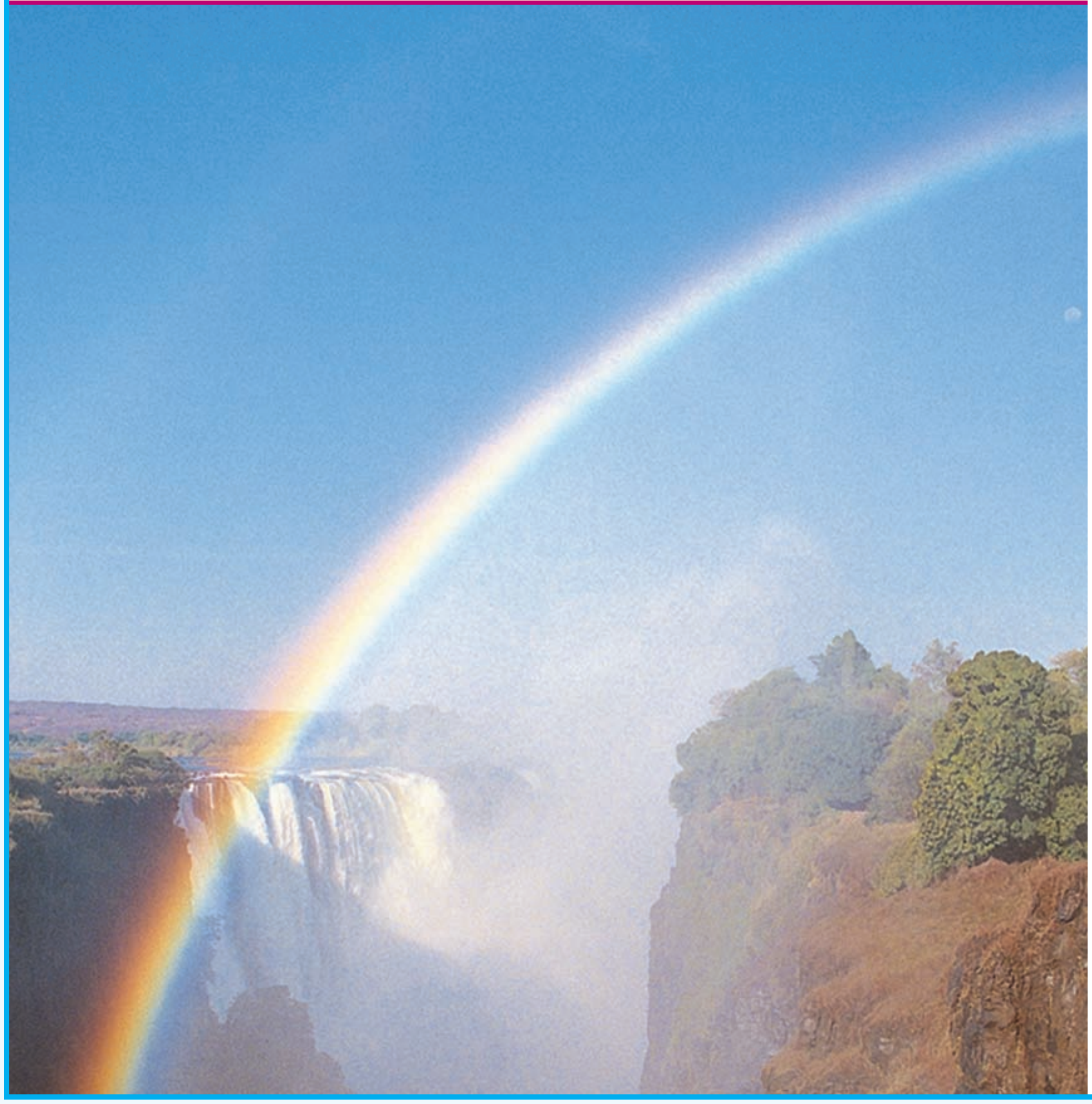
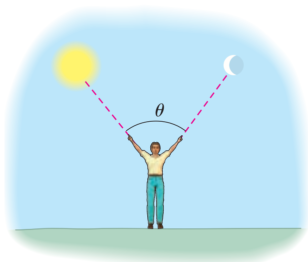
6 Trigonometric Functions of Angles



Chapter Overview

The trigonometric functions can be defined in two different but equivalent ways—as functions of real numbers (Chapter 5) or as functions of angles (Chapter 6). The two approaches to trigonometry are independent of each other, so either Chapter 5 or Chapter 6 may be studied first. We study both approaches because different applications require that we view these functions differently. The approach in this chapter lends itself to geometric problems involving finding angles and distances.

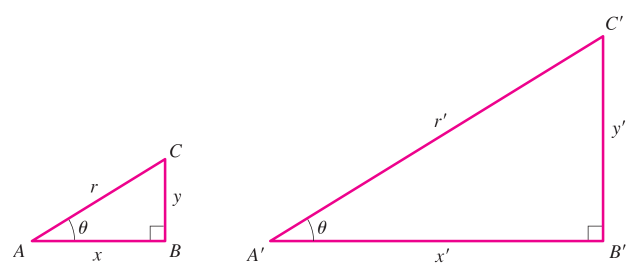


Suppose we want to find the distance to the sun. Using a tape measure is of course impractical, so we need something besides simple measurement to tackle this problem. Angles are easy to measure—for example, we can find the angle formed by the sun, earth, and moon by simply pointing to the sun with one arm and the moon with the other and estimating the angle between them. The key idea then is to find a relationship between angles and distances. So if we had a way to determine distances from angles, we'd be able to find the distance to the sun without going there. The trigonometric functions provide us with just the tools we need.

If ABC is a right triangle with acute angle θ as in the figure, then we define $\sin\theta$ to be the ratio y/r. Triangle $A'B'C'$ is similar to triangle ABC, so

$$\frac{y}{r} = \frac{y'}{r'}$$

Although the distances y' and r' are different from y and r, the given ratio is the same. Thus, in *any* right triangle with acute angle θ, the ratio of the side opposite angle θ to the hypotenuse is the same and is called $\sin\theta$. The other trigonometric ratios are defined in a similar fashion.



In this chapter we learn how trigonometric functions can be used to measure distances on the earth and in space. In Exercises 61 and 62 on page 487, we actually determine the distance to the sun using trigonometry. Right triangle trigonometry has many other applications, from determining the optimal cell structure in a beehive (Exercise 67, page 497) to explaining the shape of a rainbow (Exercise 69, page 498). In the *Focus on Modeling*, pages 522–523, we see how a surveyor uses trigonometry to map a town.

SUGGESTED TIME AND EMPHASIS

1 class.
Essential material.

POINTS TO STRESS

1. Degree and radian measurements of angles.
2. Coterminal angles.
3. Arc lengths.
4. Linear and angular speed.

6.1 Angle Measure

An **angle** AOB consists of two rays R_1 and R_2 with a common vertex O (see Figure 1). We often interpret an angle as a rotation of the ray R_1 onto R_2. In this case, R_1 is called the **initial side**, and R_2 is called the **terminal side** of the angle. If the rotation is counterclockwise, the angle is considered **positive**, and if the rotation is clockwise, the angle is considered **negative**.

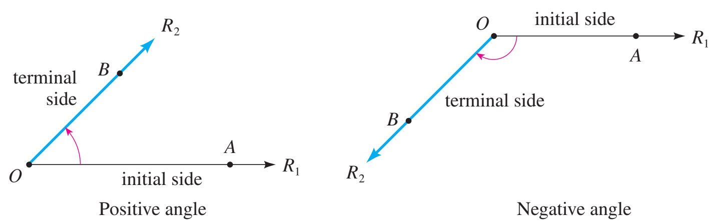


Figure 1

Angle Measure

The **measure** of an angle is the amount of rotation about the vertex required to move R_1 onto R_2. Intuitively, this is how much the angle "opens." One unit of measurement for angles is the **degree**. An angle of measure 1 degree is formed by rotating the initial side $\frac{1}{360}$ of a complete revolution. In calculus and other branches of mathematics, a more natural method of measuring angles is used—*radian measure*. The amount an angle opens is measured along the arc of a circle of radius 1 with its center at the vertex of the angle.

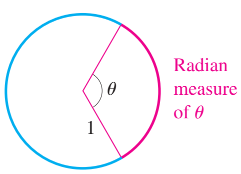


Figure 2

Definition of Radian Measure

If a circle of radius 1 is drawn with the vertex of an angle at its center, then the measure of this angle in **radians** (abbreviated **rad**) is the length of the arc that subtends the angle (see Figure 2).

The circumference of the circle of radius 1 is 2π and so a complete revolution has measure 2π rad, a straight angle has measure π rad, and a right angle has measure

SAMPLE QUESTIONS

Text Questions

(a) Which two of the following angles (given in radian measure) are coterminal?

$$0 \qquad \frac{\pi}{2} \qquad \pi \qquad \frac{3\pi}{2} \qquad 2\pi$$

(b) Which two of the following angles are coterminal?

$$45° \qquad -45° \qquad 180° \qquad -180° \qquad 0°$$

Answers

(a) 0 and 2π
(b) 180° and −180°

$\pi/2$ rad. An angle that is subtended by an arc of length 2 along the unit circle has radian measure 2 (see Figure 3).

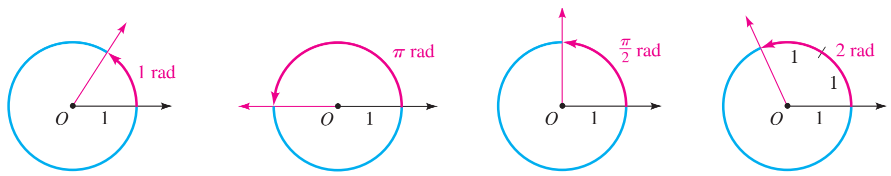


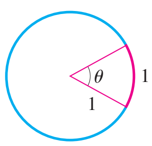
Figure 3
Radian measure

Since a complete revolution measured in degrees is 360° and measured in radians is 2π rad, we get the following simple relationship between these two methods of angle measurement.

Relationship between Degrees and Radians

$$180° = \pi \text{ rad} \qquad 1 \text{ rad} = \left(\frac{180}{\pi}\right)^{\circ} \qquad 1° = \frac{\pi}{180} \text{ rad}$$

1. To convert degrees to radians, multiply by $\frac{\pi}{180}$.
2. To convert radians to degrees, multiply by $\frac{180}{\pi}$.



Measure of $\theta = 1$ rad
Measure of $\theta \approx 57.296°$

Figure 4

To get some idea of the size of a radian, notice that

$$1 \text{ rad} \approx 57.296° \qquad \text{and} \qquad 1° \approx 0.01745 \text{ rad}$$

An angle θ of measure 1 rad is shown in Figure 4.

Example 1 Converting between Radians and Degrees

(a) Express 60° in radians. (b) Express $\frac{\pi}{6}$ rad in degrees.

Solution The relationship between degrees and radians gives

(a) $60° = 60\left(\frac{\pi}{180}\right) \text{ rad} = \frac{\pi}{3} \text{ rad}$ (b) $\frac{\pi}{6} \text{ rad} = \left(\frac{\pi}{6}\right)\left(\frac{180}{\pi}\right) = 30°$ ■

A note on terminology: We often use a phrase such as "a 30° angle" to mean *an angle whose measure is* 30°. Also, for an angle θ, we write $\theta = 30°$ or $\theta = \pi/6$ to mean *the measure of* θ *is* 30° *or* $\pi/6$ *rad*. When no unit is given, the angle is assumed to be measured in radians.

DRILL QUESTION

Find an angle with measure (in radians) between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ that is coterminal with the angle of measure $\frac{11\pi}{3}$ in standard position.

Answer

$-\frac{\pi}{3}$

ALTERNATE EXAMPLE 1a
Express 20° in radians.

ANSWER
$\frac{\pi}{9}$

ALTERNATE EXAMPLE 1b
Express $\frac{\pi}{9}$ rad in degrees.

ANSWER
20°

IN-CLASS MATERIALS

Make an analogy between angle measurement and clock measurement. In 48 hours 15 minutes, how far will the minute hand have moved? One answer is that it will go around the clock face $48\frac{1}{4}$ times. Another answer is that it moved one quarter of the way around the clock face, relative to where it started.

Angles in Standard Position

An angle is in **standard position** if it is drawn in the xy-plane with its vertex at the origin and its initial side on the positive x-axis. Figure 5 gives examples of angles in standard position.

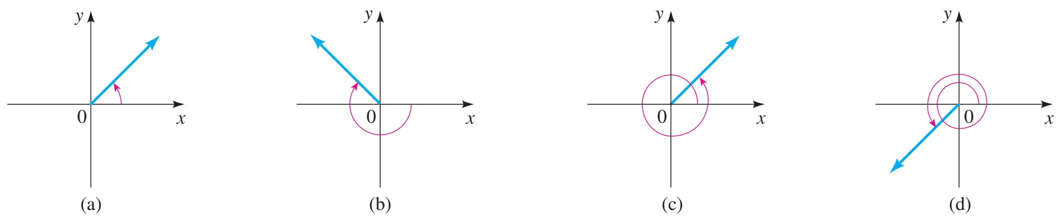


Figure 5
Angles in standard position

Two angles in standard position are **coterminal** if their sides coincide. In Figure 5 the angles in (a) and (c) are coterminal.

Example 2 Coterminal Angles

(a) Find angles that are coterminal with the angle $\theta = 30°$ in standard position.

(b) Find angles that are coterminal with the angle $\theta = \dfrac{\pi}{3}$ in standard position.

Solution

(a) To find positive angles that are coterminal with θ, we add any multiple of 360°. Thus

$$30° + 360° = 390° \qquad \text{and} \qquad 30° + 720° = 750°$$

are coterminal with $\theta = 30°$. To find negative angles that are coterminal with θ, we subtract any multiple of 360°. Thus

$$30° - 360° = -330° \qquad \text{and} \qquad 30° - 720° = -690°$$

are coterminal with θ. (See Figure 6.)

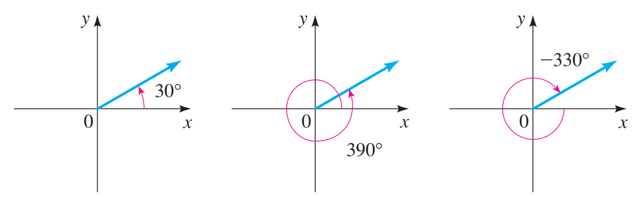


Figure 6

(b) To find positive angles that are coterminal with θ, we add any multiple of 2π. Thus

$$\frac{\pi}{3} + 2\pi = \frac{7\pi}{3} \qquad \text{and} \qquad \frac{\pi}{3} + 4\pi = \frac{13\pi}{3}$$

EXAMPLE
Find an angle between 0 and 2π that is coterminal with 100.

ANSWER
$100 - 30\pi \approx 5.7522$

ALTERNATE EXAMPLE 2

(a) Find angles that are coterminal with the angle $\theta = 62°$ in standard position.

(b) Find angles that are coterminal with the angle $\theta = \dfrac{5\pi}{6}$ in standard position.

ANSWERS

(a) We add any positive or negative multiple of 360° to 62°. Answers include 422° and –298°.

(b) We add any positive or negative multiple of 2π to $\dfrac{5\pi}{6}$. Answers include $\dfrac{17\pi}{6}$ and $-\dfrac{7\pi}{6}$.

EXAMPLE
Find an angle between 0° and 360° that is coterminal with −3624°.

ANSWER
336°

EXAMPLE
We can approximate the shape of the Earth by a sphere of radius 3960 miles. If we wanted to walk far enough to traverse exactly one degree in latitude, how far a trip would we have to take?

ANSWER
$3960 \cdot 1 \cdot \dfrac{\pi}{180} \approx 69$ miles

are coterminal with $\theta = \pi/3$. To find negative angles that are coterminal with θ, we subtract any multiple of 2π. Thus

$$\frac{\pi}{3} - 2\pi = -\frac{5\pi}{3} \quad \text{and} \quad \frac{\pi}{3} - 4\pi = -\frac{11\pi}{3}$$

are coterminal with θ. (See Figure 7.)

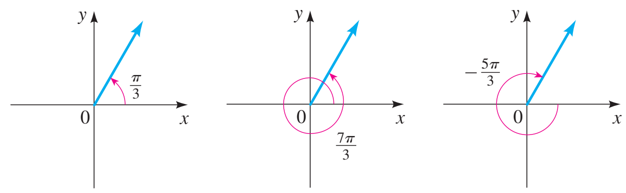


Figure 7

Example 3 Coterminal Angles

Find an angle with measure between 0° and 360° that is coterminal with the angle of measure 1290° in standard position.

Solution We can subtract 360° as many times as we wish from 1290°, and the resulting angle will be coterminal with 1290°. Thus, 1290° − 360° = 930° is coterminal with 1290°, and so is the angle 1290° − 2(360)° = 570°.

To find the angle we want between 0° and 360°, we subtract 360° from 1290° as many times as necessary. An efficient way to do this is to determine how many times 360° goes into 1290°, that is, divide 1290 by 360, and the remainder will be the angle we are looking for. We see that 360 goes into 1290 three times with a remainder of 210. Thus, 210° is the desired angle (see Figure 8).

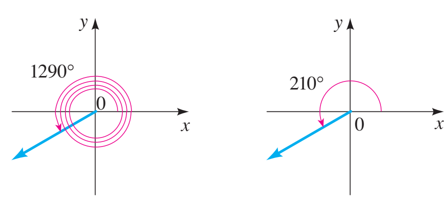


Figure 8

ALTERNATE EXAMPLE 3
Find an angle with measure between 0° and 360° that is coterminal with the angle of measure 1635° in standard position.

ANSWER
195°

EXAMPLE
Find an angle between 0 and 2π that is coterminal with $\frac{88\pi}{3}$.

ANSWER
$\frac{4\pi}{3}$

Length of a Circular Arc

An angle whose radian measure is θ is subtended by an arc that is the fraction $\theta/(2\pi)$ of the circumference of a circle. Thus, in a circle of radius r, the length s of an arc that subtends the angle θ (see Figure 9) is

$$s = \frac{\theta}{2\pi} \times \text{circumference of circle}$$

$$= \frac{\theta}{2\pi}(2\pi r) = \theta r$$

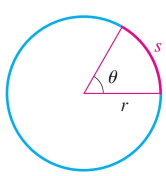


Figure 9
$s = \theta r$

IN-CLASS MATERIALS

Students often get the false impression that angles with integer measure are always in degrees, and that angles with measures that are rational multiples of π are in radians. Ask students to sketch a 1° angle and then a 1 radian angle. Similarly, ask students to sketch angles with measures $\frac{\pi}{2}$ radians and $\frac{\pi}{2}$ degrees.

IN-CLASS MATERIALS

Show that the familiar formulas $A = \pi r^2$ and $C = 2\pi r$ are actually special cases of formulas in this section.

Length of a Circular Arc

In a circle of radius r, the length s of an arc that subtends a central angle of θ radians is

$$s = r\theta$$

Solving for θ, we get the important formula

$$\boxed{\theta = \frac{s}{r}}$$

This formula allows us to define radian measure using a circle of any radius r: The radian measure of an angle θ is s/r, where s is the length of the circular arc that subtends θ in a circle of radius r (see Figure 10).

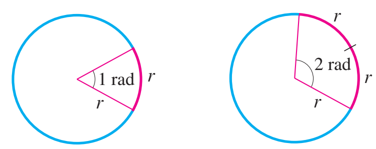


Figure 10
The radian measure of θ is the number of "radiuses" that can fit in the arc that subtends θ; hence the term *radian*.

ALTERNATE EXAMPLE 4a
Find the length of an arc of a circle with radius 21 m that subtends a central angle of 15°.

ANSWER
$\frac{7\pi}{4}$

ALTERNATE EXAMPLE 4b
A central angle θ in a circle of radius 9 m is subtended by an arc of length 12 m. Find the measure of θ in radians.

ANSWER
$\frac{4}{3}$

Example 4 Arc Length and Angle Measure

(a) Find the length of an arc of a circle with radius 10 m that subtends a central angle of 30°.

(b) A central angle θ in a circle of radius 4 m is subtended by an arc of length 6 m. Find the measure of θ in radians.

Solution

(a) From Example 1(b) we see that $30° = \pi/6$ rad. So the length of the arc is

$$s = r\theta = (10)\frac{\pi}{6} = \frac{5\pi}{3} \text{ m}$$

The formula $s = r\theta$ is true only when θ is measured in radians.

(b) By the formula $\theta = s/r$, we have

$$\theta = \frac{s}{r} = \frac{6}{4} = \frac{3}{2} \text{ rad}$$ ■

Area of a Circular Sector

The area of a circle of radius r is $A = \pi r^2$. A sector of this circle with central angle θ has an area that is the fraction $\theta/(2\pi)$ of the area of the entire circle (see Figure 11). So the area of this sector is

$$A = \frac{\theta}{2\pi} \times \text{area of circle}$$

$$= \frac{\theta}{2\pi}(\pi r^2) = \frac{1}{2}r^2\theta$$

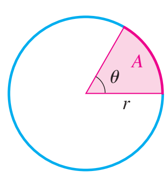


Figure 11
$A = \frac{1}{2}r^2\theta$

IN-CLASS MATERIALS

The formula for the length of a circular arc can be verified experimentally. Have students either draw circles or use trash can lids, soup cans, plastic cups, pie tins, baking pans, or what have you. Measure arc length and radius for 90° angles, 45° angles, and so on, verifying that $s = r\theta$.

Area of a Circular Sector

In a circle of radius r, the area A of a sector with a central angle of θ radians is

$$A = \frac{1}{2}r^2\theta$$

Example 5 Area of a Sector

Find the area of a sector of a circle with central angle 60° if the radius of the circle is 3 m.

ALTERNATE EXAMPLE 5
Find the area of a sector of a circle with central angle 4° if the radius of the circle is 45 m.

ANSWER
$\frac{45\pi}{2}$

Solution To use the formula for the area of a circular sector, we must find the central angle of the sector in radians: $60° = 60(\pi/180)$ rad $= \pi/3$ rad. Thus, the area of the sector is

$$A = \frac{1}{2}r^2\theta = \frac{1}{2}(3)^2\left(\frac{\pi}{3}\right) = \frac{3\pi}{2}\text{ m}^2$$ ■

The formula $A = \frac{1}{2}r^2\theta$ is true only when θ is measured in radians.

Circular Motion

Suppose a point moves along a circle as shown in Figure 12. There are two ways to describe the motion of the point—linear speed and angular speed. **Linear speed** is the rate at which the distance traveled is changing, so linear speed is the distance traveled divided by the time elapsed. **Angular speed** is the rate at which the central angle θ is changing, so angular speed is the number of radians this angle changes divided by the time elapsed.

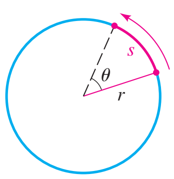


Figure 12

Linear Speed and Angular Speed

Suppose a point moves along a circle of radius r and the ray from the center of the circle to the point traverses θ radians in time t. Let $s = r\theta$ be the distance the point travels in time t. Then the speed of the object is given by

Angular speed $\qquad \omega = \dfrac{\theta}{t}$

Linear speed $\qquad v = \dfrac{s}{t}$

The symbol ω is the Greek letter "omega."

Example 6 Finding Linear and Angular Speed

A boy rotates a stone in a 3-ft-long sling at the rate of 15 revolutions every 10 seconds. Find the angular and linear velocities of the stone.

Solution In 10 s, the angle θ changes by $15 \cdot 2\pi = 30\pi$ radians. So the *angular speed* of the stone is

$$\omega = \frac{\theta}{t} = \frac{30\pi\text{ rad}}{10\text{ s}} = 3\pi\text{ rad/s}$$

ALTERNATE EXAMPLE 6
A disk with a 12-inch diameter spins at the rate of 45 revolutions per minute. Find the angular and linear velocities of a point at the edge of the disk in radians per second and inches per second, respectively.

ANSWER
angular speed $= \frac{3}{2}\pi$ rad/s
linear speed $= 18\pi$ in/s

IN-CLASS MATERIALS

One can demonstrate the relationship between linear and angular speed this way: The teacher sits in a rotating office chair, and spins at a rate of, say, 2π radians per five seconds. Now have one student stand close to the professor, and one stand farther away, and have them both try to run in circles at that rate. Note that the one farther away has to travel much faster, demonstrating that if the angular speed is held constant, and r is increased, then v is increased as well.

ALTERNATE EXAMPLE 7
A woman is riding a bicycle whose wheels are 30 inches in diameter. If the wheels rotate at 150 revolutions per minute, find the speed at which she is traveling in miles per hour.

ANSWER
13.4 mph

The distance traveled by the stone in 10 s is $s = 15 \cdot 2\pi r = 15 \cdot 2\pi \cdot 3 = 90\pi$ ft. So the *linear speed* of the stone is

$$v = \frac{s}{t} = \frac{90\pi \text{ ft}}{10 \text{ s}} = 9\pi \text{ ft/s}$$ ■

Notice that angular speed does *not* depend on the radius of the circle, but only on the angle θ. However, if we know the angular speed ω and the radius r, we can find linear speed as follows: $v = s/t = r\theta/t = r(\theta/t) = r\omega$.

Relationship between Linear and Angular Speed

If a point moves along a circle of radius r with angular speed ω, then its linear speed v is given by

$$v = r\omega$$

Example 7 Finding Linear Speed from Angular Speed

A woman is riding a bicycle whose wheels are 26 inches in diameter. If the wheels rotate at 125 revolutions per minute (rpm), find the speed at which she is traveling, in mi/h.

Solution The angular speed of the wheels is $2\pi \cdot 125 = 250\pi$ rad/min. Since the wheels have radius 13 in. (half the diameter), the linear speed is

$$v = r\omega = 13 \cdot 250\pi \approx 10{,}210.2 \text{ in./min}$$

Since there are 12 inches per foot, 5280 feet per mile, and 60 minutes per hour, her speed in miles per hour is

$$\frac{10{,}210.2 \text{ in./min} \times 60 \text{ min/h}}{12 \text{ in./ft} \times 5280 \text{ ft/mi}} = \frac{612{,}612 \text{ in./h}}{63{,}360 \text{ in./mi}}$$
$$\approx 9.7 \text{ mi/h}$$ ■

6.1 Exercises

1–12 ■ Find the radian measure of the angle with the given degree measure.

1. $72°$ **2.** $54°$ **3.** $-45°$
4. $-60°$ **5.** $-75°$ **6.** $-300°$
7. $1080°$ **8.** $3960°$ **9.** $96°$
10. $15°$ **11.** $7.5°$ **12.** $202.5°$

13–24 ■ Find the degree measure of the angle with the given radian measure.

13. $\frac{7\pi}{6}$ **14.** $\frac{11\pi}{3}$ **15.** $-\frac{5\pi}{4}$
16. $-\frac{3\pi}{2}$ **17.** 3 **18.** -2
19. -1.2 **20.** 3.4 **21.** $\frac{\pi}{10}$
22. $\frac{5\pi}{18}$ **23.** $-\frac{2\pi}{15}$ **24.** $-\frac{13\pi}{12}$

25–30 ■ The measure of an angle in standard position is given. Find two positive angles and two negative angles that are coterminal with the given angle.

25. $50°$ **26.** $135°$ **27.** $\frac{3\pi}{4}$

IN-CLASS MATERIALS

If the students have taken some physics, present them with this paradox: We know that Einstein said that the speed of an object cannot exceed 186,000 miles per second, regardless of reference frame. Let the reference frame be a basketball. Have them calculate how fast the sun is moving if the basketball is spinning on a finger, relative to the frame of reference of the basketball. Then have them calculate how fast the stars are moving. These numbers will be faster than the speed of light! This “paradox” is actually a misunderstanding of relativity. The rules about the speed of light being a maximum, and most of relativity in general, applies to what is called an “inertial” reference frame—one where objects in motion remain in motion, and objects at rest remain at rest. The reference frames of a spinning basketball or an accelerating train are not inertial reference frames.

28. $\dfrac{11\pi}{6}$ **29.** $-\dfrac{\pi}{4}$ **30.** $-45°$

31–36 ■ The measures of two angles in standard position are given. Determine whether the angles are coterminal.

31. $70°$, $430°$ **32.** $-30°$, $330°$

33. $\dfrac{5\pi}{6}$, $\dfrac{17\pi}{6}$ **34.** $\dfrac{32\pi}{3}$, $\dfrac{11\pi}{3}$

35. $155°$, $875°$ **36.** $50°$, $340°$

37–42 ■ Find an angle between 0° and 360° that is coterminal with the given angle.

37. $733°$ **38.** $361°$

39. $1110°$ **40.** $-100°$

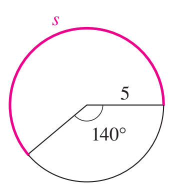
41. $-800°$ **42.** $1270°$

43–48 ■ Find an angle between 0 and 2π that is coterminal with the given angle.

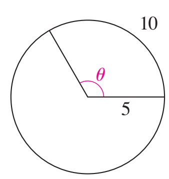
43. $\dfrac{17\pi}{6}$ **44.** $-\dfrac{7\pi}{3}$ **45.** 87π

46. 10 **47.** $\dfrac{17\pi}{4}$ **48.** $\dfrac{51\pi}{2}$

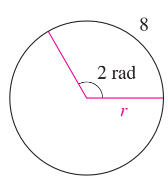
49. Find the length of the arc s in the figure.



50. Find the angle θ in the figure.



51. Find the radius r of the circle in the figure.



52. Find the length of an arc that subtends a central angle of 45° in a circle of radius 10 m.

53. Find the length of an arc that subtends a central angle of 2 rad in a circle of radius 2 mi.

54. A central angle θ in a circle of radius 5 m is subtended by an arc of length 6 m. Find the measure of θ in degrees and in radians.

55. An arc of length 100 m subtends a central angle θ in a circle of radius 50 m. Find the measure of θ in degrees and in radians.

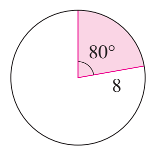
56. A circular arc of length 3 ft subtends a central angle of 25°. Find the radius of the circle.

57. Find the radius of the circle if an arc of length 6 m on the circle subtends a central angle of $\pi/6$ rad.

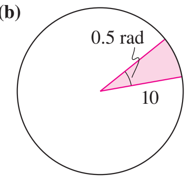
58. Find the radius of the circle if an arc of length 4 ft on the circle subtends a central angle of 135°.

59. Find the area of the sector shown in each figure.

(a)

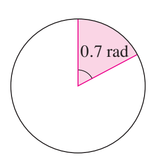


(b)

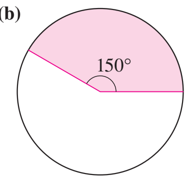


60. Find the radius of each circle if the area of the sector is 12.

(a)



(b)



61. Find the area of a sector with central angle 1 rad in a circle of radius 10 m.

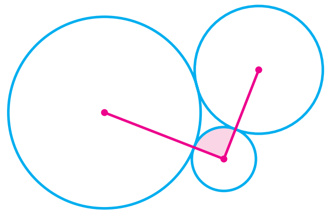
62. A sector of a circle has a central angle of 60°. Find the area of the sector if the radius of the circle is 3 mi.

63. The area of a sector of a circle with a central angle of 2 rad is 16 m^2. Find the radius of the circle.

64. A sector of a circle of radius 24 mi has an area of 288 mi^2. Find the central angle of the sector.

65. The area of a circle is 72 cm^2. Find the area of a sector of this circle that subtends a central angle of $\pi/6$ rad.

66. Three circles with radii 1, 2, and 3 ft are externally tangent to one another, as shown in the figure on the next page. Find the area of the sector of the circle of radius 1 that is cut off by the line segments joining the center of that circle to the centers of the other two circles.

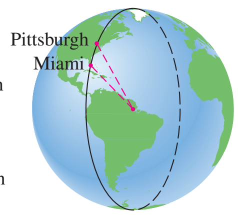


Applications

67. Travel Distance A car's wheels are 28 in. in diameter. How far (in miles) will the car travel if its wheels revolve 10,000 times without slipping?

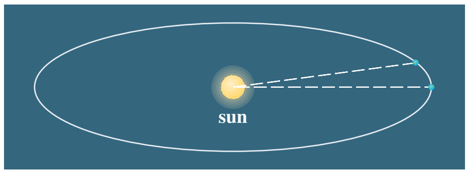
68. Wheel Revolutions How many revolutions will a car wheel of diameter 30 in. make as the car travels a distance of one mile?

69. Latitudes Pittsburgh, Pennsylvania, and Miami, Florida, lie approximately on the same meridian. Pittsburgh has a latitude of 40.5° N and Miami, 25.5° N. Find the distance between these two cities. (The radius of the earth is 3960 mi.)

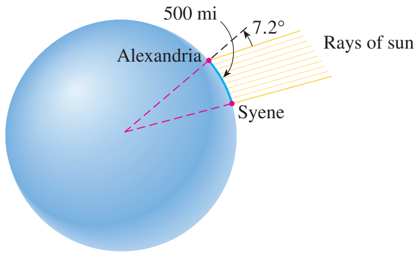


70. Latitudes Memphis, Tennessee, and New Orleans, Louisiana, lie approximately on the same meridian. Memphis has latitude 35° N and New Orleans, 30° N. Find the distance between these two cities. (The radius of the earth is 3960 mi.)

71. Orbit of the Earth Find the distance that the earth travels in one day in its path around the sun. Assume that a year has 365 days and that the path of the earth around the sun is a circle of radius 93 million miles. [The path of the earth around the sun is actually an *ellipse* with the sun at one focus (see Section 10.2). This ellipse, however, has very small eccentricity, so it is nearly circular.]

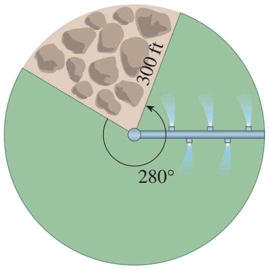


72. Circumference of the Earth The Greek mathematician Eratosthenes (ca. 276–195 B.C.) measured the circumference of the earth from the following observations. He noticed that on a certain day the sun shone directly down a deep well in Syene (modern Aswan). At the same time in Alexandria, 500 miles north (on the same meridian), the rays of the sun shone at an angle of 7.2° to the zenith. Use this information and the figure to find the radius and circumference of the earth.

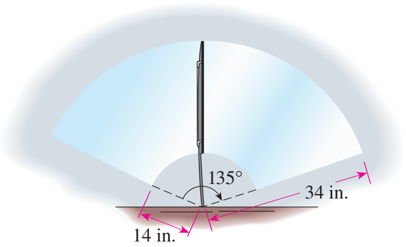


73. Nautical Miles Find the distance along an arc on the surface of the earth that subtends a central angle of 1 minute (1 minute $= \frac{1}{60}$ degree). This distance is called a *nautical mile*. (The radius of the earth is 3960 mi.)

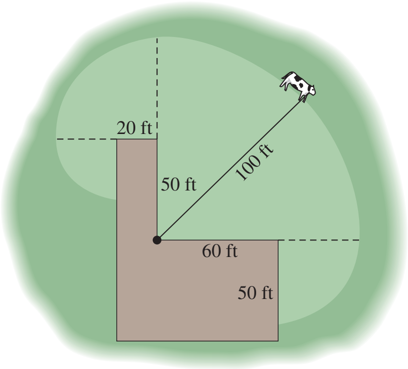
74. Irrigation An irrigation system uses a straight sprinkler pipe 300 ft long that pivots around a central point as shown. Due to an obstacle the pipe is allowed to pivot through 280° only. Find the area irrigated by this system.



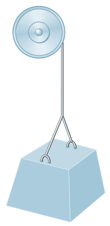
75. Windshield Wipers The top and bottom ends of a windshield wiper blade are 34 in. and 14 in. from the pivot point, respectively. While in operation the wiper sweeps through 135°. Find the area swept by the blade.



76. The Tethered Cow A cow is tethered by a 100-ft rope to the inside corner of an L-shaped building, as shown in the figure. Find the area that the cow can graze.



77. Winch A winch of radius 2 ft is used to lift heavy loads. If the winch makes 8 revolutions every 15 s, find the speed at which the load is rising.



78. Fan A ceiling fan with 16-in. blades rotates at 45 rpm.

(a) Find the angular speed of the fan in rad/min.

(b) Find the linear speed of the tips of the blades in in./min.

79. Radial Saw A radial saw has a blade with a 6-in. radius. Suppose that the blade spins at 1000 rpm.

(a) Find the angular speed of the blade in rad/min.

(b) Find the linear speed of the sawteeth in ft/s.

80. Speed at Equator The earth rotates about its axis once every 23 h 56 min 4 s, and the radius of the earth is 3960 mi. Find the linear speed of a point on the equator in mi/h.

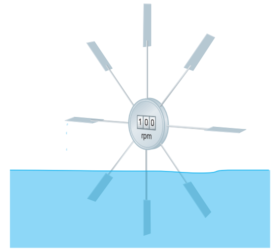
81. Speed of a Car The wheels of a car have radius 11 in. and are rotating at 600 rpm. Find the speed of the car in mi/h.

82. Truck Wheels A truck with 48-in.-diameter wheels is traveling at 50 mi/h.

(a) Find the angular speed of the wheels in rad/min.

(b) How many revolutions per minute do the wheels make?

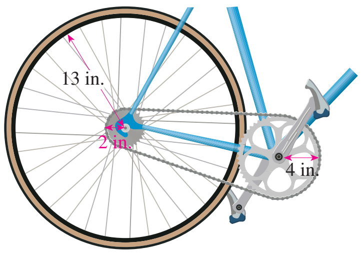
83. Speed of a Current To measure the speed of a current, scientists place a paddle wheel in the stream and observe the rate at which it rotates. If the paddle wheel has radius 0.20 m and rotates at 100 rpm, find the speed of the current in m/s.



84. Bicycle Wheel The sprockets and chain of a bicycle are shown in the figure. The pedal sprocket has a radius of 4 in., the wheel sprocket a radius of 2 in., and the wheel a radius of 13 in. The cyclist pedals at 40 rpm.

(a) Find the angular speed of the wheel sprocket.

(b) Find the speed of the bicycle. (Assume that the wheel turns at the same rate as the wheel sprocket.)



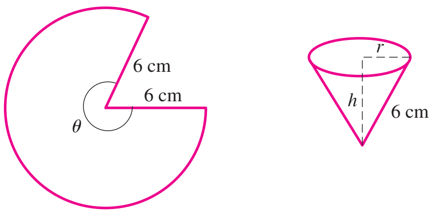
85. Conical Cup A conical cup is made from a circular piece of paper with radius 6 cm by cutting out a sector and joining the edges as shown. Suppose $\theta = 5\pi/3$.

(a) Find the circumference C of the opening of the cup.

(b) Find the radius r of the opening of the cup. [*Hint*: Use $C = 2\pi r$.]

(c) Find the height h of the cup. [*Hint*: Use the Pythagorean Theorem.]

(d) Find the volume of the cup.



86. Conical Cup In this exercise we find the volume of the conical cup in Exercise 85 for any angle θ.

(a) Follow the steps in Exercise 85 to show that the volume of the cup as a function of θ is

$$V(\theta) = \frac{9}{\pi^2}\theta^2\sqrt{4\pi^2 - \theta^2}, \qquad 0 < \theta < 2\pi$$

(b) Graph the function V.

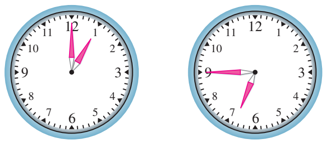
(c) For what angle θ is the volume of the cup a maximum?

Discovery • Discussion

87. Different Ways of Measuring Angles The custom of measuring angles using degrees, with 360° in a circle, dates back to the ancient Babylonians, who used a number system based on groups of 60. Another system of measuring angles divides the circle into 400 units, called *grads*. In this system a right angle is 100 grad, so this fits in with our base 10 number system.

Write a short essay comparing the advantages and disadvantages of these two systems and the radian system of measuring angles. Which system do you prefer?

88. Clocks and Angles In one hour, the minute hand on a clock moves through a complete circle, and the hour hand moves through $\frac{1}{12}$ of a circle. Through how many radians do the minute and the hour hand move between 1:00 P.M. and 6:45 P.M. (on the same day)?



SUGGESTED TIME AND EMPHASIS

$\frac{1}{2}$–1 class.
Essential material.

POINTS TO STRESS

1. Definition of the six trigonometric functions as ratios of sides of right triangles.
2. Special triangles: 30°–60°–90° and 45°–45°–90°.
3. Applications that involve solving right triangles.

6.2 Trigonometry of Right Triangles

In this section we study certain ratios of the sides of right triangles, called trigonometric ratios, and give several applications.

Trigonometric Ratios

Consider a right triangle with θ as one of its acute angles. The trigonometric ratios are defined as follows (see Figure 1).

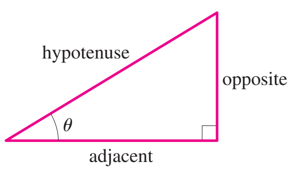


Figure 1

The Trigonometric Ratios

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

$$\csc\theta = \frac{\text{hypotenuse}}{\text{opposite}} \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}} \qquad \cot\theta = \frac{\text{adjacent}}{\text{opposite}}$$

The symbols we use for these ratios are abbreviations for their full names: **sine**, **cosine**, **tangent**, **cosecant**, **secant**, **cotangent**. Since any two right triangles with

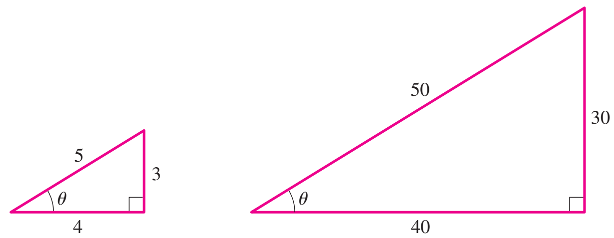
IN-CLASS MATERIALS

Note: If Chapter 5 was covered, this point may have already been discussed.

The etymology of the word *sine* is fairly interesting and not that well-known. It starts with the Indian word *jya*, meaning "cord of a bowstring." The Arabs translated this word as *jiba*. The written language had no vowels, so it looked like this: jb. In 1145, the Spanish translator Robert of Cheste had to figure out what vowels to put in. Due to the shape of the curve, he thought the word was *jaib*, which meant the opening of a garment that shows a woman's cleavage. So he used the Latin word for the cavity formed by a curve: *sinus*. (This word exists in the English language today—right behind your nose are the sinus cavities.)

Hipparchus (circa 140 B.C.) is considered the founder of trigonometry. He constructed tables for a function closely related to the modern sine function and evaluated for angles at half-degree intervals. These are considered the first trigonometric tables. He used his tables mainly to calculate the paths of the planets through the heavens.

angle θ are similar, these ratios are the same, regardless of the size of the triangle; the trigonometric ratios depend only on the angle θ (see Figure 2).



$\sin\theta = \frac{3}{5}$ $\qquad$ $\sin\theta = \frac{30}{50} = \frac{3}{5}$

$\cos\theta = \frac{4}{5}$ $\qquad$ $\cos\theta = \frac{40}{50} = \frac{4}{5}$

Figure 2

Example 1 Finding Trigonometric Ratios

Find the six trigonometric ratios of the angle θ in Figure 3.

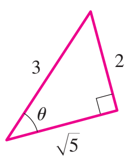


Figure 3

Solution

$$\sin\theta = \frac{2}{3} \qquad \cos\theta = \frac{\sqrt{5}}{3} \qquad \tan\theta = \frac{2}{\sqrt{5}}$$

$$\csc\theta = \frac{3}{2} \qquad \sec\theta = \frac{3}{\sqrt{5}} \qquad \cot\theta = \frac{\sqrt{5}}{2}$$

■

Example 2 Finding Trigonometric Ratios

If $\cos\alpha = \frac{3}{4}$, sketch a right triangle with acute angle α, and find the other five trigonometric ratios of α.

Solution Since $\cos\alpha$ is defined as the ratio of the adjacent side to the hypotenuse, we sketch a triangle with hypotenuse of length 4 and a side of length 3 adjacent to α. If the opposite side is x, then by the Pythagorean Theorem, $3^2 + x^2 = 4^2$ or $x^2 = 7$, so $x = \sqrt{7}$. We then use the triangle in Figure 4 to find the ratios.

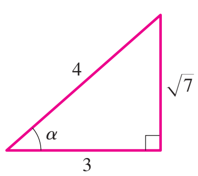


Figure 4

$$\sin\alpha = \frac{\sqrt{7}}{4} \qquad \cos\alpha = \frac{3}{4} \qquad \tan\alpha = \frac{\sqrt{7}}{3}$$

$$\csc\alpha = \frac{4}{\sqrt{7}} \qquad \sec\alpha = \frac{4}{3} \qquad \cot\alpha = \frac{3}{\sqrt{7}}$$

■

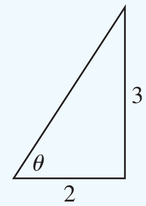
Special Triangles

Certain right triangles have ratios that can be calculated easily from the Pythagorean Theorem. Since they are used frequently, we mention them here.

The first triangle is obtained by drawing a diagonal in a square of side 1 (see Figure 5 on page 480). By the Pythagorean Theorem this diagonal has length $\sqrt{2}$. The

DRILL QUESTION

Consider this triangle:



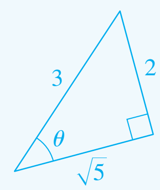
Find $\sin\theta$, $\sec\theta$, and $\tan\theta$.

Answer

$\frac{3}{\sqrt{13}}, \frac{\sqrt{13}}{2}, \frac{3}{2}$

ALTERNATE EXAMPLE 1

Find the six trigonometric ratios of the angle θ in the figure below.



ANSWER

$\sin(\theta) = \frac{2}{3}$, $\cos(\theta) = \frac{\sqrt{5}}{3}$, $\tan(\theta) = \frac{2}{\sqrt{5}}$, $\cot(\theta) = \frac{\sqrt{5}}{2}$, $\sec(\theta) = \frac{3}{\sqrt{5}}$, $\csc(\theta) = \frac{3}{2}$

ALTERNATE EXAMPLE 2

Consider a right triangle with α as one of its acute angles. If $\cos\alpha = \frac{7}{8}$, find the other five trigonometric ratios of α.

ANSWER

$\sin(\alpha) = \frac{\sqrt{15}}{8}$, $\tan(\alpha) = \frac{\sqrt{15}}{7}$, $\csc(\alpha) = \frac{8}{\sqrt{15}}$, $\sec(\alpha) = \frac{8}{7}$, $\cot(\alpha) = \frac{7}{\sqrt{15}}$

IN-CLASS MATERIALS

Note: If Chapter 5 was covered, this point may have already been discussed.

There is an inconsistency in mathematical notation that can be made explicit at this time. We say that $\frac{1}{3} = 3^{-1}$. Similarly, we say that $\frac{1}{\pi} = \pi^{-1}$. But $\frac{1}{\sin x}$ is written as $\csc x$. Unfortunately, there is a symbol, $\sin^{-1} x$, that means something entirely different—the arcsine of x. Even worse, as noted in the section, $\sin^2 x = (\sin x)^2$ and $\sin^3 x = (\sin x)^3$. The exponent -1 is a notational anomaly.

resulting triangle has angles 45°, 45°, and 90° (or $\pi/4$, $\pi/4$, and $\pi/2$). To get the second triangle, we start with an equilateral triangle *ABC* of side 2 and draw the perpendicular bisector *DB* of the base, as in Figure 6. By the Pythagorean Theorem the length of *DB* is $\sqrt{3}$. Since *DB* bisects angle *ABC*, we obtain a triangle with angles 30°, 60°, and 90° (or $\pi/6$, $\pi/3$, and $\pi/2$).

Aristarchus of Samos (310–230 B.C.) was a famous Greek scientist, musician, astronomer, and geometer. In his book *On the Sizes and Distances of the Sun and the Moon*, he estimated the distance to the sun by observing that when the moon is exactly half full, the triangle formed by the sun, moon, and the earth has a right angle at the moon. His method was similar to the one described in Exercise 61 in this section. Aristarchus was the first to advance the theory that the earth and planets move around the sun, an idea that did not gain full acceptance until after the time of Copernicus, 1800 years later. For this reason he is often called the "Copernicus of antiquity."

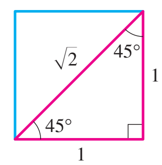


Figure 5

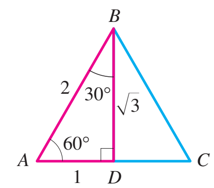


Figure 6

We can now use the special triangles in Figures 5 and 6 to calculate the trigonometric ratios for angles with measures 30°, 45°, and 60° (or $\pi/6$, $\pi/4$, and $\pi/3$). These are listed in Table 1.

Table 1 Values of the trigonometric ratios for special angles

θ in degrees	θ in radians	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$

It's useful to remember these special trigonometric ratios because they occur often. Of course, they can be recalled easily if we remember the triangles from which they are obtained.

For an explanation of numerical methods, see the margin note on page 436.

To find the values of the trigonometric ratios for other angles, we use a calculator. Mathematical methods (called *numerical methods*) used in finding the trigonometric ratios are programmed directly into scientific calculators. For instance, when the SIN key is pressed, the calculator computes an approximation to the value of the sine of the given angle. Calculators give the values of sine, cosine, and tangent; the other ratios can be easily calculated from these using the following *reciprocal relations*:

$$\csc t = \frac{1}{\sin t} \qquad \sec t = \frac{1}{\cos t} \qquad \cot t = \frac{1}{\tan t}$$

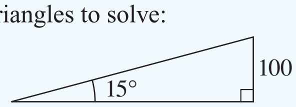
You should check that these relations follow immediately from the definitions of the trigonometric ratios.

We follow the convention that when we write $\sin t$, *we mean the sine of the angle whose radian measure is* t. For instance, sin 1 means the sine of the angle whose ra-

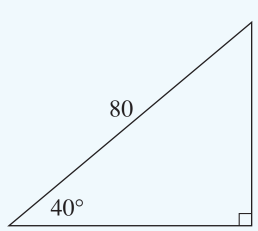
EXAMPLES

Triangles to solve:

1.

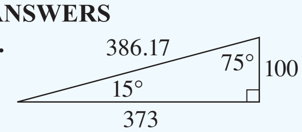


2.

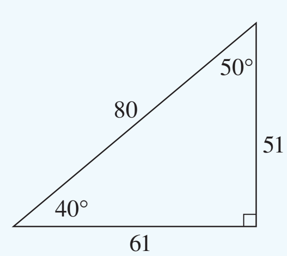


ANSWERS

1.



2.



dian measure is 1. When using a calculator to find an approximate value for this number, set your calculator to radian mode; you will find that

$$\sin 1 \approx 0.841471$$

If you want to find the sine of the angle whose measure is 1°, set your calculator to degree mode; you will find that

$$\sin 1° \approx 0.0174524$$

Example 3 Using a Calculator to Find Trigonometric Ratios

With our calculator in degree mode, and writing the results correct to five decimal places, we find

$$\sin 17° \approx 0.29237 \qquad \sec 88° = \frac{1}{\cos 88°} \approx 28.65371$$

With our calculator in radian mode, and writing the results correct to five decimal places, we find

$$\cos 1.2 \approx 0.36236 \qquad \cot 1.54 = \frac{1}{\tan 1.54} \approx 0.03081$$

■

ALTERNATE EXAMPLE 3
Use a calculator to find the trigonometric ratio: cot 1.83. Please give the answer to five decimal places.

ANSWER
–0.26517

Applications of Trigonometry of Right Triangles

A triangle has six parts: three angles and three sides. To **solve a triangle** means to determine all of its parts from the information known about the triangle, that is, to determine the lengths of the three sides and the measures of the three angles.

Example 4 Solving a Right Triangle

Solve triangle ABC, shown in Figure 7.

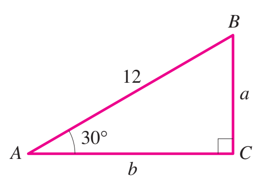


Figure 7

Solution It's clear that $\angle B = 60°$. To find a, we look for an equation that relates a to the lengths and angles we already know. In this case, we have $\sin 30° = a/12$, so

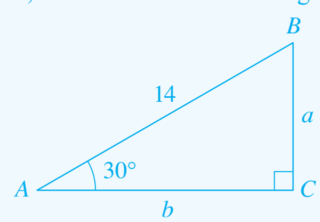
$$a = 12 \sin 30° = 12\left(\tfrac{1}{2}\right) = 6$$

Similarly, $\cos 30° = b/12$, so

$$b = 12 \cos 30° = 12\left(\frac{\sqrt{3}}{2}\right) = 6\sqrt{3}$$

■

ALTERNATE EXAMPLE 4
Solve triangle ABC, shown in the figure below (find the sides a and b, and the third unknown angle).



ANSWER
$7, 7\sqrt{3}, 60$

It's very useful to know that, using the information given in Figure 8, the lengths of the legs of a right triangle are

$$a = r \sin\theta \qquad \text{and} \qquad b = r\cos\theta$$

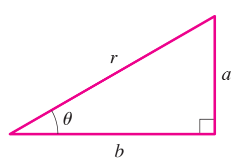


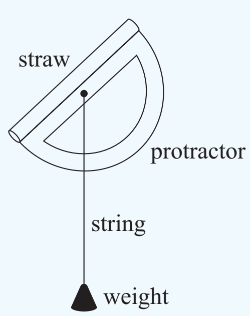
Figure 8
$a = r\sin\theta$
$b = r\cos\theta$

The ability to solve right triangles using the trigonometric ratios is fundamental to many problems in navigation, surveying, astronomy, and the measurement of distances. The applications we consider in this section always involve right triangles but, as we will see in the next three sections, trigonometry is also useful in solving triangles that are not right triangles.

To discuss the next examples, we need some terminology. If an observer is looking at an object, then the line from the eye of the observer to the object is called

IN-CLASS MATERIALS

It is relatively easy to make a device to measure angle of elevation. Tape a soda straw to the bottom of a protractor, and tie a string with a weight to the center. When you sight an object through the straw, you can figure out the angle by noticing where the string falls.



SAMPLE QUESTION
Text Question
Define "angle of elevation."

Answer
Answers will vary.

ALTERNATE EXAMPLE 5
A giant redwood tree casts a shadow 452 ft long. Find the height of the tree if the angle of elevation of the sun is 12.3°.

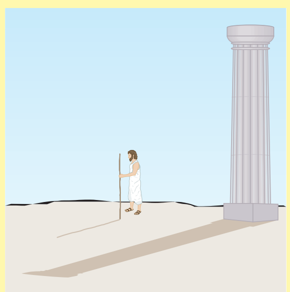
ANSWER
176 ft

Thales of Miletus (circa 625–547 B.C.) is the legendary founder of Greek geometry. It is said that he calculated the height of a Greek column by comparing the length of the shadow of his staff with that of the column. Using properties of similar triangles, he argued that the ratio of the height h of the column to the height h' of his staff was equal to the ratio of the length s of the column's shadow to the length s' of the staff's shadow:

$$\frac{h}{h'} = \frac{s}{s'}$$

Since three of these quantities are known, Thales was able to calculate the height of the column.

According to legend, Thales used a similar method to find the height of the Great Pyramid in Egypt, a feat that impressed Egypt's king. Plutarch wrote that "although he [the king of Egypt] admired you [Thales] for other things, yet he particularly liked the manner by which you measured the height of the pyramid without any trouble or instrument." The principle Thales used, the fact that ratios of corresponding sides of similar triangles are equal, is the foundation of the subject of trigonometry.



the **line of sight** (Figure 9). If the object being observed is above the horizontal, then the angle between the line of sight and the horizontal is called the **angle of elevation**. If the object is below the horizontal, then the angle between the line of sight and the horizontal is called the **angle of depression**. In many of the examples and exercises in this chapter, angles of elevation and depression will be given for a hypothetical observer at ground level. If the line of sight follows a physical object, such as an inclined plane or a hillside, we use the term **angle of inclination**.

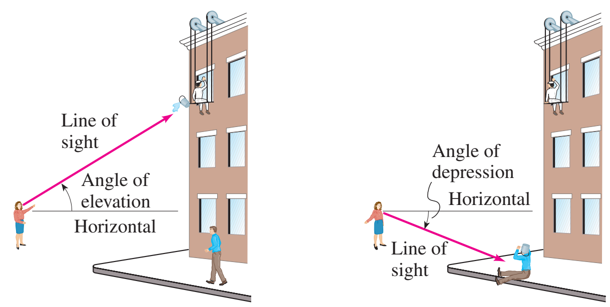


Figure 9

The next example gives an important application of trigonometry to the problem of measurement: We measure the height of a tall tree without having to climb it! Although the example is simple, the result is fundamental to understanding how the trigonometric ratios are applied to such problems.

Example 5 Finding the Height of a Tree

A giant redwood tree casts a shadow 532 ft long. Find the height of the tree if the angle of elevation of the sun is 25.7°.

Solution Let the height of the tree be h. From Figure 10 we see that

$$\frac{h}{532} = \tan 25.7° \qquad \text{Definition of tangent}$$

$$h = 532 \tan 25.7° \qquad \text{Multiply by 532}$$

$$\approx 532(0.48127) \approx 256 \qquad \text{Use a calculator}$$

Therefore, the height of the tree is about 256 ft.

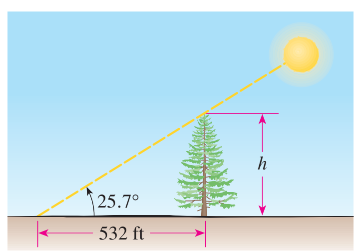


Figure 10

■

IN-CLASS MATERIALS

Example 5 is a good outline for an experiment that can be done in real life. Students can estimate the height of their school, their favorite roller coaster, a flagpole, or anything else. If students measure the distance of ten of their paces, they can "pace off" the distance of shadows, which makes it easier to measure them.

Example 6 A Problem Involving Right Triangles

From a point on the ground 500 ft from the base of a building, an observer finds that the angle of elevation to the top of the building is 24° and that the angle of elevation to the top of a flagpole atop the building is 27°. Find the height of the building and the length of the flagpole.

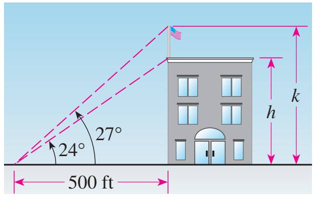


Figure 11

Solution Figure 11 illustrates the situation. The height of the building is found in the same way that we found the height of the tree in Example 5.

$$\frac{h}{500} = \tan 24^\circ \qquad \text{Definition of tangent}$$

$$h = 500 \tan 24^\circ \qquad \text{Multiply by 500}$$

$$\approx 500(0.4452) \approx 223 \qquad \text{Use a calculator}$$

The height of the building is approximately 223 ft.

To find the length of the flagpole, let's first find the height from the ground to the top of the pole:

$$\frac{k}{500} = \tan 27^\circ$$

$$k = 500 \tan 27^\circ$$

$$\approx 500(0.5095)$$

$$\approx 255$$

To find the length of the flagpole, we subtract h from k. So the length of the pole is approximately $255 - 223 = 32$ ft. ■

The key labels $\boxed{\text{SIN}^{-1}}$ or $\boxed{\text{INV}}$ $\boxed{\text{SIN}}$ stand for "inverse sine." We study the inverse trigonometric functions in Section 7.4.

In some problems we need to find an angle in a right triangle whose sides are given. To do this, we use Table 1 (page 480) "backward"; that is, we find the *angle* with the specified trigonometric ratio. For example, if $\sin\theta = \frac{1}{2}$, what is the angle θ? From Table 1 we can tell that $\theta = 30°$. To find an angle whose sine is not given in the table, we use the $\boxed{\text{SIN}^{-1}}$ or $\boxed{\text{INV}}$ $\boxed{\text{SIN}}$ or $\boxed{\text{ARCSIN}}$ keys on a calculator. For example, if $\sin\theta = 0.8$, we apply the $\boxed{\text{SIN}^{-1}}$ key to 0.8 to get $\theta = 53.13°$ or 0.927 rad. The calculator also gives angles whose cosine or tangent are known, using the $\boxed{\text{COS}^{-1}}$ or $\boxed{\text{TAN}^{-1}}$ key.

Example 7 Solving for an Angle in a Right Triangle

A 40-ft ladder leans against a building. If the base of the ladder is 6 ft from the base of the building, what is the angle formed by the ladder and the building?

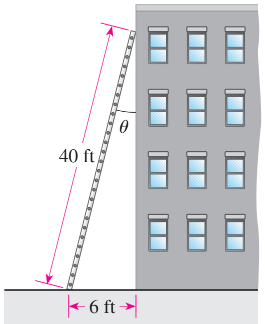


Figure 12

Solution First we sketch a diagram as in Figure 12. If θ is the angle between the ladder and the building, then

$$\sin\theta = \frac{6}{40} = 0.15$$

So θ is the angle whose sine is 0.15. To find the angle θ, we use the $\boxed{\text{SIN}^{-1}}$ key on a calculator. With our calculator in degree mode, we get

$$\theta \approx 8.6^\circ$$ ■

ALTERNATE EXAMPLE 6
From a point on the ground 700 ft from the base of a building, it is observed that the angle of elevation to the top of the building is 21° and the angle of elevation to the top of a flagpole atop the building is 23°. Find the height of the building and the length of the flagpole.

ANSWER
269 ft, 28 ft

ALTERNATE EXAMPLE 7
A 50-ft ladder leans against a building. If the base of the ladder is 7 ft from the base of the building, what is the angle formed by the ladder and the building?

ANSWER
8°

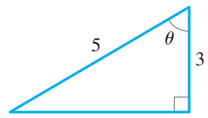
IN-CLASS MATERIALS

In addition to figuring out the height of an object, we can also figure out distances. If you know a building is, for example, 700 ft high, you can tell how far away you are by measuring the angle of elevation from where you are standing.

6.2 Exercises

1–6 ■ Find the exact values of the six trigonometric ratios of the angle θ in the triangle.

1.



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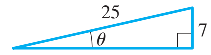

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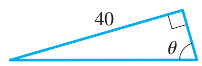

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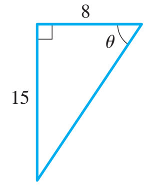

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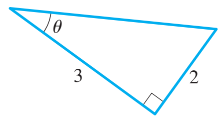

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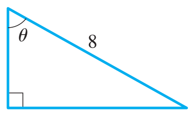
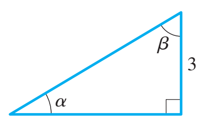


7–8 ■ Find (a) $\sin\alpha$ and $\cos\beta$, (b) $\tan\alpha$ and $\cot\beta$, and (c) $\sec\alpha$ and $\csc\beta$.

7.



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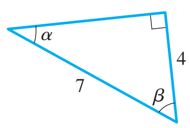
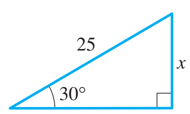


9–14 ■ Find the side labeled x. In Exercises 13 and 14 state your answer correct to five decimal places.

9.



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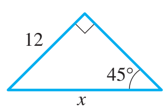

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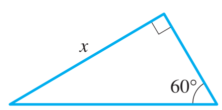

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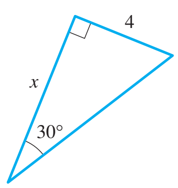

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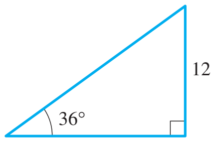

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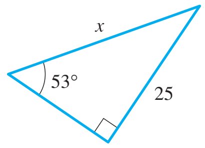
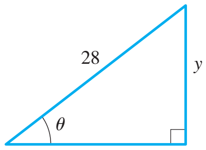


15–16 ■ Express x and y in terms of trigonometric ratios of θ.

15.



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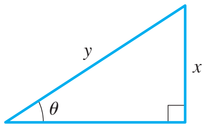


17–22 ■ Sketch a triangle that has acute angle θ, and find the other five trigonometric ratios of θ.

17. $\sin\theta = \frac{3}{5}$

18. $\cos\theta = \frac{9}{40}$

19. $\cot\theta = 1$

20. $\tan\theta = \sqrt{3}$

21. $\sec\theta = \frac{7}{2}$

22. $\csc\theta = \frac{13}{12}$

23–28 ■ Evaluate the expression without using a calculator.

23. $\sin\dfrac{\pi}{6} + \cos\dfrac{\pi}{6}$

24. $\sin 30° \csc 30°$

25. $\sin 30° \cos 60° + \sin 60° \cos 30°$

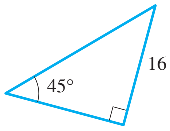
26. $(\sin 60°)^2 + (\cos 60°)^2$

27. $(\cos 30°)^2 - (\sin 30°)^2$

28. $\left(\sin\dfrac{\pi}{3}\cos\dfrac{\pi}{4} - \sin\dfrac{\pi}{4}\cos\dfrac{\pi}{3}\right)^2$

29–36 ■ Solve the right triangle.

29.



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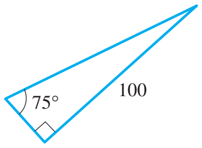

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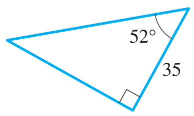

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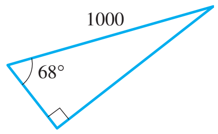

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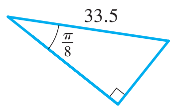

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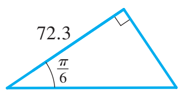

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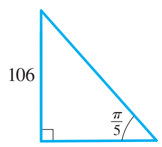

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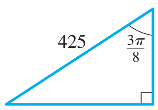
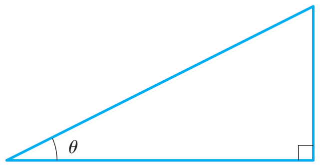


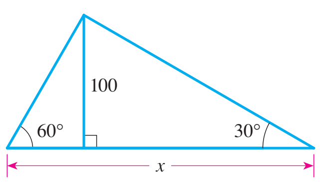
37. Use a ruler to carefully measure the sides of the triangle, and then use your measurements to estimate the six trigonometric ratios of θ.



38. Using a protractor, sketch a right triangle that has the acute angle 40°. Measure the sides carefully, and use your results to estimate the six trigonometric ratios of 40°.

39–42 ■ Find x correct to one decimal place.

39.



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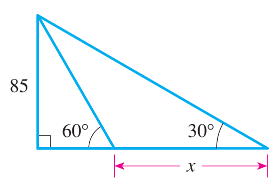

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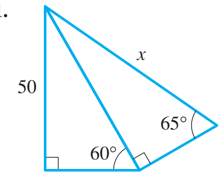

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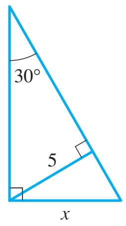
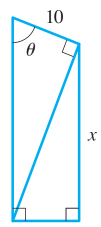
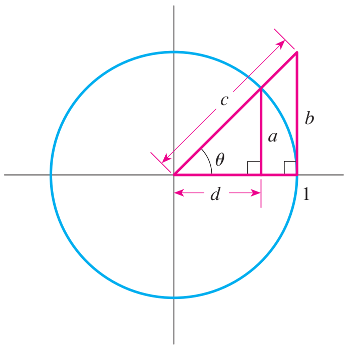


43. Express the length x in terms of the trigonometric ratios of θ.



44. Express the length a, b, c, and d in the figure in terms of the trigonometric ratios of θ.



Applications

45. Height of a Building The angle of elevation to the top of the Empire State Building in New York is found to be 11° from the ground at a distance of 1 mi from the base of the building. Using this information, find the height of the Empire State Building.

46. Gateway Arch A plane is flying within sight of the Gateway Arch in St. Louis, Missouri, at an elevation of 35,000 ft. The pilot would like to estimate her distance from the Gateway Arch. She finds that the angle of depression to a point on the ground below the arch is 22°.

(a) What is the distance between the plane and the arch?

(b) What is the distance between a point on the ground directly below the plane and the arch?

47. Deviation of a Laser Beam A laser beam is to be directed toward the center of the moon, but the beam strays 0.5° from its intended path.

(a) How far has the beam diverged from its assigned target when it reaches the moon? (The distance from the earth to the moon is 240,000 mi.)

(b) The radius of the moon is about 1000 mi. Will the beam strike the moon?

48. Distance at Sea From the top of a 200-ft lighthouse, the angle of depression to a ship in the ocean is 23°. How far is the ship from the base of the lighthouse?

49. Leaning Ladder A 20-ft ladder leans against a building so that the angle between the ground and the ladder is 72°. How high does the ladder reach on the building?

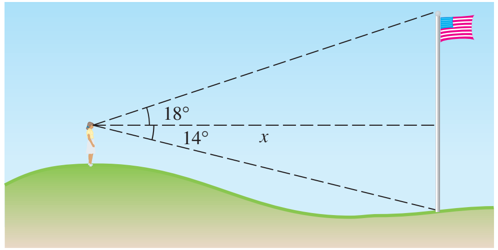
50. Leaning Ladder A 20-ft ladder is leaning against a building. If the base of the ladder is 6 ft from the base of the building, what is the angle of elevation of the ladder? How high does the ladder reach on the building?

51. Angle of the Sun A 96-ft tree casts a shadow that is 120 ft long. What is the angle of elevation of the sun?

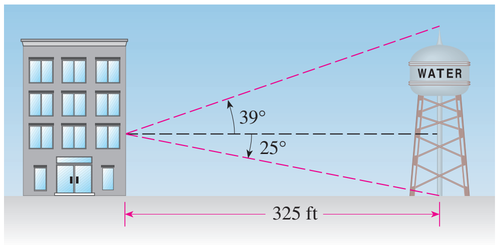
52. Height of a Tower A 600-ft guy wire is attached to the top of a communications tower. If the wire makes an angle of 65° with the ground, how tall is the communications tower?

53. Elevation of a Kite A man is lying on the beach, flying a kite. He holds the end of the kite string at ground level, and estimates the angle of elevation of the kite to be 50°. If the string is 450 ft long, how high is the kite above the ground?

54. Determining a Distance A woman standing on a hill sees a flagpole that she knows is 60 ft tall. The angle of depression to the bottom of the pole is 14°, and the angle of elevation to the top of the pole is 18°. Find her distance x from the pole.



55. Height of a Tower A water tower is located 325 ft from a building (see the figure). From a window in the building, an observer notes that the angle of elevation to the top of the tower is 39° and that the angle of depression to the bottom of the tower is 25°. How tall is the tower? How high is the window?



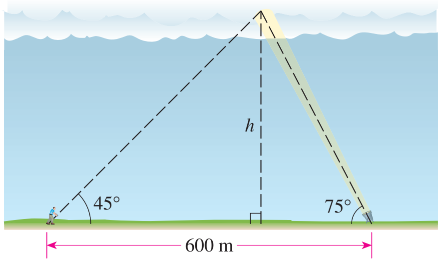
56. Determining a Distance An airplane is flying at an elevation of 5150 ft, directly above a straight highway. Two motorists are driving cars on the highway on opposite sides of the plane, and the angle of depression to one car is 35° and to the other is 52°. How far apart are the cars?

57. Determining a Distance If both cars in Exercise 56 are on one side of the plane and if the angle of depression to one car is 38° and to the other car is 52°, how far apart are the cars?

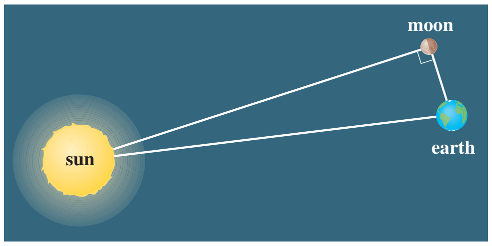
58. Height of a Balloon A hot-air balloon is floating above a straight road. To estimate their height above the ground, the balloonists simultaneously measure the angle of depression to two consecutive mileposts on the road on the same side of the balloon. The angles of depression are found to be 20° and 22°. How high is the balloon?

59. Height of a Mountain To estimate the height of a mountain above a level plain, the angle of elevation to the top of the mountain is measured to be 32°. One thousand feet closer to the mountain along the plain, it is found that the angle of elevation is 35°. Estimate the height of the mountain.

60. Height of Cloud Cover To measure the height of the cloud cover at an airport, a worker shines a spotlight upward at an angle 75° from the horizontal. An observer 600 m away measures the angle of elevation to the spot of light to be 45°. Find the height h of the cloud cover.



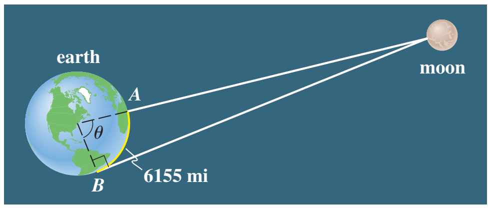
61. Distance to the Sun When the moon is exactly half full, the earth, moon, and sun form a right angle (see the figure). At that time the angle formed by the sun, earth, and moon is measured to be 89.85°. If the distance from the earth to the moon is 240,000 mi, estimate the distance from the earth to the sun.



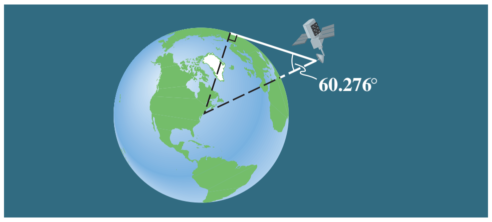
62. Distance to the Moon To find the distance to the sun as in Exercise 61, we needed to know the distance to the moon. Here is a way to estimate that distance: When the moon is seen at its zenith at a point A on the earth, it is observed to be at the horizon from point B (see the figure). Points A and B are 6155 mi apart, and the radius of the earth is 3960 mi.

(a) Find the angle θ in degrees.

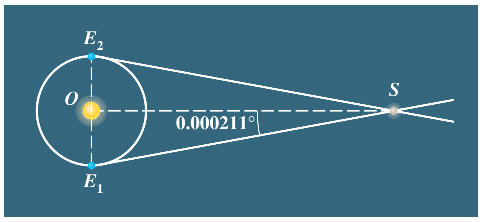
(b) Estimate the distance from point A to the moon.



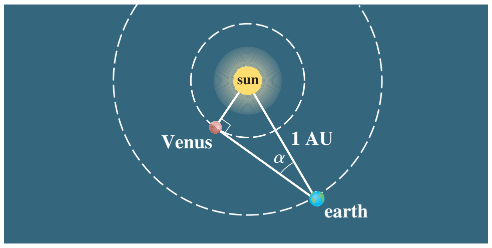
63. Radius of the Earth In Exercise 72 of Section 6.1 a method was given for finding the radius of the earth. Here is a more modern method: From a satellite 600 mi above the earth, it is observed that the angle formed by the vertical and the line of sight to the horizon is 60.276°. Use this information to find the radius of the earth.



64. Parallax To find the distance to nearby stars, the method of parallax is used. The idea is to find a triangle with the star at one vertex and with a base as large as possible. To do this, the star is observed at two different times exactly 6 months apart, and its apparent change in position is recorded. From these two observations, $\angle E_1SE_2$ can be calculated. (The times are chosen so that $\angle E_1SE_2$ is as large as possible, which guarantees that $\angle E_1OS$ is 90°.) The angle E_1SO is called the *parallax* of the star. Alpha Centauri, the star nearest the earth, has a parallax of 0.000211°. Estimate the distance to this star. (Take the distance from the earth to the sun to be 9.3×10^7 mi.)



65. Distance from Venus to the Sun The **elongation** α of a planet is the angle formed by the planet, earth, and sun (see the figure). When Venus achieves its maximum elongation of 46.3°, the earth, Venus, and the sun form a triangle with a right angle at Venus. Find the distance between Venus and the sun in Astronomical Units (AU). (By definition, the distance between the earth and the sun is 1 AU.)



Discovery • Discussion

66. Similar Triangles If two triangles are similar, what properties do they share? Explain how these properties make it possible to define the trigonometric ratios without regard to the size of the triangle.

SUGGESTED TIME AND EMPHASIS

1 class.
Essential material.

6.3 Trigonometric Functions of Angles

In the preceding section we defined the trigonometric ratios for acute angles. Here we extend the trigonometric ratios to all angles by defining the trigonometric functions of angles. With these functions we can solve practical problems that involve angles which are not necessarily acute.

Trigonometric Functions of Angles

Let POQ be a right triangle with acute angle θ as shown in Figure 1(a). Place θ in standard position as shown in Figure 1(b).

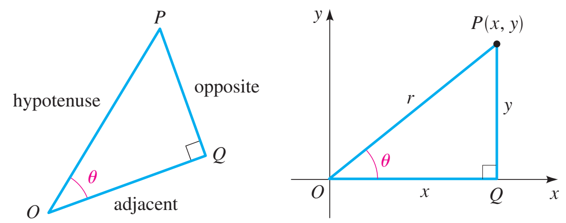


Figure 1 (a) (b)

Then $P = P(x, y)$ is a point on the terminal side of θ. In triangle POQ, the opposite side has length y and the adjacent side has length x. Using the Pythagorean Theorem, we see that the hypotenuse has length $r = \sqrt{x^2 + y^2}$. So

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x}$$

The other trigonometric ratios can be found in the same way.

These observations allow us to extend the trigonometric ratios to any angle. We define the trigonometric functions of angles as follows (see Figure 2).

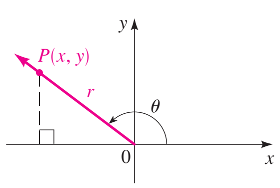


Figure 2

Definition of the Trigonometric Functions

Let θ be an angle in standard position and let $P(x, y)$ be a point on the terminal side. If $r = \sqrt{x^2 + y^2}$ is the distance from the origin to the point $P(x, y)$, then

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x} \quad (x \neq 0)$$

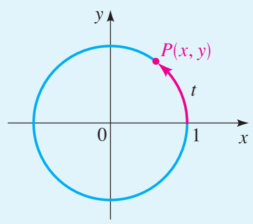
$$\csc\theta = \frac{r}{y} \quad (y \neq 0) \qquad \sec\theta = \frac{r}{x} \quad (x \neq 0) \qquad \cot\theta = \frac{x}{y} \quad (y \neq 0)$$

POINTS TO STRESS

1. Finding the reference angle for a given angle.
2. Using the reference angle to evaluate trigonometric functions.
3. The Pythagorean and reciprocal identities of the trigonometric functions.

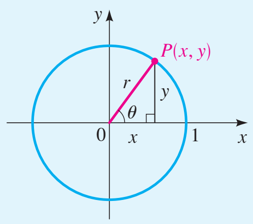
Relationship to the Trigonometric Functions of Real Numbers

You may have already studied the trigonometric functions defined using the unit circle (Chapter 5). To see how they relate to the trigonometric functions of an *angle*, let's start with the unit circle in the coordinate plan.



$P(x, y)$ is the terminal point determined by t.

Let $P(x, y)$ be the terminal point determined by an arc of length t on the unit circle. Then t subtends an angle θ at the center of the circle. If we drop a perpendicular from P onto the point Q on the x-axis, then triangle ΔOPQ is a right triangle with legs of length x and y, as shown in the figure.



Triangle OPQ is a right triangle.

Now, by the definition of the trigonometric functions of the *real number* t, we have

$$\sin t = y$$

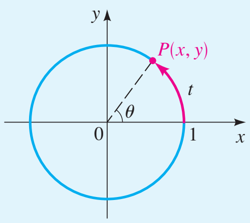
$$\cos t = x$$

By the definition of the trigonometric functions of the *angle* θ, we have

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{y}{1} = y$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{x}{1} = x$$

If θ is measured in radians, then $\theta = t$. (See the figure below.) Comparing the two ways of defining the trigonometric functions, we see that they are identical. In other words, as functions, they assign identical values to a given real number (the real number is the radian measure of θ in one case or the length t of an arc in the other).



The radian measure of angle θ is t.

Why then do we study trigonometry in two different ways? Because different applications require that we view the trigonometric functions differently. (See *Focus on Modeling*, pages 459, 522, and 575, and Sections 6.2, 6.4, and 6.5.)

IN-CLASS MATERIALS

Students often make the mistake of assuming that the trig functions are linear functions. Since the class will probably involve computing a few trig functions, one can kill two birds with one stone by computing $\sin\left(\frac{\pi}{2} + \frac{\pi}{3}\right)$, $\sin\frac{\pi}{2}$, and $\sin\frac{\pi}{3}$ to demonstrate that, for example, $\sin\left(\frac{\pi}{2} + \frac{\pi}{3}\right) \neq \sin\frac{\pi}{2} + \sin\frac{\pi}{3}$. Return to this point several times; they will be grateful when they avoid that mistake in later courses.

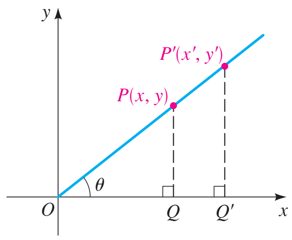


Figure 3

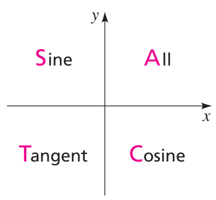
Since division by 0 is an undefined operation, certain trigonometric functions are not defined for certain angles. For example, $\tan 90° = y/x$ is undefined because $x = 0$. The angles for which the trigonometric functions may be undefined are the angles for which either the x- or y-coordinate of a point on the terminal side of the angle is 0. These are **quadrantal angles**—angles that are coterminal with the coordinate axes.

It is a crucial fact that the values of the trigonometric functions do *not* depend on the choice of the point $P(x, y)$. This is because if $P'(x', y')$ is any other point on the terminal side, as in Figure 3, then triangles POQ and $P'OQ'$ are similar.

Evaluating Trigonometric Functions at Any Angle

From the definition we see that the values of the trigonometric functions are all positive if the angle θ has its terminal side in quadrant I. This is because x and y are positive in this quadrant. [Of course, r is always positive, since it is simply the distance from the origin to the point $P(x, y)$.] If the terminal side of θ is in quadrant II, however, then x is negative and y is positive. Thus, in quadrant II the functions $\sin\theta$ and $\csc\theta$ are positive, and all the other trigonometric functions have negative values. You can check the other entries in the following table.

The following mnemonic device can be used to remember which trigonometric functions are positive in each quadrant: **A**ll of them, **S**ine, **T**angent, or **C**osine.



You can remember this as "**A**ll **S**tudents **T**ake **C**alculus."

Signs of the Trigonometric Functions

Quadrant	Positive functions	Negative functions
I	all	none
II	sin, csc	cos, sec, tan, cot
III	tan, cot	sin, csc, cos, sec
IV	cos, sec	sin, csc, tan, cot

We now turn our attention to finding the values of the trigonometric functions for angles that are not acute.

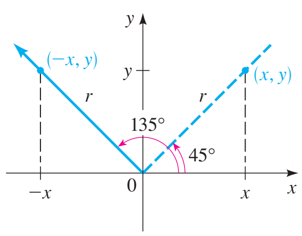


Figure 4

Example 1 Finding Trigonometric Functions of Angles

Find (a) $\cos 135°$ and (b) $\tan 390°$.

Solution

(a) From Figure 4 we see that $\cos 135° = -x/r$. But $\cos 45° = x/r$, and since $\cos 45° = \sqrt{2}/2$, we have

$$\cos 135° = -\frac{\sqrt{2}}{2}$$

(b) The angles 390° and 30° are coterminal. From Figure 5 it's clear that $\tan 390° = \tan 30°$ and, since $\tan 30° = \sqrt{3}/3$, we have

$$\tan 390° = \frac{\sqrt{3}}{3}$$

■

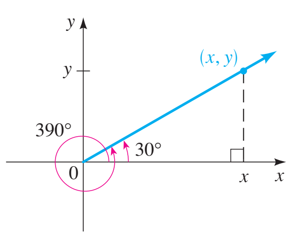


Figure 5

EXAMPLES

- $\sin\dfrac{\pi}{3} = \dfrac{\sqrt{3}}{2}$
- $\cos\dfrac{7\pi}{6} = -\dfrac{\sqrt{3}}{2}$
- $\tan\dfrac{11\pi}{4} = -1$
- $\sec\dfrac{17\pi}{3} = 2$
- $\csc\dfrac{17\pi}{2} = 1$
- $\cot\dfrac{121\pi}{6} = \sqrt{3}$

ALTERNATE EXAMPLE 1a
Find cos 60°.

ANSWER
$\dfrac{1}{2}$

ALTERNATE EXAMPLE 1b
Find tan 240°.

ANSWER
$\sqrt{3}$

IN-CLASS MATERIALS

There is the danger that students will use the mnemonic given in the text (All Students Take Calculus) as a way of avoiding all understanding of the sign of the various trigonometric functions in different quadrants. Make sure that they are able to articulate why, for example, the cosine function is positive in quadrants I and IV and negative in quadrants II and III.

From Example 1 we see that the trigonometric functions for angles that aren't acute have the same value, except possibly for sign, as the corresponding trigonometric functions of an acute angle. That acute angle will be called the *reference angle*.

Reference Angle

Let θ be an angle in standard position. The **reference angle** $\bar{\theta}$ associated with θ is the acute angle formed by the terminal side of θ and the x-axis.

Figure 6 shows that to find a reference angle it's useful to know the quadrant in which the terminal side of the angle lies.

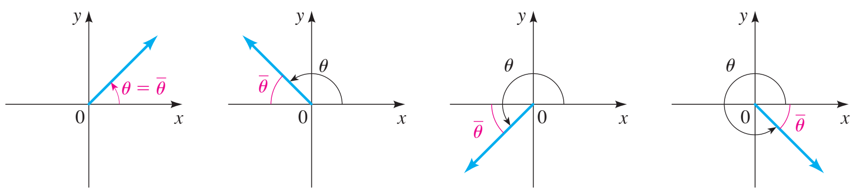


Figure 6
The reference angle $\bar{\theta}$ for an angle θ

Example 2 Finding Reference Angles

Find the reference angle for (a) $\theta = \dfrac{5\pi}{3}$ and (b) $\theta = 870°$.

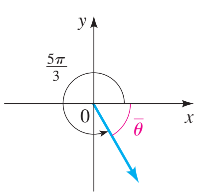


Figure 7

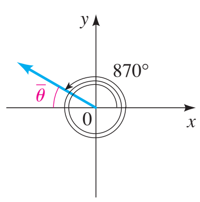


Figure 8

Solution

(a) The reference angle is the acute angle formed by the terminal side of the angle $5\pi/3$ and the x-axis (see Figure 7). Since the terminal side of this angle is in quadrant IV, the reference angle is

$$\bar{\theta} = 2\pi - \frac{5\pi}{3} = \frac{\pi}{3}$$

(b) The angles 870° and 150° are coterminal [because $870 - 2(360) = 150$]. Thus, the terminal side of this angle is in quadrant II (see Figure 8). So the reference angle is

$$\bar{\theta} = 180° - 150° = 30°$$

■

Evaluating Trigonometric Functions for Any Angle

To find the values of the trigonometric functions for any angle θ, we carry out the following steps.

1. Find the reference angle $\bar{\theta}$ associated with the angle θ.
2. Determine the sign of the trigonometric function of θ by noting the quadrant in which θ lies.
3. The value of the trigonometric function of θ is the same, except possibly for sign, as the value of the trigonometric function of $\bar{\theta}$.

DRILL QUESTION

If $\theta = \dfrac{17\pi}{6}$, find its reference angle $\bar{\theta}$. In what quadrant is θ?

Answer

$\bar{\theta} = \dfrac{\pi}{6}$. θ is in the second quadrant.

ALTERNATE EXAMPLE 2a

Find the reference angle for $\theta = \dfrac{13\pi}{7}$.

ANSWER

$\dfrac{\pi}{7}$

SAMPLE QUESTION
Text Question

Is it possible for two different angles to have the same reference angle?

Answer

Yes

ALTERNATE EXAMPLE 3a
Find sin 120°.

ANSWER
$-\frac{1}{2}$

ALTERNATE EXAMPLE 4a
Find $\sin\frac{43\pi}{6}$.

ANSWER
$-\frac{1}{2}$

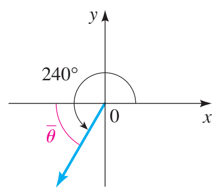


Figure 9

S | A
T | C

sin 240° is *negative*.

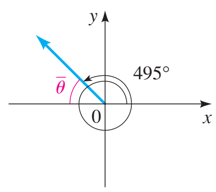


Figure 10

S | A
T | C

tan 495° *is negative,*
so cot 495° *is negative.*

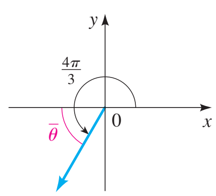


Figure 11

S | A
T | C

sin $\frac{16\pi}{3}$ *is negative.*

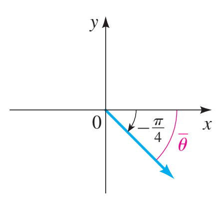


Figure 12

S | A
T | C

$\cos(-\frac{\pi}{4})$ *is positive,*
so $\sec(-\frac{\pi}{4})$ *is positive.*

Example 3 Using the Reference Angle to Evaluate Trigonometric Functions

Find (a) sin 240° and (b) cot 495°.

Solution

(a) This angle has its terminal side in quadrant III, as shown in Figure 9. The reference angle is therefore 240° − 180° = 60°, and the value of sin 240° is negative. Thus

$$\sin 240° = \underbrace{-}_{\text{Sign}}\underbrace{\sin 60°}_{\text{Reference angle}} = -\frac{\sqrt{3}}{2}$$

(b) The angle 495° is coterminal with the angle 135°, and the terminal side of this angle is in quadrant II, as shown in Figure 10. So the reference angle is 180° − 135° = 45°, and the value of cot 495° is negative. We have

$$\cot 495° \underbrace{=}_{\text{Coterminal angles}} \cot 135° = \underbrace{-}_{\text{Sign}}\underbrace{\cot 45°}_{\text{Reference angle}} = -1$$

■

Example 4 Using the Reference Angle to Evaluate Trigonometric Functions

Find (a) $\sin\frac{16\pi}{3}$ and (b) $\sec\left(-\frac{\pi}{4}\right)$.

Solution

(a) The angle $16\pi/3$ is coterminal with $4\pi/3$, and these angles are in quadrant III (see Figure 11). Thus, the reference angle is $(4\pi/3) - \pi = \pi/3$. Since the value of sine is negative in quadrant III, we have

$$\sin\frac{16\pi}{3} \underbrace{=}_{\text{Coterminal angles}} \sin\frac{4\pi}{3} = \underbrace{-}_{\text{Sign}}\underbrace{\sin\frac{\pi}{3}}_{\text{Reference angle}} = -\frac{\sqrt{3}}{2}$$

(b) The angle $-\pi/4$ is in quadrant IV, and its reference angle is $\pi/4$ (see Figure 12). Since secant is positive in this quadrant, we get

$$\sec\left(-\frac{\pi}{4}\right) = \underbrace{+}_{\text{Sign}}\underbrace{\sec\frac{\pi}{4}}_{\text{Reference angle}} = \frac{\sqrt{2}}{2}$$

■

Trigonometric Identities

The trigonometric functions of angles are related to each other through several important equations called **trigonometric identities**. We've already encountered the reciprocal identities. These identities continue to hold for any angle θ, provided both sides of the equation are defined. The Pythagorean identities are a consequence of the Pythagorean Theorem.*

Fundamental Identities

Reciprocal Identities

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1 \qquad \tan^2\theta + 1 = \sec^2\theta \qquad 1 + \cot^2\theta = \csc^2\theta$$

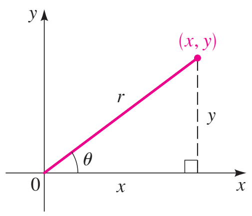


Figure 13

■ **Proof** Let's prove the first Pythagorean identity. Using $x^2 + y^2 = r^2$ (the Pythagorean Theorem) in Figure 13, we have

$$\sin^2\theta + \cos^2\theta = \left(\frac{y}{r}\right)^2 + \left(\frac{x}{r}\right)^2 = \frac{x^2 + y^2}{r^2} = \frac{r^2}{r^2} = 1$$

Thus, $\sin^2\theta + \cos^2\theta = 1$. (Although the figure indicates an acute angle, you should check that the proof holds for all angles θ.) ■

See Exercises 59 and 60 for the proofs of the other two Pythagorean identities.

Example 5 Expressing One Trigonometric Function in Terms of Another

(a) Express $\sin\theta$ in terms of $\cos\theta$.

(b) Express $\tan\theta$ in terms of $\sin\theta$, where θ is in quadrant II.

ALTERNATE EXAMPLE 5b
Express $\tan\theta$ in terms of $\sin\theta$, where θ is in quadrant III.

ANSWER

$$-\frac{\sin(\theta)}{\sqrt{1 - \sin^2(\theta)}}$$

Solution

(a) From the first Pythagorean identity we get

$$\sin\theta = \pm\sqrt{1 - \cos^2\theta}$$

where the sign depends on the quadrant. If θ is in quadrant I or II, then $\sin\theta$ is positive, and hence

$$\sin\theta = \sqrt{1 - \cos^2\theta}$$

whereas if θ is in quadrant III or IV, $\sin\theta$ is negative and so

$$\sin\theta = -\sqrt{1 - \cos^2\theta}$$

* We follow the usual convention of writing $\sin^2\theta$ for $(\sin\theta)^2$. In general, we write $\sin^n\theta$ for $(\sin\theta)^n$ for all integers n except $n = -1$. The exponent $n = -1$ will be assigned another meaning in Section 7.4. Of course, the same convention applies to the other five trigonometric functions.

ALTERNATE EXAMPLE 6
If $\tan\theta = \frac{1}{2}$ and θ is in quadrant IV, find $\cos\theta$.

ANSWER
$\dfrac{2}{\sqrt{5}}$

ALTERNATE EXAMPLE 7
If $\sec\theta = 6$ and θ is in quadrant IV, find the other five trigonometric functions of θ.

ANSWER
$\sin(\theta) = -\dfrac{\sqrt{35}}{6}$, $\cos(\theta) = \dfrac{1}{6}$, $\tan(\theta) = -\sqrt{35}$, $\cot(\theta) = -\dfrac{1}{\sqrt{35}}$, $\csc(\theta) = -\dfrac{6}{\sqrt{35}}$

(b) Since $\tan\theta = \sin\theta/\cos\theta$, we need to write $\cos\theta$ in terms of $\sin\theta$. By part (a)

$$\cos\theta = \pm\sqrt{1 - \sin^2\theta}$$

and since $\cos\theta$ is negative in quadrant II, the negative sign applies here. Thus

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sin\theta}{-\sqrt{1-\sin^2\theta}}$$

■

Example 6 Evaluating a Trigonometric Function

If $\tan\theta = \frac{2}{3}$ and θ is in quadrant III, find $\cos\theta$.

Solution 1 We need to write $\cos\theta$ in terms of $\tan\theta$. From the identity $\tan^2\theta + 1 = \sec^2\theta$, we get $\sec\theta = \pm\sqrt{\tan^2\theta + 1}$. In quadrant III, $\sec\theta$ is negative, so

$$\sec\theta = -\sqrt{\tan^2\theta + 1}$$

Thus

$$\cos\theta = \frac{1}{\sec\theta} = \frac{1}{-\sqrt{\tan^2\theta+1}}$$
$$= \frac{1}{-\sqrt{\left(\frac{2}{3}\right)^2+1}} = \frac{1}{-\sqrt{\frac{13}{9}}} = -\frac{3}{\sqrt{13}}$$

If you wish to rationalize the denominator, you can express $\cos\theta$ as

$$-\frac{3}{\sqrt{13}}\cdot\frac{\sqrt{13}}{\sqrt{13}} = -\frac{3\sqrt{13}}{13}$$

Solution 2 This problem can be solved more easily using the method of Example 2 of Section 6.2. Recall that, except for sign, the values of the trigonometric functions of any angle are the same as those of an acute angle (the reference angle). So, ignoring the sign for the moment, let's sketch a right triangle with an acute angle $\bar{\theta}$ satisfying $\tan\bar{\theta} = \frac{2}{3}$ (see Figure 14). By the Pythagorean Theorem the hypotenuse of this triangle has length $\sqrt{13}$. From the triangle in Figure 14 we immediately see that $\cos\bar{\theta} = 3/\sqrt{13}$. Since θ is in quadrant III, $\cos\theta$ is negative and so

$$\cos\theta = -\frac{3}{\sqrt{13}}$$

■

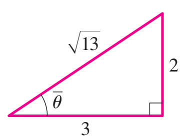


Figure 14

Example 7 Evaluating Trigonometric Functions

If $\sec\theta = 2$ and θ is in quadrant IV, find the other five trigonometric functions of θ.

Solution We sketch a triangle as in Figure 15 so that $\sec\bar{\theta} = 2$. Taking into account the fact that θ is in quadrant IV, we get

$$\sin\theta = -\frac{\sqrt{3}}{2} \qquad \cos\theta = \frac{1}{2} \qquad \tan\theta = -\sqrt{3}$$
$$\csc\theta = -\frac{2}{\sqrt{3}} \qquad \sec\theta = 2 \qquad \cot\theta = -\frac{1}{\sqrt{3}}$$

■

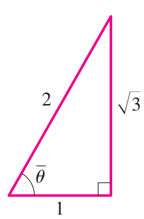


Figure 15

Areas of Triangles

We conclude this section with an application of the trigonometric functions that involves angles that are not necessarily acute. More extensive applications appear in the next two sections.

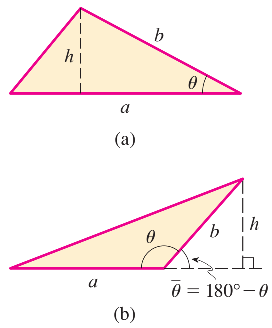


Figure 16

The area of a triangle is $\mathcal{A} = \frac{1}{2} \times \text{base} \times \text{height}$. If we know two sides and the included angle of a triangle, then we can find the height using the trigonometric functions, and from this we can find the area.

If θ is an acute angle, then the height of the triangle in Figure 16(a) is given by $h = b \sin \theta$. Thus, the area is

$$\mathcal{A} = \tfrac{1}{2} \times \text{base} \times \text{height} = \tfrac{1}{2}ab \sin \theta$$

If the angle θ is not acute, then from Figure 16(b) we see that the height of the triangle is

$$h = b \sin(180^\circ - \theta) = b \sin \theta$$

This is so because the reference angle of θ is the angle $180^\circ - \theta$. Thus, in this case also, the area of the triangle is

$$\mathcal{A} = \tfrac{1}{2} \times \text{base} \times \text{height} = \tfrac{1}{2}ab \sin \theta$$

Area of a Triangle

The area $\mathcal{A}$ of a triangle with sides of lengths a and b and with included angle θ is

$$\mathcal{A} = \tfrac{1}{2}ab \sin \theta$$

Example 8 Finding the Area of a Triangle

Find the area of triangle ABC shown in Figure 17.

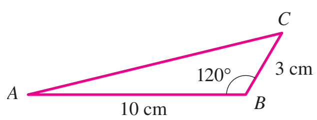


Figure 17

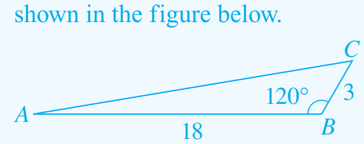
Solution The triangle has sides of length 10 cm and 3 cm, with included angle 120°. Therefore

$$\begin{aligned}\mathcal{A} &= \tfrac{1}{2}ab \sin \theta \\ &= \tfrac{1}{2}(10)(3) \sin 120^\circ \\ &= 15 \sin 60^\circ && \text{Reference angle} \\ &= 15\frac{\sqrt{3}}{2} \approx 13 \text{ cm}^2\end{aligned}$$

■

ALTERNATE EXAMPLE 8

Find the area of triangle *ABC* shown in the figure below.



ANSWER

$27 \cdot \frac{\sqrt{3}}{2}$

6.3 Exercises

1–8 ■ Find the reference angle for the given angle.

1. (a) 150° **(b)** 330° **(c)** −30°

2. (a) 120° **(b)** −210° **(c)** 780°

3. (a) 225° **(b)** 810° **(c)** −105°

4. (a) 99° **(b)** −199° **(c)** 359°

5. (a) $\frac{11\pi}{4}$ **(b)** $-\frac{11\pi}{6}$ **(c)** $\frac{11\pi}{3}$

6. (a) $\frac{4\pi}{3}$ **(b)** $\frac{33\pi}{4}$ **(c)** $-\frac{23\pi}{6}$

7. (a) $\frac{5\pi}{7}$ **(b)** -1.4π **(c)** 1.4

8. (a) 2.3π **(b)** 2.3 **(c)** -10π

9–32 ■ Find the exact value of the trigonometric function.

9. $\sin 150°$ **10.** $\sin 225°$ **11.** $\cos 135°$

12. $\cos(-60°)$ **13.** $\tan(-60°)$ **14.** $\sec 300°$

15. $\csc(-630°)$ **16.** $\cot 210°$ **17.** $\cos 570°$

18. $\sec 120°$ **19.** $\tan 750°$ **20.** $\cos 660°$

21. $\sin \dfrac{2\pi}{3}$ **22.** $\sin \dfrac{5\pi}{3}$ **23.** $\sin \dfrac{3\pi}{2}$

24. $\cos \dfrac{7\pi}{3}$ **25.** $\cos\left(-\dfrac{7\pi}{3}\right)$ **26.** $\tan \dfrac{5\pi}{6}$

27. $\sec \dfrac{17\pi}{3}$ **28.** $\csc \dfrac{5\pi}{4}$ **29.** $\cot\left(-\dfrac{\pi}{4}\right)$

30. $\cos \dfrac{7\pi}{4}$ **31.** $\tan \dfrac{5\pi}{2}$ **32.** $\sin \dfrac{11\pi}{6}$

33–36 ■ Find the quadrant in which θ lies from the information given.

33. $\sin\theta < 0$ and $\cos\theta < 0$

34. $\tan\theta < 0$ and $\sin\theta < 0$

35. $\sec\theta > 0$ and $\tan\theta < 0$

36. $\csc\theta > 0$ and $\cos\theta < 0$

37–42 ■ Write the first trigonometric function in terms of the second for θ in the given quadrant.

37. $\tan\theta$, $\cos\theta$; θ in quadrant III

38. $\cot\theta$, $\sin\theta$; θ in quadrant II

39. $\cos\theta$, $\sin\theta$; θ in quadrant IV

40. $\sec\theta$, $\sin\theta$; θ in quadrant I

41. $\sec\theta$, $\tan\theta$; θ in quadrant II

42. $\csc\theta$, $\cot\theta$; θ in quadrant III

43–50 ■ Find the values of the trigonometric functions of θ from the information given.

43. $\sin\theta = \frac{3}{5}$, θ in quadrant II

44. $\cos\theta = -\frac{7}{12}$, θ in quadrant III

45. $\tan\theta = -\frac{3}{4}$, $\cos\theta > 0$

46. $\sec\theta = 5$, $\sin\theta < 0$

47. $\csc\theta = 2$, θ in quadrant I

48. $\cot\theta = \frac{1}{4}$, $\sin\theta < 0$

49. $\cos\theta = -\frac{2}{7}$, $\tan\theta < 0$

50. $\tan\theta = -4$, $\sin\theta > 0$

51. If $\theta = \pi/3$, find the value of each expression.
(a) $\sin 2\theta$, $2\sin\theta$ (b) $\sin\frac{1}{2}\theta$, $\frac{1}{2}\sin\theta$
(c) $\sin^2\theta$, $\sin(\theta^2)$

52. Find the area of a triangle with sides of length 7 and 9 and included angle 72°.

53. Find the area of a triangle with sides of length 10 and 22 and included angle 10°.

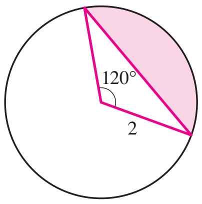
54. Find the area of an equilateral triangle with side of length 10.

55. A triangle has an area of 16 in^2, and two of the sides of the triangle have lengths 5 in. and 7 in. Find the angle included by these two sides.

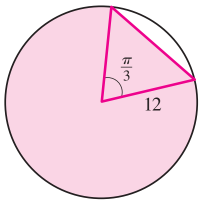
56. An isosceles triangle has an area of 24 cm^2, and the angle between the two equal sides is $5\pi/6$. What is the length of the two equal sides?

57–58 ■ Find the area of the shaded region in the figure.

57.



58.



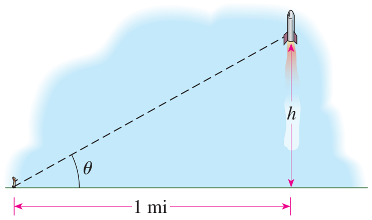
59. Use the first Pythagorean identity to prove the second. [*Hint:* Divide by $\cos^2\theta$.]

60. Use the first Pythagorean identity to prove the third.

Applications

61. Height of a Rocket A rocket fired straight up is tracked by an observer on the ground a mile away.
(a) Show that when the angle of elevation is θ, the height of the rocket in feet is $h = 5280\tan\theta$.
(b) Complete the table to find the height of the rocket at the given angles of elevation.

θ	20°	60°	80°	85°
h				



IN-CLASS MATERIALS

Problems such as Exercises 55–58 are crucial when students take calculus. It is important that they go beyond memorizing the answers and truly understand where the answers come from.

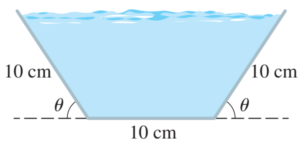
62. Rain Gutter A rain gutter is to be constructed from a metal sheet of width 30 cm by bending up one-third of the sheet on each side through an angle θ.

(a) Show that the cross-sectional area of the gutter is modeled by the function

$$A(\theta) = 100 \sin \theta + 100 \sin \theta \cos \theta$$

(b) Graph the function A for $0 \leq \theta \leq \pi/2$.

(c) For what angle θ is the largest cross-sectional area achieved?

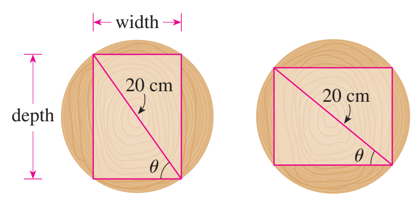


63. Wooden Beam A rectangular beam is to be cut from a cylindrical log of diameter 20 cm. The figures show different ways this can be done.

(a) Express the cross-sectional area of the beam as a function of the angle θ in the figures.

(b) Graph the function you found in part (a).

(c) Find the dimensions of the beam with largest cross-sectional area.



64. Strength of a Beam The strength of a beam is proportional to the width and the square of the depth. A beam is cut from a log as in Exercise 63. Express the strength of the beam as a function of the angle θ in the figures.

65. Throwing a Shot Put The range R and height H of a shot put thrown with an initial velocity of v_0 ft/s at an angle θ are given by

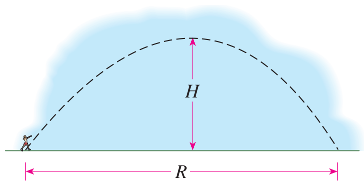
$$R = \frac{v_0^2 \sin(2\theta)}{g}$$

$$H = \frac{v_0^2 \sin^2\theta}{2g}$$

On the earth $g = 32$ ft/s^2 and on the moon $g = 5.2$ ft/s^2. Find the range and height of a shot put thrown under the given conditions.

(a) On the earth with $v_0 = 12$ ft/s and $\theta = \pi/6$

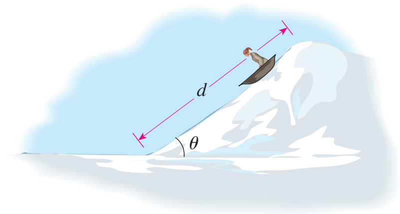
(b) On the moon with $v_0 = 12$ ft/s and $\theta = \pi/6$



66. Sledding The time in seconds that it takes for a sled to slide down a hillside inclined at an angle θ is

$$t = \sqrt{\frac{d}{16 \sin \theta}}$$

where d is the length of the slope in feet. Find the time it takes to slide down a 2000-ft slope inclined at 30°.



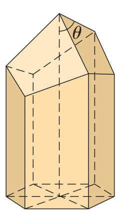
67. Beehives In a beehive each cell is a regular hexagonal prism, as shown in the figure. The amount of wax W in the cell depends on the apex angle θ and is given by

$$W = 3.02 - 0.38 \cot \theta + 0.65 \csc \theta$$

Bees instinctively choose θ so as to use the least amount of wax possible.

(a) Use a graphing device to graph W as a function of θ for $0 < \theta < \pi$.

(b) For what value of θ does W have its minimum value? [*Note*: Biologists have discovered that bees rarely deviate from this value by more than a degree or two.]



68. Turning a Corner A steel pipe is being carried down a hallway 9 ft wide. At the end of the hall there is a right-angled turn into a narrower hallway 6 ft wide.

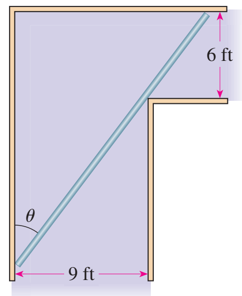
(a) Show that the length of the pipe in the figure is modeled by the function

$$L(\theta) = 9 \csc \theta + 6 \sec \theta$$

(b) Graph the function L for $0 < \theta < \pi/2$.

(c) Find the minimum value of the function L.

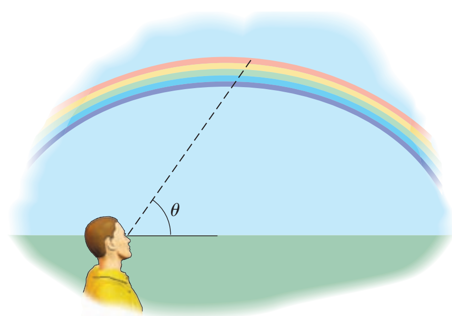
(d) Explain why the value of L you found in part (c) is the length of the longest pipe that can be carried around the corner.



69. Rainbows Rainbows are created when sunlight of different wavelengths (colors) is refracted and reflected in raindrops. The angle of elevation θ of a rainbow is always the same. It can be shown that $\theta = 4\beta - 2\alpha$ where

$$\sin \alpha = k \sin \beta$$

and $\alpha = 59.4°$ and $k = 1.33$ is the index of refraction of water. Use the given information to find the angle of elevation θ of a rainbow. (For a mathematical explanation of rainbows see *Calculus,* 5th Edition, by James Stewart, pages 288–289.)



Discovery • Discussion

70. Using a Calculator To solve a certain problem, you need to find the sine of 4 rad. Your study partner uses his calculator and tells you that

$$\sin 4 = 0.0697564737$$

On your calculator you get

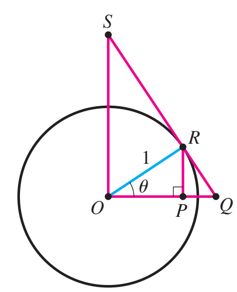
$$\sin 4 = -0.7568024953$$

What is wrong? What mistake did your partner make?

71. Viète's Trigonometric Diagram In the 16th century, the French mathematician François Viète (see page 49) published the following remarkable diagram. Each of the six trigonometric functions of θ is equal to the length of a line segment in the figure. For instance, $\sin \theta = |PR|$, since from $\triangle OPR$ we see that

$$\begin{aligned} \sin \theta &= \frac{\text{opp}}{\text{hyp}} \\ &= \frac{|PR|}{|OR|} \\ &= \frac{|PR|}{1} \\ &= |PR| \end{aligned}$$

For each of the five other trigonometric functions, find a line segment in the figure whose length equals the value of the function at θ. (*Note:* The radius of the circle is 1, the center is O, segment QS is tangent to the circle at R, and $\angle SOQ$ is a right angle.)



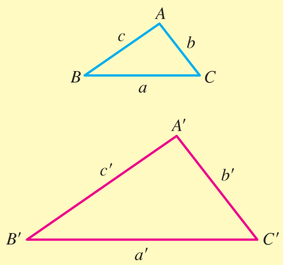
DISCOVERY PROJECT

Similarity

In geometry you learned that two triangles are similar if they have the same angles. In this case, the ratios of corresponding sides are equal. Triangles ABC and $A'B'C'$ in the margin are similar, so

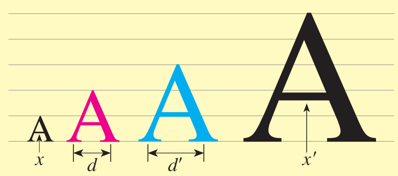
$$\frac{a'}{a} = \frac{b'}{b} = \frac{c'}{c}$$

Similarity is the crucial idea underlying trigonometry. We can define $\sin\theta$ as the ratio of the opposite side to the hypotenuse in *any* right triangle with an angle θ, because all such right triangles are similar. So the ratio represented by $\sin\theta$ does not depend on the size of the right triangle but only on the angle θ. This is a powerful idea because angles are often easier to measure than distances. For example, the angle formed by the sun, earth, and moon can be measured from the earth. The secret to finding the distance to the sun is that the trigonometric ratios are the same for the huge triangle formed by the sun, earth, and moon as for any other similar triangle (see Exercise 61 in Section 6.2).



Thales used similar triangles to find the height of a tall column. (See page 482.)

In general, two objects are **similar** if they have the same shape even though they may not be the same size.* For example, we recognize the following as representations of the letter A because they are all similar.



If two figures are similar, then the distances between corresponding points in the figures are proportional. The blue and red A's above are similar—the ratio of distances between corresponding points is $\frac{3}{2}$. We say that the **similarity ratio** is $s = \frac{3}{2}$. To obtain the distance d' between any two points in the blue A, we multiply the corresponding distance d in the red A by $\frac{3}{2}$. So

$$d' = sd \qquad \text{or} \qquad d' = \tfrac{3}{2}d$$

Likewise, the similarity ratio between the first and last letters is $s = 5$, so $x' = 5x$.

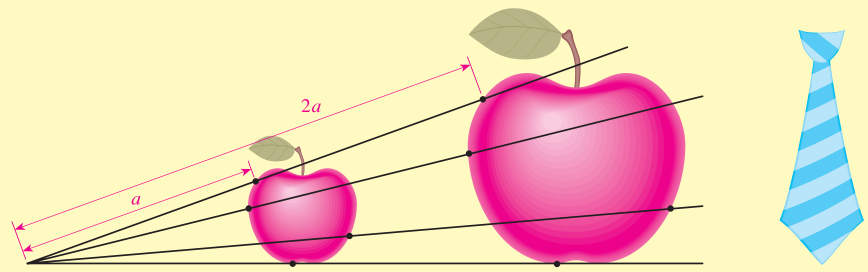
1. Write a short paragraph explaining how the concept of similarity is used to define the trigonometric ratios.
2. How is similarity used in map making? How are distances on a city road map related to actual distances?
3. How is your yearbook photograph similar to you? Compare distances between different points on your face (such as distance between ears, length of

---

* If they have the same shape *and* size, they are congruent, which is a special case of similarity.

nose, distance between eyes, and so on) to the corresponding distances in a photograph. What is the similarity ratio?

4. The figure illustrates a method for drawing an apple twice the size of a given apple. Use the method to draw a tie 3 times the size (similarity ratio 3) of the blue tie.

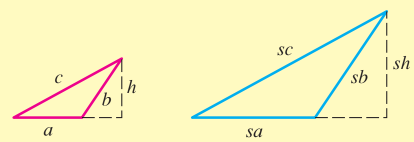


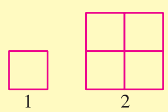
5. Give conditions under which two rectangles are similar to each other. Do the same for two isosceles triangles.

6. Suppose that two similar triangles have similarity ratio s.

(a) How are the perimeters of the triangles related?

(b) How are the areas of the triangles related?



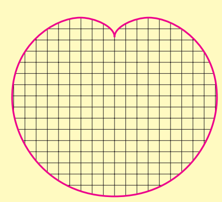


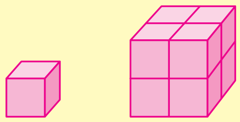
If the side of a square is doubled, its area is multiplied by 2^2.

7. (a) If two squares have similarity ratio s, show that their areas A_1 and A_2 have the property that $A_2 = s^2A_1$.

(b) If the side of a square is tripled, its area is multiplied by what factor?

(c) A plane figure can be approximated by squares (as shown). Explain how we can conclude that for any two plane figures with similarity ratio s, their areas satisfy $A_2 = s^2A_1$. (Use part (a).)





1 2

If the side of a cube is doubled, its volume is multiplied by 2^3.

8. (a) If two cubes have similarity ratio s, show that their volumes V_1 and V_2 have the property that $V_2 = s^3V_1$.

(b) If the side of a cube is multiplied by 10, by what factor is the volume multiplied?

(c) How can we use the fact that a solid object can be "filled" by little cubes to show that for any two solids with similarity ratio s, the volumes satisfy $V_2 = s^3V_1$?

9. King Kong is 10 times as tall as Joe, a normal-sized 300-lb gorilla. Assuming that King Kong and Joe are similar, use the results from Problems 7 and 8 to answer the following questions.

(a) How much does King Kong weigh?

(b) If Joe's hand is 13 in. long, how long is King Kong's hand?

(c) If it takes 2 square yards of material to make a shirt for Joe, how much material would a shirt for King Kong require?

6.4 The Law of Sines

SUGGESTED TIME AND EMPHASIS

$\frac{1}{2}$–1 class.

Essential material.

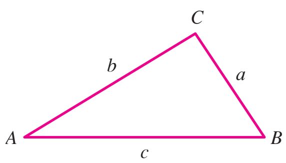
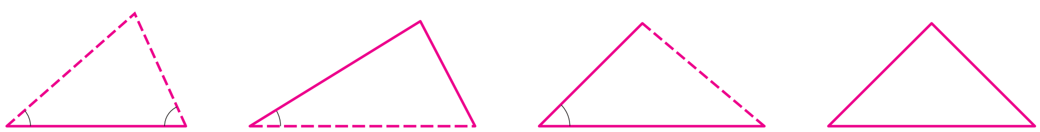


Figure 1

In Section 6.2 we used the trigonometric ratios to solve right triangles. The trigonometric functions can also be used to solve *oblique triangles*, that is, triangles with no right angles. To do this, we first study the Law of Sines here and then the Law of Cosines in the next section. To state these laws (or formulas) more easily, we follow the convention of labeling the angles of a triangle as A, B, C, and the lengths of the corresponding opposite sides as a, b, c, as in Figure 1.

To solve a triangle, we need to know certain information about its sides and angles. To decide whether we have enough information, it's often helpful to make a sketch. For instance, if we are given two angles and the included side, then it's clear that one and only one triangle can be formed (see Figure 2(a)). Similarly, if two sides and the included angle are known, then a unique triangle is determined (Figure 2(c)). But if we know all three angles and no sides, we cannot uniquely determine the triangle because many triangles can have the same three angles. (All these triangles would be similar, of course.) So we won't consider this last case.



(a) ASA or SAA (b) SSA (c) SAS (d) SSS

Figure 2

In general, a triangle is determined by three of its six parts (angles and sides) as long as at least one of these three parts is a side. So, the possibilities, illustrated in Figure 2, are as follows.

POINTS TO STRESS

1. Using the Law of Sines to solve triangles.

2. Understanding which cases are ambiguous and which cases are unambiguous.

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Case 1 One side and two angles (ASA or SAA)
Case 2 Two sides and the angle opposite one of those sides (SSA)
Case 3 Two sides and the included angle (SAS)
Case 4 Three sides (SSS)

Cases 1 and 2 are solved using the Law of Sines; Cases 3 and 4 require the Law of Cosines.

The Law of Sines

The **Law of Sines** says that in any triangle the lengths of the sides are proportional to the sines of the corresponding opposite angles.

The Law of Sines

In triangle ABC we have

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

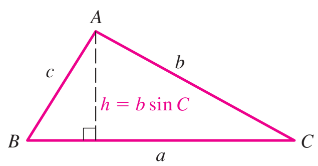


Figure 3

■ **Proof** To see why the Law of Sines is true, refer to Figure 3. By the formula in Section 6.3 the area of triangle ABC is $\frac{1}{2}ab \sin C$. By the same formula the area of this triangle is also $\frac{1}{2}ac \sin B$ and $\frac{1}{2}bc \sin A$. Thus

$$\tfrac{1}{2}bc \sin A = \tfrac{1}{2}ac \sin B = \tfrac{1}{2}ab \sin C$$

Multiplying by $2/(abc)$ gives the Law of Sines. ■

Example 1 Tracking a Satellite (ASA)

A satellite orbiting the earth passes directly overhead at observation stations in Phoenix and Los Angeles, 340 mi apart. At an instant when the satellite is between these two stations, its angle of elevation is simultaneously observed to be 60° at Phoenix and 75° at Los Angeles. How far is the satellite from Los Angeles? In other words, find the distance AC in Figure 4.

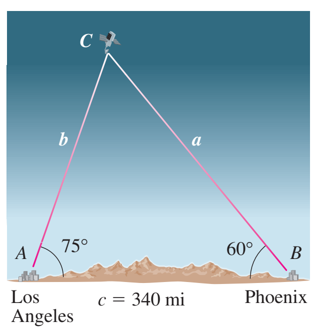


Figure 4

Solution Whenever two angles in a triangle are known, the third angle can be determined immediately because the sum of the angles of a triangle is 180°. In this case, $\angle C = 180° - (75° + 60°) = 45°$ (see Figure 4), so we have

$$\frac{\sin B}{b} = \frac{\sin C}{c} \qquad \text{Law of Sines}$$

$$\frac{\sin 60°}{b} = \frac{\sin 45°}{340} \qquad \text{Substitute}$$

$$b = \frac{340 \sin 60°}{\sin 45°} \approx 416 \qquad \text{Solve for } b$$

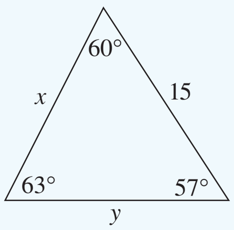
The distance of the satellite from Los Angeles is approximately 416 mi. ■

IN-CLASS MATERIALS

There is an interesting bit of trivia about the Mount Everest expedition described in the text: Sir George Everest measured the peak of the mountain to be 29,000 ft exactly. He didn't want people to think it was an estimate (to only two decimal places of accuracy) so he lied and said he measured it to be 29,002 ft. See also Section 1.2 of this book.

DRILL QUESTION

Find x in the following diagram:



Answer

$x \approx 14.119$

ALTERNATE EXAMPLE 1

Two people who live 20 miles apart look at an overhead airplane simultaneously. The angle of elevation relative to the first person is 80 degrees, and the angle of elevation relative to the second person is 15 degrees. What is the distance between the first person and the airplane?

ANSWER

5.2 miles

EXAMPLES

■ **ASA**

1. $A = 30°, B = 40°, c = 100$
2. $A = 0.75$ rad, $B = 0.8$ rad, $c = 100$

ANSWERS

1. $A = 30°, B = 40°, C = 110°, a \approx 53.2, b \approx 68.4, c = 100$
2. $A = 0.75$ rad, $B = 0.8$ rad, $C \approx \pi - 1.55 \approx 1.59$ rad, $a \approx 68.18, b \approx 71.75, c = 100$

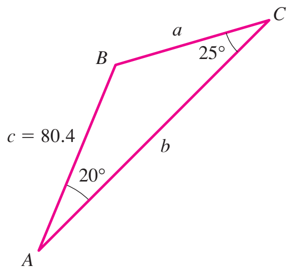


Figure 5

Example 2 Solving a Triangle (SAA)

Solve the triangle in Figure 5.

Solution First, $\angle B = 180° - (20° + 25°) = 135°$. Since side c is known, to find side a we use the relation

$$\frac{\sin A}{a} = \frac{\sin C}{c} \qquad \text{Law of Sines}$$

$$a = \frac{c \sin A}{\sin C} = \frac{80.4 \sin 20°}{\sin 25°} \approx 65.1 \qquad \text{Solve for } a$$

Similarly, to find b we use

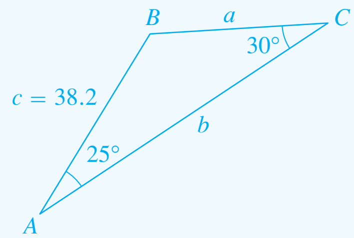
$$\frac{\sin B}{b} = \frac{\sin C}{c} \qquad \text{Law of Sines}$$

$$b = \frac{c \sin B}{\sin C} = \frac{80.4 \sin 135°}{\sin 25°} \approx 134.5 \qquad \text{Solve for } b$$

■

ALTERNATE EXAMPLE 2

Solve the triangle. Find the solution for the value of $\angle B$, and the solutions for the values of the sides a and b.



ANSWER

125, 32.3, 62.6

The Ambiguous Case

In Examples 1 and 2 a unique triangle was determined by the information given. This is always true of Case 1 (ASA or SAA). But in Case 2 (SSA) there may be two triangles, one triangle, or no triangle with the given properties. For this reason, Case 2 is sometimes called the **ambiguous case**. To see why this is so, we show in Figure 6 the possibilities when angle A and sides a and b are given. In part (a) no solution is possible, since side a is too short to complete the triangle. In part (b) the solution is a right triangle. In part (c) two solutions are possible, and in part (d) there is a unique triangle with the given properties. We illustrate the possibilities of Case 2 in the following examples.

SAMPLE QUESTION

Text Question

Why does the book say that SSA is the ambiguous case?

Answer

It is possible to form two different triangles given two sides and an opposite angle.

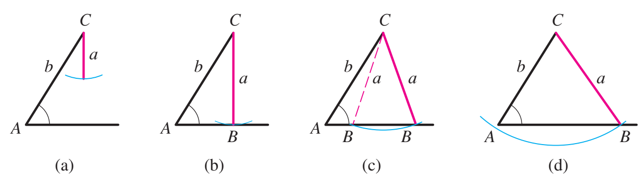


Figure 6
The ambiguous case

Example 3 SSA, the One-Solution Case

Solve triangle ABC, where $\angle A = 45°$, $a = 7\sqrt{2}$, and $b = 7$.

Solution We first sketch the triangle with the information we have (see Figure 7). Our sketch is necessarily tentative, since we don't yet know the other angles. Nevertheless, we can now see the possibilities.

We first find $\angle B$.

$$\frac{\sin A}{a} = \frac{\sin B}{b} \qquad \text{Law of Sines}$$

$$\sin B = \frac{b \sin A}{a} = \frac{7}{7\sqrt{2}} \sin 45° = \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{2}}{2}\right) = \frac{1}{2} \qquad \text{Solve for } \sin B$$

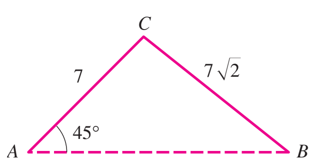


Figure 7

ALTERNATE EXAMPLE 3

Solve triangle ABC, where $\angle A = 45°$, $a = 9\sqrt{2}$, and $b = 9.0$. Find the length of the side c, and the values of angles B and C.

ANSWER

$B = 30°$, $C = 105$, $c = 17.39$

IN-CLASS MATERIALS

One can demonstrate the ambiguous case physically, using two rulers taped together to make a fixed angle, and a piece of string. See Figure 6 in the text.

IN-CLASS MATERIALS

It is tempting to do all examples in degrees. Students should also be exposed to the relatively unfamiliar radian, if possible.

We consider only angles smaller than 180°, since no triangle can contain an angle of 180° or larger.

Which angles B have $\sin B = \frac{1}{2}$? From the preceding section we know that there are two such angles smaller than 180° (they are 30° and 150°). Which of these angles is compatible with what we know about triangle ABC? Since $\angle A = 45°$, we cannot have $\angle B = 150°$, because $45° + 150° > 180°$. So $\angle B = 30°$, and the remaining angle is $\angle C = 180° - (30° + 45°) = 105°$.

Now we can find side c.

$$\frac{\sin B}{b} = \frac{\sin C}{c} \qquad \text{Law of Sines}$$

$$c = \frac{b \sin C}{\sin B} = \frac{7 \sin 105°}{\sin 30°} = \frac{7 \sin 105°}{\frac{1}{2}} \approx 13.5 \qquad \text{Solve for } c \quad ■$$

In Example 3 there were two possibilities for angle B, and one of these was not compatible with the rest of the information. In general, if $\sin A < 1$, we must check the angle and its supplement as possibilities, because any angle smaller than 180° can be in the triangle. To decide whether either possibility works, we check to see whether the resulting sum of the angles exceeds 180°. It can happen, as in Figure 6(c), that both possibilities are compatible with the given information. In that case, two different triangles are solutions to the problem.

The *supplement* of an angle θ (where $0 \le \theta \le 180°$) is the angle $180° - \theta$.

Alan Oddie/PhotoEdit

Surveying is a method of land measurement used for mapmaking. Surveyors use a process called *triangulation* in which a network of thousands of interlocking triangles is created on the area to be mapped. The process is started by measuring the length of a *baseline* between two surveying stations. Then, using an instrument called a *theodolite*, the angles between these two stations and a third station are measured. The Law of Sines is then used to calculate the two other sides of the triangle formed by the three stations. The calculated sides are used as baselines, and the process is repeated over and over to create a network of triangles. In this method, the only distance measured is the initial baseline; all (*continued*)

ALTERNATE EXAMPLE 4

Solve triangle ABC if $\angle A = 45.3°$, $a = 167.1$, and $b = 185.2$.

ANSWER

(52, 82.7, 233.2), (128, 6.7, 27.4)

Example 4 SSA, the Two-Solution Case

Solve triangle ABC if $\angle A = 43.1°$, $a = 186.2$, and $b = 248.6$.

Solution From the given information we sketch the triangle shown in Figure 8. Note that side a may be drawn in two possible positions to complete the triangle. From the Law of Sines

$$\sin B = \frac{b \sin A}{a} = \frac{248.6 \sin 43.1°}{186.2} \approx 0.91225$$

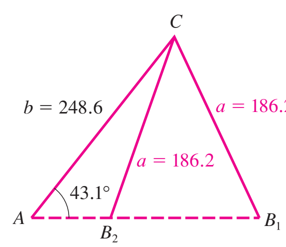


Figure 8

There are two possible angles B between 0° and 180° such that $\sin B = 0.91225$. Using the `SIN⁻¹` key on a calculator (or `INV` `SIN` or `ARCSIN`), we find that one of these angles is approximately 65.8°. The other is approximately $180° - 65.8° = 114.2°$. We denote these two angles by B_1 and B_2 so that

$$\angle B_1 \approx 65.8° \qquad \text{and} \qquad \angle B_2 \approx 114.2°$$

EXAMPLES

■ **SSA:** For each of the following, have students draw triangles to try to guess how many solutions there are.

1. $A = 80°$, $b = 100$, $a = 10$

2. $A = 80°$, $b = 10$, $a = 100$

3. $A = 80°$, $b = 121$, $a = 120$

4. $A = 0.2$ rad, $b = 50$, $a = 40$

ANSWERS

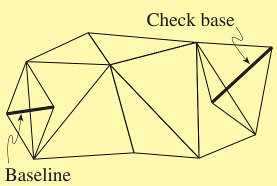
1. No solution

2. One solution:
$A = 80°$, $B \approx 5.65°$, $C \approx 94.35°$, $a = 100$, $b = 10$, $c \approx 101.25$

3. Two solutions:
$A = 80°$, $B \approx 83.2°$, $C \approx 16.8°$, $a = 120$, $b = 121$, $c \approx 35.2$;
$A = 80°$, $B \approx 96.8°$, $C \approx 3.2°$, $a = 120$, $b = 121$, $c \approx 6.9$

4. Two solutions:
$A = 0.2$ rad, $B \approx 0.25$ rad, $C \approx 2.69$ rad, $a = 40$, $b = 50$, $c \approx 87.8$;
$A = 0.2$ rad, $B \approx 2.89$ rad, $C \approx 0.05$ rad, $a = 40$, $b = 50$, $c \approx 10.3$

other distances are calculated from the Law of Sines. This method is practical because it is much easier to measure angles than distances.



One of the most ambitious map-making efforts of all time was the Great Trigonometric Survey of India (see Problem 8, page 525) which required several expeditions and took over a century to complete. The famous expedition of 1823, led by **Sir George Everest**, lasted 20 years. Ranging over treacherous terrain and encountering the dreaded malaria-carrying mosquitoes, this expedition reached the foothills of the Himalayas. A later expedition, using triangulation, calculated the height of the highest peak of the Himalayas to be 29,002 ft. The peak was named in honor of Sir George Everest.

Today, using satellites, the height of Mt. Everest is estimated to be 29,028 ft. The very close agreement of these two estimates shows the great accuracy of the trigonometric method.

Thus, two triangles satisfy the given conditions: triangle $A_1B_1C_1$ and triangle $A_2B_2C_2$.

Solve triangle $A_1B_1C_1$:

$$\angle C_1 \approx 180° - (43.1° + 65.8°) = 71.1° \qquad \text{Find } \angle C_1$$

Thus $$c_1 = \frac{a_1 \sin C_1}{\sin A_1} \approx \frac{186.2 \sin 71.1°}{\sin 43.1°} \approx 257.8 \qquad \text{Law of Sines}$$

Solve triangle $A_2B_2C_2$:

$$\angle C_2 \approx 180° - (43.1° + 114.2°) = 22.7° \qquad \text{Find } \angle C_2$$

Thus $$c_2 = \frac{a_2 \sin C_2}{\sin A_2} \approx \frac{186.2 \sin 22.7°}{\sin 43.1°} \approx 105.2 \qquad \text{Law of Sines}$$

Triangles $A_1B_1C_1$ and $A_2B_2C_2$ are shown in Figure 9.

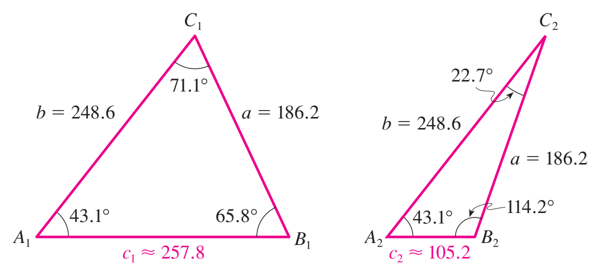


Figure 9

The next example presents a situation for which no triangle is compatible with the given data.

Example 5 SSA, the No-Solution Case

Solve triangle ABC, where $\angle A = 42°$, $a = 70$, and $b = 122$.

Solution To organize the given information, we sketch the diagram in Figure 10. Let's try to find $\angle B$. We have

$$\frac{\sin A}{a} = \frac{\sin B}{b} \qquad \text{Law of Sines}$$

$$\sin B = \frac{b \sin A}{a} = \frac{122 \sin 42°}{70} \approx 1.17 \qquad \text{Solve for } \sin B$$

Since the sine of an angle is never greater than 1, we conclude that no triangle satisfies the conditions given in this problem. ■

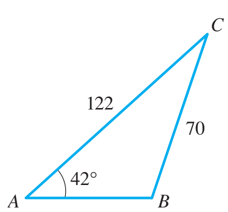


Figure 10

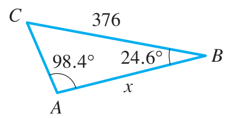
ALTERNATE EXAMPLE 5
Solve triangle ABC, where $\angle A = 41°$, $a = 40$, and $b = 89$. Find the length of the side c and the values of angles B and C.

ANSWER
No solution

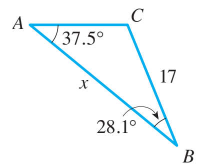
6.4 Exercises

1–6 ■ Use the Law of Sines to find the indicated side x or angle θ.

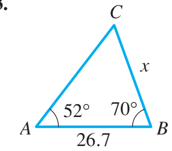
1.



2.



3.



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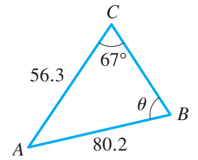

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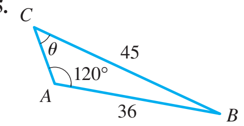
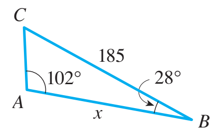
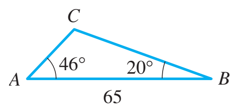


6.

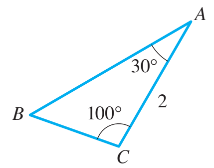


7–10 ■ Solve the triangle using the Law of Sines.

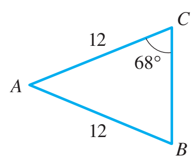
7.



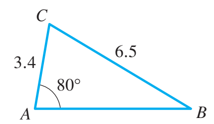
8.



9.



10.



11–16 ■ Sketch each triangle and then solve the triangle using the Law of Sines.

11. $\angle A = 50°$, $\angle B = 68°$, $c = 230$

12. $\angle A = 23°$, $\angle B = 110°$, $c = 50$

13. $\angle A = 30°$, $\angle C = 65°$, $b = 10$

14. $\angle A = 22°$, $\angle B = 95°$, $a = 420$

15. $\angle B = 29°$, $\angle C = 51°$, $b = 44$

16. $\angle B = 10°$, $\angle C = 100°$, $c = 115$

17–26 ■ Use the Law of Sines to solve for all possible triangles that satisfy the given conditions.

17. $a = 28$, $b = 15$, $\angle A = 110°$

18. $a = 30$, $c = 40$, $\angle A = 37°$

19. $a = 20$, $c = 45$, $\angle A = 125°$

20. $b = 45$, $c = 42$, $\angle C = 38°$

21. $b = 25$, $c = 30$, $\angle B = 25°$

22. $a = 75$, $b = 100$, $\angle A = 30°$

23. $a = 50$, $b = 100$, $\angle A = 50°$

24. $a = 100$, $b = 80$, $\angle A = 135°$

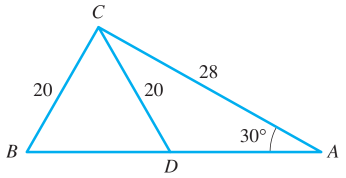
25. $a = 26$, $c = 15$, $\angle C = 29°$

26. $b = 73$, $c = 82$, $\angle B = 58°$

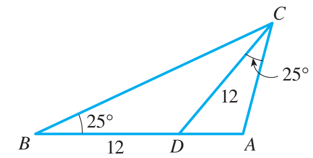
27. For the triangle shown, find

(a) $\angle BCD$ and

(b) $\angle DCA$.



28. For the triangle shown, find the length AD.



29. In triangle ABC, $\angle A = 40°$, $a = 15$, and $b = 20$.

(a) Show that there are two triangles, ABC and $A'B'C'$, that satisfy these conditions.

(b) Show that the areas of the triangles in part (a) are proportional to the sines of the angles C and C', that is,

$$\frac{\text{area of } \triangle ABC}{\text{area of } \triangle A'B'C'} = \frac{\sin C}{\sin C'}$$

30. Show that, given the three angles A, B, C of a triangle and one side, say a, the area of the triangle is

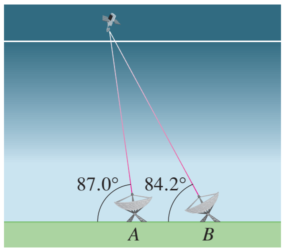
$$\text{area} = \frac{a^2 \sin B \sin C}{2 \sin A}$$

Applications

31. Tracking a Satellite The path of a satellite orbiting the earth causes it to pass directly over two tracking stations A and B, which are 50 mi apart. When the satellite is on one side of the two stations, the angles of elevation at A and B are measured to be 87.0° and 84.2°, respectively.

(a) How far is the satellite from station A?

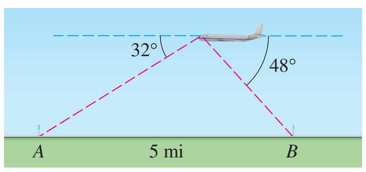
(b) How high is the satellite above the ground?



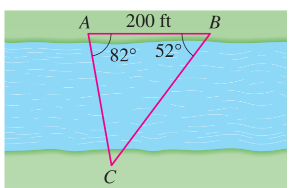
32. Flight of a Plane A pilot is flying over a straight highway. He determines the angles of depression to two mileposts, 5 mi apart, to be 32° and 48°, as shown in the figure.

(a) Find the distance of the plane from point A.

(b) Find the elevation of the plane.



33. Distance Across a River To find the distance across a river, a surveyor chooses points A and B, which are 200 ft apart on one side of the river (see the figure). She then chooses a reference point C on the opposite side of the river and finds that $\angle BAC \approx 82°$ and $\angle ABC \approx 52°$. Approximate the distance from A to C.

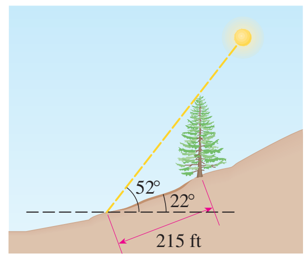


34. Distance Across a Lake Points A and B are separated by a lake. To find the distance between them, a surveyor locates a point C on land such that $\angle CAB = 48.6°$. He also measures CA as 312 ft and CB as 527 ft. Find the distance between A and B.

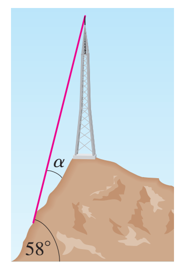
35. The Leaning Tower of Pisa The bell tower of the cathedral in Pisa, Italy, leans 5.6° from the vertical. A tourist stands 105 m from its base, with the tower leaning directly toward her. She measures the angle of elevation to the top of the tower to be 29.2°. Find the length of the tower to the nearest meter.

36. Radio Antenna A short-wave radio antenna is supported by two guy wires, 165 ft and 180 ft long. Each wire is attached to the top of the antenna and anchored to the ground, at two anchor points on opposite sides of the antenna. The shorter wire makes an angle of 67° with the ground. How far apart are the anchor points?

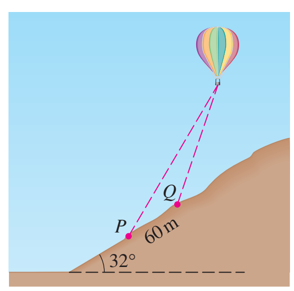
37. Height of a Tree A tree on a hillside casts a shadow 215 ft down the hill. If the angle of inclination of the hillside is 22° to the horizontal and the angle of elevation of the sun is 52°, find the height of the tree.



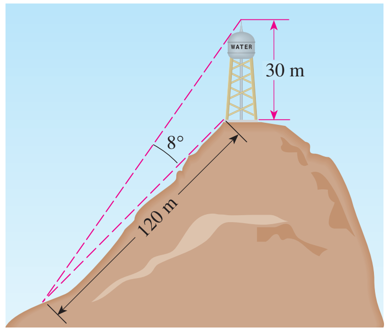
38. Length of a Guy Wire A communications tower is located at the top of a steep hill, as shown. The angle of inclination of the hill is 58°. A guy wire is to be attached to the top of the tower and to the ground, 100 m downhill from the base of the tower. The angle α in the figure is determined to be 12°. Find the length of cable required for the guy wire.



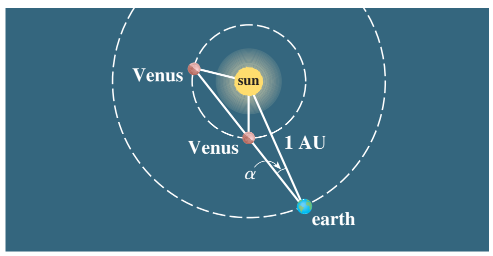
39. Calculating a Distance Observers at P and Q are located on the side of a hill that is inclined 32° to the horizontal, as shown. The observer at P determines the angle of elevation to a hot-air balloon to be 62°. At the same instant, the observer at Q measures the angle of elevation to the balloon to be 71°. If P is 60 m down the hill from Q, find the distance from Q to the balloon.



40. Calculating an Angle A water tower 30 m tall is located at the top of a hill. From a distance of 120 m down the hill, it is observed that the angle formed between the top and base of the tower is 8°. Find the angle of inclination of the hill.



41. Distances to Venus The *elongation* α of a planet is the angle formed by the planet, earth, and sun (see the figure). It is known that the distance from the sun to Venus is 0.723 AU (see Exercise 65 in Section 6.2). At a certain time the elongation of Venus is found to be 39.4°. Find the possible distances from the earth to Venus at that time in Astronomical Units (AU).



42. Soap Bubbles When two bubbles cling together in midair, their common surface is part of a sphere whose center D lies on the line passing throught the centers of the bubbles (see the figure). Also, angles ACB and ACD each have measure 60°.

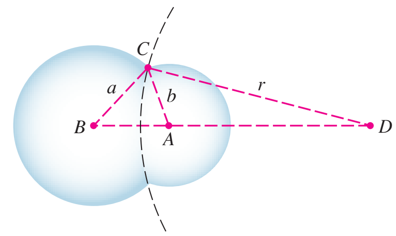
(a) Show that the radius r of the common face is given by

$$r = \frac{ab}{a - b}$$

[*Hint:* Use the Law of Sines together with the fact that an angle θ and its supplement $180° - \theta$ have the same sine.]

(b) Find the radius of the common face if the radii of the bubbles are 4 cm and 3 cm.

(c) What shape does the common face take if the two bubbles have equal radii?



Discovery • Discussion

43. Number of Solutions in the Ambiguous Case We have seen that when using the Law of Sines to solve a triangle in the SSA case, there may be two, one, or no solution(s). Sketch triangles like those in Figure 6 to verify the criteria in the table for the number of solutions if you are given $\angle A$ and sides a and b.

Criterion	Number of Solutions
$a \ge b$	1
$b > a > b \sin A$	2
$a = b \sin A$	1
$a < b \sin A$	0

If $\angle A = 30°$ and $b = 100$, use these criteria to find the range of values of a for which the triangle ABC has two solutions, one solution, or no solution.

6.5 The Law of Cosines

The Law of Sines cannot be used directly to solve triangles if we know two sides and the angle between them or if we know all three sides (these are Cases 3 and 4 of the preceding section). In these two cases, the **Law of Cosines** applies.

SUGGESTED TIME AND EMPHASIS

$\frac{1}{2}$–1 class.
Essential material.

POINTS TO STRESS

1. Using the Law of Cosines to solve triangles.
2. Using Heron's Formula to find the area of a triangle.

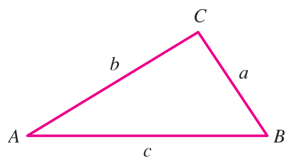


Figure 1

The Law of Cosines

In any triangle ABC (see Figure 1), we have

$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$b^2 = a^2 + c^2 - 2ac \cos B$$
$$c^2 = a^2 + b^2 - 2ab \cos C$$

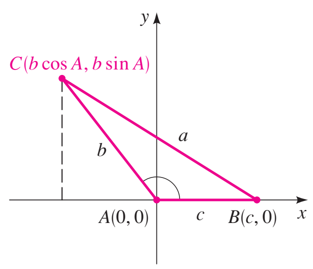


Figure 2

■ **Proof** To prove the Law of Cosines, place triangle ABC so that $\angle A$ is at the origin, as shown in Figure 2. The coordinates of the vertices B and C are $(c, 0)$ and $(b \cos A, b \sin A)$, respectively. (You should check that the coordinates of these points will be the same if we draw angle A as an acute angle.) Using the Distance Formula, we get

$$\begin{aligned} a^2 &= (b \cos A - c)^2 + (b \sin A - 0)^2 \\ &= b^2 \cos^2 A - 2bc \cos A + c^2 + b^2 \sin^2 A \\ &= b^2(\cos^2 A + \sin^2 A) - 2bc \cos A + c^2 \\ &= b^2 + c^2 - 2bc \cos A \qquad \text{Because } \sin^2 A + \cos^2 A = 1 \end{aligned}$$

This proves the first formula. The other two formulas are obtained in the same way by placing each of the other vertices of the triangle at the origin and repeating the preceding argument. ■

In words, the Law of Cosines says that the square of any side of a triangle is equal to the sum of the squares of the other two sides, minus twice the product of those two sides times the cosine of the included angle.

If one of the angles of a triangle, say $\angle C$, is a right angle, then $\cos C = 0$ and the Law of Cosines reduces to the Pythagorean Theorem, $c^2 = a^2 + b^2$. Thus, the Pythagorean Theorem is a special case of the Law of Cosines.

Example 1 Length of a Tunnel

A tunnel is to be built through a mountain. To estimate the length of the tunnel, a surveyor makes the measurements shown in Figure 3. Use the surveyor's data to approximate the length of the tunnel.

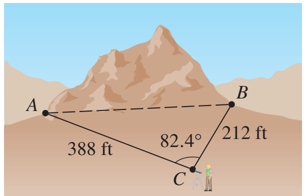


Figure 3

Solution To approximate the length c of the tunnel, we use the Law of Cosines:

$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab \cos C && \text{Law of Cosines} \\ &= 388^2 + 212^2 - 2(388)(212) \cos 82.4° && \text{Substitute} \\ &\approx 173730.2367 && \text{Use a calculator} \\ c &\approx \sqrt{173730.2367} \approx 416.8 && \text{Take square roots} \end{aligned}$$

Thus, the tunnel will be approximately 417 ft long. ■

DRILL QUESTION

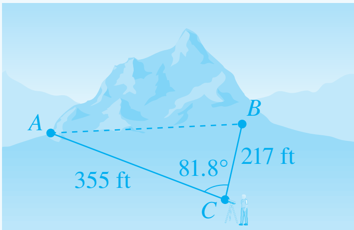
The sides of a triangle are $a = 5$, $b = 8$, and $c = 12$. Find the angle opposite side a.

Answer

0.307

ALTERNATE EXAMPLE 1

A tunnel is to be built through a mountain. To estimate the length of the tunnel, a surveyor makes the measurements shown in the figure below. Use the surveyor's data to approximate the length of the tunnel.



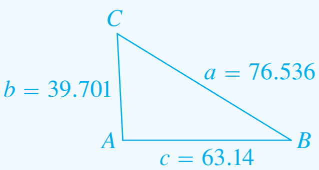
ANSWER

388.8

IN-CLASS MATERIALS

Have the class apply the Law of Cosines to the special case where C is a right angle. Note that the Law of Cosines is actually a generalization of the Pythagorean Theorem.

ALTERNATE EXAMPLE 2
The sides of a triangle are $a = 76.536$, $b = 39.701$, and $c = 63.14$. Find the angles of the triangle.



ANSWER
$A = 93.371$, $B = 31.187$, $C = 55.442$

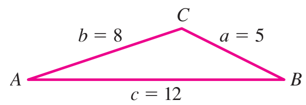


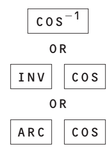
Figure 4

Example 2 SSS, the Law of Cosines

The sides of a triangle are $a = 5$, $b = 8$, and $c = 12$ (see Figure 4). Find the angles of the triangle.

Solution We first find $\angle A$. From the Law of Cosines, we have $a^2 = b^2 + c^2 - 2bc \cos A$. Solving for $\cos A$, we get

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{8^2 + 12^2 - 5^2}{2(8)(12)} = \frac{183}{192} = 0.953125$$



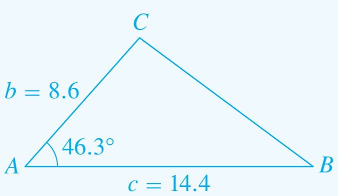
Using a calculator, we find that $\angle A \approx 18°$. In the same way the equations

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{5^2 + 12^2 - 8^2}{2(5)(12)} = 0.875$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{5^2 + 8^2 - 12^2}{2(5)(8)} = -0.6875$$

give $\angle B \approx 29°$ and $\angle C \approx 133°$. Of course, once two angles are calculated, the third can more easily be found from the fact that the sum of the angles of a triangle is 180°. However, it's a good idea to calculate all three angles using the Law of Cosines and add the three angles as a check on your computations. ■

ALTERNATE EXAMPLE 3
Solve triangle ABC, where $\angle A = 46.3$, $b = 8.6$, and $c = 14.4$ (see the figure below).



ANSWER
10.5, 36.3, 97.4

Example 3 SAS, the Law of Cosines

Solve triangle ABC, where $\angle A = 46.5°$, $b = 10.5$, and $c = 18.0$.

Solution We can find a using the Law of Cosines.

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc \cos A \\ &= (10.5)^2 + (18.0)^2 - 2(10.5)(18.0)(\cos 46.5°) \approx 174.05 \end{aligned}$$

Thus, $a \approx \sqrt{174.05} \approx 13.2$. We also use the Law of Cosines to find $\angle B$ and $\angle C$, as in Example 2.

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{13.2^2 + 18.0^2 - 10.5^2}{2(13.2)(18.0)} \approx 0.816477$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{13.2^2 + 10.5^2 - 18.0^2}{2(13.2)(10.5)} \approx -0.142532$$

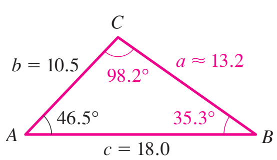


Figure 5

Using a calculator, we find that $\angle B \approx 35.3°$ and $\angle C \approx 98.2°$.
To summarize: $\angle B \approx 35.3°$, $\angle C \approx 98.2°$, and $a \approx 13.2$. (See Figure 5.) ■

We could have used the Law of Sines to find $\angle B$ and $\angle C$ in Example 3, since we knew all three sides and an angle in the triangle. But knowing the sine of an angle does not uniquely specify the angle, since an angle θ and its supplement $180° - \theta$ both have the same sine. Thus we would need to decide which of the two angles is the correct choice. This ambiguity does not arise when we use the Law of Cosines, because every angle between 0° and 180° has a unique cosine. So using only the Law of Cosines is preferable in problems like Example 3.

EXAMPLE
SAS: $A = 25°$, $b = 10$, $c = 20$

ANSWER
$a \approx 11.725$, $B \approx 21.1°$, $C \approx 133.9°$

EXAMPLE
SSS: $a = 4$, $b = 8$, $c = 11$

ANSWER
$A \approx 16.2°$, $B \approx 33.9°$, $C \approx 129.8°$

Navigation: Heading and Bearing

In navigation a direction is often given as a **bearing**, that is, as an acute angle measured from due north or due south. The bearing N 30° E, for example, indicates a direction that points 30° to the east of due north (see Figure 6).

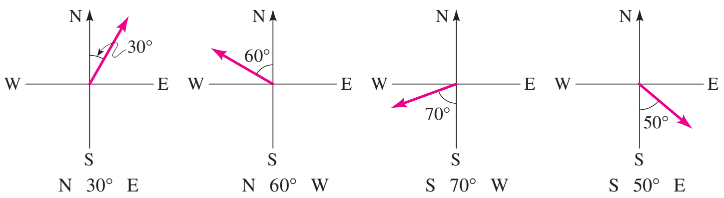


Figure 6

Example 4 Navigation

A pilot sets out from an airport and heads in the direction N 20° E, flying at 200 mi/h. After one hour, he makes a course correction and heads in the direction N 40° E. Half an hour after that, engine trouble forces him to make an emergency landing.

(a) Find the distance between the airport and his final landing point.

(b) Find the bearing from the airport to his final landing point.

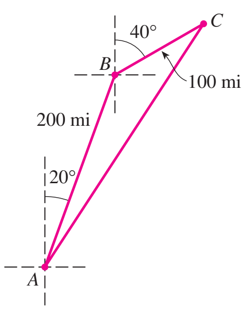


Figure 7

Solution

(a) In one hour the plane travels 200 mi, and in half an hour it travels 100 mi, so we can plot the pilot's course as in Figure 7. When he makes his course correction, he turns 20° to the right, so the angle between the two legs of his trip is $180° - 20° = 160°$. So by the Law of Cosines we have

$$b^2 = 200^2 + 100^2 - 2 \cdot 200 \cdot 100 \cos 160°$$
$$\approx 87{,}587.70$$

Thus, $b \approx 295.95$. The pilot lands about 296 mi from his starting point.

(b) We first use the Law of Sines to find $\angle A$.

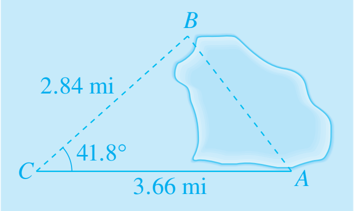
$$\frac{\sin A}{100} = \frac{\sin 160°}{295.95}$$
$$\sin A = 100 \cdot \frac{\sin 160°}{295.95}$$
$$\approx 0.11557$$

Using the $\boxed{\text{SIN}^{-1}}$ key on a calculator, we find that $\angle A \approx 6.636°$. From Figure 7 we see that the line from the airport to the final landing site points in the direction $20° + 6.636° = 26.636°$ east of due north. Thus, the bearing is about N 26.6° E. ■

Another angle with sine 0.11557 is $180° - 6.636° = 173.364°$. But this is clearly too large to be $\angle A$ in $\angle ABC$.

ALTERNATE EXAMPLE 4

To find the distance across a small lake, a surveyor has taken the measurements shown in the figure below. Find the distance across the lake (in miles) using this information.



ANSWER

2.44 miles

SAMPLE QUESTION

Text Question

Is it possible to find the area of a triangle if the side lengths are known, but the angles are not?

Answer

Yes

The Area of a Triangle

An interesting application of the Law of Cosines involves a formula for finding the area of a triangle from the lengths of its three sides (see Figure 8).

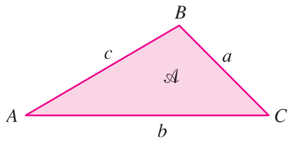


Figure 8

Heron's Formula

The area $\mathcal{A}$ of triangle ABC is given by

$$\mathcal{A} = \sqrt{s(s-a)(s-b)(s-c)}$$

where $s = \frac{1}{2}(a+b+c)$ is the **semiperimeter** of the triangle; that is, s is half the perimeter.

■ **Proof** We start with the formula $\mathcal{A} = \frac{1}{2}ab\sin C$ from Section 6.3. Thus

$$\begin{aligned}
\mathcal{A}^2 &= \tfrac{1}{4}a^2b^2\sin^2 C \\
&= \tfrac{1}{4}a^2b^2(1-\cos^2 C) && \text{Pythagorean identity} \\
&= \tfrac{1}{4}a^2b^2(1-\cos C)(1+\cos C) && \text{Factor}
\end{aligned}$$

Next, we write the expressions $1-\cos C$ and $1+\cos C$ in terms of a, b and c. By the Law of Cosines we have

$$\begin{aligned}
\cos C &= \frac{a^2+b^2-c^2}{2ab} && \text{Law of Cosines} \\
1+\cos C &= 1+\frac{a^2+b^2-c^2}{2ab} && \text{Add 1} \\
&= \frac{2ab+a^2+b^2-c^2}{2ab} && \text{Common denominator} \\
&= \frac{(a+b)^2-c^2}{2ab} && \text{Factor} \\
&= \frac{(a+b+c)(a+b-c)}{2ab} && \text{Difference of squares}
\end{aligned}$$

Similarly

$$1-\cos C = \frac{(c+a-b)(c-a+b)}{2ab}$$

Substituting these expressions in the formula we obtained for $\mathcal{A}^2$ gives

$$\begin{aligned}
\mathcal{A}^2 &= \tfrac{1}{4}a^2b^2\frac{(a+b+c)(a+b-c)}{2ab}\,\frac{(c+a-b)(c-a+b)}{2ab} \\
&= \frac{(a+b+c)}{2}\,\frac{(a+b-c)}{2}\,\frac{(c+a-b)}{2}\,\frac{(c-a+b)}{2} \\
&= s(s-c)(s-b)(s-a)
\end{aligned}$$

To see that the factors in the last two products are equal, note for example that

$$\begin{aligned}
\frac{a+b-c}{2} &= \frac{a+b+c}{2} - c \\
&= s-c
\end{aligned}$$

Heron's Formula now follows by taking the square root of each side. ■

IN-CLASS MATERIALS

Notice that the ability to easily find areas of triangles can be used to help find areas of polygons, because they can be decomposed into triangles.

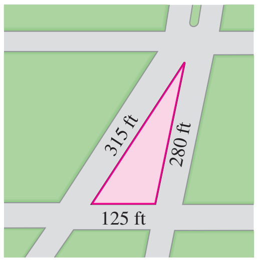


Figure 9

Example 5 Area of a Lot

A businessman wishes to buy a triangular lot in a busy downtown location (see Figure 9). The lot frontages on the three adjacent streets are 125, 280, and 315 ft. Find the area of the lot.

Solution The semiperimeter of the lot is

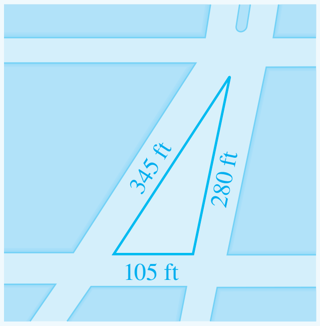
$$s = \frac{125 + 280 + 315}{2} = 360$$

By Heron's Formula the area is

$$\mathscr{A} = \sqrt{360(360 - 125)(360 - 280)(360 - 315)} \approx 17{,}451.6$$

Thus, the area is approximately 17,452 ft^2. ■

ALTERNATE EXAMPLE 5
A businessman wishes to buy a triangular lot in a busy downtown location (see the figure below). The lot frontages on the three adjacent streets are 105, 280, and 345 ft. Find the area of the lot.

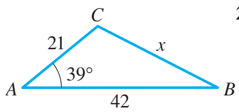


ANSWER
12,702 ft^2

6.5 Exercises

1–8 ■ Use the Law of Cosines to determine the indicated side x or angle θ.

1.



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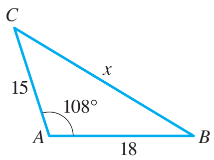

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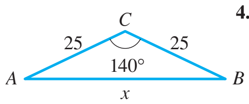

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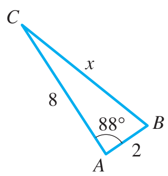

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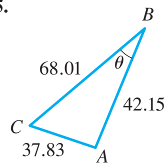

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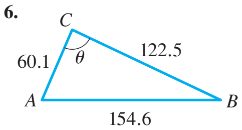

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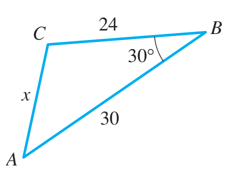

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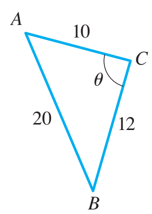
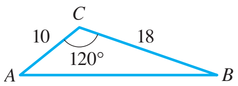
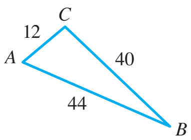


9–18 ■ Solve triangle ABC.

9.



10.



11. $a = 3.0, \quad b = 4.0, \quad \angle C = 53°$

12. $b = 60, \quad c = 30, \quad \angle A = 70°$

13. $a = 20, \quad b = 25, \quad c = 22$

14. $a = 10, \quad b = 12, \quad c = 16$

15. $b = 125, \quad c = 162, \quad \angle B = 40°$

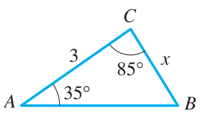
16. $a = 65, \quad c = 50, \quad \angle C = 52°$

17. $a = 50, \quad b = 65, \quad \angle A = 55°$

18. $a = 73.5, \quad \angle B = 61°, \quad \angle C = 83°$

19–26 ■ Find the indicated side x or angle θ. (Use either the Law of Sines or the Law of Cosines, as appropriate.)

19.



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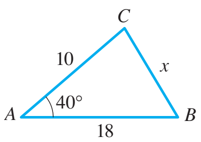

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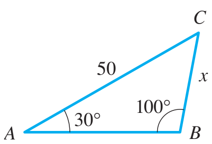

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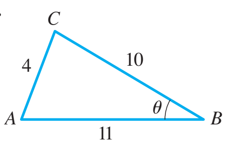

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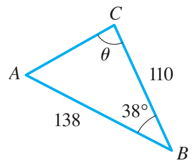

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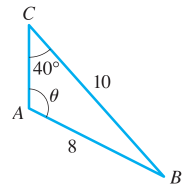

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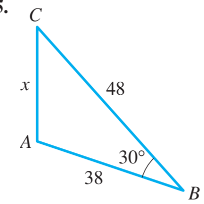

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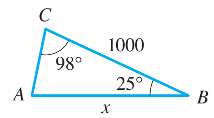


27–30 ■ Find the area of the triangle whose sides have the given lengths.

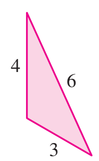
27. $a = 9, \quad b = 12, \quad c = 15$ **28.** $a = 1, \quad b = 2, \quad c = 2$

29. $a = 7, \quad b = 8, \quad c = 9$

30. $a = 11, \quad b = 100, \quad c = 101$

31–34 ■ Find the area of the shaded figure, correct to two decimals.

31.



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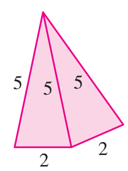

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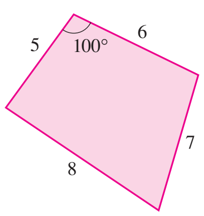

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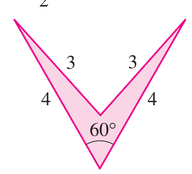
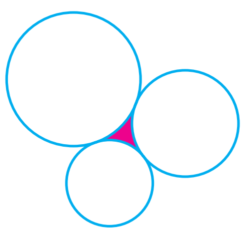


35. Three circles of radii 4, 5, and 6 cm are mutually tangent. Find the shaded area enclosed between the circles.



36. Prove that in triangle ABC

$$a = b \cos C + c \cos B$$

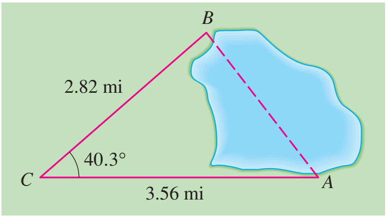
$$b = c \cos A + a \cos C$$

$$c = a \cos B + b \cos A$$

These are called the *Projection Laws*. [*Hint:* To get the first equation, add the second and third equations in the Law of Cosines and solve for a.]

Applications

37. Surveying To find the distance across a small lake, a surveyor has taken the measurements shown. Find the distance across the lake using this information.



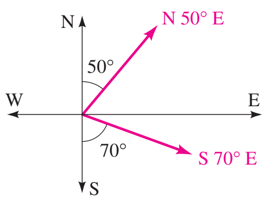
38. Geometry A parallelogram has sides of lengths 3 and 5, and one angle is 50°. Find the lengths of the diagonals.

39. Calculating Distance Two straight roads diverge at an angle of 65°. Two cars leave the intersection at 2:00 P.M., one traveling at 50 mi/h and the other at 30 mi/h. How far apart are the cars at 2:30 P.M.?

40. Calculating Distance A car travels along a straight road, heading east for 1 h, then traveling for 30 min on another road that leads northeast. If the car has maintained a constant speed of 40 mi/h, how far is it from its starting position?

41. Dead Reckoning A pilot flies in a straight path for 1 h 30 min. She then makes a course correction, heading 10° to the right of her original course, and flies 2 h in the new direction. If she maintains a constant speed of 625 mi/h, how far is she from her starting position?

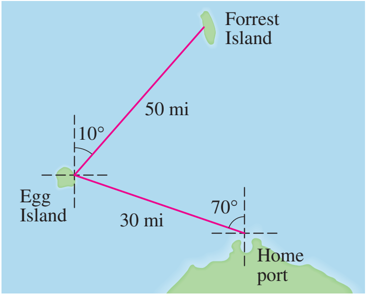
42. Navigation Two boats leave the same port at the same time. One travels at a speed of 30 mi/h in the direction N 50° E and the other travels at a speed of 26 mi/h in a direction S 70° E (see the figure). How far apart are the two boats after one hour?



43. Navigation A fisherman leaves his home port and heads in the direction N 70° W. He travels 30 mi and reaches Egg Island. The next day he sails N 10° E for 50 mi, reaching Forrest Island.

(a) Find the distance between the fisherman's home port and Forrest Island.

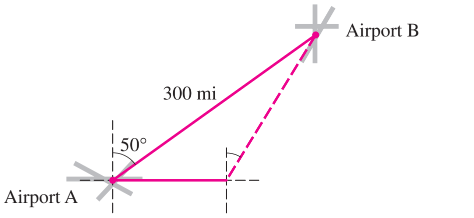
(b) Find the bearing from Forrest Island back to his home port.



44. Navigation Airport B is 300 mi from airport A at a bearing N 50° E (see the figure). A pilot wishing to fly from A to B mistakenly flies due east at 200 mi/h for 30 minutes, when he notices his error.

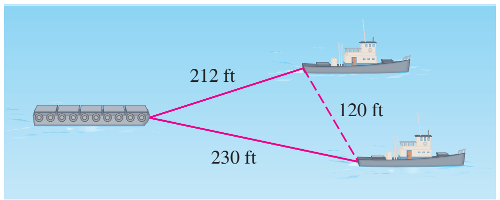
(a) How far is the pilot from his destination at the time he notices the error?

(b) What bearing should he head his plane in order to arrive at airport B?

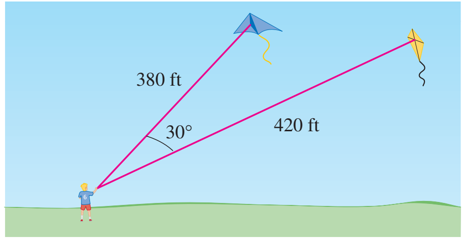


45. Triangular Field A triangular field has sides of lengths 22, 36, and 44 yd. Find the largest angle.

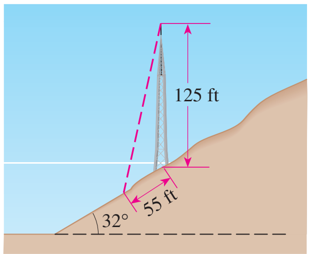
46. Towing a Barge Two tugboats that are 120 ft apart pull a barge, as shown. If the length of one cable is 212 ft and the length of the other is 230 ft, find the angle formed by the two cables.



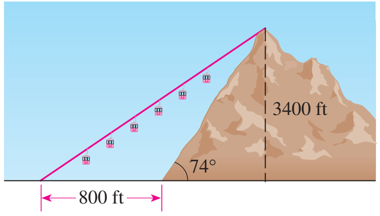
47. Flying Kites A boy is flying two kites at the same time. He has 380 ft of line out to one kite and 420 ft to the other. He estimates the angle between the two lines to be 30°. Approximate the distance between the kites.



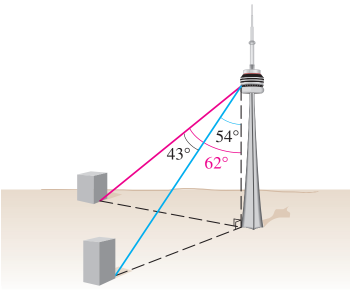
48. Securing a Tower A 125-ft tower is located on the side of a mountain that is inclined 32° to the horizontal. A guy wire is to be attached to the top of the tower and anchored at a point 55 ft downhill from the base of the tower. Find the shortest length of wire needed.



49. Cable Car A steep mountain is inclined 74° to the horizontal and rises 3400 ft above the surrounding plain. A cable car is to be installed from a point 800 ft from the base to the top of the mountain, as shown. Find the shortest length of cable needed.



50. CN Tower The CN Tower in Toronto, Canada, is the tallest free-standing structure in the world. A woman on the observation deck, 1150 ft above the ground, wants to determine the distance between two landmarks on the ground below. She observes that the angle formed by the lines of sight to these two landmarks is 43°. She also observes that the angle between the vertical and the line of sight to one of the landmarks is 62° and to the other landmark is 54°. Find the distance between the two landmarks.



51. Land Value Land in downtown Columbia is valued at $20 a square foot. What is the value of a triangular lot with sides of lengths 112, 148, and 190 ft?

Discovery • Discussion

52. Solving for the Angles in a Triangle The paragraph that follows the solution of Example 3 on page 510 explains an alternative method for finding $\angle B$ and $\angle C$, using the Law of Sines. Use this method to solve the triangle in the example, finding $\angle B$ first and then $\angle C$. Explain how you chose the appropriate value for the measure of $\angle B$. Which method do you prefer for solving an SAS triangle problem, the one explained in Example 3 or the one you used in this exercise?

6 Review

Concept Check

1. **(a)** Explain the difference between a positive angle and a negative angle.
   (b) How is an angle of measure 1 degree formed?
   (c) How is an angle of measure 1 radian formed?
   (d) How is the radian measure of an angle θ defined?
   (e) How do you convert from degrees to radians?
   (f) How do you convert from radians to degrees?
2. **(a)** When is an angle in standard position?
   (b) When are two angles coterminal?
3. **(a)** What is the length s of an arc of a circle with radius r that subtends a central angle of θ radians?
   (b) What is the area A of a sector of a circle with radius r and central angle θ radians?
4. If θ is an acute angle in a right triangle, define the six trigonometric ratios in terms of the adjacent and opposite sides and the hypotenuse.
5. What does it mean to solve a triangle?
6. If θ is an angle in standard position, $P(x, y)$ is a point on the terminal side, and r is the distance from the origin to P, write expressions for the six trigonometric functions of θ.
7. Which trigonometric functions are positive in quadrants I, II, III, and IV?
8. If θ is an angle in standard position, what is its reference angle $\bar{\theta}$?
9. **(a)** State the reciprocal identities.
   (b) State the Pythagorean identities.
10. **(a)** What is the area of a triangle with sides of length a and b and with included angle θ?
    (b) What is the area of a triangle with sides of length a, b, and c?
11. **(a)** State the Law of Sines.
    (b) State the Law of Cosines.
12. Explain the ambiguous case in the Law of Sines.

Exercises

1–2 ■ Find the radian measure that corresponds to the given degree measure.

1. **(a)** 60° **(b)** 330° **(c)** $-135°$ **(d)** $-90°$
2. **(a)** 24° **(b)** $-330°$ **(c)** 750° **(d)** 5°

3–4 ■ Find the degree measure that corresponds to the given radian measure.

3. **(a)** $\frac{5\pi}{2}$ **(b)** $-\frac{\pi}{6}$ **(c)** $\frac{9\pi}{4}$ **(d)** 3.1

4. **(a)** 8 **(b)** $-\frac{5}{2}$ **(c)** $\frac{11\pi}{6}$ **(d)** $\frac{3\pi}{5}$

5. Find the length of an arc of a circle of radius 8 m if the arc subtends a central angle of 1 rad.

6. Find the measure of a central angle θ in a circle of radius 5 ft if the angle is subtended by an arc of length 7 ft.

7. A circular arc of length 100 ft subtends a central angle of 70°. Find the radius of the circle.

8. How many revolutions will a car wheel of diameter 28 in. make over a period of half an hour if the car is traveling at 60 mi/h?

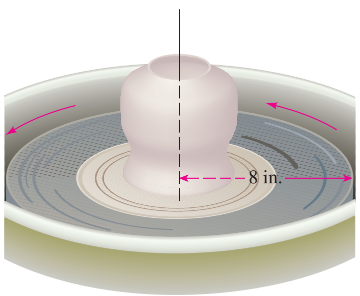
9. New York and Los Angeles are 2450 mi apart. Find the angle that the arc between these two cities subtends at the center of the earth. (The radius of the earth is 3960 mi.)

10. Find the area of a sector with central angle 2 rad in a circle of radius 5 m.

11. Find the area of a sector with central angle 52° in a circle of radius 200 ft.

12. A sector in a circle of radius 25 ft has an area of 125 ft^2. Find the central angle of the sector.

13. A potter's wheel with radius 8 in. spins at 150 rpm. Find the angular and linear speeds of a point on the rim of the wheel.



14. In an automobile transmission a *gear ratio* g is the ratio

$$g = \frac{\text{angular speed of engine}}{\text{angular speed of wheels}}$$

The angular speed of the engine is shown on the tachometer (in rpm).

A certain sports car has wheels with radius 11 in. Its gear ratios are shown in the following table. Suppose the car is in fourth gear and the tachometer reads 3500 rpm.

(a) Find the angular speed of the engine.

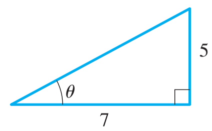
(b) Find the angular speed of the wheels.

(c) How fast (in mi/h) is the car traveling?

Gear	Ratio
1st	4.1
2nd	3.0
3rd	1.6
4th	0.9
5th	0.7

15–16 ■ Find the values of the six trigonometric ratios of θ.

15.



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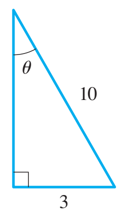
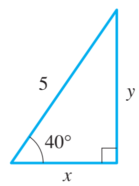


17–20 ■ Find the sides labeled x and y, correct to two decimal places.

17.



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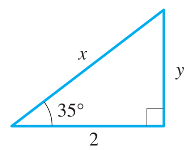

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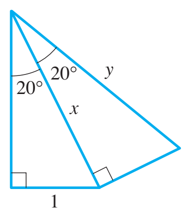

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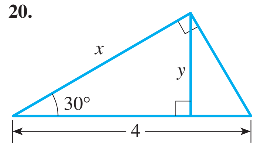
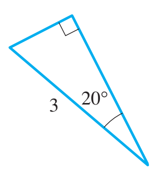


21–22 ■ Solve the triangle.

21.



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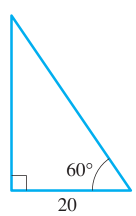
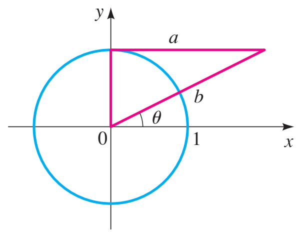


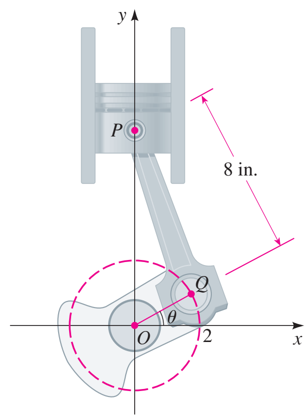
23. Express the lengths a and b in the figure in terms of the trigonometric ratios of θ.



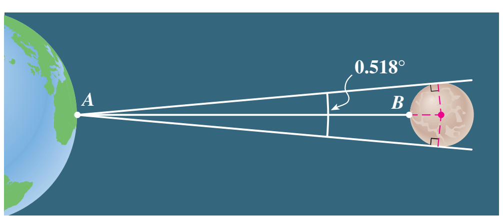
24. The highest free-standing tower in the world is the CN Tower in Toronto, Canada. From a distance of 1 km from its base, the angle of elevation to the top of the tower is 28.81°. Find the height of the tower.

25. Find the perimeter of a regular hexagon that is inscribed in a circle of radius 8 m.

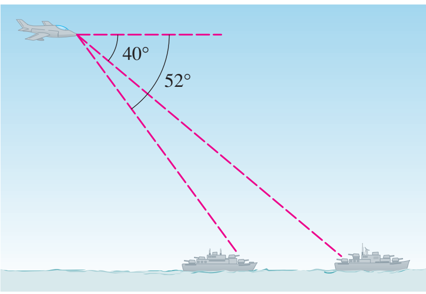
26. The pistons in a car engine move up and down repeatedly to turn the crankshaft, as shown. Find the height of the point P above the center O of the crankshaft in terms of the angle θ.



27. As viewed from the earth, the angle subtended by the full moon is 0.518°. Use this information and the fact that the distance AB from the earth to the moon is 236,900 mi to find the radius of the moon.



28. A pilot measures the angles of depression to two ships to be 40° and 52° (see the figure). If the pilot is flying at an elevation of 35,000 ft, find the distance between the two ships.



29–40 ■ Find the exact value.

29. $\sin 315°$

30. $\csc \dfrac{9\pi}{4}$

31. $\tan(-135°)$

32. $\cos \dfrac{5\pi}{6}$

33. $\cot\left(-\dfrac{22\pi}{3}\right)$

34. $\sin 405°$

35. $\cos 585°$

36. $\sec \dfrac{22\pi}{3}$

37. $\csc \dfrac{8\pi}{3}$

38. $\sec \dfrac{13\pi}{6}$

39. $\cot(-390°)$

40. $\tan \dfrac{23\pi}{4}$

41. Find the values of the six trigonometric ratios of the angle θ in standard position if the point $(-5, 12)$ is on the terminal side of θ.

42. Find $\sin\theta$ if θ is in standard position and its terminal side intersects the circle of radius 1 centered at the origin at the point $(-\sqrt{3}/2, \frac{1}{2})$.

43. Find the acute angle that is formed by the line $y - \sqrt{3}x + 1 = 0$ and the x-axis.

44. Find the six trigonometric ratios of the angle θ in standard position if its terminal side is in quadrant III and is parallel to the line $4y - 2x - 1 = 0$.

45–48 ■ Write the first expression in terms of the second, for θ in the given quadrant.

45. $\tan\theta$, $\cos\theta$; θ in quadrant II

46. $\sec\theta$, $\sin\theta$; θ in quadrant III

47. $\tan^2\theta$, $\sin\theta$; θ in any quadrant

48. $\csc^2\theta \cos^2\theta$, $\sin\theta$; θ in any quadrant

49–52 ■ Find the values of the six trigonometric functions of θ from the information given.

49. $\tan\theta = \sqrt{7}/3$, $\sec\theta = \frac{4}{3}$ **50.** $\sec\theta = \frac{41}{40}$, $\csc\theta = -\frac{41}{9}$

51. $\sin\theta = \frac{3}{5}$, $\cos\theta < 0$ **52.** $\sec\theta = -\frac{13}{5}$, $\tan\theta > 0$

53. If $\tan\theta = -\frac{1}{2}$ for θ in quadrant II, find $\sin\theta + \cos\theta$.

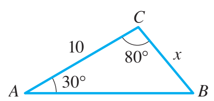
54. If $\sin\theta = \frac{1}{2}$ for θ in quadrant I, find $\tan\theta + \sec\theta$.

55. If $\tan\theta = -1$, find $\sin^2\theta + \cos^2\theta$.

56. If $\cos\theta = -\sqrt{3}/2$ and $\pi/2 < \theta < \pi$, find $\sin 2\theta$.

57–62 ■ Find the side labeled x.

57.



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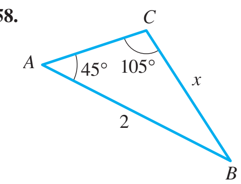

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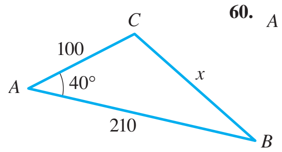

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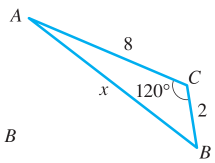

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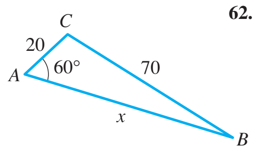

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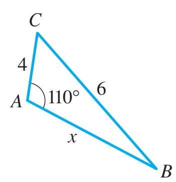
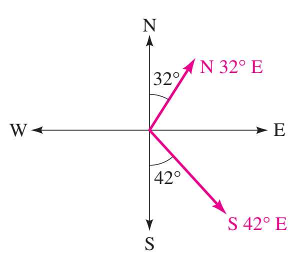
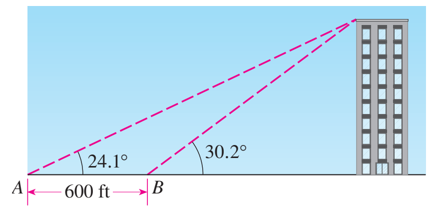


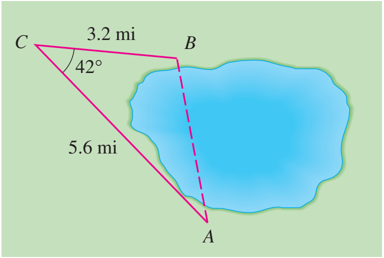
63. Two ships leave a port at the same time. One travels at 20 mi/h in a direction N 32° E, and the other travels at 28 mi/h in a direction S 42° E (see the figure). How far apart are the two ships after 2 h?



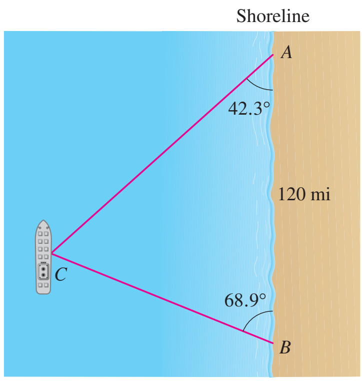
64. From a point A on the ground, the angle of elevation to the top of a tall building is 24.1°. From a point B, which is 600 ft closer to the building, the angle of elevation is measured to be 30.2°. Find the height of the building.



65. Find the distance between points A and B on opposite sides of a lake from the information shown.



66. A boat is cruising the ocean off a straight shoreline. Points A and B are 120 mi apart on the shore, as shown. It is found that $\angle A = 42.3°$ and $\angle B = 68.9°$. Find the shortest distance from the boat to the shore.



67. Find the area of a triangle with sides of length 8 and 14 and included angle 35°.

68. Find the area of a triangle with sides of length 5, 6, and 8.

6 Test

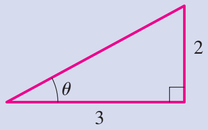
1. Find the radian measures that correspond to the degree measures 330° and $-135°$.

2. Find the degree measures that correspond to the radian measures $\dfrac{4\pi}{3}$ and -1.3.

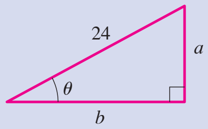
3. The rotor blades of a helicopter are 16 ft long and are rotating at 120 rpm.
  (a) Find the angular speed of the rotor.
  (b) Find the linear speed of a point on the tip of a blade.

4. Find the exact value of each of the following.
  (a) $\sin 405°$ (b) $\tan(-150°)$
  (c) $\sec \dfrac{5\pi}{3}$ (d) $\csc \dfrac{5\pi}{2}$

5. Find $\tan\theta + \sin\theta$ for the angle θ shown.



6. Express the lengths a and b shown in the figure in terms of θ.

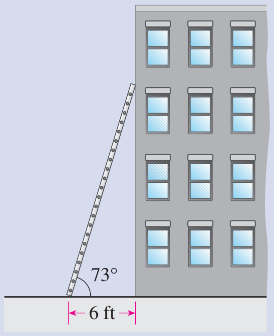


7. If $\cos\theta = -\frac{1}{3}$ and θ is in quadrant III, find $\tan\theta\cot\theta + \csc\theta$.

8. If $\sin\theta = \frac{5}{13}$ and $\tan\theta = -\frac{5}{12}$, find $\sec\theta$.

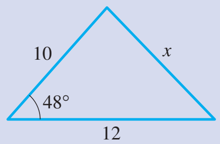
9. Express $\tan\theta$ in terms of $\sec\theta$ for θ in quadrant II.

10. The base of the ladder in the figure is 6 ft from the building, and the angle formed by the ladder and the ground is 73°. How high up the building does the ladder touch?



11–14 ■ Find the side labeled x.

11.



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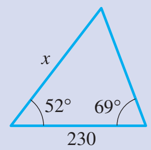

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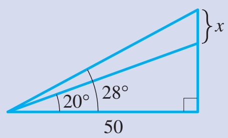

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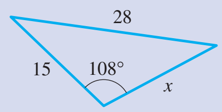
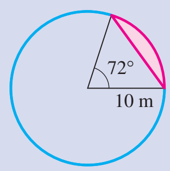


15. Refer to the figure below.

(a) Find the area of the shaded region.

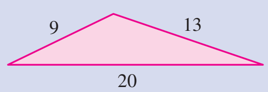
(b) Find the perimeter of the shaded region.



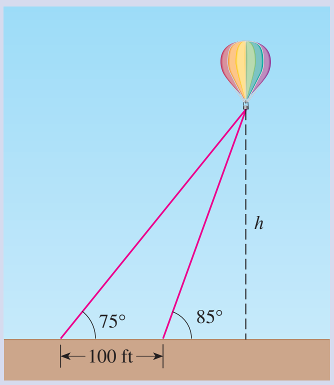
16. Refer to the figure below.

(a) Find the angle opposite the longest side.

(b) Find the area of the triangle.



17. Two wires tether a balloon to the ground, as shown. How high is the balloon above the ground?



Focus on Modeling
Surveying

How can we measure the height of a mountain, or the distance across a lake? Obviously it may be difficult, inconvenient, or impossible to measure these distances directly (that is, using a tape measure or a yard stick). On the other hand, it is easy to measure *angles* to distant objects. That's where trigonometry comes in—the trigonometric ratios relate angles to distances, so they can be used to *calculate* distances from the *measured* angles. In this *Focus* we examine how trigonometry is used to map a town. Modern map making methods use satellites and the Global Positioning System, but mathematics remains at the core of the process.

Mapping a Town

A student wants to draw a map of his hometown. To construct an accurate map (or scale model), he needs to find distances between various landmarks in the town. The student makes the measurements shown in Figure 1. Note that only one distance is measured, between City Hall and the first bridge. All other measurements are angles.

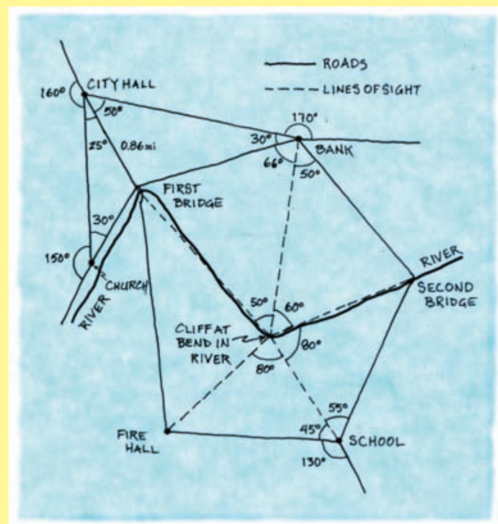


Figure 1

The distances between other landmarks can now be found using the Law of Sines. For example, the distance x from the bank to the first bridge is calculated by applying the Law of Sines to the triangle with vertices at City Hall, the bank, and the first bridge:

$$\frac{x}{\sin 50°} = \frac{0.86}{\sin 30°} \qquad \text{Law of Sines}$$

$$x = \frac{0.86 \sin 50°}{\sin 30°} \qquad \text{Solve for } x$$

$$\approx 1.32 \text{ mi} \qquad \text{Calculator}$$

So the distance between the bank and the first bridge is 1.32 mi.

The distance we just found can now be used to find other distances. For instance, we find the distance y between the bank and the cliff as follows:

$$\frac{y}{\sin 64°} = \frac{1.32}{\sin 50°} \qquad \text{Law of Sines}$$

$$y = \frac{1.32 \sin 64°}{\sin 50°} \qquad \text{Solve for } y$$

$$\approx 1.55 \text{ mi} \qquad \text{Calculator}$$

Continuing in this fashion, we can calculate all the distances between the landmarks shown in the rough sketch in Figure 1. We can use this information to draw the map shown in Figure 2.

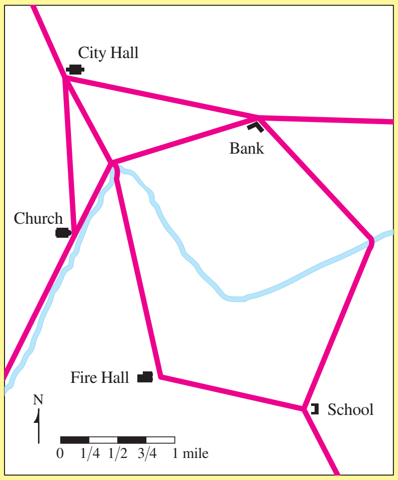


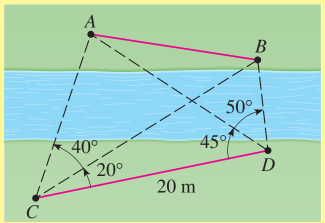
Figure 2

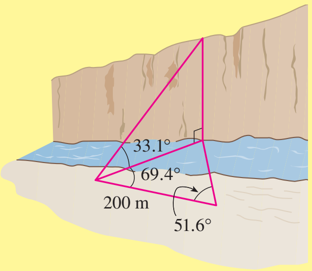
To make a topographic map, we need to measure elevation. This concept is explored in Problems 4–6.

Problems

1. **Completing the Map** Find the distance between the church and City Hall.
2. **Completing the Map** Find the distance between the fire hall and the school. (You will need to find other distances first.)

3. Determining a Distance A surveyor on one side of a river wishes to find the distance between points A and B on the opposite side of the river. On her side, she chooses points C and D, which are 20 m apart, and measures the angles shown in the figure below. Find the distance between A and B.





4. Height of a Cliff To measure the height of an inaccessible cliff on the opposite side of a river, a surveyor makes the measurements shown in the figure at the left. Find the height of the cliff.

5. Height of a Mountain To calculate the height h of a mountain, angle α, β, and distance d are measured, as shown in the figure below.

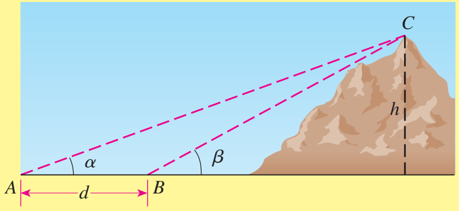
(a) Show that

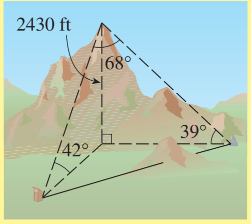
$$h = \frac{d}{\cot \alpha - \cot \beta}$$

(b) Show that

$$h = d \frac{\sin \alpha \sin \beta}{\sin(\beta - \alpha)}$$

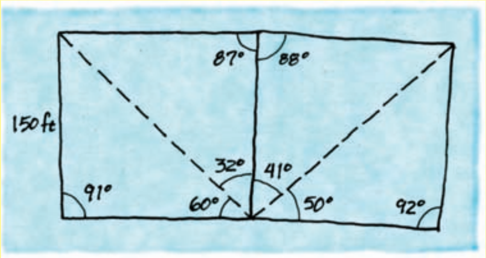
(c) Use the formulas from parts (a) and (b) to find the height of a mountain if $\alpha = 25°$, $\beta = 29°$, and $d = 800$ ft. Do you get the same answer from each formula?





6. Determining a Distance A surveyor has determined that a mountain is 2430 ft high. From the top of the mountain he measures the angles of depression to two landmarks at the base of the mountain, and finds them to be 42° and 39°. (Observe that these are the same as the angles of elevation from the landmarks as shown in the figure at the left.) The angle between the lines of sight to the landmarks is 68°. Calculate the distance between the two landmarks.

7. Surveying Building Lots A surveyor surveys two adjacent lots and makes the following rough sketch showing his measurements. Calculate all the distances shown in the figure and use your result to draw an accurate map of the two lots.



8. Great Survey of India The Great Trigonometric Survey of India was one of the most massive mapping projects ever undertaken (see the margin note on page 504). Do some research at your library or on the Internet to learn more about the Survey, and write a report on your findings.

British Library