the World

The method used to survey and map a town (page 522) works well for small areas. But mapping the whole world would introduce a new difficulty: How do we represent the *spherical* world by a *flat* map? Several ingenious methods have been developed.

Cylindrical Projection

One method is the **cylindrical projection**. In this method we imagine a cylinder "wrapped" around the earth at the equator as in Figure 1. Each point on the earth is projected onto the cylinder by a ray emanating from the center of the earth. The "unwrapped" cylinder is the desired flat map of the world. The process is illustrated in Figure 2.

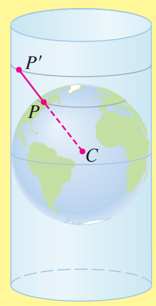


Figure 1
Point P on the earth is projected onto point P' on the cylinder by a ray from the center of the earth C.

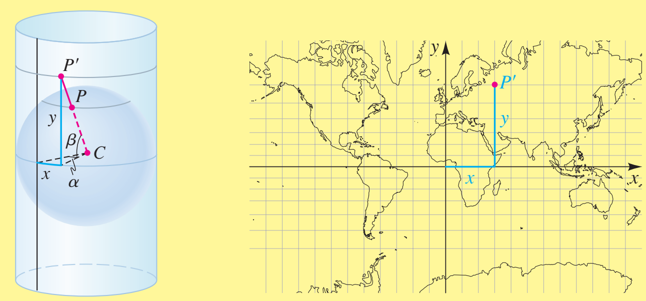


Figure 2 (a) Cylindrical projection (b) Cylindrical projection map

Of course, we cannot actually wrap a large piece of paper around the world, so this whole process must be done mathematically, and the tool we need is trigonometry. On the unwrapped cylinder we take the x-axis to correspond to the equator and the y-axis to the meridian through Greenwich, England (0° longitude). Let R be the radius of the earth and let P be the point on the earth at $\alpha°$ E longitude and $\beta°$ N latitude. The point P is projected to the point $P'(x, y)$ on the cylinder (viewed as part of the coordinate plane) where

$$x = \left(\frac{\pi}{180}\right)\alpha R \qquad \text{Formula for length of a circular arc}$$

$$y = R \tan \beta \qquad \text{Definition of tangent}$$

See Figure 2(a). These formulas can then be used to draw the map. (Note that West longitude and South latitude correspond to negative values of α and β, respectively.) Of course, using R as the radius of the earth would produce a huge map, so we replace R by a smaller value to get a map at an appropriate scale as in Figure 2(b).

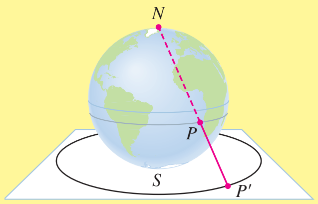


Figure 3
Point P on the earth's surface is projected onto point P' on the plane by a ray from the north pole.

Stereographic Projection

In the **stereographic projection** we imagine the earth placed on the coordinate plane with the south pole at the origin. Points on the earth are projected onto the plane by rays emanating from the north pole (see Figure 3). The earth is placed so that the prime meridian (0° longitude) corresponds to the polar axis. As shown in Figure 4(a), a point P on the earth at $\alpha°$ E longitude and $\beta°$ N latitude is projected onto the point $P'(r, \theta)$ whose polar coordinates are

$$r = 2R\tan\left(\frac{\beta}{2} + 45°\right)$$

$$\theta = \alpha$$

Figure 4(b) shows how the first of these formulas is obtained

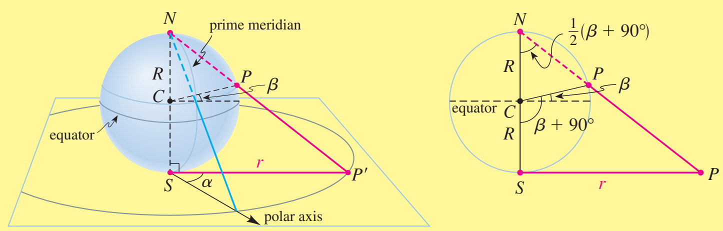


Figure 4
(a) Stereographic projection
(b) Cross-section of stereographic projection

Figure 5 shows a stereographic map of the southern hemisphere.

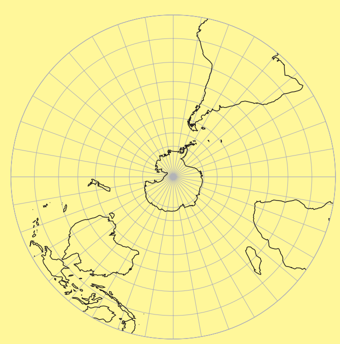


Figure 5
Stereographic projection of the southern hemisphere

Problems

1. Cylindrical Projection A map maker wishes to map the earth using a cylindrical projection. The map is to be 36 inches wide. Thus, the equator is mapped onto a horizontal 36-inch line segment. The radius of the earth is 3960 miles.

(a) What value of R should he use in the cylindrical projection formulas?

(b) How many miles does one inch on the map represent at the equator?

2. Cylindrical Projection To map the entire world using the cylindrical projection, the cylinder must extend infinitely far in the vertical direction. So a practical cylindrical map cannot extend all the way to the poles. The map maker in Problem 1 decides that his map should show the earth between 70° N and 70° S latitudes. How tall should his map be?

3. Cylindrical Projection The map maker in Problem 1 places the y-axis (0° longitude) at the center of the map as shown in Figure 2(b). Find the x- and y-coordinates of the following cities on the map.

(a) Seattle, Washington; 47.6° N, 122.3° W

(b) Moscow, Russia; 55.8° N, 37.6° E

(c) Sydney, Australia; 33.9° S, 151.2° E

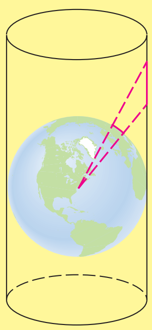
(d) Rio de Janeiro, Brazil; 22.9° S, 43.1° W

4. Stereographic Projection A map maker makes a stereographic projection of the southern hemisphere, from the south pole to the equator. The map is to have a radius of 20 in.

(a) What value of R should he use in the stereographic projection formulas?

(b) Find the polar coordinates of Sydney, Australia (33.9° S, 151.2° E) on his map.

5–6 ■ The cylindrical projection stretches distances between points not on the equator— the farther from the equator, the more the distances are stretched. In these problems we find the factors by which distances are distorted on the cylindrical projection at various locations.



5. Projected Distances Find the ratio of the projected distance on the cylinder to the actual distance on the sphere between the given latitudes along a meridian (see the figure at the left).

(a) Between 20° and 21° N latitude

(b) Between 40° and 41° N latitude

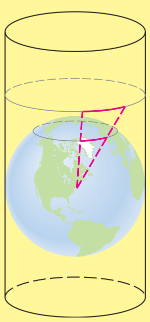
(c) Between 80° and 81° N latitude

6. Projected Distances Find the ratio of the projected distance on the cylinder to the distance on the sphere along the given parallel of latitude between two points that are 1° longitude apart (see the figure below).

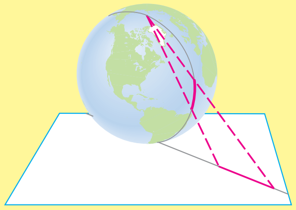
(a) 20° N latitude

(b) 40° N latitude

(c) 80° N latitude



7–8 ■ The stereographic projection also stretches distances—the farther from the south pole, the more distances are stretched. In these problems we find the factors by which distances are distorted on the stereographic projection at various locations.



7. Projected Distances Find the ratio of the projected distance on the plane to the actual distance on the sphere between the given latitudes along a meridian (see the figure at the left).

(a) Between 20° and 21° S latitude

(b) Between 40° and 41° S latitude

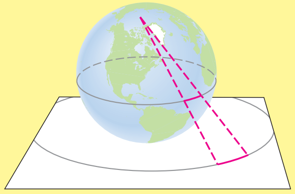
(c) Between 80° and 81° S latitude

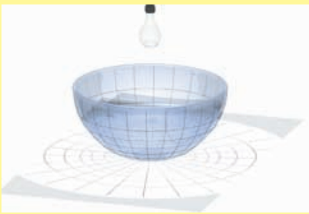
8. Projected Distances Find the ratio of the projected distance on the plane to the distance on the sphere along the given parallel of latitude between two points that are 1° longitude apart (see the figure).

(a) 20° S latitude

(b) 40° S latitude

(c) 80° S latitude

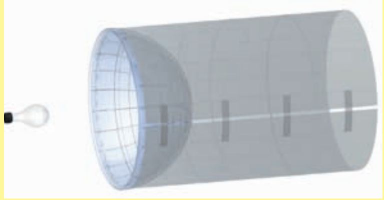




9. Lines of Latitude and Longitude In this project we see how projection transfers lines of latitude and longitude from a sphere to a flat surface. You will need a round glass bowl, tracing paper, and a light source (a small transparent light bulb). Use a black marker to draw equally spaced lines of latitude and longitude on the outside of the bowl.

(a) To model the stereographic projection, place the bowl on a sheet of tracing paper and use the light source as shown in the figure at the left.

(b) To model the cylindrical projection, wrap the tracing paper around the bowl and use the light source as shown in the figure below.



10. Other Projections There are many other map projections, such as the Albers Conic Projection, the Azimuthal Projection, the Behrmann Cylindrical Equal-Area Projection, the Gall Isographic and Orthographic Projections, the Gnomonic Projection, the Lambert Equal-Area Projection, the Mercator Projection, the Mollweide Projection, the Rectangular Projection, and the Sinusoidal Projection. Research one of these projections in your library or on the Internet and write a report explaining how the map is constructed, and describing its advantages and disadvantages.