

5

Trigonometric Functions of Real Numbers

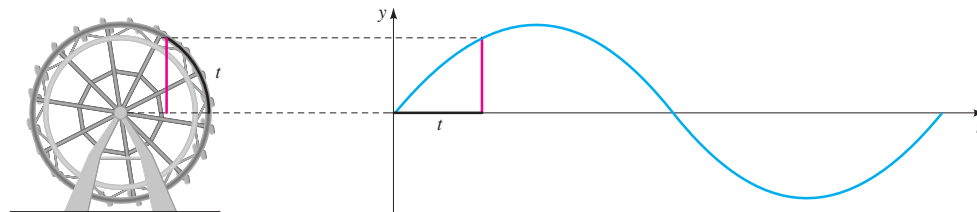


- 5.1 The Unit Circle
- 5.2 Trigonometric Functions of Real Numbers
- 5.3 Trigonometric Graphs
- 5.4 More Trigonometric Graphs
- 5.5 Modeling Harmonic Motion

Chapter Overview

In this chapter and the next we introduce new functions called the trigonometric functions. The trigonometric functions can be defined in two different but equivalent ways—as functions of angles (Chapter 6) or functions of real numbers (Chapter 5). **The two approaches to trigonometry are independent of each other, so either Chapter 5 or Chapter 6 may be studied first.** We study both approaches because different applications require that we view these functions differently. The approach in this chapter lends itself to modeling periodic motion.

If you've ever taken a ferris wheel ride, then you know about periodic motion—that is, motion that repeats over and over. This type of motion is common in nature. Think about the daily rising and setting of the sun (day, night, day, night, . . .), the daily variation in tide levels (high, low, high, low, . . .), the vibrations of a leaf in the wind (left, right, left, right, . . .), or the pressure in the cylinders of a car engine (high, low, high, low, . . .). To describe such motion mathematically we need a function whose values increase, then decrease, then increase, . . . , repeating this pattern indefinitely. To understand how to define such a function, let's look at the ferris wheel again. A person riding on the wheel goes up and down, up and down, The graph shows how high the person is above the center of the ferris wheel at time t . Notice that as the wheel turns the graph goes up and down repeatedly.



We define the trigonometric function called *sine* in a similar way. We start with a circle of radius 1, and for each distance t along the arc of the circle ending at (x, y) we define the value of the function $\sin t$ to be the height (or y -coordinate) of that point. To apply this function to real-world situations we use the transformations we learned in Chapter 2 to stretch, shrink, or shift the function to fit the variation we are modeling.

There are six trigonometric functions, each with its special properties. In this

chapter we study their definitions, graphs, and applications. In Section 5.5 we see how trigonometric functions can be used to model harmonic motion.

SUGGESTED TIME AND EMPHASIS

$\frac{1}{2}$ -1 class.
Essential material.

POINTS TO STRESS

1. Finding the reference number for a given real number.
2. Using reference numbers to find terminal points.

ALTERNATE EXAMPLE 1

Is the point $P\left(\frac{\sqrt{6}}{6}, \frac{\sqrt{7}}{6}\right)$ on the unit circle?

ANSWER

No

ALTERNATE EXAMPLE 2

The point $P(\sqrt{35}/6, y)$ is on the unit circle in quadrant I. Find its y -coordinate.

ANSWER

$\frac{1}{6}$

5.1 The Unit Circle

In this section we explore some properties of the circle of radius 1 centered at the origin. These properties are used in the next section to define the trigonometric functions.

The Unit Circle

The set of points at a distance 1 from the origin is a circle of radius 1 (see Figure 1). In Section 1.8 we learned that the equation of this circle is $x^2 + y^2 = 1$.

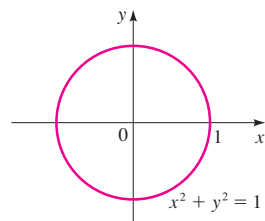


Figure 1
The unit circle

The Unit Circle

The **unit circle** is the circle of radius 1 centered at the origin in the xy -plane. Its equation is

$$x^2 + y^2 = 1$$

Example 1 A Point on the Unit Circle

Show that the point $P\left(\frac{\sqrt{3}}{3}, \frac{\sqrt{6}}{3}\right)$ is on the unit circle.

Solution We need to show that this point satisfies the equation of the unit circle, that is, $x^2 + y^2 = 1$. Since

$$\left(\frac{\sqrt{3}}{3}\right)^2 + \left(\frac{\sqrt{6}}{3}\right)^2 = \frac{3}{9} + \frac{6}{9} = 1$$

P is on the unit circle. ■

Example 2 Locating a Point on the Unit Circle

The point $P(\sqrt{3}/2, y)$ is on the unit circle in quadrant IV. Find its y -coordinate.

Solution Since the point is on the unit circle, we have

$$\left(\frac{\sqrt{3}}{2}\right)^2 + y^2 = 1$$

$$y^2 = 1 - \frac{3}{4} = \frac{1}{4}$$

$$y = \pm \frac{1}{2}$$

Since the point is in quadrant IV, its y -coordinate must be negative, so $y = -\frac{1}{2}$. ■

Terminal Points on the Unit Circle

Suppose t is a real number. Let's mark off a distance t along the unit circle, starting at the point $(1, 0)$ and moving in a counterclockwise direction if t is positive or in a clockwise direction if t is negative (Figure 2). In this way we arrive at a point $P(x, y)$ on the unit circle. The point $P(x, y)$ obtained in this way is called the **terminal point** determined by the real number t .

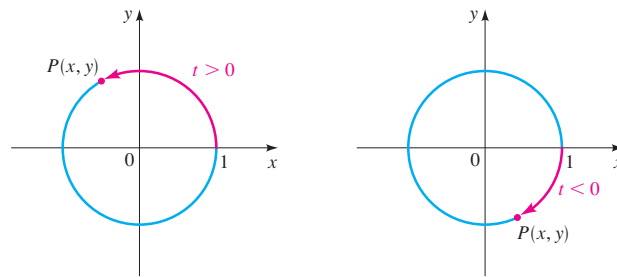


Figure 2 (a) Terminal point $P(x, y)$ determined by $t > 0$ (b) Terminal point $P(x, y)$ determined by $t < 0$

The circumference of the unit circle is $C = 2\pi(1) = 2\pi$. So, if a point starts at $(1, 0)$ and moves counterclockwise all the way around the unit circle and returns to $(1, 0)$, it travels a distance of 2π . To move halfway around the circle, it travels a distance of $\frac{1}{2}(2\pi) = \pi$. To move a quarter of the distance around the circle, it travels a distance of $\frac{1}{4}(2\pi) = \pi/2$. Where does the point end up when it travels these distances along the circle? From Figure 3 we see, for example, that when it travels a distance of π starting at $(1, 0)$, its terminal point is $(-1, 0)$.

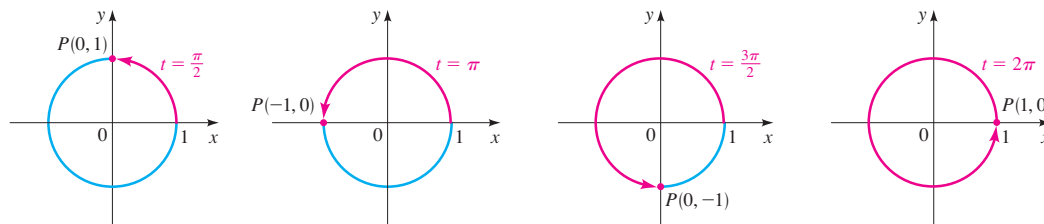


Figure 3
Terminal points determined by $t = \frac{\pi}{2}$, π , $\frac{3\pi}{2}$, and 2π

Example 3 Finding Terminal Points

Find the terminal point on the unit circle determined by each real number t .

- (a) $t = 3\pi$ (b) $t = -\pi$ (c) $t = -\frac{\pi}{2}$

Solution From Figure 4 we get the following.

- (a) The terminal point determined by 3π is $(-1, 0)$.
 (b) The terminal point determined by $-\pi$ is $(-1, 0)$.



ALTERNATE EXAMPLE 3c

Find the terminal point on the unit circle determined by the real number t :

$$t = \frac{3\pi}{2}$$

ANSWER

$(0, -1)$

IN-CLASS MATERIALS

Look at the time, and then ask the class what time it will be 8 hours from now. Then ask about 12 hours from now, 13 hours from now, 25 hours from now, and 50 hours from now. Then ask what time it was 12 hours ago, 11 hours ago, and 13 hours ago. Draw the analogy of the type of arithmetic you are doing when computing future and past times to that of finding reference angles, only instead of freely adding and subtracting 12, we are freely adding and subtracting 2π .

(c) The terminal point determined by $-\pi/2$ is $(0, -1)$.

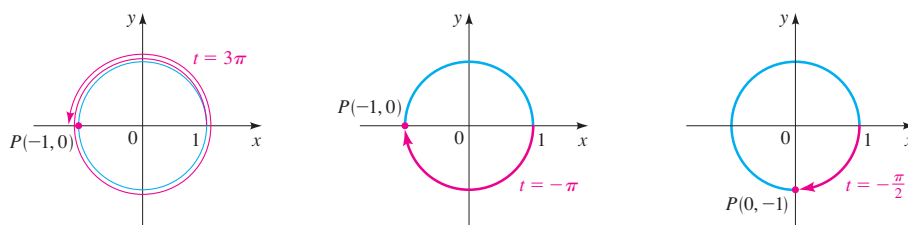


Figure 4

Notice that different values of t can determine the same terminal point. ■

The terminal point $P(x, y)$ determined by $t = \pi/4$ is the same distance from $(1, 0)$ as from $(0, 1)$ along the unit circle (see Figure 5).

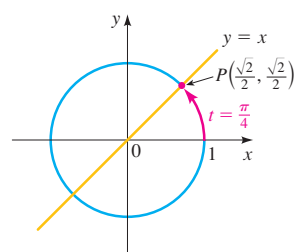


Figure 5

Since the unit circle is symmetric with respect to the line $y = x$, it follows that P lies on the line $y = x$. So P is the point of intersection (in the first quadrant) of the circle $x^2 + y^2 = 1$ and the line $y = x$. Substituting x for y in the equation of the circle, we get

$$\begin{aligned} x^2 + x^2 &= 1 \\ 2x^2 &= 1 && \text{Combine like terms} \\ x^2 &= \frac{1}{2} && \text{Divide by 2} \\ x &= \pm \frac{1}{\sqrt{2}} && \text{Take square roots} \end{aligned}$$

Since P is in the first quadrant, $x = 1/\sqrt{2}$ and since $y = x$, we have $y = 1/\sqrt{2}$ also. Thus, the terminal point determined by $\pi/4$ is

$$P\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = P\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$

Similar methods can be used to find the terminal points determined by $t = \pi/6$ and $t = \pi/3$ (see Exercises 55 and 56). Table 1 and Figure 6 give the terminal points for some special values of t .

Table 1

t	Terminal point determined by t
0	(1, 0)
$\frac{\pi}{6}$	$(\frac{\sqrt{3}}{2}, \frac{1}{2})$
$\frac{\pi}{4}$	$(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$
$\frac{\pi}{3}$	$(\frac{1}{2}, \frac{\sqrt{3}}{2})$
$\frac{\pi}{2}$	(0, 1)

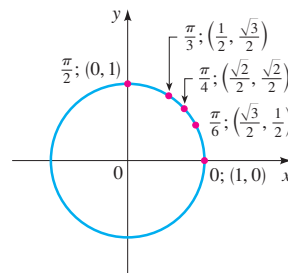


Figure 6

Example 4 Finding Terminal Points

Find the terminal point determined by each given real number t .

(a) $t = -\frac{\pi}{4}$ (b) $t = \frac{3\pi}{4}$ (c) $t = -\frac{5\pi}{6}$

Solution

(a) Let P be the terminal point determined by $-\pi/4$, and let Q be the terminal point determined by $\pi/4$. From Figure 7(a) we see that the point P has the same coordinates as Q except for sign. Since P is in quadrant IV, its x -coordinate is positive and its y -coordinate is negative. Thus, the terminal point is $P(\sqrt{2}/2, -\sqrt{2}/2)$.

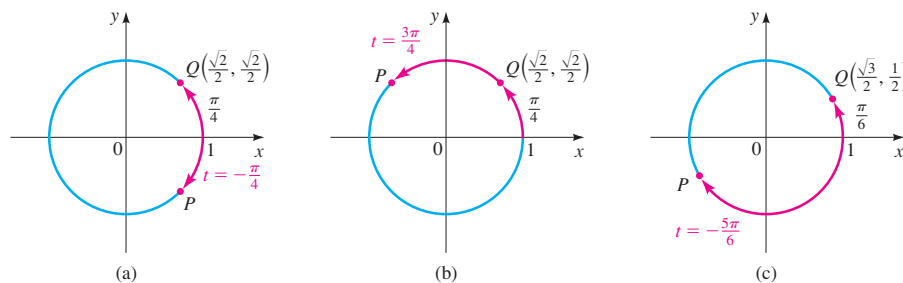


Figure 7

(b) Let P be the terminal point determined by $3\pi/4$, and let Q be the terminal point determined by $\pi/4$. From Figure 7(b) we see that the point P has the same coordinates as Q except for sign. Since P is in quadrant II, its x -coordinate is negative and its y -coordinate is positive. Thus, the terminal point is $P(-\sqrt{2}/2, \sqrt{2}/2)$.

ALTERNATE EXAMPLE 4b

Find the terminal point determined by the given real number t :

$$t = \frac{5\pi}{4}$$

ANSWER

$$\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$$

ALTERNATE EXAMPLE 4c

Find the terminal point determined by the given real number t :

$$t = -\frac{\pi}{6}$$

ANSWER

$$\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$$

IN-CLASS MATERIALS

Point out that finding terminal points is much more difficult than finding reference numbers. If the reference number is not given in Table 1, the students don't have a good method for finding the terminal point. Have them try to estimate a few by measurement. For example, have the students draw a circle with a 1-inch radius, and have them draw arcs of length 2 and 2.5 (by "rolling" the ruler along the circle) and measuring the coordinates of the relevant terminal point. $t = 2$ should yield a terminal point at approximately $(-0.4, 0.9)$. $t = 2.5$ should yield a terminal point at approximately $(-0.8, 0.6)$.

- (c) Let P be the terminal point determined by $-5\pi/6$, and let Q be the terminal point determined by $\pi/6$. From Figure 7(c) we see that the point P has the same coordinates as Q except for sign. Since P is in quadrant III, its coordinates are both negative. Thus, the terminal point is $P(-\sqrt{3}/2, -\frac{1}{2})$. ■

The Reference Number

From Examples 3 and 4, we see that to find a terminal point in any quadrant we need only know the “corresponding” terminal point in the first quadrant. We use the idea of the *reference number* to help us find terminal points.

Reference Number

Let t be a real number. The **reference number** \bar{t} associated with t is the shortest distance along the unit circle between the terminal point determined by t and the x -axis.

Figure 8 shows that to find the reference number \bar{t} it's helpful to know the quadrant in which the terminal point determined by t lies. If the terminal point lies in quadrants I or IV, where x is positive, we find \bar{t} by moving along the circle to the *positive* x -axis. If it lies in quadrants II or III, where x is negative, we find \bar{t} by moving along the circle to the *negative* x -axis.

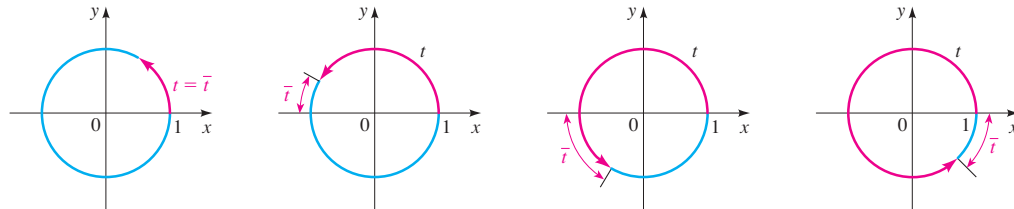


Figure 8
The reference number \bar{t} for t

IN-CLASS MATERIALS

Figure 8 is very important, because the way we find \bar{t} depends upon the quadrant we are in. Perhaps find the reference numbers for $t = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{3\pi}{3} = \pi, \frac{4\pi}{3}, \frac{5\pi}{3}$ and $\frac{6\pi}{3} = 2\pi$ all in a row to illustrate this principle.

ALTERNATE EXAMPLE 5d

Find the reference number determined by the given real number t :
 $t = 5.29$

Please give the answer to two decimal places.

ANSWER

0.99

Example 5 Finding Reference Numbers

Find the reference number for each value of t .

- (a) $t = \frac{5\pi}{6}$ (b) $t = \frac{7\pi}{4}$ (c) $t = -\frac{2\pi}{3}$ (d) $t = 5.80$

Solution From Figure 9 we find the reference numbers as follows.

- (a) $\bar{t} = \pi - \frac{5\pi}{6} = \frac{\pi}{6}$
(b) $\bar{t} = 2\pi - \frac{7\pi}{4} = \frac{\pi}{4}$

IN-CLASS MATERIALS

The text uses integer multiples of $\frac{\pi}{6}$ and $\frac{\pi}{4}$ for most of its examples of finding reference numbers. After the students understand these examples, try having them work with different rational multiples of π , such as $-\frac{121}{35}\pi$ or $\frac{10,001}{5}\pi$. Transition them to numbers whose terminal points lie in quadrants that are easy to figure out, such as $t = 3$, 6 , or 1.5 . Finally, have them try to find reference numbers of arbitrary real numbers such as 22 and -13 .

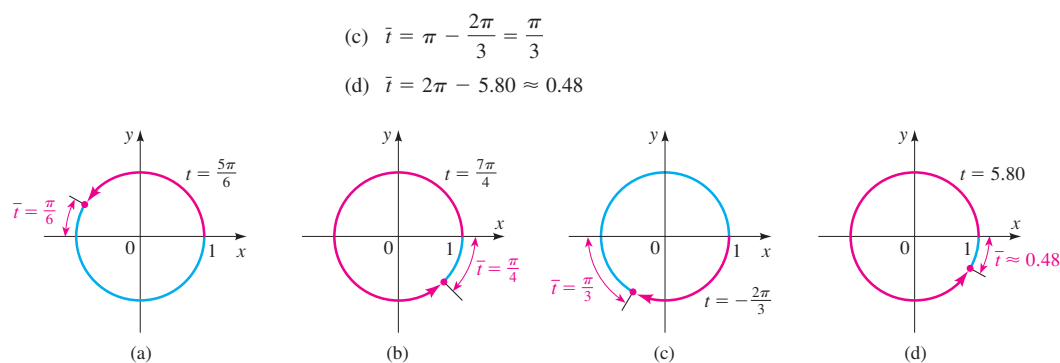


Figure 9

Using Reference Numbers to Find Terminal Points

To find the terminal point P determined by any value of t , we use the following steps:

1. Find the reference number \bar{t} .
2. Find the terminal point $Q(a, b)$ determined by \bar{t} .
3. The terminal point determined by t is $P(\pm a, \pm b)$, where the signs are chosen according to the quadrant in which this terminal point lies.

Example 6 Using Reference Numbers to Find Terminal Points

Find the terminal point determined by each given real number t .

(a) $t = \frac{5\pi}{6}$ (b) $t = \frac{7\pi}{4}$ (c) $t = -\frac{2\pi}{3}$

Solution The reference numbers associated with these values of t were found in Example 5.

- (a) The reference number is $\bar{t} = \pi/6$, which determines the terminal point $(\sqrt{3}/2, 1/2)$ from Table 1. Since the terminal point determined by t is in quadrant II, its x -coordinate is negative and its y -coordinate is positive. Thus, the desired terminal point is

$$\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$$

- (b) The reference number is $\bar{t} = \pi/4$, which determines the terminal point $(\sqrt{2}/2, \sqrt{2}/2)$ from Table 1. Since the terminal point is in quadrant IV, its

DRILL QUESTION

Find the reference number \bar{t} for

$$t = \frac{17\pi}{6}.$$

Answer

$$\bar{t} = \frac{17\pi}{6}$$

ALTERNATE EXAMPLE 6a

Find the terminal point determined by the given real number t :

$$t = -\frac{7\pi}{6}$$

ANSWER

$$\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$$

ALTERNATE EXAMPLE 6c

Find the terminal point determined by the given real number t :

$$t = -\frac{4\pi}{3}$$

ANSWER

$$\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$$

SAMPLE QUESTIONS

Text Question

Is it possible for two different real numbers to have the same reference number?

Answer

Yes

x -coordinate is positive and its y -coordinate is negative. Thus, the desired terminal point is

$$\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$$

- (c) The reference number is $\bar{t} = \pi/3$, which determines the terminal point $(\frac{1}{2}, \sqrt{3}/2)$ from Table 1. Since the terminal point determined by t is in quadrant III, its coordinates are both negative. Thus, the desired terminal point is

$$\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$$

Since the circumference of the unit circle is 2π , the terminal point determined by t is the same as that determined by $t + 2\pi$ or $t - 2\pi$. In general, we can add or subtract 2π any number of times without changing the terminal point determined by t . We use this observation in the next example to find terminal points for large t .

Example 7 Finding the Terminal Point for Large t

Find the terminal point determined by $t = \frac{29\pi}{6}$.

Solution Since

$$t = \frac{29\pi}{6} = 4\pi + \frac{5\pi}{6}$$

we see that the terminal point of t is the same as that of $5\pi/6$ (that is, we subtract 4π). So by Example 6(a) the terminal point is $(-\sqrt{3}/2, \frac{1}{2})$. (See Figure 10.)

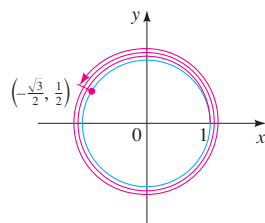


Figure 10

ALTERNATE EXAMPLE 7

Find the terminal point determined by $t = \frac{55\pi}{6}$.

ANSWER

$$\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$$

5.1 Exercises

1–6 ■ Show that the point is on the unit circle.

1. $\left(\frac{4}{5}, -\frac{3}{5}\right)$ 2. $\left(-\frac{5}{13}, \frac{12}{13}\right)$ 3. $\left(\frac{7}{25}, \frac{24}{25}\right)$

4. $\left(-\frac{5}{7}, -\frac{2\sqrt{6}}{7}\right)$ 5. $\left(-\frac{\sqrt{5}}{3}, \frac{2}{3}\right)$ 6. $\left(\frac{\sqrt{11}}{6}, \frac{5}{6}\right)$

7–12 ■ Find the missing coordinate of P , using the fact that P lies on the unit circle in the given quadrant.

Coordinates	Quadrant
7. $P\left(-\frac{3}{5}, \quad\right)$	III
8. $P\left(\quad, -\frac{7}{25}\right)$	IV
9. $P\left(\quad, \frac{1}{3}\right)$	II
10. $P\left(\frac{2}{5}, \quad\right)$	I
11. $P\left(\quad, -\frac{2}{7}\right)$	IV
12. $P\left(-\frac{2}{3}, \quad\right)$	II

13–18 ■ The point P is on the unit circle. Find $P(x, y)$ from the given information.

13. The x -coordinate of P is $\frac{4}{5}$ and the y -coordinate is positive.

14. The y -coordinate of P is $-\frac{1}{3}$ and the x -coordinate is positive.

15. The y -coordinate of P is $\frac{2}{3}$ and the x -coordinate is negative.

16. The x -coordinate of P is positive and the y -coordinate of P is $-\sqrt{5}/5$.

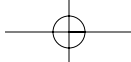
17. The x -coordinate of P is $-\sqrt{2}/3$ and P lies below the x -axis.

18. The x -coordinate of P is $-\frac{2}{3}$ and P lies above the x -axis.

EXAMPLE

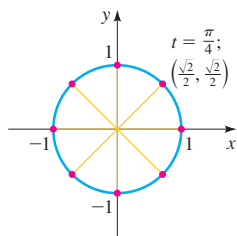
Some values of t , their reference numbers, and their terminal points.

t	\bar{t}	Terminal Point
$\frac{33\pi}{2}$	$\frac{\pi}{2}$	$(0, -1)$
$\frac{55\pi}{4}$	$\frac{\pi}{4}$	$\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$
$\frac{48\pi}{6}$	0	$(1, 0)$
$\frac{16\pi}{3}$	$\frac{\pi}{3}$	$\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$
10	$10 - 3\pi \approx 0.57522204$	$(-0.839, -0.544)$

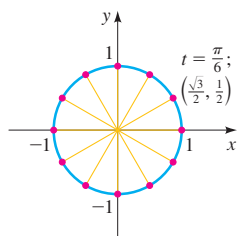


19–20 ■ Find t and the terminal point determined by t for each point in the figure. In Exercise 19, t increases in increments of $\pi/4$; in Exercise 20, t increases in increments of $\pi/6$.

19.



20.



21–30 ■ Find the terminal point $P(x, y)$ on the unit circle determined by the given value of t .

21. $t = \frac{\pi}{2}$

22. $t = \frac{3\pi}{2}$

23. $t = \frac{5\pi}{6}$

24. $t = \frac{7\pi}{6}$

25. $t = -\frac{\pi}{3}$

26. $t = \frac{5\pi}{3}$

27. $t = \frac{2\pi}{3}$

28. $t = -\frac{\pi}{2}$

29. $t = -\frac{3\pi}{4}$

30. $t = \frac{11\pi}{6}$

31. Suppose that the terminal point determined by t is the point $(\frac{3}{5}, \frac{4}{5})$ on the unit circle. Find the terminal point determined by each of the following.

(a) $\pi - t$

(b) $-t$

(c) $\pi + t$

(d) $2\pi + t$

32. Suppose that the terminal point determined by t is the point $(\frac{3}{4}, \sqrt{7}/4)$ on the unit circle. Find the terminal point determined by each of the following.

(a) $-t$

(b) $4\pi + t$

(c) $\pi - t$

(d) $t - \pi$

33–36 ■ Find the reference number for each value of t .

33. (a) $t = \frac{5\pi}{4}$

(b) $t = \frac{7\pi}{3}$

(c) $t = -\frac{4\pi}{3}$

(d) $t = \frac{\pi}{6}$

34. (a) $t = \frac{5\pi}{6}$

(b) $t = \frac{7\pi}{6}$

(c) $t = \frac{11\pi}{3}$

(d) $t = -\frac{7\pi}{4}$

35. (a) $t = \frac{5\pi}{7}$

(b) $t = -\frac{7\pi}{9}$

(c) $t = -3$

(d) $t = 5$

36. (a) $t = \frac{11\pi}{5}$

(b) $t = -\frac{9\pi}{7}$

(c) $t = 6$

(d) $t = -7$

37–50 ■ Find (a) the reference number for each value of t , and (b) the terminal point determined by t .

37. $t = \frac{2\pi}{3}$

38. $t = \frac{4\pi}{3}$

39. $t = \frac{3\pi}{4}$

40. $t = \frac{7\pi}{3}$

41. $t = -\frac{2\pi}{3}$

42. $t = -\frac{7\pi}{6}$

43. $t = \frac{13\pi}{4}$

44. $t = \frac{13\pi}{6}$

45. $t = \frac{7\pi}{6}$

46. $t = \frac{17\pi}{4}$

47. $t = -\frac{11\pi}{3}$

48. $t = \frac{31\pi}{6}$

49. $t = \frac{16\pi}{3}$

50. $t = -\frac{41\pi}{4}$

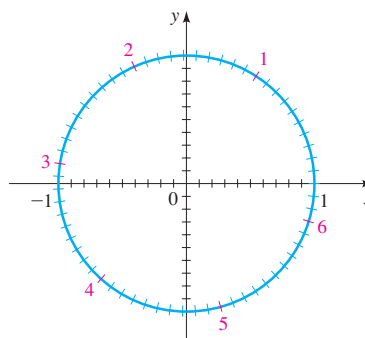
51–54 ■ Use the figure to find the terminal point determined by the real number t , with coordinates correct to one decimal place.

51. $t = 1$

52. $t = 2.5$

53. $t = -1.1$

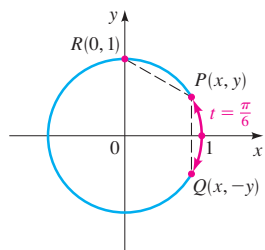
54. $t = 4.2$



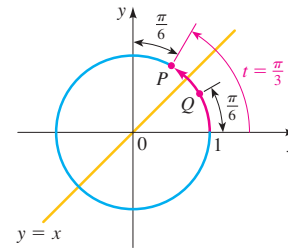
Discovery • Discussion

55. Finding the Terminal Point for $\pi/6$ Suppose the terminal point determined by $t = \pi/6$ is $P(x, y)$ and the points Q and R are as shown in the figure on the next page. Why are the distances PQ and PR the same? Use this fact, together with the Distance Formula, to show that the coordinates of

P satisfy the equation $2y = \sqrt{x^2 + (y - 1)^2}$. Simplify this equation using the fact that $x^2 + y^2 = 1$. Solve the simplified equation to find $P(x, y)$.



56. Finding the Terminal Point for $\pi/3$ Now that you know the terminal point determined by $t = \pi/6$, use symmetry to find the terminal point determined by $t = \pi/3$ (see the figure). Explain your reasoning.



SUGGESTED TIME AND EMPHASIS

$\frac{1}{2}$ -1 class.
Essential material.

POINTS TO STRESS

1. Computation of the trigonometric functions, given the terminal point of a real number.
2. Properties of the trigonometric functions, including their domains, and parity based on the quadrant of the terminal point.
3. The Pythagorean and reciprocal identities of the trigonometric functions.

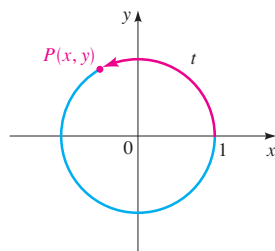


Figure 1

5.2 Trigonometric Functions of Real Numbers

A function is a rule that assigns to each real number another real number. In this section we use properties of the unit circle from the preceding section to define the trigonometric functions.

The Trigonometric Functions

Recall that to find the terminal point $P(x, y)$ for a given real number t , we move a distance t along the unit circle, starting at the point $(1, 0)$. We move in a counterclockwise direction if t is positive and in a clockwise direction if t is negative (see Figure 1). We now use the x - and y -coordinates of the point $P(x, y)$ to define several functions. For instance, we define the function called *sine* by assigning to each real number t the y -coordinate of the terminal point $P(x, y)$ determined by t . The functions *cosine*, *tangent*, *cosecant*, *secant*, and *cotangent* are also defined using the coordinates of $P(x, y)$.

Definition of the Trigonometric Functions

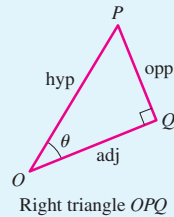
Let t be any real number and let $P(x, y)$ be the terminal point on the unit circle determined by t . We define

$$\begin{aligned} \sin t &= y & \cos t &= x & \tan t &= \frac{y}{x} \quad (x \neq 0) \\ \csc t &= \frac{1}{y} \quad (y \neq 0) & \sec t &= \frac{1}{x} \quad (x \neq 0) & \cot t &= \frac{x}{y} \quad (y \neq 0) \end{aligned}$$

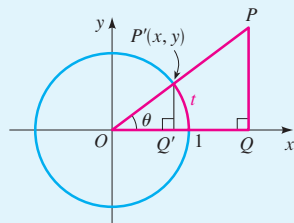
Because the trigonometric functions can be defined in terms of the unit circle, they are sometimes called the **circular functions**.

Relationship to the Trigonometric Functions of Angles

If you have previously studied trigonometry of right triangles (Chapter 6), you are probably wondering how the sine and cosine of an *angle* relate to those of this section. To see how, let's start with a right triangle, $\triangle OPQ$.



Place the triangle in the coordinate plane as shown, with angle θ in standard position.



$P'(x, y)$ is the terminal point determined by t .

The point $P'(x, y)$ in the figure is the terminal point determined by the arc t . Note that triangle OPQ is similar to the small triangle $OP'Q'$ whose legs have lengths x and y .

Now, by the definition of the trigonometric functions of the *angle* θ we have

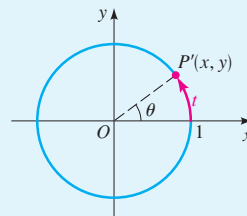
$$\begin{aligned}\sin \theta &= \frac{\text{opp}}{\text{hyp}} = \frac{PQ}{OP} = \frac{P'Q'}{OP'} \\ &= \frac{y}{1} = y\end{aligned}$$

$$\begin{aligned}\cos \theta &= \frac{\text{adj}}{\text{hyp}} = \frac{OQ}{OP} = \frac{OQ'}{OP'} \\ &= \frac{x}{1} = x\end{aligned}$$

By the definition of the trigonometric functions of the *real number* t , we have

$$\sin t = y \quad \cos t = x$$

Now, if θ is measured in radians, then $\theta = t$ (see the figure). So the trigonometric functions of the angle with radian measure θ are exactly the same as the trigonometric functions defined in terms of the terminal point determined by the real number t .



The radian measure of angle θ is t .

Why then study trigonometry in two different ways? Because different applications require that we view the trigonometric functions differently. (Compare Section 5.5 with Sections 6.2, 6.4, and 6.5.)

ALTERNATE EXAMPLE 1a

Find the six trigonometric functions of the given real number

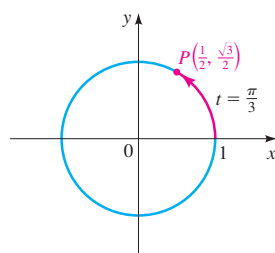
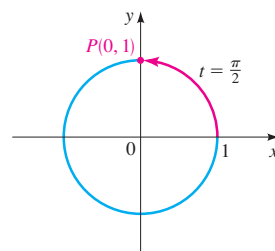
$$t = \frac{\pi}{4}.$$

ANSWER

$$\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 1, \sqrt{2}, \sqrt{2}, 1$$

EXAMPLES

- $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$
- $\cos \frac{7\pi}{6} = -\frac{\sqrt{3}}{2}$
- $\tan \frac{11\pi}{4} = -1$
- $\sec \frac{17\pi}{3} = 2$
- $\csc \frac{17\pi}{2} = 1$
- $\cot \frac{121\pi}{6} = \sqrt{3}$

**Figure 2****Figure 3**

We can easily remember the sines and cosines of the basic angles by writing them in the form $\sqrt{\square}/2$:

t	$\sin t$	$\cos t$
0	$\sqrt{0}/2$	$\sqrt{4}/2$
$\pi/6$	$\sqrt{1}/2$	$\sqrt{3}/2$
$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$
$\pi/3$	$\sqrt{3}/2$	$\sqrt{1}/2$
$\pi/2$	$\sqrt{4}/2$	$\sqrt{0}/2$

Example 1 Evaluating Trigonometric Functions

Find the six trigonometric functions of each given real number t .

(a) $t = \frac{\pi}{3}$ (b) $t = \frac{\pi}{2}$

Solution

(a) From Table 1 on page 403, we see that the terminal point determined by $t = \pi/3$ is $P(\frac{1}{2}, \sqrt{3}/2)$. (See Figure 2.) Since the coordinates are $x = \frac{1}{2}$ and $y = \sqrt{3}/2$, we have

$$\begin{aligned} \sin \frac{\pi}{3} &= \frac{\sqrt{3}}{2} & \cos \frac{\pi}{3} &= \frac{1}{2} & \tan \frac{\pi}{3} &= \frac{\sqrt{3}/2}{1/2} = \sqrt{3} \\ \csc \frac{\pi}{3} &= \frac{2\sqrt{3}}{3} & \sec \frac{\pi}{3} &= 2 & \cot \frac{\pi}{3} &= \frac{1/2}{\sqrt{3}/2} = \frac{\sqrt{3}}{3} \end{aligned}$$

(b) The terminal point determined by $\pi/2$ is $P(0, 1)$. (See Figure 3.) So

$$\sin \frac{\pi}{2} = 1 \quad \cos \frac{\pi}{2} = 0 \quad \csc \frac{\pi}{2} = \frac{1}{1} = 1 \quad \cot \frac{\pi}{2} = \frac{0}{1} = 0$$

But $\tan \pi/2$ and $\sec \pi/2$ are undefined because $x = 0$ appears in the denominator in each of their definitions. ■

Some special values of the trigonometric functions are listed in Table 1. This table is easily obtained from Table 1 of Section 5.1, together with the definitions of the trigonometric functions.

Table 1 Special values of the trigonometric functions

t	$\sin t$	$\cos t$	$\tan t$	$\csc t$	$\sec t$	$\cot t$
0	0	1	0	—	1	—
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{2}$	1	0	—	1	—	0

Example 1 shows that some of the trigonometric functions fail to be defined for certain real numbers. So we need to determine their domains. The functions sine and cosine are defined for all values of t . Since the functions cotangent and cosecant have y in the denominator of their definitions, they are not defined whenever the y -coordinate of the terminal point $P(x, y)$ determined by t is 0. This happens when $t = n\pi$ for any integer n , so their domains do not include these points. The functions tangent and secant have x in the denominator in their definitions, so they are not defined whenever $x = 0$. This happens when $t = (\pi/2) + n\pi$ for any integer n .

Domains of the Trigonometric Functions

Function	Domain
sin, cos	All real numbers
tan, sec	All real numbers other than $\frac{\pi}{2} + n\pi$ for any integer n
cot, csc	All real numbers other than $n\pi$ for any integer, n

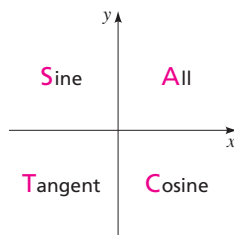
Values of the Trigonometric Functions

To compute other values of the trigonometric functions, we first determine their signs. The signs of the trigonometric functions depend on the quadrant in which the terminal point of t lies. For example, if the terminal point $P(x, y)$ determined by t lies in quadrant III, then its coordinates are both negative. So $\sin t$, $\cos t$, $\csc t$, and $\sec t$ are all negative, whereas $\tan t$ and $\cot t$ are positive. You can check the other entries in the following box.

Signs of the Trigonometric Functions

Quadrant	Positive Functions	Negative functions
I	all	none
II	sin, csc	cos, sec, tan, cot
III	tan, cot	sin, csc, cos, sec
IV	cos, sec	sin, csc, tan, cot

The following mnemonic device will help you remember which trigonometric functions are positive in each quadrant: All of them, Sine, Tangent, or Cosine.



You can remember this as “All Students Take Calculus.”

Example 2 Determining the Sign of a Trigonometric Function

- $\cos \frac{\pi}{3} > 0$, because the terminal point of $t = \frac{\pi}{3}$ is in quadrant I.
- $\tan 4 > 0$, because the terminal point of $t = 4$ is in quadrant III.
- If $\cos t < 0$ and $\sin t > 0$, then the terminal point of t must be in quadrant II.

In Section 5.1 we used the reference number to find the terminal point determined by a real number t . Since the trigonometric functions are defined in terms of the coordinates of terminal points, we can use the reference number to find values of the trigonometric functions. Suppose that \bar{t} is the reference number for t . Then the terminal point of \bar{t} has the same coordinates, except possibly for sign, as the terminal point of t . So the values of the trigonometric functions at t are the same, except possibly for sign, as their values at \bar{t} . We illustrate this procedure in the next example.

IN-CLASS MATERIALS

The etymology of the word *sine* is fairly interesting and not that well-known. It starts with the Indian word *jya*, meaning “cord of a bowstring.” The Arabs translated this word as *jiba*. The written language had no vowels, so it looked like this: *jb*. In 1145, the Spanish translator Robert of Cheste had to figure out what vowels to put in. Due to the shape of the curve, he thought the word was *jaib*, which meant the opening of a garment that shows a woman’s cleavage. So he used the Latin word for the cavity formed by a curve: *sinus*.

DRILL QUESTION

If $\sin t = \frac{3}{5}$ and the terminal point of t is in the second quadrant, what is the value of $\cos t$?

Answer

$$-\frac{4}{5}$$

ALTERNATE EXAMPLE 2a

Determine the sign of the trigonometric function $\cos \frac{7\pi}{4}$.

ANSWER

Positive

ALTERNATE EXAMPLE 2b

Determine the sign of the trigonometric function $\tan 1$.

ANSWER

Positive

ALTERNATE EXAMPLE 3aFind the value of $\sin \frac{2\pi}{3}$.**ANSWER**

$$\frac{\sqrt{3}}{2}$$

ALTERNATE EXAMPLE 3bFind the value of $\tan \left(-\frac{4\pi}{3}\right)$.**ANSWER**

$$-\sqrt{3}$$

ALTERNATE EXAMPLE 3cFind the value of $\cos \frac{14\pi}{3}$.**ANSWER**

$$-\frac{1}{2}$$

Example 3 Evaluating Trigonometric Functions

Find each value.

(a) $\cos \frac{2\pi}{3}$ (b) $\tan \left(-\frac{\pi}{3}\right)$ (c) $\sin \frac{19\pi}{4}$

Solution

- (a) The reference number for
- $2\pi/3$
- is
- $\pi/3$
- (see Figure 4(a)). Since the terminal point of
- $2\pi/3$
- is in quadrant II,
- $\cos(2\pi/3)$
- is negative. Thus

$$\cos \frac{2\pi}{3} = -\cos \frac{\pi}{3} = -\frac{1}{2}$$

Sign Reference number From Table 1

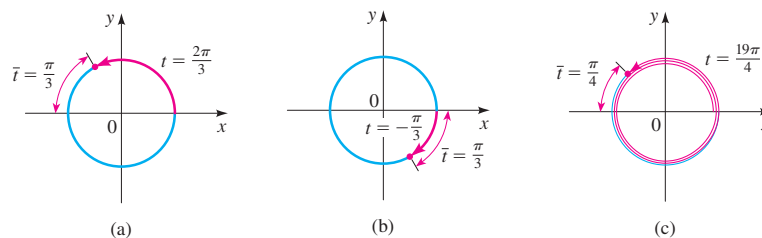
- (b) The reference number for
- $-\pi/3$
- is
- $\pi/3$
- (see Figure 4(b)). Since the terminal point of
- $-\pi/3$
- is in quadrant IV,
- $\tan(-\pi/3)$
- is negative. Thus

$$\tan \left(-\frac{\pi}{3}\right) = -\tan \frac{\pi}{3} = -\sqrt{3}$$

Sign Reference number From Table 1

- (c) Since
- $(19\pi/4) - 4\pi = 3\pi/4$
- , the terminal points determined by
- $19\pi/4$
- and
- $3\pi/4$
- are the same. The reference number for
- $3\pi/4$
- is
- $\pi/4$
- (see Figure 4(c)). Since the terminal point of
- $3\pi/4$
- is in quadrant II,
- $\sin(3\pi/4)$
- is positive. Thus

$$\sin \frac{19\pi}{4} = \sin \frac{3\pi}{4} = +\sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

Subtract 4π Sign Reference number From Table 1**Figure 4**

So far we have been able to compute the values of the trigonometric functions only for certain values of t . In fact, we can compute the values of the trigonometric functions whenever t is a multiple of $\pi/6$, $\pi/4$, $\pi/3$, and $\pi/2$. How can we compute the trigonometric functions for other values of t ? For example, how can we find $\sin 1.5$? One way is to carefully sketch a diagram and read the value (see Exercises 37–44); however, this method is not very accurate. Fortunately, programmed directly into scientific calculators are mathematical procedures (see the margin note on page 436) that find the values of *sine*, *cosine*, and *tangent* correct to the number of digits in the

display. **The calculator must be put in radian mode to evaluate these functions.** To find values of cosecant, secant, and cotangent using a calculator, we need to use the following *reciprocal relations*:

$$\csc t = \frac{1}{\sin t} \quad \sec t = \frac{1}{\cos t} \quad \cot t = \frac{1}{\tan t}$$

These identities follow from the definitions of the trigonometric functions. For instance, since $\sin t = y$ and $\csc t = 1/y$, we have $\csc t = 1/y = 1/(\sin t)$. The others follow similarly.

Example 4 Using a Calculator to Evaluate Trigonometric Functions

Making sure our calculator is set to radian mode and rounding the results to six decimal places, we get

$$\begin{aligned} \text{(a)} \quad \sin 2.2 &\approx 0.808496 & \text{(b)} \quad \cos 1.1 &\approx 0.453596 \\ \text{(c)} \quad \cot 28 &= \frac{1}{\tan 28} \approx -3.553286 & \text{(d)} \quad \csc 0.98 &= \frac{1}{\sin 0.98} \approx 1.204098 \quad \blacksquare \end{aligned}$$

Let's consider the relationship between the trigonometric functions of t and those of $-t$. From Figure 5 we see that

$$\begin{aligned} \sin(-t) &= -y = -\sin t \\ \cos(-t) &= x = \cos t \\ \tan(-t) &= \frac{-y}{x} = -\frac{y}{x} = -\tan t \end{aligned}$$

These equations show that sine and tangent are odd functions, whereas cosine is an even function. It's easy to see that the reciprocal of an even function is even and the reciprocal of an odd function is odd. This fact, together with the reciprocal relations, completes our knowledge of the even-odd properties for all the trigonometric functions.

Even-Odd Properties

Sine, cosecant, tangent, and cotangent are odd functions; cosine and secant are even functions.

$$\begin{aligned} \sin(-t) &= -\sin t & \cos(-t) &= \cos t & \tan(-t) &= -\tan t \\ \csc(-t) &= -\csc t & \sec(-t) &= \sec t & \cot(-t) &= -\cot t \end{aligned}$$

Example 5 Even and Odd Trigonometric Functions

Use the even-odd properties of the trigonometric functions to determine each value.

$$\text{(a)} \quad \sin\left(-\frac{\pi}{6}\right) \quad \text{(b)} \quad \cos\left(-\frac{\pi}{4}\right)$$

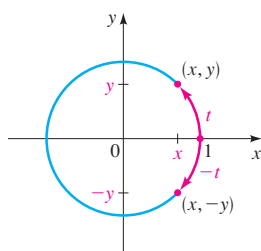


Figure 5

Even and odd functions are defined in Section 2.4.

ALTERNATE EXAMPLE 4

Approximate the following with a calculator:

- (a) $\sin(0.5236)$
- (b) $\cos(3.2)$
- (c) $\csc(37)$
- (d) $\cot(1.7)$

ANSWERS

(all answers rounded to 5 significant figures)

- (a) 0.50000
- (b) -0.99829
- (c) -1.5539
- (d) -0.12993

ALTERNATE EXAMPLE 5

Use the even-odd properties of the trigonometric functions to

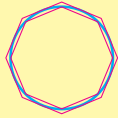
determine the value of $\cos\left(-\frac{\pi}{3}\right)$.

ANSWER

$$\frac{1}{2}$$

The Value of π

The number π is the ratio of the circumference of a circle to its diameter. It has been known since ancient times that this ratio is the same for all circles. The first systematic effort to find a numerical approximation for π was made by Archimedes (ca. 240 B.C.), who proved that $\frac{22}{7} < \pi < \frac{223}{71}$ by finding the perimeters of regular polygons inscribed in and circumscribed about a circle.



In about A.D. 480, the Chinese physicist Tsu Ch'ung-chih gave the approximation

$$\pi \approx \frac{355}{113} = 3.141592\dots$$

which is correct to six decimals. This remained the most accurate estimation of π until the Dutch mathematician Adrianus Romanus (1593) used polygons with more than a billion sides to compute π correct to 15 decimals. In the 17th century, mathematicians began to use infinite series and trigonometric identities in the quest for π . The Englishman William Shanks spent 15 years (1858–1873) using these methods to compute π to 707 decimals, but in 1946 it was found that his figures were wrong beginning with the 528th decimal. Today, with the aid of computers, mathematicians routinely determine π correct to millions of decimals.

Solution By the even-odd properties and Table 1, we have

$$(a) \sin\left(-\frac{\pi}{6}\right) = -\sin\frac{\pi}{6} = -\frac{1}{2} \quad \text{Sine is odd}$$

$$(b) \cos\left(-\frac{\pi}{4}\right) = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} \quad \text{Cosine is even}$$

Fundamental Identities

The trigonometric functions are related to each other through equations called **trigonometric identities**. We give the most important ones in the following box.*

Fundamental Identities**Reciprocal Identities**

$$\csc t = \frac{1}{\sin t} \quad \sec t = \frac{1}{\cos t} \quad \cot t = \frac{1}{\tan t}$$

$$\tan t = \frac{\sin t}{\cos t} \quad \cot t = \frac{\cos t}{\sin t}$$

Pythagorean Identities

$$\sin^2 t + \cos^2 t = 1 \quad \tan^2 t + 1 = \sec^2 t \quad 1 + \cot^2 t = \csc^2 t$$

■ **Proof** The reciprocal identities follow immediately from the definition on page 408. We now prove the Pythagorean identities. By definition, $\cos t = x$ and $\sin t = y$, where x and y are the coordinates of a point $P(x, y)$ on the unit circle. Since $P(x, y)$ is on the unit circle, we have $x^2 + y^2 = 1$. Thus

$$\sin^2 t + \cos^2 t = 1$$

Dividing both sides by $\cos^2 t$ (provided $\cos t \neq 0$), we get

$$\begin{aligned} \frac{\sin^2 t}{\cos^2 t} + \frac{\cos^2 t}{\cos^2 t} &= \frac{1}{\cos^2 t} \\ \left(\frac{\sin t}{\cos t}\right)^2 + 1 &= \left(\frac{1}{\cos t}\right)^2 \\ \tan^2 t + 1 &= \sec^2 t \end{aligned}$$

We have used the reciprocal identities $\sin t/\cos t = \tan t$ and $1/\cos t = \sec t$. Similarly, dividing both sides of the first Pythagorean identity by $\sin^2 t$ (provided $\sin t \neq 0$) gives us $1 + \cot^2 t = \csc^2 t$. ■

*We follow the usual convention of writing $\sin^2 t$ for $(\sin t)^2$. In general, we write $\sin^n t$ for $(\sin t)^n$ for all integers n except $n = -1$. The exponent $n = -1$ will be assigned another meaning in Section 7.4. Of course, the same convention applies to the other five trigonometric functions.

IN-CLASS MATERIALS

There is an inconsistency in mathematical notation that can be made explicit at this time. We say that

$\frac{1}{3} = 3^{-1}$. Similarly, we say that $\frac{1}{\pi} = \pi^{-1}$. But $\frac{1}{\sin x}$ is written as $\csc x$. Unfortunately, there is a symbol, $\sin^{-1} x$, that means something entirely different—the arcsine of x . Even worse, as noted in the section, $\sin^2 x = (\sin x)^2$ and $\sin^3 x = (\sin x)^3$. The exponent -1 is a notational anomaly.

As their name indicates, the fundamental identities play a central role in trigonometry because we can use them to relate any trigonometric function to any other. So, if we know the value of any one of the trigonometric functions at t , then we can find the values of all the others at t .

Example 6 Finding All Trigonometric Functions from the Value of One



If $\cos t = \frac{3}{5}$ and t is in quadrant IV, find the values of all the trigonometric functions at t .

Solution From the Pythagorean identities we have

$$\begin{aligned}\sin^2 t + \cos^2 t &= 1 \\ \sin^2 t + \left(\frac{3}{5}\right)^2 &= 1 && \text{Substitute } \cos t = \frac{3}{5} \\ \sin^2 t &= 1 - \frac{9}{25} = \frac{16}{25} && \text{Solve for } \sin^2 t \\ \sin t &= \pm \frac{4}{5} && \text{Take square roots}\end{aligned}$$

Since this point is in quadrant IV, $\sin t$ is negative, so $\sin t = -\frac{4}{5}$. Now that we know both $\sin t$ and $\cos t$, we can find the values of the other trigonometric functions using the reciprocal identities:

$$\begin{aligned}\sin t &= -\frac{4}{5} & \cos t &= \frac{3}{5} & \tan t &= \frac{\sin t}{\cos t} = \frac{-\frac{4}{5}}{\frac{3}{5}} = -\frac{4}{3} \\ \csc t &= \frac{1}{\sin t} = -\frac{5}{4} & \sec t &= \frac{1}{\cos t} = \frac{5}{3} & \cot t &= \frac{1}{\tan t} = -\frac{3}{4}\end{aligned}$$

Example 7 Writing One Trigonometric Function in Terms of Another

Write $\tan t$ in terms of $\cos t$, where t is in quadrant III.

Solution Since $\tan t = \sin t / \cos t$, we need to write $\sin t$ in terms of $\cos t$. By the Pythagorean identities we have

$$\begin{aligned}\sin^2 t + \cos^2 t &= 1 \\ \sin^2 t &= 1 - \cos^2 t && \text{Solve for } \sin^2 t \\ \sin t &= \pm \sqrt{1 - \cos^2 t} && \text{Take square roots}\end{aligned}$$

Since $\sin t$ is negative in quadrant III, the negative sign applies here. Thus

$$\tan t = \frac{\sin t}{\cos t} = \frac{-\sqrt{1 - \cos^2 t}}{\cos t}$$

ALTERNATE EXAMPLE 6

If $\cos t = -\frac{5}{13}$ and t is in quadrant II, find the values of all the trigonometric functions at t .

ANSWER

$$\frac{12}{13}, -\frac{5}{13}, -\frac{12}{13}, \frac{13}{12}, -\frac{13}{5}, -\frac{5}{12}$$

ALTERNATE EXAMPLE 7

Write $\tan t$ in terms of $\cos t$, where t is in quadrant I.

ANSWER

$$\tan(t) = \frac{\sqrt{1 - \cos^2(t)}}{\cos(t)}$$

SAMPLE QUESTION

Text Question

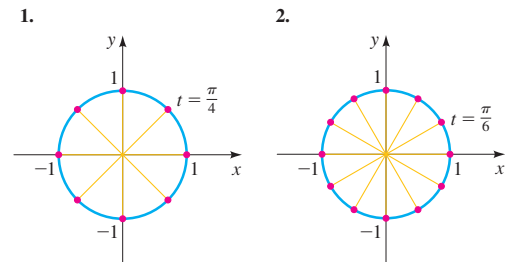
If $\sin t = \frac{3}{5}$, what is the value of $\csc t$?

Answer

$$\frac{5}{3}$$

5.2 Exercises

1–2 ■ Find $\sin t$ and $\cos t$ for the values of t whose terminal points are shown on the unit circle in the figure. In Exercise 1, t increases in increments of $\pi/4$; in Exercise 2, t increases in increments of $\pi/6$. (See Exercises 19 and 20 in Section 5.1.)



3–22 ■ Find the exact value of the trigonometric function at the given real number.

3. (a) $\sin \frac{2\pi}{3}$ (b) $\cos \frac{2\pi}{3}$ (c) $\tan \frac{2\pi}{3}$
 4. (a) $\sin \frac{5\pi}{6}$ (b) $\cos \frac{5\pi}{6}$ (c) $\tan \frac{5\pi}{6}$
 5. (a) $\sin \frac{7\pi}{6}$ (b) $\sin\left(-\frac{\pi}{6}\right)$ (c) $\sin \frac{11\pi}{6}$
 6. (a) $\cos \frac{5\pi}{3}$ (b) $\cos\left(-\frac{5\pi}{3}\right)$ (c) $\cos \frac{7\pi}{3}$
 7. (a) $\cos \frac{3\pi}{4}$ (b) $\cos \frac{5\pi}{4}$ (c) $\cos \frac{7\pi}{4}$
 8. (a) $\sin \frac{3\pi}{4}$ (b) $\sin \frac{5\pi}{4}$ (c) $\sin \frac{7\pi}{4}$
 9. (a) $\sin \frac{7\pi}{3}$ (b) $\csc \frac{7\pi}{3}$ (c) $\cot \frac{7\pi}{3}$
 10. (a) $\cos\left(-\frac{\pi}{3}\right)$ (b) $\sec\left(-\frac{\pi}{3}\right)$ (c) $\tan\left(-\frac{\pi}{3}\right)$
 11. (a) $\sin\left(-\frac{\pi}{2}\right)$ (b) $\cos\left(-\frac{\pi}{2}\right)$ (c) $\cot\left(-\frac{\pi}{2}\right)$
 12. (a) $\sin\left(-\frac{3\pi}{2}\right)$ (b) $\cos\left(-\frac{3\pi}{2}\right)$ (c) $\cot\left(-\frac{3\pi}{2}\right)$
 13. (a) $\sec \frac{11\pi}{3}$ (b) $\csc \frac{11\pi}{3}$ (c) $\sec\left(-\frac{\pi}{3}\right)$
 14. (a) $\cos \frac{7\pi}{6}$ (b) $\sec \frac{7\pi}{6}$ (c) $\csc \frac{7\pi}{6}$

15. (a) $\tan \frac{5\pi}{6}$ (b) $\tan \frac{7\pi}{6}$ (c) $\tan \frac{11\pi}{6}$
 16. (a) $\cot\left(-\frac{\pi}{3}\right)$ (b) $\cot \frac{2\pi}{3}$ (c) $\cot \frac{5\pi}{3}$
 17. (a) $\cos\left(-\frac{\pi}{4}\right)$ (b) $\csc\left(-\frac{\pi}{4}\right)$ (c) $\cot\left(-\frac{\pi}{4}\right)$
 18. (a) $\sin \frac{5\pi}{4}$ (b) $\sec \frac{5\pi}{4}$ (c) $\tan \frac{5\pi}{4}$
 19. (a) $\csc\left(-\frac{\pi}{2}\right)$ (b) $\csc \frac{\pi}{2}$ (c) $\csc \frac{3\pi}{2}$
 20. (a) $\sec(-\pi)$ (b) $\sec \pi$ (c) $\sec 4\pi$
 21. (a) $\sin 13\pi$ (b) $\cos 14\pi$ (c) $\tan 15\pi$
 22. (a) $\sin \frac{25\pi}{2}$ (b) $\cos \frac{25\pi}{2}$ (c) $\cot \frac{25\pi}{2}$

23–26 ■ Find the value of each of the six trigonometric functions (if it is defined) at the given real number t . Use your answers to complete the table.

23. $t = 0$ 24. $t = \frac{\pi}{2}$ 25. $t = \pi$ 26. $t = \frac{3\pi}{2}$

t	$\sin t$	$\cos t$	$\tan t$	$\csc t$	$\sec t$	$\cot t$
0	0	1		undefined		
$\frac{\pi}{2}$						
π			0			undefined
$\frac{3\pi}{2}$						

27–36 ■ The terminal point $P(x, y)$ determined by a real number t is given. Find $\sin t$, $\cos t$, and $\tan t$.

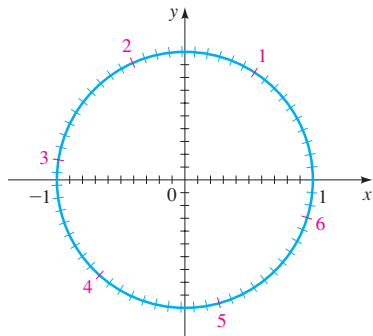
27. $\left(\frac{3}{5}, \frac{4}{5}\right)$ 28. $\left(-\frac{3}{5}, \frac{4}{5}\right)$
 29. $\left(\frac{\sqrt{5}}{4}, -\frac{\sqrt{11}}{4}\right)$ 30. $\left(-\frac{1}{3}, -\frac{2\sqrt{2}}{3}\right)$
 31. $\left(-\frac{6}{7}, \frac{\sqrt{13}}{7}\right)$ 32. $\left(\frac{40}{41}, \frac{9}{41}\right)$
 33. $\left(-\frac{5}{13}, -\frac{12}{13}\right)$ 34. $\left(\frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5}\right)$
 35. $\left(-\frac{20}{29}, \frac{21}{29}\right)$ 36. $\left(\frac{24}{25}, -\frac{7}{25}\right)$

IN-CLASS MATERIALS

Problems such as Exercises 23–26 are crucial when students take calculus. It is vitally important that they go beyond memorizing the answers and truly understand where they come from.

37–44 ■ Find the approximate value of the given trigonometric function by using (a) the figure and (b) a calculator. Compare the two values.

37. $\sin 1$
 38. $\cos 0.8$
 39. $\sin 1.2$
 40. $\cos 5$
 41. $\tan 0.8$
 42. $\tan(-1.3)$
 43. $\cos 4.1$
 44. $\sin(-5.2)$



45–48 ■ Find the sign of the expression if the terminal point determined by t is in the given quadrant.

45. $\sin t \cos t$, quadrant II 46. $\tan t \sec t$, quadrant IV
 47. $\frac{\tan t \sin t}{\cot t}$, quadrant III 48. $\cos t \sec t$, any quadrant

49–52 ■ From the information given, find the quadrant in which the terminal point determined by t lies.

49. $\sin t > 0$ and $\cos t < 0$ 50. $\tan t > 0$ and $\sin t < 0$
 51. $\csc t > 0$ and $\sec t < 0$ 52. $\cos t < 0$ and $\cot t < 0$

53–62 ■ Write the first expression in terms of the second if the terminal point determined by t is in the given quadrant.

53. $\sin t, \cos t$; quadrant II 54. $\cos t, \sin t$; quadrant IV
 55. $\tan t, \sin t$; quadrant IV 56. $\tan t, \cos t$; quadrant III
 57. $\sec t, \tan t$; quadrant II 58. $\csc t, \cot t$; quadrant III
 59. $\tan t, \sec t$; quadrant III 60. $\sin t, \sec t$; quadrant IV
 61. $\tan^2 t, \sin t$; any quadrant
 62. $\sec^2 t \sin^2 t, \cos t$; any quadrant

63–70 ■ Find the values of the trigonometric functions of t from the given information.

63. $\sin t = \frac{3}{5}$, terminal point of t is in quadrant II
 64. $\cos t = -\frac{4}{5}$, terminal point of t is in quadrant III
 65. $\sec t = 3$, terminal point of t is in quadrant IV
 66. $\tan t = \frac{1}{4}$, terminal point of t is in quadrant III
 67. $\tan t = -\frac{3}{4}$, $\cos t > 0$ 68. $\sec t = 2$, $\sin t < 0$
 69. $\sin t = -\frac{1}{4}$, $\sec t < 0$ 70. $\tan t = -4$, $\csc t > 0$

71–78 ■ Determine whether the function is even, odd, or neither.

71. $f(x) = x^2 \sin x$ 72. $f(x) = x^2 \cos 2x$
 73. $f(x) = \sin x \cos x$ 74. $f(x) = \sin x + \cos x$
 75. $f(x) = |x| \cos x$ 76. $f(x) = x \sin^3 x$
 77. $f(x) = x^3 + \cos x$ 78. $f(x) = \cos(\sin x)$

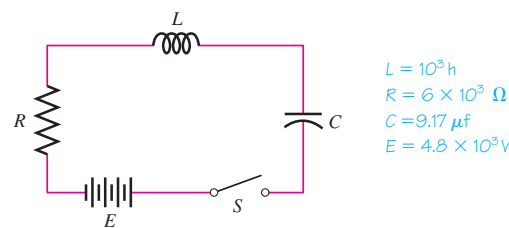
Applications

79. Harmonic Motion The displacement from equilibrium of an oscillating mass attached to a spring is given by $y(t) = 4 \cos 3\pi t$ where y is measured in inches and t in seconds. Find the displacement at the times indicated in the table.

t	$y(t)$
0	
0.25	
0.50	
0.75	
1.00	
1.25	

80. Circadian Rhythms Everybody's blood pressure varies over the course of the day. In a certain individual the resting diastolic blood pressure at time t is given by $B(t) = 80 + 7 \sin(\pi t/12)$, where t is measured in hours since midnight and $B(t)$ in mmHg (millimeters of mercury). Find this person's diastolic blood pressure at
 (a) 6:00 A.M. (b) 10:30 A.M. (c) Noon (d) 8:00 P.M.

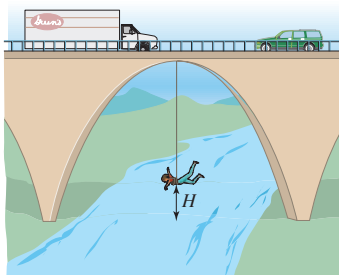
81. Electric Circuit After the switch is closed in the circuit shown, the current t seconds later is $I(t) = 0.8e^{-3t} \sin 10t$. Find the current at the times
 (a) $t = 0.1$ s and (b) $t = 0.5$ s.



82. Bungee Jumping A bungee jumper plummets from a high bridge to the river below and then bounces back over and over again. At time t seconds after her jump, her height H (in meters) above the river is given by

$H(t) = 100 + 75e^{-t/20}\cos(\frac{\pi}{4}t)$. Find her height at the times indicated in the table.

t	$H(t)$
0	
1	
2	
4	
6	
8	
12	

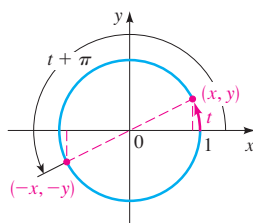


Discovery • Discussion

83. Reduction Formulas A reduction formula is one that can be used to “reduce” the number of terms in the input for a trigonometric function. Explain how the figure shows that the following reduction formulas are valid:

$$\sin(t + \pi) = -\sin t \quad \cos(t + \pi) = -\cos t$$

$$\tan(t + \pi) = \tan t$$

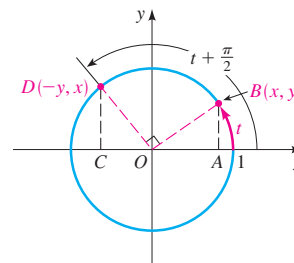


84. More Reduction Formulas By the “Angle-Side-Angle” theorem from elementary geometry, triangles CDO and AOB in the figure are congruent. Explain how this proves that if B has coordinates (x, y) , then D has coordinates $(-y, x)$. Then explain how the figure shows that the following reduction formulas are valid:

$$\sin\left(t + \frac{\pi}{2}\right) = \cos t$$

$$\cos\left(t + \frac{\pi}{2}\right) = -\sin t$$

$$\tan\left(t + \frac{\pi}{2}\right) = -\cot t$$



SUGGESTED TIME AND EMPHASIS

1 class.
Essential material.

POINTS TO STRESS

- The graphs of sine and cosine, their periodicity, and their transformations.
- Sine and cosine curves with variable amplitudes.

5.3

Trigonometric Graphs

The graph of a function gives us a better idea of its behavior. So, in this section we graph the sine and cosine functions and certain transformations of these functions. The other trigonometric functions are graphed in the next section.

Graphs of the Sine and Cosine Functions

To help us graph the sine and cosine functions, we first observe that these functions repeat their values in a regular fashion. To see exactly how this happens, recall that the circumference of the unit circle is 2π . It follows that the terminal point $P(x, y)$ determined by the real number t is the same as that determined by $t + 2\pi$. Since the sine and cosine functions are defined in terms of the coordinates of $P(x, y)$, it follows that their values are unchanged by the addition of any integer multiple of 2π . In other words,

$$\sin(t + 2n\pi) = \sin t \quad \text{for any integer } n$$

$$\cos(t + 2n\pi) = \cos t \quad \text{for any integer } n$$

Thus, the sine and cosine functions are *periodic* according to the following definition: A function f is **periodic** if there is a positive number p such that $f(t + p) = f(t)$ for every t . The least such positive number (if it exists) is the **period** of f . If f has period p , then the graph of f on any interval of length p is called **one complete period** of f .

Periodic Properties of Sine and Cosine

The functions sine and cosine have period 2π :

$$\sin(t + 2\pi) = \sin t \qquad \cos(t + 2\pi) = \cos t$$

Table 1

t	$\sin t$	$\cos t$
$0 \rightarrow \frac{\pi}{2}$	$0 \rightarrow 1$	$1 \rightarrow 0$
$\frac{\pi}{2} \rightarrow \pi$	$1 \rightarrow 0$	$0 \rightarrow -1$
$\pi \rightarrow \frac{3\pi}{2}$	$0 \rightarrow -1$	$-1 \rightarrow 0$
$\frac{3\pi}{2} \rightarrow 2\pi$	$-1 \rightarrow 0$	$0 \rightarrow 1$

So the sine and cosine functions repeat their values in any interval of length 2π . To sketch their graphs, we first graph one period. To sketch the graphs on the interval $0 \leq t \leq 2\pi$, we could try to make a table of values and use those points to draw the graph. Since no such table can be complete, let's look more closely at the definitions of these functions.

Recall that $\sin t$ is the y -coordinate of the terminal point $P(x, y)$ on the unit circle determined by the real number t . How does the y -coordinate of this point vary as t increases? It's easy to see that the y -coordinate of $P(x, y)$ increases to 1, then decreases to -1 repeatedly as the point $P(x, y)$ travels around the unit circle. (See Figure 1.) In fact, as t increases from 0 to $\pi/2$, $y = \sin t$ increases from 0 to 1. As t increases from $\pi/2$ to π , the value of $y = \sin t$ decreases from 1 to 0. Table 1 shows the variation of the sine and cosine functions for t between 0 and 2π .

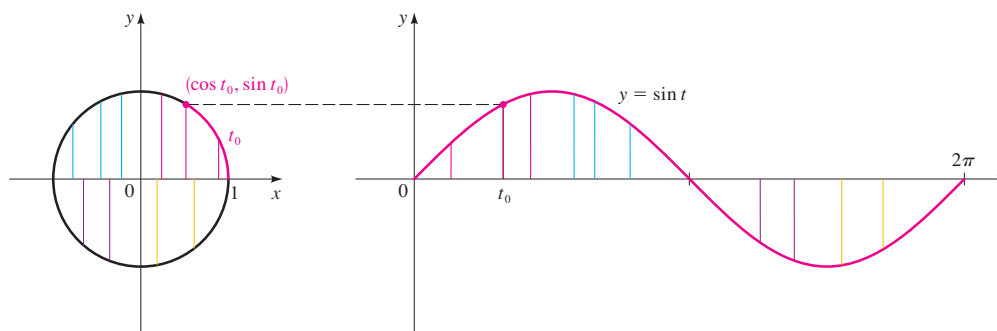


Figure 1

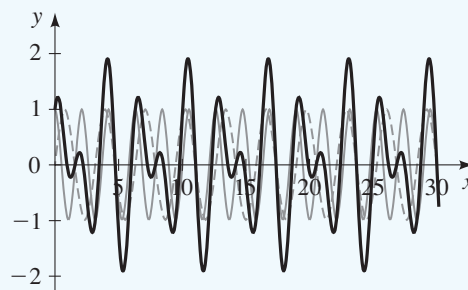
To draw the graphs more accurately, we find a few other values of $\sin t$ and $\cos t$ in Table 2. We could find still other values with the aid of a calculator.

Table 2

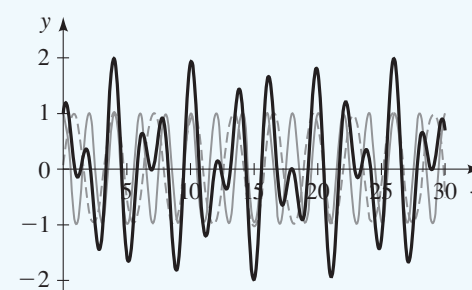
t	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$\sin t$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0
$\cos t$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1

IN-CLASS MATERIALS

Ask the class if it is true that the sum of two periodic curves is always periodic. It turns out that although this seems a reasonable proposition, it is actually false. It is a true statement if you add the stipulation that the periods of the two curves are rational multiples of each other.



$\sin 2x + \cos 3x$ is periodic



$\sin 2x + \cos \pi x$ is not periodic

Now we use this information to graph the functions $\sin t$ and $\cos t$ for t between 0 and 2π in Figures 2 and 3. These are the graphs of one period. Using the fact that these functions are periodic with period 2π , we get their complete graphs by continuing the same pattern to the left and to the right in every successive interval of length 2π .

The graph of the sine function is symmetric with respect to the origin. This is as expected, since sine is an odd function. Since the cosine function is an even function, its graph is symmetric with respect to the y -axis.

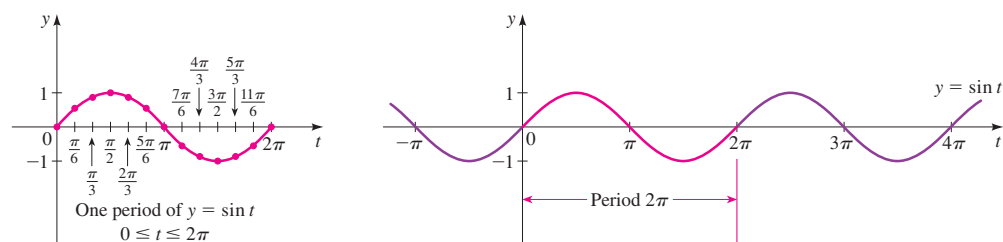


Figure 2
Graph of $\sin t$

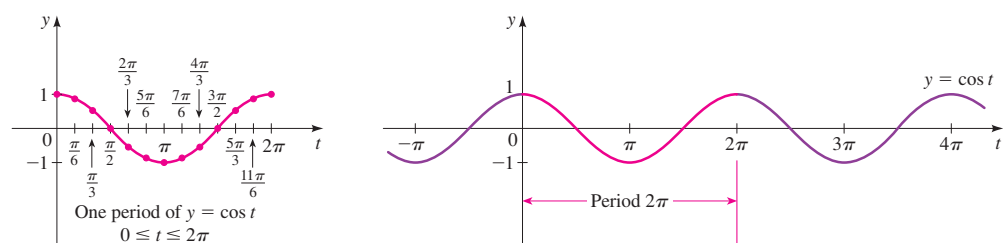


Figure 3
Graph of $\cos t$

Graphs of Transformations of Sine and Cosine

We now consider graphs of functions that are transformations of the sine and cosine functions. Thus, the graphing techniques of Section 2.4 are very useful here. The graphs we obtain are important for understanding applications to physical situations such as harmonic motion (see Section 5.5), but some of them are beautiful graphs that are interesting in their own right.

It's traditional to use the letter x to denote the variable in the domain of a function. So, from here on we use the letter x and write $y = \sin x$, $y = \cos x$, $y = \tan x$, and so on to denote these functions.

Example 1 Cosine Curves

Sketch the graph of each function.

- (a) $f(x) = 2 + \cos x$ (b) $g(x) = -\cos x$

Solution

- (a) The graph of $y = 2 + \cos x$ is the same as the graph of $y = \cos x$, but shifted up 2 units (see Figure 4(a)).

ALTERNATE EXAMPLE 1a
In what direction and how many units should the graph of $y = \cos x$ be shifted to obtain the graph of $y = 5 + \cos x$?

ANSWER
Up, 5

(b) The graph of $y = -\cos x$ in Figure 4(b) is the reflection of the graph of $y = \cos x$ in the x -axis.

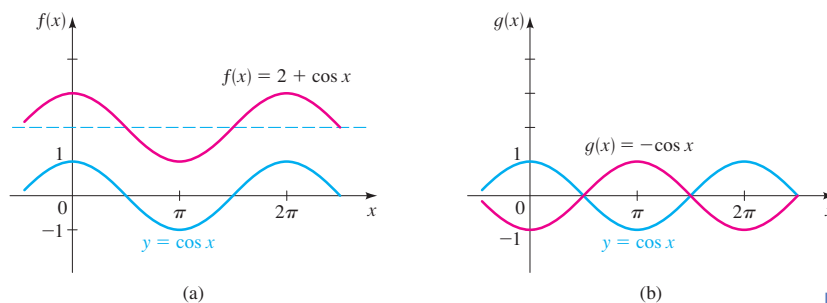


Figure 4

Let's graph $y = 2 \sin x$. We start with the graph of $y = \sin x$ and multiply the y -coordinate of each point by 2. This has the effect of stretching the graph vertically by a factor of 2. To graph $y = \frac{1}{2} \sin x$, we start with the graph of $y = \sin x$ and multiply the y -coordinate of each point by $\frac{1}{2}$. This has the effect of shrinking the graph vertically by a factor of $\frac{1}{2}$ (see Figure 5).

Vertical stretching and shrinking of graphs is discussed in Section 2.4.

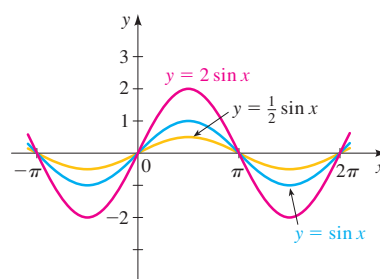


Figure 5

In general, for the functions

$$y = a \sin x \quad \text{and} \quad y = a \cos x$$

the number $|a|$ is called the **amplitude** and is the largest value these functions attain. Graphs of $y = a \sin x$ for several values of a are shown in Figure 6.

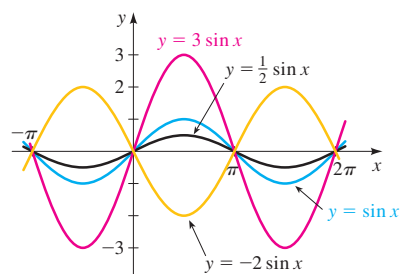


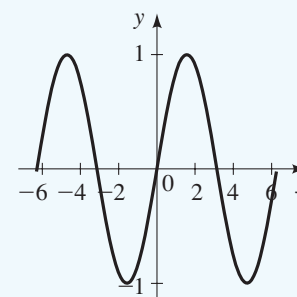
Figure 6

SAMPLE QUESTION

Text Question

Sketch $y = \sin x$.

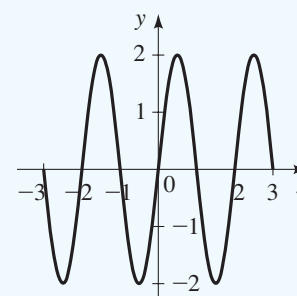
Answer



DRILL QUESTION

Sketch $y = 2 \sin \pi x$.

Answer

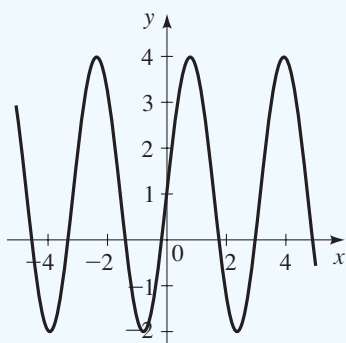


IN-CLASS MATERIALS

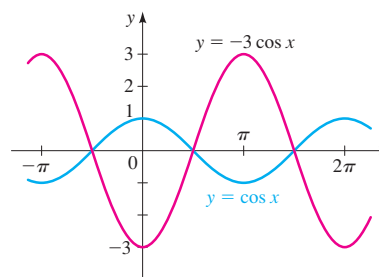
Let $f(x) = \sin x + \frac{1}{100} \cos x$. Have students graph this function with their calculators. In the default viewing window, the graph will be indistinguishable from that of $\sin x$. Show that when they zoom in they can see the “bumps” in the function.

ALTERNATE EXAMPLE 2Find the amplitude of $y = \pi \cos x$.**ANSWER** π **EXAMPLE**

Graph of variants of sine and cosine curves: $y = 3 \sin 2x + 1$ has amplitude 3, period π , and vertical shift 1.

**Example 2** Stretching a Cosine CurveFind the amplitude of $y = -3 \cos x$ and sketch its graph.

Solution The amplitude is $|-3| = 3$, so the largest value the graph attains is 3 and the smallest value is -3 . To sketch the graph, we begin with the graph of $y = \cos x$, stretch the graph vertically by a factor of 3, and reflect in the x -axis, arriving at the graph in Figure 7.

**Figure 7**

Since the sine and cosine functions have period 2π , the functions

$$y = a \sin kx \quad \text{and} \quad y = a \cos kx \quad (k > 0)$$

complete one period as kx varies from 0 to 2π , that is, for $0 \leq kx \leq 2\pi$ or for $0 \leq x \leq 2\pi/k$. So these functions complete one period as x varies between 0 and $2\pi/k$ and thus have period $2\pi/k$. The graphs of these functions are called **sine curves** and **cosine curves**, respectively. (Collectively, sine and cosine curves are often referred to as **sinusoidal curves**.)

Sine and Cosine Curves

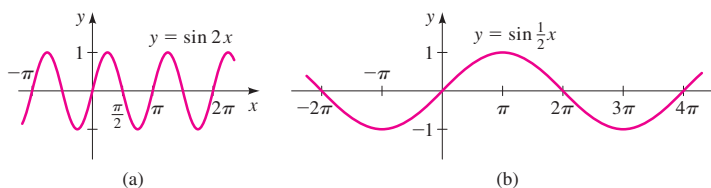
The sine and cosine curves

$$y = a \sin kx \quad \text{and} \quad y = a \cos kx \quad (k > 0)$$

have amplitude $|a|$ and period $2\pi/k$.

An appropriate interval on which to graph one complete period is $[0, 2\pi/k]$.

To see how the value of k affects the graph of $y = \sin kx$, let's graph the sine curve $y = \sin 2x$. Since the period is $2\pi/2 = \pi$, the graph completes one period in the interval $0 \leq x \leq \pi$ (see Figure 8(a)). For the sine curve $y = \sin \frac{1}{2}x$, the period is $2\pi \div \frac{1}{2} = 4\pi$, and so the graph completes one period in the interval $0 \leq x \leq 4\pi$ (see Figure 8(b)). We see that the effect is to *shrink* the graph horizontally if $k > 1$ or to *stretch* the graph horizontally if $k < 1$.

**Figure 8**

Horizontal stretching and shrinking of graphs is discussed in Section 2.4.

IN-CLASS MATERIALS

Demonstrate simple harmonic motion by bringing a spring (like a slinky, or another kind of spring, perhaps with a weight attached at the end) and letting it oscillate. Notice how the graph of the height of the end of the spring versus time is a sine (or cosine) curve. Similarly, if you chart the height with respect to time of a point on a spinning disk whose axis is parallel to the floor, you will get a sine or cosine curve.

For comparison, in Figure 9 we show the graphs of one period of the sine curve $y = a \sin kx$ for several values of k .

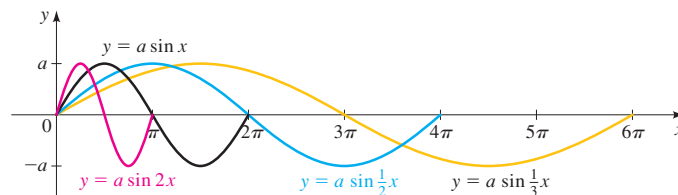


Figure 9

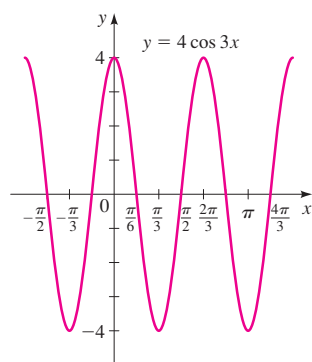


Figure 10

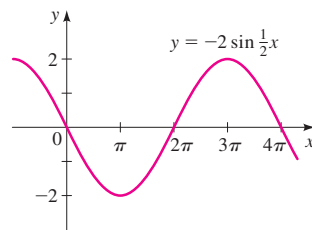


Figure 11

Example 3 Amplitude and Period

Find the amplitude and period of each function, and sketch its graph.

- (a) $y = 4 \cos 3x$ (b) $y = -2 \sin \frac{1}{2}x$

Solution

- (a) We get the amplitude and period from the form of the function as follows:

$$\text{amplitude} = |a| = 4$$

$$y = 4 \cos 3x$$

$$\text{period} = \frac{2\pi}{k} = \frac{2\pi}{3}$$

The amplitude is 4 and the period is $2\pi/3$. The graph is shown in Figure 10.

- (b) For $y = -2 \sin \frac{1}{2}x$,

$$\text{amplitude} = |a| = |-2| = 2$$

$$\text{period} = \frac{2\pi}{\frac{1}{2}} = 4\pi$$

The graph is shown in Figure 11. ■

The graphs of functions of the form $y = a \sin k(x - b)$ and $y = a \cos k(x - b)$ are simply sine and cosine curves shifted horizontally by an amount $|b|$. They are shifted to the right if $b > 0$ or to the left if $b < 0$. The number b is the **phase shift**. We summarize the properties of these functions in the following box.

Shifted Sine and Cosine Curves

The sine and cosine curves

$$y = a \sin k(x - b) \quad \text{and} \quad y = a \cos k(x - b) \quad (k > 0)$$

have amplitude $|a|$, period $2\pi/k$, and phase shift b .

An appropriate interval on which to graph one complete period is $[b, b + (2\pi/k)]$.

ALTERNATE EXAMPLE 3a

Find the amplitude and the period of $y = 2 \cos 3x$.

ANSWER

$$2, \frac{2\pi}{3}$$

ALTERNATE EXAMPLE 3b

Find the amplitude and the period of $y = -5 \sin \frac{1}{3}x$.

ANSWER

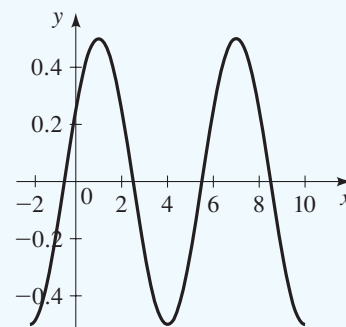
$$5, 6\pi$$

EXAMPLE

A graph of a transformed cosine curve:

$$y = \frac{1}{2} \cos\left(\frac{\pi}{3}(x - 1)\right)$$

has amplitude $\frac{1}{2}$, period 6, and horizontal shift 1.



IN-CLASS MATERIALS

Have students obtain a table of time of sunrise versus date for your city. Perhaps have each student write down the information for a given month. (Websites such as http://aa.usno.navy.mil/data/docs/RS_OneYear.html will do these calculations for you.) Have them pool and plot the data, and see how the resulting graph resembles a sine or cosine curve.

The graphs of $y = \sin\left(x - \frac{\pi}{3}\right)$ and $y = \sin\left(x + \frac{\pi}{6}\right)$ are shown in Figure 12.

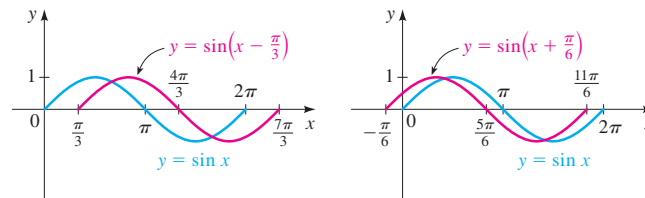


Figure 12

ALTERNATE EXAMPLE 4
Find the amplitude, the period,
and the phase shift of

$$y = -2 \sin 4(x - 1.3)$$

ANSWER

Amplitude = 2, period = $\pi/2$,
phase shift = 1.3 (to the right)

Here is another way to find an appropriate interval on which to graph one complete period. Since the period of $y = \sin x$ is 2π , the function $y = 3 \sin 2\left(x - \frac{\pi}{4}\right)$ will go through one complete period as $2\left(x - \frac{\pi}{4}\right)$ varies from 0 to 2π .

Start of period: End of period:

$$\begin{array}{ll} 2\left(x - \frac{\pi}{4}\right) = 0 & 2\left(x - \frac{\pi}{4}\right) = 2\pi \\ x - \frac{\pi}{4} = 0 & x - \frac{\pi}{4} = \pi \\ x = \frac{\pi}{4} & x = \frac{5\pi}{4} \end{array}$$

So we graph one period on the interval $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$.

Example 4 A Shifted Sine Curve

Find the amplitude, period, and phase shift of $y = 3 \sin 2\left(x - \frac{\pi}{4}\right)$, and graph one complete period.

Solution We get the amplitude, period, and phase shift from the form of the function as follows:

$$\text{amplitude} = |a| = 3 \qquad \text{period} = \frac{2\pi}{k} = \frac{2\pi}{2} = \pi$$

$$y = 3 \sin 2\left(x - \frac{\pi}{4}\right)$$

$$\text{phase shift} = \frac{\pi}{4} \text{ (to the right)}$$

Since the phase shift is $\pi/4$ and the period is π , one complete period occurs on the interval

$$\left[\frac{\pi}{4}, \frac{\pi}{4} + \pi\right] = \left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$$

As an aid in sketching the graph, we divide this interval into four equal parts, then graph a sine curve with amplitude 3 as in Figure 13.

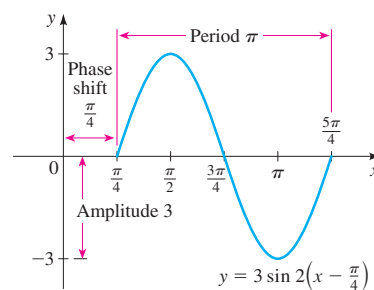


Figure 13

Example 5 A Shifted Cosine Curve

Find the amplitude, period, and phase shift of

$$y = \frac{3}{4} \cos\left(2x + \frac{2\pi}{3}\right)$$

and graph one complete period.

Solution We first write this function in the form $y = a \cos k(x - b)$. To do this, we factor 2 from the expression $2x + \frac{2\pi}{3}$ to get

$$y = \frac{3}{4} \cos 2\left[x - \left(-\frac{\pi}{3}\right)\right]$$

Thus, we have

$$\text{amplitude} = |a| = \frac{3}{4}$$

$$\text{period} = \frac{2\pi}{k} = \frac{2\pi}{2} = \pi$$

$$\text{phase shift} = b = -\frac{\pi}{3} \quad \text{Shift } \frac{\pi}{3} \text{ to the left}$$

From this information it follows that one period of this cosine curve begins at $-\pi/3$ and ends at $(-\pi/3) + \pi = 2\pi/3$. To sketch the graph over the interval $[-\pi/3, 2\pi/3]$, we divide this interval into four equal parts and graph a cosine curve with amplitude $\frac{3}{4}$ as shown in Figure 14.

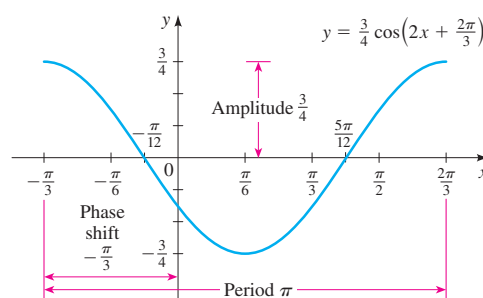


Figure 14

Using Graphing Devices to Graph Trigonometric Functions

When using a graphing calculator or a computer to graph a function, it is important to choose the viewing rectangle carefully in order to produce a reasonable graph of the function. This is especially true for trigonometric functions; the next example shows that, if care is not taken, it's easy to produce a very misleading graph of a trigonometric function.

We can also find one complete period as follows:

Start of period: End of period:

$$2x + \frac{2\pi}{3} = 0 \qquad 2x + \frac{2\pi}{3} = 2\pi$$

$$2x = -\frac{2\pi}{3} \qquad 2x = \frac{4\pi}{3}$$

$$x = -\frac{\pi}{3} \qquad x = \frac{2\pi}{3}$$

So we graph one period on the interval $[-\frac{\pi}{3}, \frac{2\pi}{3}]$.

See Section 1.9 for guidelines on choosing an appropriate viewing rectangle.

ALTERNATE EXAMPLE 5

Find the amplitude, period, and phase shift of

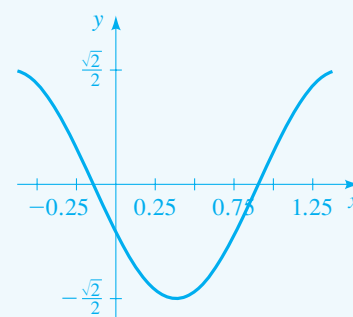
$$y = \frac{\sqrt{2}}{2} \cos(3x + 2)$$

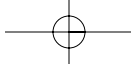
and graph one complete period.

ANSWER

$$\text{Amplitude} = \frac{\sqrt{2}}{2},$$

$$\text{period} = \frac{2\pi}{3},$$

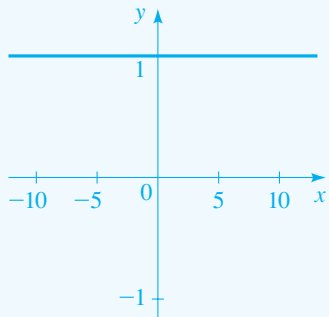
phase shift = $2/3$ to the left

**ALTERNATE EXAMPLE 6**

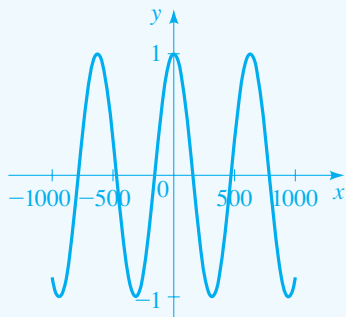
Graph the function $f(x) = \cos(x/100)$ in an appropriate viewing rectangle.

ANSWER

Bad viewing window:



Good viewing window:



The appearance of the graphs in Figure 15 depends on the machine used. The graphs you get with your own graphing device might not look like these figures, but they will also be quite inaccurate.

Example 6 Choosing the Viewing Rectangle

Graph the function $f(x) = \sin 50x$ in an appropriate viewing rectangle.

Solution Figure 15(a) shows the graph of f produced by a graphing calculator using the viewing rectangle $[-12, 12]$ by $[-1.5, 1.5]$. At first glance the graph appears to be reasonable. But if we change the viewing rectangle to the ones shown in Figure 15, the graphs look very different. Something strange is happening.

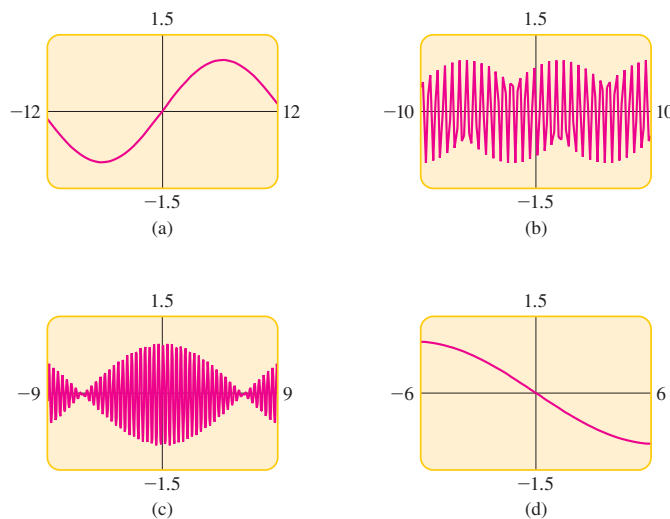


Figure 15
Graphs of $f(x) = \sin 50x$ in different viewing rectangles

To explain the big differences in appearance of these graphs and to find an appropriate viewing rectangle, we need to find the period of the function $y = \sin 50x$:

$$\text{period} = \frac{2\pi}{50} = \frac{\pi}{25} \approx 0.126$$

This suggests that we should deal only with small values of x in order to show just a few oscillations of the graph. If we choose the viewing rectangle $[-0.25, 0.25]$ by $[-1.5, 1.5]$, we get the graph shown in Figure 16.

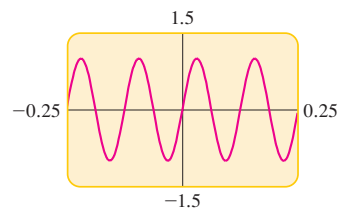
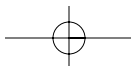


Figure 16
 $f(x) = \sin 50x$

Now we see what went wrong in Figure 15. The oscillations of $y = \sin 50x$ are so rapid that when the calculator plots points and joins them, it misses most of the



The function h in Example 7 is **periodic** with period 2π . In general, functions that are sums of functions from the following list

$$1, \cos kx, \cos 2kx, \cos 3kx, \dots \\ \sin kx, \sin 2kx, \sin 3kx, \dots$$

are periodic. Although these functions appear to be special, they are actually fundamental to describing all periodic functions that arise in practice. The French mathematician J. B. J. Fourier (see page 536) discovered that nearly every periodic function can be written as a sum (usually an infinite sum) of these functions. This is remarkable because it means that any situation in which periodic variation occurs can be described mathematically using the functions sine and cosine. A modern application of Fourier's discovery is the digital encoding of sound on compact discs.

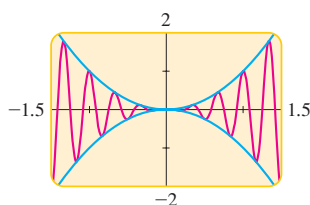


Figure 18
 $y = x^2 \cos 6\pi x$

maximum and minimum points and therefore gives a very misleading impression of the graph. ■

Example 7 A Sum of Sine and Cosine Curves

Graph $f(x) = 2 \cos x$, $g(x) = \sin 2x$, and $h(x) = 2 \cos x + \sin 2x$ on a common screen to illustrate the method of graphical addition.

Solution Notice that $h = f + g$, so its graph is obtained by adding the corresponding y -coordinates of the graphs of f and g . The graphs of f , g , and h are shown in Figure 17.

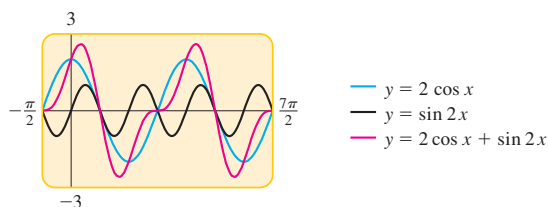


Figure 17

Example 8 A Cosine Curve with Variable Amplitude

Graph the functions $y = x^2$, $y = -x^2$, and $y = x^2 \cos 6\pi x$ on a common screen. Comment on and explain the relationship among the graphs.

Solution Figure 18 shows all three graphs in the viewing rectangle $[-1.5, 1.5]$ by $[-2, 2]$. It appears that the graph of $y = x^2 \cos 6\pi x$ lies between the graphs of the functions $y = x^2$ and $y = -x^2$.

To understand this, recall that the values of $\cos 6\pi x$ lie between -1 and 1 , that is,

$$-1 \leq \cos 6\pi x \leq 1$$

for all values of x . Multiplying the inequalities by x^2 , and noting that $x^2 \geq 0$, we get

$$-x^2 \leq x^2 \cos 6\pi x \leq x^2$$

This explains why the functions $y = x^2$ and $y = -x^2$ form a boundary for the graph of $y = x^2 \cos 6\pi x$. (Note that the graphs touch when $\cos 6\pi x = \pm 1$.) ■

Example 8 shows that the function $y = x^2$ controls the amplitude of the graph of $y = x^2 \cos 6\pi x$. In general, if $f(x) = a(x) \sin kx$ or $f(x) = a(x) \cos kx$, the function a determines how the amplitude of f varies, and the graph of f lies between the graphs of $y = -a(x)$ and $y = a(x)$. Here is another example.

Example 9 A Cosine Curve with Variable Amplitude

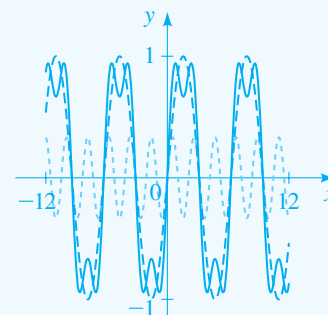
Graph the function $f(x) = \cos 2\pi x \cos 16\pi x$.

Solution The graph is shown in Figure 19 on the next page. Although it was drawn by a computer, we could have drawn it by hand, by first sketching the bound-

ALTERNATE EXAMPLE 7

Graph $f(x) = \sin(x)$, $g(x) = \frac{1}{3} \sin(3x)$, and $h(x) = \sin(x) + \frac{1}{3} \sin(3x)$ on a common screen to illustrate the method of graphical addition.

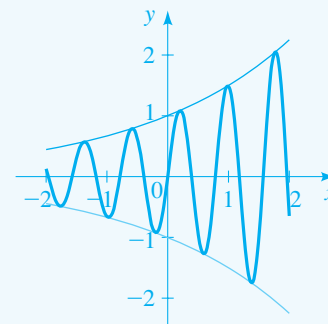
ANSWER



ALTERNATE EXAMPLE 8

Graph $y = 1.5^x$, $y = -1.5^x$, and $y = 1.5^x \sin(8\pi x)$ on the same screen and notice the relationship among the graphs.

ANSWER

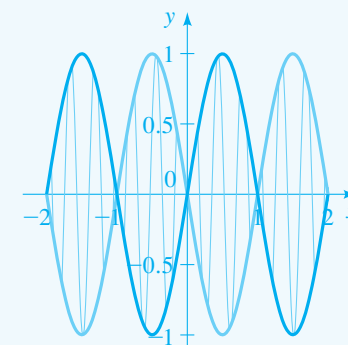


ALTERNATE EXAMPLE 9

Graph the function $f(x) = \sin(\pi x) \sin(18x)$.

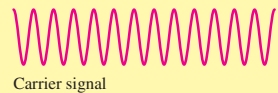
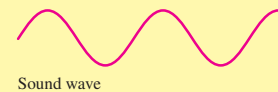
ANSWER

We graph $\sin(\pi x)$ and $-\sin(\pi x)$ to see how f lies between these two curves.

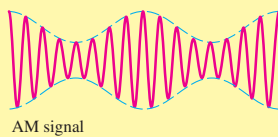


AM and FM Radio

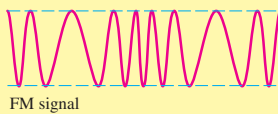
Radio transmissions consist of sound waves superimposed on a harmonic electromagnetic wave form called the **carrier signal**.



There are two types of radio transmission, called **amplitude modulation (AM)** and **frequency modulation (FM)**. In AM broadcasting the sound wave changes, or **modulates**, the amplitude of the carrier, but the frequency remains unchanged.



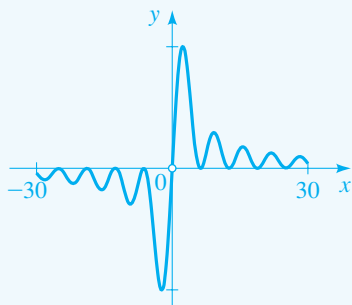
In FM broadcasting the sound wave modulates the frequency, but the amplitude remains the same.

**ALTERNATE EXAMPLE 10**

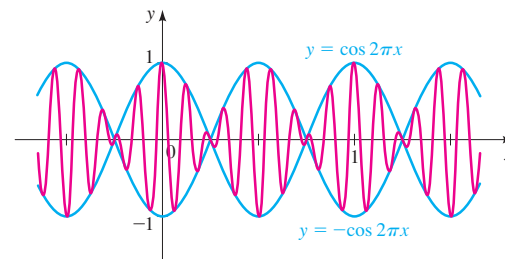
The function $f(x) = \frac{1 - \cos x}{x}$ is important in calculus. Graph this function and comment on its behavior when x is close to zero.

ANSWER

When x is close to zero, $f(x)$ is close to zero.



ary curves $y = \cos 2\pi x$ and $y = -\cos 2\pi x$. The graph of f is a cosine curve that lies between the graphs of these two functions.

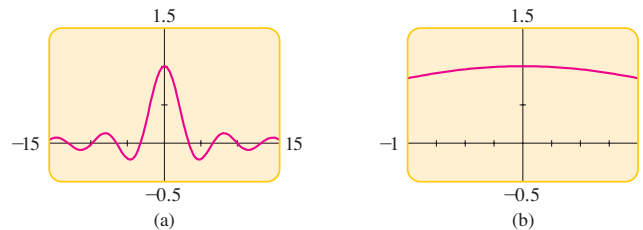
**Figure 19**

$$f(x) = \cos 2\pi x \cos 16\pi x$$

Example 10 A Sine Curve with Decaying Amplitude

The function $f(x) = \frac{\sin x}{x}$ is important in calculus. Graph this function and comment on its behavior when x is close to 0.

Solution The viewing rectangle $[-15, 15]$ by $[-0.5, 1.5]$ shown in Figure 20(a) gives a good global view of the graph of f . The viewing rectangle $[-1, 1]$ by $[-0.5, 1.5]$ in Figure 20(b) focuses on the behavior of f when $x \approx 0$. Notice that although $f(x)$ is not defined when $x = 0$ (in other words, 0 is not in the domain of f), the values of f seem to approach 1 when x gets close to 0. This fact is crucial in calculus.

**Figure 20**

$$f(x) = \frac{\sin x}{x}$$

The function in Example 10 can be written as

$$f(x) = \frac{1}{x} \sin x$$

and may thus be viewed as a sine function whose amplitude is controlled by the function $a(x) = 1/x$.

5.3 Exercises

1–14 ■ Graph the function.

1. $f(x) = 1 + \cos x$
2. $f(x) = 3 + \sin x$
3. $f(x) = -\sin x$
4. $f(x) = 2 - \cos x$
5. $f(x) = -2 + \sin x$
6. $f(x) = -1 + \cos x$
7. $g(x) = 3 \cos x$
8. $g(x) = 2 \sin x$
9. $g(x) = -\frac{1}{2} \sin x$
10. $g(x) = -\frac{2}{3} \cos x$
11. $g(x) = 3 + 3 \cos x$
12. $g(x) = 4 - 2 \sin x$
13. $h(x) = |\cos x|$
14. $h(x) = |\sin x|$

15–26 ■ Find the amplitude and period of the function, and sketch its graph.

15. $y = \cos 2x$
16. $y = -\sin 2x$
17. $y = -3 \sin 3x$
18. $y = \frac{1}{2} \cos 4x$
19. $y = 10 \sin \frac{1}{2}x$
20. $y = 5 \cos \frac{1}{4}x$
21. $y = -\frac{1}{3} \cos \frac{1}{3}x$
22. $y = 4 \sin(-2x)$
23. $y = -2 \sin 2\pi x$
24. $y = -3 \sin \pi x$
25. $y = 1 + \frac{1}{2} \cos \pi x$
26. $y = -2 + \cos 4\pi x$

27–40 ■ Find the amplitude, period, and phase shift of the function, and graph one complete period.

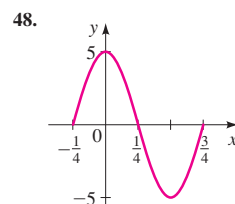
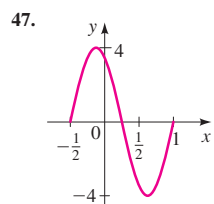
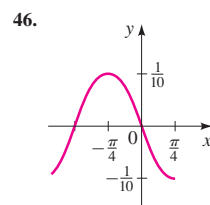
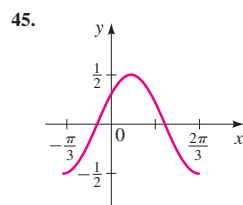
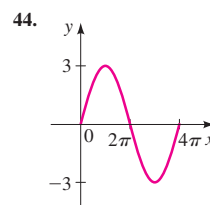
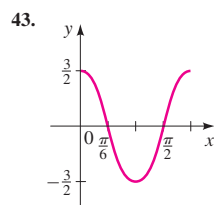
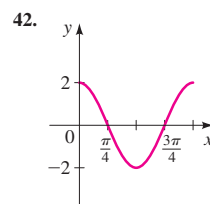
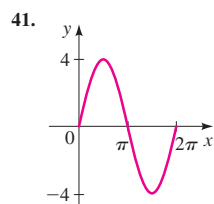
27. $y = \cos\left(x - \frac{\pi}{2}\right)$
28. $y = 2 \sin\left(x - \frac{\pi}{3}\right)$
29. $y = -2 \sin\left(x - \frac{\pi}{6}\right)$
30. $y = 3 \cos\left(x + \frac{\pi}{4}\right)$
31. $y = -4 \sin 2\left(x + \frac{\pi}{2}\right)$
32. $y = \sin \frac{1}{2}\left(x + \frac{\pi}{4}\right)$
33. $y = 5 \cos\left(3x - \frac{\pi}{4}\right)$
34. $y = 2 \sin\left(\frac{2}{3}x - \frac{\pi}{6}\right)$
35. $y = \frac{1}{2} - \frac{1}{2} \cos\left(2x - \frac{\pi}{3}\right)$
36. $y = 1 + \cos\left(3x + \frac{\pi}{2}\right)$
37. $y = 3 \cos \pi\left(x + \frac{1}{2}\right)$
38. $y = 3 + 2 \sin 3(x + 1)$
39. $y = \sin(\pi + 3x)$
40. $y = \cos\left(\frac{\pi}{2} - x\right)$

41–48 ■ The graph of one complete period of a sine or cosine curve is given.

(a) Find the amplitude, period, and phase shift.

(b) Write an equation that represents the curve in the form

$$y = a \sin k(x - b) \quad \text{or} \quad y = a \cos k(x - b)$$




49–56 ■ Determine an appropriate viewing rectangle for each function, and use it to draw the graph.

49. $f(x) = \cos 100x$
50. $f(x) = 3 \sin 120x$
51. $f(x) = \sin(x/40)$
52. $f(x) = \cos(x/80)$


$$53. y = \tan 25x \qquad 54. y = \csc 40x$$

$$55. y = \sin^2 20x \qquad 56. y = \sqrt{\tan 10\pi x}$$

 **57–58** ■ Graph f , g , and $f + g$ on a common screen to illustrate graphical addition.

$$57. f(x) = x, \quad g(x) = \sin x$$

$$58. f(x) = \sin x, \quad g(x) = \sin 2x$$

 **59–64** ■ Graph the three functions on a common screen. How are the graphs related?

$$59. y = x^2, \quad y = -x^2, \quad y = x^2 \sin x$$


$$60. y = x, \quad y = -x, \quad y = x \cos x$$

$$61. y = \sqrt{x}, \quad y = -\sqrt{x}, \quad y = \sqrt{x} \sin 5\pi x$$

$$62. y = \frac{1}{1+x^2}, \quad y = -\frac{1}{1+x^2}, \quad y = \frac{\cos 2\pi x}{1+x^2}$$

$$63. y = \cos 3\pi x, \quad y = -\cos 3\pi x, \quad y = \cos 3\pi x \cos 21\pi x$$

$$64. y = \sin 2\pi x, \quad y = -\sin 2\pi x, \quad y = \sin 2\pi x \sin 10\pi x$$


 **65–68** ■ Find the maximum and minimum values of the function.

$$65. y = \sin x + \sin 2x$$

$$66. y = x - 2 \sin x, \quad 0 \leq x \leq 2\pi$$

$$67. y = 2 \sin x + \sin^2 x$$

$$68. y = \frac{\cos x}{2 + \sin x}$$

 **69–72** ■ Find all solutions of the equation that lie in the interval $[0, \pi]$. State each answer correct to two decimal places.

$$69. \cos x = 0.4$$

$$70. \tan x = 2$$

$$71. \csc x = 3$$

$$72. \cos x = x$$

 **73–74** ■ A function f is given.

(a) Is f even, odd, or neither?

(b) Find the x -intercepts of the graph of f .

(c) Graph f in an appropriate viewing rectangle.

(d) Describe the behavior of the function as $x \rightarrow \pm\infty$.

(e) Notice that $f(x)$ is not defined when $x = 0$. What happens as x approaches 0?

$$73. f(x) = \frac{1 - \cos x}{x}$$

$$74. f(x) = \frac{\sin 4x}{2x}$$

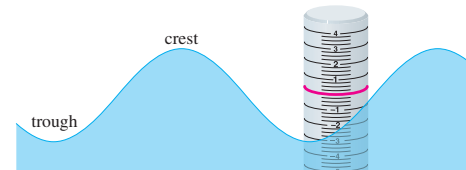
Applications

75. Height of a Wave As a wave passes by an offshore piling, the height of the water is modeled by the function

$$h(t) = 3 \cos\left(\frac{\pi}{10}t\right)$$

where $h(t)$ is the height in feet above mean sea level at time t seconds.

- Find the period of the wave.
- Find the wave height, that is, the vertical distance between the trough and the crest of the wave.



76. Sound Vibrations A tuning fork is struck, producing a pure tone as its tines vibrate. The vibrations are modeled by the function

$$v(t) = 0.7 \sin(880\pi t)$$

where $v(t)$ is the displacement of the tines in millimeters at time t seconds.

- Find the period of the vibration.
- Find the frequency of the vibration, that is, the number of times the fork vibrates per second.
- Graph the function v .

77. Blood Pressure Each time your heart beats, your blood pressure first increases and then decreases as the heart rests between beats. The maximum and minimum blood pressures are called the *systolic* and *diastolic* pressures, respectively. Your *blood pressure reading* is written as systolic/diastolic. A reading of 120/80 is considered normal.

A certain person's blood pressure is modeled by the function

$$p(t) = 115 + 25 \sin(160\pi t)$$

where $p(t)$ is the pressure in mmHg, at time t measured in minutes.

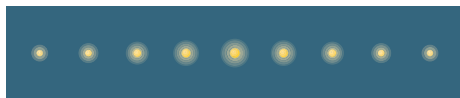
- Find the period of p .
- Find the number of heartbeats per minute.
- Graph the function p .
- Find the blood pressure reading. How does this compare to normal blood pressure?

- 78. Variable Stars** Variable stars are ones whose brightness varies periodically. One of the most visible is R Leonis; its brightness is modeled by the function

$$b(t) = 7.9 - 2.1 \cos\left(\frac{\pi}{156}t\right)$$

where t is measured in days.

- Find the period of R Leonis.
- Find the maximum and minimum brightness.
- Graph the function b .



Discovery • Discussion

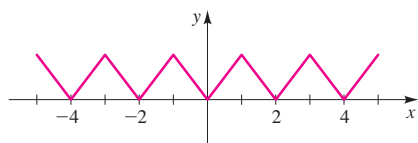
79. Compositions Involving Trigonometric Functions

This exercise explores the effect of the inner function g on a composite function $y = f(g(x))$.

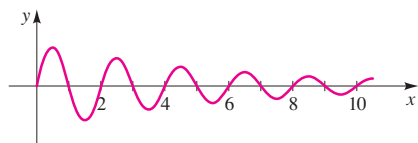
- Graph the function $y = \sin\sqrt{x}$ using the viewing rectangle $[0, 400]$ by $[-1.5, 1.5]$. In what ways does this graph differ from the graph of the sine function?
- Graph the function $y = \sin(x^2)$ using the viewing rectangle $[-5, 5]$ by $[-1.5, 1.5]$. In what ways does this graph differ from the graph of the sine function?

- 80. Periodic Functions I** Recall that a function f is *periodic* if there is a positive number p such that $f(t + p) = f(t)$ for every t , and the least such p (if it exists) is the *period* of f . The graph of a function of period p looks the same on each interval of length p , so we can easily determine the period from the graph. Determine whether the function whose graph is shown is periodic; if it is periodic, find the period.

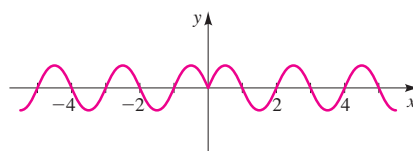
(a)



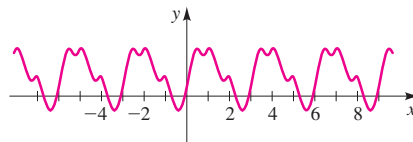
(b)



(c)



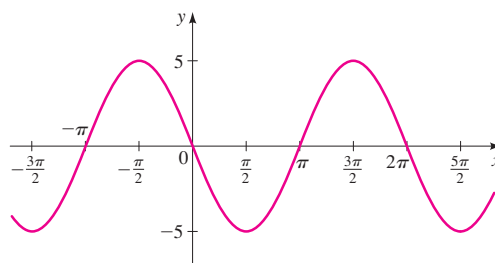
(d)



- 81. Periodic Functions II** Use a graphing device to graph the following functions. From the graph, determine whether the function is periodic; if it is periodic find the period. (See page 162 for the definition of $\llbracket x \rrbracket$.)

- $y = |\sin x|$
- $y = \sin|x|$
- $y = 2^{\cos x}$
- $y = x - \llbracket x \rrbracket$
- $y = \cos(\sin x)$
- $y = \cos(x^2)$

- 82. Sinusoidal Curves** The graph of $y = \sin x$ is the same as the graph of $y = \cos x$ shifted to the right $\pi/2$ units. So the sine curve $y = \sin x$ is also at the same time a cosine curve: $y = \cos(x - \pi/2)$. In fact, any sine curve is also a cosine curve with a different phase shift, and any cosine curve is also a sine curve. Sine and cosine curves are collectively referred to as *sinusoidal*. For the curve whose graph is shown, find all possible ways of expressing it as a sine curve $y = a \sin(x - b)$ or as a cosine curve $y = a \cos(x - b)$. Explain why you think you have found all possible choices for a and b in each case.





**DISCOVERY
PROJECT**



Jeff Lepore/Photo Researchers, Inc.

Predator/Prey Models

Sine and cosine functions are used primarily in physics and engineering to model oscillatory behavior, such as the motion of a pendulum or the current in an AC electrical circuit. (See Section 5.5.) But these functions also arise in the other sciences. In this project, we consider an application to biology—we use sine functions to model the population of a predator and its prey.

An isolated island is inhabited by two species of mammals: lynx and hares. The lynx are *predators* who feed on the hares, their *prey*. The lynx and hare populations change cyclically, as graphed in Figure 1. In part A of the graph, hares are abundant, so the lynx have plenty to eat and their population increases. By the time portrayed in part B, so many lynx are feeding on the hares that the hare population declines. In part C, the hare population has declined so much that there is not enough food for the lynx, so the lynx population starts to decrease. In part D, so many lynx have died that the hares have few enemies, and their population increases again. This takes us back to where we started, and the cycle repeats over and over again.

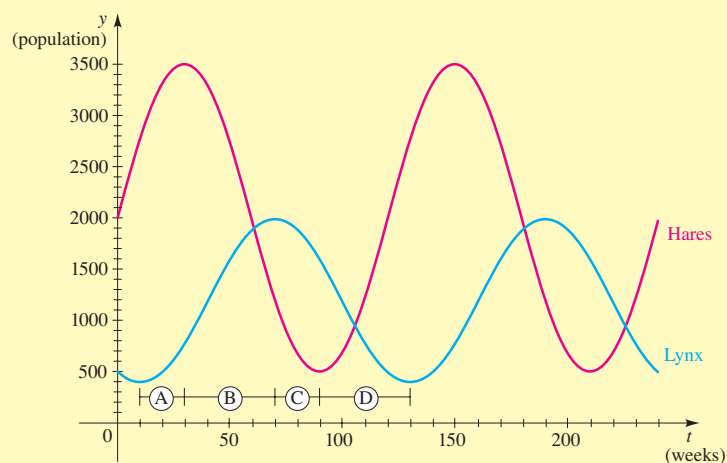


Figure 1

The graphs in Figure 1 are sine curves that have been shifted upward, so they are graphs of functions of the form

$$y = a \sin k(t - b) + c$$

Here c is the amount by which the sine curve has been shifted vertically (see Section 2.4). Note that c is the average value of the function, halfway between the highest and lowest values on the graph. The amplitude $|a|$ is

the amount by which the graph varies above and below the average value (see Figure 2).

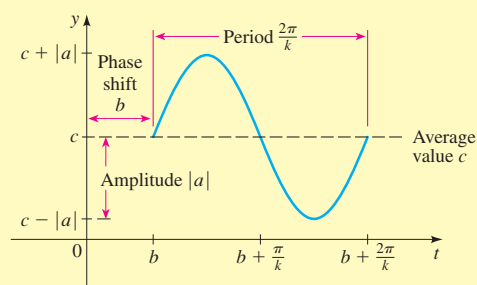


Figure 2
 $y = a \sin k(t - b) + c$

- Find functions of the form $y = a \sin k(t - b) + c$ that model the lynx and hare populations graphed in Figure 1. Graph both functions on your calculator and compare to Figure 1 to verify that your functions are the right ones.
- Add the lynx and hare population functions to get a new function that models the total *mammal* population on this island. Graph this function on your calculator, and find its average value, amplitude, period, and phase shift. How are the average value and period of the mammal population function related to the average value and period of the lynx and hare population functions?
- A small lake on the island contains two species of fish: hake and redfish. The hake are predators that eat the redfish. The fish population in the lake varies periodically with period 180 days. The number of hake varies between 500 and 1500, and the number of redfish varies between 1000 and 3000. The hake reach their maximum population 30 days after the redfish have reached *their* maximum population in the cycle.
 - Sketch a graph (like the one in Figure 1) that shows two complete periods of the population cycle for these species of fish. Assume that $t = 0$ corresponds to a time when the redfish population is at a maximum.
 - Find cosine functions of the form $y = a \cos k(t - b) + c$ that model the hake and redfish populations in the lake.
- In real life, most predator/prey populations do not behave as simply as the examples we have described here. In most cases, the populations of predator and prey oscillate, but the amplitude of the oscillations gets smaller and smaller, so that eventually both populations stabilize near a constant value. Sketch a rough graph that illustrates how the populations of predator and prey might behave in this case.

SUGGESTED TIME AND EMPHASIS

$\frac{1}{2}$ -class.
Essential material.

POINT TO STRESS

The graphs of tangent, cotangent, secant, and cosecant, their periodicity, and their transformations.

5.4 More Trigonometric Graphs

In this section we graph the tangent, cotangent, secant, and cosecant functions, and transformations of these functions.

Graphs of the Tangent, Cotangent, Secant, and Cosecant Function

We begin by stating the periodic properties of these functions. Recall that sine and cosine have period 2π . Since cosecant and secant are the reciprocals of sine and cosine, respectively, they also have period 2π (see Exercise 53). Tangent and cotangent, however, have period π (see Exercise 83 of Section 5.2).

Periodic Properties

The functions tangent and cotangent have period π :

$$\tan(x + \pi) = \tan x \quad \cot(x + \pi) = \cot x$$

The functions cosecant and secant have period 2π :

$$\csc(x + 2\pi) = \csc x \quad \sec(x + 2\pi) = \sec x$$

x	$\tan x$
0	0
$\frac{\pi}{6}$	0.58
$\frac{\pi}{4}$	1.00
$\frac{\pi}{3}$	1.73
1.4	5.80
1.5	14.10
1.55	48.08
1.57	1,255.77
1.5707	10,381.33

Arrow notation is discussed in Section 3.6.

Asymptotes are discussed in Section 3.6.

We first sketch the graph of tangent. Since it has period π , we need only sketch the graph on any interval of length π and then repeat the pattern to the left and to the right. We sketch the graph on the interval $(-\pi/2, \pi/2)$. Since $\tan \pi/2$ and $\tan(-\pi/2)$ aren't defined, we need to be careful in sketching the graph at points near $\pi/2$ and $-\pi/2$. As x gets near $\pi/2$ through values less than $\pi/2$, the value of $\tan x$ becomes large. To see this, notice that as x gets close to $\pi/2$, $\cos x$ approaches 0 and $\sin x$ approaches 1 and so $\tan x = \sin x / \cos x$ is large. A table of values of $\tan x$ for x close to $\pi/2$ (≈ 1.570796) is shown in the margin.

Thus, by choosing x close enough to $\pi/2$ through values less than $\pi/2$, we can make the value of $\tan x$ larger than any given positive number. We express this by writing

$$\tan x \rightarrow \infty \quad \text{as} \quad x \rightarrow \frac{\pi}{2}^-$$

This is read “ $\tan x$ approaches infinity as x approaches $\pi/2$ from the left.”

In a similar way, by choosing x close to $-\pi/2$ through values greater than $-\pi/2$, we can make $\tan x$ smaller than any given negative number. We write this as

$$\tan x \rightarrow -\infty \quad \text{as} \quad x \rightarrow -\frac{\pi}{2}^+$$

This is read “ $\tan x$ approaches negative infinity as x approaches $-\pi/2$ from the right.”

Thus, the graph of $y = \tan x$ approaches the vertical lines $x = \pi/2$ and $x = -\pi/2$. So these lines are **vertical asymptotes**. With the information we have so far, we sketch the graph of $y = \tan x$ for $-\pi/2 < x < \pi/2$ in Figure 1. The complete graph

of tangent (see Figure 5(a) on page 436) is now obtained using the fact that tangent is periodic with period π .

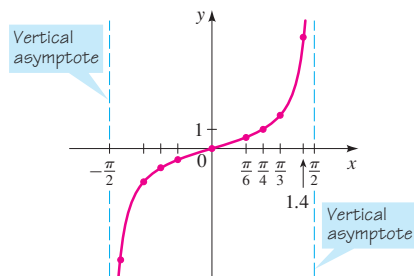


Figure 1
One period of $y = \tan x$

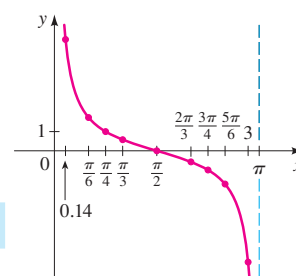


Figure 2
One period of $y = \cot x$

The function $y = \cot x$ is graphed on the interval $(0, \pi)$ by a similar analysis (see Figure 2). Since $\cot x$ is undefined for $x = n\pi$ with n an integer, its complete graph (in Figure 5(b) on page 436) has vertical asymptotes at these values.

To graph the cosecant and secant functions, we use the reciprocal identities

$$\csc x = \frac{1}{\sin x} \quad \text{and} \quad \sec x = \frac{1}{\cos x}$$

So, to graph $y = \csc x$, we take the reciprocals of the y -coordinates of the points of the graph of $y = \sin x$. (See Figure 3.) Similarly, to graph $y = \sec x$, we take the reciprocals of the y -coordinates of the points of the graph of $y = \cos x$. (See Figure 4.)

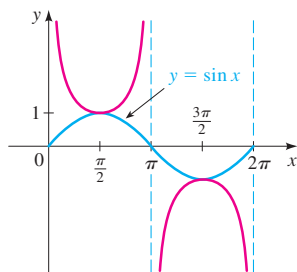


Figure 3
One period of $y = \csc x$

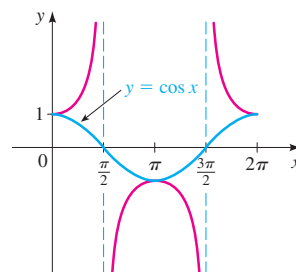


Figure 4
One period of $y = \sec x$

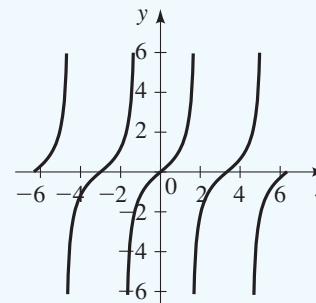
Let's consider more closely the graph of the function $y = \csc x$ on the interval $0 < x < \pi$. We need to examine the values of the function near 0 and π since at these values $\sin x = 0$, and $\csc x$ is thus undefined. We see that

$$\begin{aligned} \csc x &\rightarrow \infty & \text{as} & \quad x \rightarrow 0^+ \\ \csc x &\rightarrow -\infty & \text{as} & \quad x \rightarrow \pi^- \end{aligned}$$

SAMPLE QUESTION Text Question

Sketch $y = \tan x$.

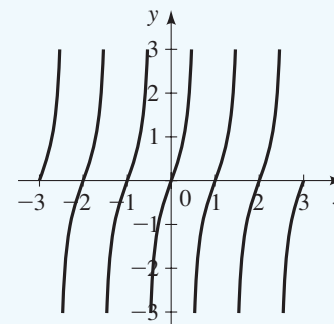
Answer



DRILL QUESTION

Sketch $y = \tan \pi x$.

Answer

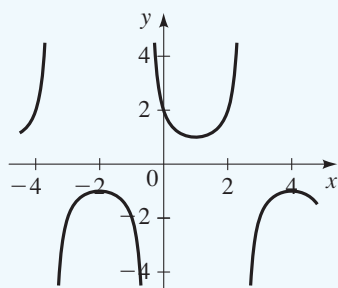


EXAMPLE

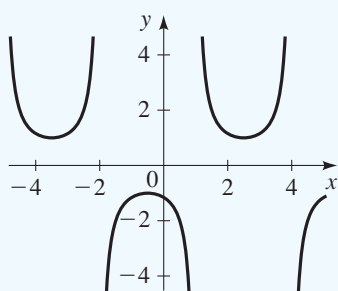
$$y = \sec\left(\frac{\pi}{3}(x-1)\right) \text{ and}$$

$$y = \csc\left(\frac{\pi}{3}(x-1)\right) \text{ have}$$

period 3 and horizontal shift 1.



$$y = \sec\left(\frac{\pi}{3}(x-1)\right)$$



$$y = \csc\left(\frac{\pi}{3}(x-1)\right)$$

Mathematics in the Modern World**Evaluating Functions on a Calculator**

How does your calculator evaluate $\sin t$, $\cos t$, e^t , $\ln t$, \sqrt{t} , and other such functions? One method is to approximate these functions by polynomials, because polynomials are easy to evaluate. For example,

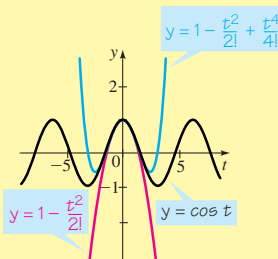
$$\sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots$$

$$\cos t = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots$$

where $n! = 1 \cdot 2 \cdot 3 \cdots n$. These remarkable formulas were found by the British mathematician Brook Taylor (1685–1731). For instance, if we use the first three terms of Taylor's series to find $\cos(0.4)$, we get

$$\begin{aligned} \cos 0.4 &\approx 1 - \frac{(0.4)^2}{2!} + \frac{(0.4)^4}{4!} \\ &\approx 0.92106667 \end{aligned}$$

(Compare this with the value you get from your calculator.) The graph shows that the more terms of the series we use, the more closely the polynomials approximate the function $\cos t$.



Thus, the lines $x = 0$ and $x = \pi$ are vertical asymptotes. In the interval $\pi < x < 2\pi$ the graph is sketched in the same way. The values of $\csc x$ in that interval are the same as those in the interval $0 < x < \pi$ except for sign (see Figure 3). The complete graph in Figure 5(c) is now obtained from the fact that the function cosecant is periodic with period 2π . Note that the graph has vertical asymptotes at the points where $\sin x = 0$, that is, at $x = n\pi$, for n an integer.

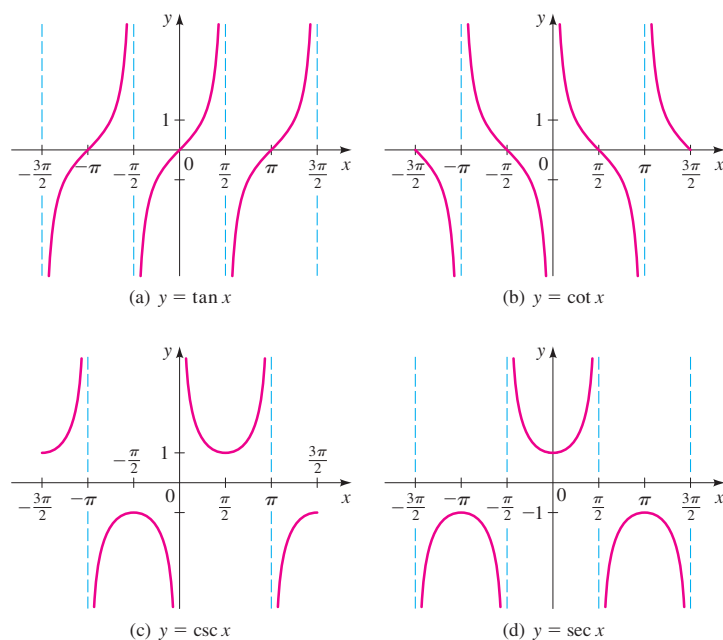


Figure 5

The graph of $y = \sec x$ is sketched in a similar manner. Observe that the domain of $\sec x$ is the set of all real numbers other than $x = (\pi/2) + n\pi$, for n an integer, so the graph has vertical asymptotes at those points. The complete graph is shown in Figure 5(d).

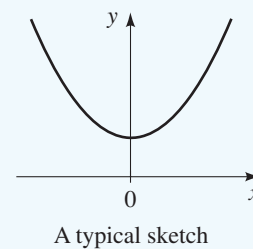
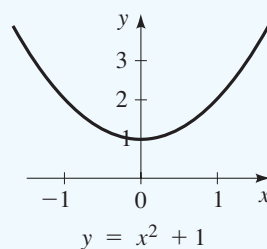
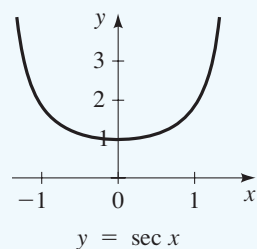
It is apparent that the graphs of $y = \tan x$, $y = \cot x$, and $y = \csc x$ are symmetric about the origin, whereas that of $y = \sec x$ is symmetric about the y -axis. This is because tangent, cotangent, and cosecant are odd functions, whereas secant is an even function.

Graphs Involving Tangent and Cotangent Functions

We now consider graphs of transformations of the tangent and cotangent functions.

IN-CLASS MATERIALS

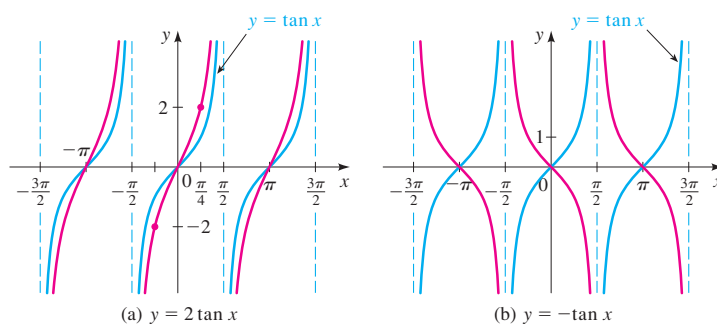
Notice that the functions $\sec x$ and $x^2 + 1$ are very different functions ($\sec x$ has vertical asymptotes, for example, and $x^2 + 1$ does not). When we casually sketch their graphs near $x = 0$, however, the sketches look similar:



Example 1 Graphing Tangent Curves

Graph each function.

(a) $y = 2 \tan x$ (b) $y = -\tan x$

Solution We first graph $y = \tan x$ and then transform it as required.(a) To graph $y = 2 \tan x$, we multiply the y -coordinate of each point on the graph of $y = \tan x$ by 2. The resulting graph is shown in Figure 6(a).(b) The graph of $y = -\tan x$ in Figure 6(b) is obtained from that of $y = \tan x$ by reflecting in the x -axis.**Figure 6**Since the tangent and cotangent functions have period π , the functions

$$y = a \tan kx \quad \text{and} \quad y = a \cot kx \quad (k > 0)$$

complete one period as kx varies from 0 to π , that is, for $0 \leq kx \leq \pi$. Solving this inequality, we get $0 \leq x \leq \pi/k$. So they each have period π/k .**Tangent and Cotangent Curves**

The functions

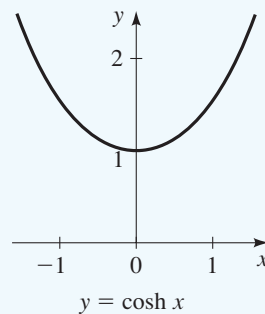
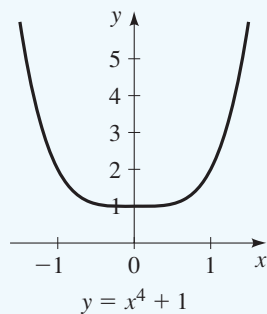
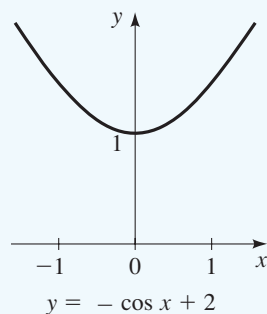
$$y = a \tan kx \quad \text{and} \quad y = a \cot kx \quad (k > 0)$$

have period π/k .Thus, one complete period of the graphs of these functions occurs on any interval of length π/k . To sketch a complete period of these graphs, it's convenient to select an interval between vertical asymptotes:To graph one period of $y = a \tan kx$, an appropriate interval is $\left(-\frac{\pi}{2k}, \frac{\pi}{2k}\right)$.To graph one period of $y = a \cot kx$, an appropriate interval is $\left(0, \frac{\pi}{k}\right)$.**ALTERNATE EXAMPLE 1a**Determine whether stretching or shrinking is needed with the graph of the function $y = \tan x$ to obtain the graph of the function $y = 4 \tan x$. Find the corresponding factor of that stretching or shrinking.**ANSWER**

Stretching, 4

ALTERNATE EXAMPLE 1bFind the axis of reflecting to obtain the graph of the function $y = -3 \tan x$ from the graph of the function $y = 3 \tan x$.**ANSWER** x **IN-CLASS MATERIALS**

Many curves have a “U” shape near zero. Have students try to come up with other examples. If they have calculators, they can experiment with functions they haven't learned about yet.



ALTERNATE EXAMPLE 2

Determine whether stretching or shrinking is needed with the graph of the function $y = \tan x$ to obtain the graph of the function $y = \tan 6x$.

Find the corresponding factor of that stretching or shrinking.

ANSWER

Stretching, $\frac{1}{6}$

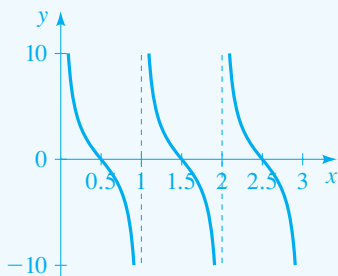
ALTERNATE EXAMPLE 3

Graph $y = 3 \cot(\pi x - \pi)$.

ANSWER

We put it in the standard form: $3 \cot \pi(x - 1)$. Now we have the graph of $3 \cot(x)$; only with period 1 and shifted to the right by 1.

We graph the interval from 0 to 3:

**Example 2 Graphing Tangent Curves**

Graph each function.

(a) $y = \tan 2x$ (b) $y = \tan 2\left(x - \frac{\pi}{4}\right)$

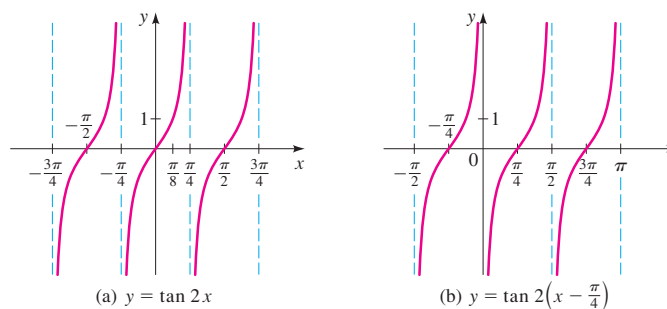
Solution

- (a) The period is $\pi/2$ and an appropriate interval is $(-\pi/4, \pi/4)$. The endpoints $x = -\pi/4$ and $x = \pi/4$ are vertical asymptotes. Thus, we graph one complete period of the function on $(-\pi/4, \pi/4)$. The graph has the same shape as that of the tangent function, but is shrunk horizontally by a factor of $\frac{1}{2}$. We then repeat that portion of the graph to the left and to the right. See Figure 7(a).
- (b) The graph is the same as that in part (a), but it is shifted to the right $\pi/4$, as shown in Figure 7(b).

Since $y = \tan x$ completes one period between $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$, the function $y = \tan 2(x - \frac{\pi}{4})$ completes one period as $2(x - \frac{\pi}{4})$ varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.

Start of period:	End of period:
$2(x - \frac{\pi}{4}) = -\frac{\pi}{2}$	$2(x - \frac{\pi}{4}) = \frac{\pi}{2}$
$x - \frac{\pi}{4} = -\frac{\pi}{4}$	$x - \frac{\pi}{4} = \frac{\pi}{4}$
$x = 0$	$x = \frac{\pi}{2}$

So we graph one period on the interval $(0, \frac{\pi}{2})$.

**Figure 7**

(a) $y = \tan 2x$

(b) $y = \tan 2\left(x - \frac{\pi}{4}\right)$

Example 3 A Shifted Cotangent Curve

Graph $y = 2 \cot\left(3x - \frac{\pi}{2}\right)$.

Solution We first put this in the form $y = a \cot k(x - b)$ by factoring 3 from the expression $3x - \frac{\pi}{2}$:

$$y = 2 \cot\left(3x - \frac{\pi}{2}\right) = 2 \cot 3\left(x - \frac{\pi}{6}\right)$$

Thus, the graph is the same as that of $y = 2 \cot 3x$, but is shifted to the right $\pi/6$. The period of $y = 2 \cot 3x$ is $\pi/3$, and an appropriate interval is $(0, \pi/3)$. To get the corresponding interval for the desired graph, we shift this interval to the right $\pi/6$. This gives

$$\left(0 + \frac{\pi}{6}, \frac{\pi}{3} + \frac{\pi}{6}\right) = \left(\frac{\pi}{6}, \frac{\pi}{2}\right)$$

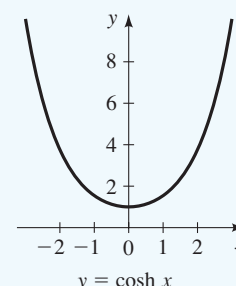
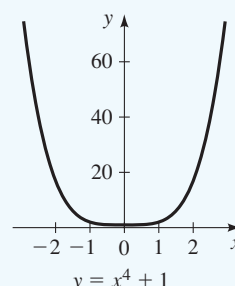
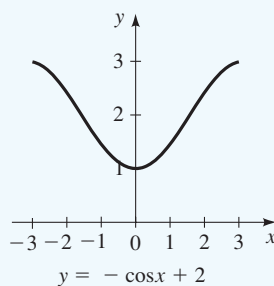
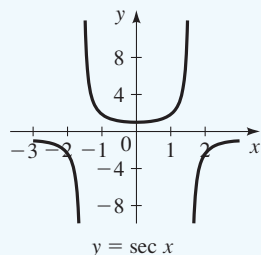
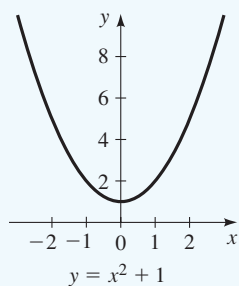
Since $y = \cot x$ completes one period between $x = 0$ and $x = \pi$, the function $y = 2 \cot(3x - \frac{\pi}{2})$ completes one period as $3x - \frac{\pi}{2}$ varies from 0 to π .

Start of period:	End of period:
$3x - \frac{\pi}{2} = 0$	$3x - \frac{\pi}{2} = \pi$
$3x = \frac{\pi}{2}$	$3x = \frac{3\pi}{2}$
$x = \frac{\pi}{6}$	$x = \frac{\pi}{2}$

So we graph one period on the interval $(\frac{\pi}{6}, \frac{\pi}{2})$.

IN-CLASS MATERIALS

If we look at the previously mentioned “U-shaped” functions’ behavior on a larger interval, then their similarity decreases, but some of them would still be hard to tell apart just by looking at the shapes of the curves.



Finally, we graph one period in the shape of cotangent on the interval $(\pi/6, \pi/2)$ and repeat that portion of the graph to the left and to the right. (See Figure 8.)

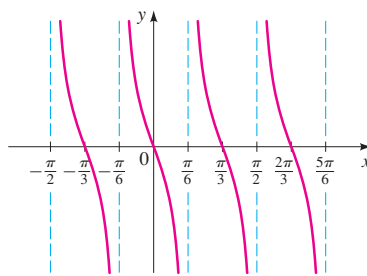


Figure 8

$$y = 2 \cot\left(3x - \frac{\pi}{2}\right)$$

Graphs Involving the Cosecant and Secant Functions

We have already observed that the cosecant and secant functions are the reciprocals of the sine and cosine functions. Thus, the following result is the counterpart of the result for sine and cosine curves in Section 5.3.

Cosecant and Secant Curves

The functions

$$y = a \csc kx \quad \text{and} \quad y = a \sec kx \quad (k > 0)$$

have period $2\pi/k$.

An appropriate interval on which to graph one complete period is $[0, 2\pi/k]$.

Example 4 Graphing Cosecant Curves

Graph each function.

$$(a) y = \frac{1}{2} \csc 2x \quad (b) y = \frac{1}{2} \csc\left(2x + \frac{\pi}{2}\right)$$

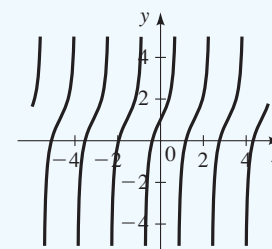
Solution

- (a) The period is $2\pi/2 = \pi$. An appropriate interval is $[0, \pi]$, and the asymptotes occur in this interval whenever $\sin 2x = 0$. So the asymptotes in this interval are $x = 0$, $x = \pi/2$, and $x = \pi$. With this information we sketch on the interval $[0, \pi]$ a graph with the same general shape as that of one period of the cosecant

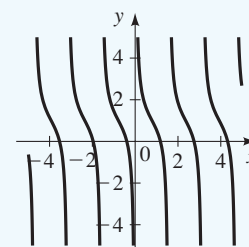
EXAMPLE

Graphs of variants of trigonometric curves:

$y = \tan 2x + 1$ and $y = \cot 2x + 1$ have period $\frac{\pi}{2}$ and vertical shift 1.



$$y = \tan 2x + 1$$



$$y = \cot 2x + 1$$

ALTERNATE EXAMPLE 4a

Let $f(x) = 2 \csc 4x$. Find the period of f , and the equation of its vertical asymptote(s) in the interval

$$\left[0, \frac{\pi}{2}\right].$$

ANSWER

$$\frac{\pi}{2}, x = 0, x = \frac{\pi}{4}, x = \frac{\pi}{2}$$

ALTERNATE EXAMPLE 4b

Find the direction and the value of a horizontal shift of the graph of

the function $y = \frac{1}{4} \tan 2x$ to

obtain the graph of the function

$$y = \frac{1}{4} \tan\left(2x + \frac{\pi}{4}\right).$$

ANSWER

$$\text{Left, } \frac{\pi}{8}$$

IN-CLASS MATERIALS

Similarly, discuss $\tan x$, x^3 , $\sin x$, and $\sinh x$. Notice that, unlike in the previous example (included in the prior In-Class Materials section), these curves don't all look alike at the origin—some are flatter than others.

function. The complete graph in Figure 9(a) is obtained by repeating this portion of the graph to the left and to the right.

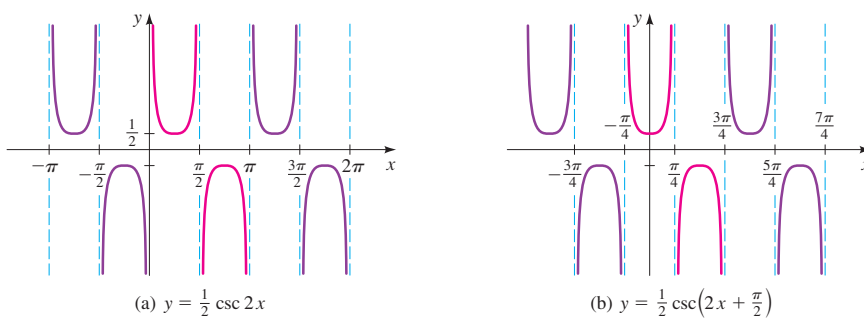


Figure 9

(a) $y = \frac{1}{2} \csc 2x$

(b) $y = \frac{1}{2} \csc\left(2x + \frac{\pi}{2}\right)$

Since $y = \csc x$ completes one period between $x = 0$ and $x = 2\pi$, the function $y = \frac{1}{2} \csc(2x + \frac{\pi}{2})$ completes one period as $2x + \frac{\pi}{2}$ varies from 0 to 2π .

Start of period:	End of period:
$2x + \frac{\pi}{2} = 0$	$2x + \frac{\pi}{2} = 2\pi$
$2x = -\frac{\pi}{2}$	$2x = \frac{3\pi}{2}$
$x = -\frac{\pi}{4}$	$x = \frac{3\pi}{4}$

So we graph one period on the interval $(-\frac{\pi}{4}, \frac{3\pi}{4})$.

(b) We first write

$$y = \frac{1}{2} \csc\left(2x + \frac{\pi}{2}\right) = \frac{1}{2} \csc 2\left(x + \frac{\pi}{4}\right)$$

From this we see that the graph is the same as that in part (a), but shifted to the left $\frac{\pi}{4}$. The graph is shown in Figure 9(b).

ALTERNATE EXAMPLE 5

Let $f(x) = 2 \sec 4x$. Find the period of f , and the equation of its vertical asymptote(s) in the interval

$$\left[0, \frac{\pi}{2}\right].$$

ANSWER

$$\frac{\pi}{4}, x = \frac{\pi}{8}, x = \frac{3\pi}{8}$$

Example 5 Graphing a Secant Curve

Graph $y = 3 \sec \frac{1}{2}x$.

Solution The period is $2\pi \div \frac{1}{2} = 4\pi$. An appropriate interval is $[0, 4\pi]$, and the asymptotes occur in this interval wherever $\cos \frac{1}{2}x = 0$. Thus, the asymptotes in this interval are $x = \pi, x = 3\pi$. With this information we sketch on the interval $[0, 4\pi]$ a graph with the same general shape as that of one period of the secant function. The complete graph in Figure 10 is obtained by repeating this portion of the graph to the left and to the right.

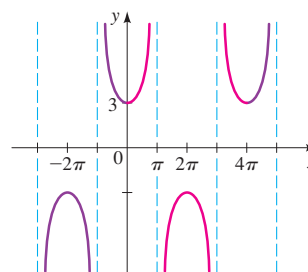
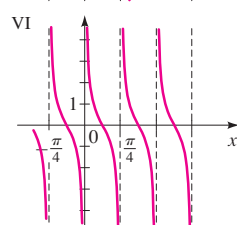
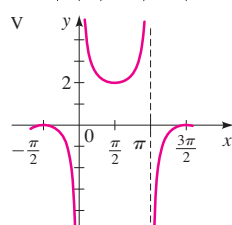
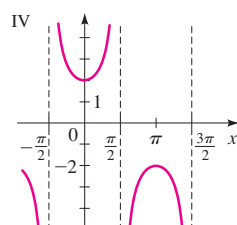
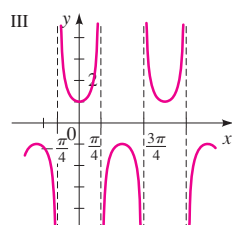
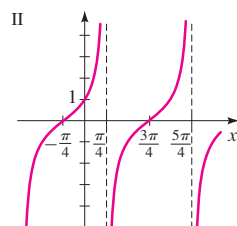
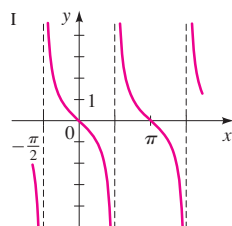


Figure 10
 $y = 3 \sec \frac{1}{2}x$

5.4 Exercises

1–6 ■ Match the trigonometric function with one of the graphs I–VI.

- | | |
|--|------------------------|
| 1. $f(x) = \tan\left(x + \frac{\pi}{4}\right)$ | 2. $f(x) = \sec 2x$ |
| 3. $f(x) = \cot 2x$ | 4. $f(x) = -\tan x$ |
| 5. $f(x) = 2 \sec x$ | 6. $f(x) = 1 + \csc x$ |



7–52 ■ Find the period and graph the function.

- | | |
|--|--|
| 7. $y = 4 \tan x$ | 8. $y = -4 \tan x$ |
| 9. $y = -\frac{1}{2} \tan x$ | 10. $y = \frac{1}{2} \tan x$ |
| 11. $y = -\cot x$ | 12. $y = 2 \cot x$ |
| 13. $y = 2 \csc x$ | 14. $y = \frac{1}{2} \csc x$ |
| 15. $y = 3 \sec x$ | 16. $y = -3 \sec x$ |
| 17. $y = \tan\left(x + \frac{\pi}{2}\right)$ | 18. $y = \tan\left(x - \frac{\pi}{4}\right)$ |
| 19. $y = \csc\left(x - \frac{\pi}{2}\right)$ | 20. $y = \sec\left(x + \frac{\pi}{4}\right)$ |

- | | |
|--|---|
| 21. $y = \cot\left(x + \frac{\pi}{4}\right)$ | 22. $y = 2 \csc\left(x - \frac{\pi}{3}\right)$ |
| 23. $y = \frac{1}{2} \sec\left(x - \frac{\pi}{6}\right)$ | 24. $y = 3 \csc\left(x + \frac{\pi}{2}\right)$ |
| 25. $y = \tan 2x$ | 26. $y = \tan \frac{1}{2}x$ |
| 27. $y = \tan \frac{\pi}{4}x$ | 28. $y = \cot \frac{\pi}{2}x$ |
| 29. $y = \sec 2x$ | 30. $y = 5 \csc 3x$ |
| 31. $y = \csc 2x$ | 32. $y = \csc \frac{1}{2}x$ |
| 33. $y = 2 \tan 3\pi x$ | 34. $y = 2 \tan \frac{\pi}{2}x$ |
| 35. $y = 5 \csc \frac{3\pi}{2}x$ | 36. $y = 5 \sec 2\pi x$ |
| 37. $y = \tan 2\left(x + \frac{\pi}{2}\right)$ | 38. $y = \csc 2\left(x + \frac{\pi}{2}\right)$ |
| 39. $y = \tan 2(x - \pi)$ | 40. $y = \sec 2\left(x - \frac{\pi}{2}\right)$ |
| 41. $y = \cot\left(2x - \frac{\pi}{2}\right)$ | 42. $y = \frac{1}{2} \tan(\pi x - \pi)$ |
| 43. $y = 2 \csc\left(\pi x - \frac{\pi}{3}\right)$ | 44. $y = 2 \sec\left(\frac{1}{2}x - \frac{\pi}{3}\right)$ |
| 45. $y = 5 \sec\left(3x - \frac{\pi}{2}\right)$ | 46. $y = \frac{1}{2} \sec(2\pi x - \pi)$ |
| 47. $y = \tan\left(\frac{2}{3}x - \frac{\pi}{6}\right)$ | 48. $y = \tan \frac{1}{2}\left(x + \frac{\pi}{4}\right)$ |
| 49. $y = 3 \sec \pi\left(x + \frac{1}{2}\right)$ | 50. $y = \sec\left(3x + \frac{\pi}{2}\right)$ |
| 51. $y = -2 \tan\left(2x - \frac{\pi}{3}\right)$ | 52. $y = 2 \csc(3x + 3)$ |

53. (a) Prove that if f is periodic with period p , then $1/f$ is also periodic with period p .

(b) Prove that cosecant and secant each have period 2π .

54. Prove that if f and g are periodic with period p , then f/g is also periodic, but its period could be smaller than p .

Applications

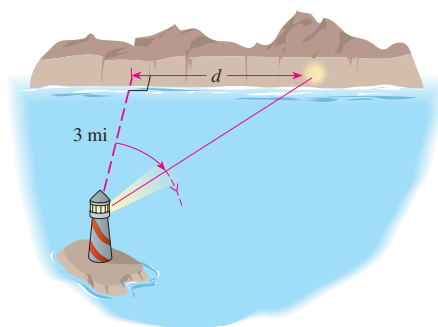
55. **Lighthouse** The beam from a lighthouse completes one rotation every two minutes. At time t , the distance d shown in the figure on the next page is

$$d(t) = 3 \tan \pi t$$

where t is measured in minutes and d in miles.

(a) Find $d(0.15)$, $d(0.25)$, and $d(0.45)$.

- (b) Sketch a graph of the function d for $0 \leq t < \frac{1}{2}$.
 (c) What happens to the distance d as t approaches $\frac{1}{2}$?



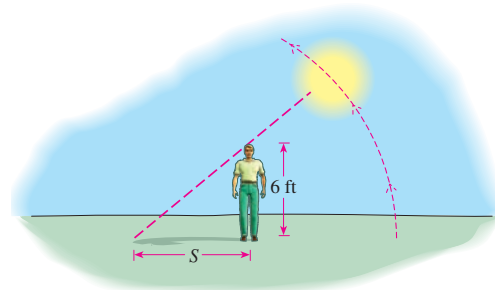
- 56. Length of a Shadow** On a day when the sun passes directly overhead at noon, a six-foot-tall man casts a shadow of length

$$S(t) = 6 \left| \cot \frac{\pi}{12} t \right|$$

where S is measured in feet and t is the number of hours since 6 A.M.

- (a) Find the length of the shadow at 8:00 A.M., noon, 2:00 P.M., and 5:45 P.M.
 (b) Sketch a graph of the function S for $0 < t < 12$.

- (c) From the graph determine the values of t at which the length of the shadow equals the man's height. To what time of day does each of these values correspond?
 (d) Explain what happens to the shadow as the time approaches 6 P.M. (that is, as $t \rightarrow 12^-$).



Discovery • Discussion

- 57. Reduction Formulas** Use the graphs in Figure 5 to explain why the following formulas are true.

$$\tan \left(x - \frac{\pi}{2} \right) = -\cot x$$

$$\sec \left(x - \frac{\pi}{2} \right) = \csc x$$

SUGGESTED TIME AND EMPHASIS

1 class.

Recommended material.

5.5

Modeling Harmonic Motion

Periodic behavior—behavior that repeats over and over again—is common in nature. Perhaps the most familiar example is the daily rising and setting of the sun, which results in the repetitive pattern of day, night, day, night, Another example is the daily variation of tide levels at the beach, which results in the repetitive pattern of high tide, low tide, high tide, low tide, Certain animal populations increase and decrease in a predictable periodic pattern: A large population exhausts the food supply, which causes the population to dwindle; this in turn results in a more plentiful food supply, which makes it possible for the population to increase; and the pattern then repeats over and over (see pages 432–433).

Other common examples of periodic behavior involve motion that is caused by vibration or oscillation. A mass suspended from a spring that has been compressed and then allowed to vibrate vertically is a simple example. This same “back and forth” motion also occurs in such diverse phenomena as sound waves, light waves, alternating electrical current, and pulsating stars, to name a few. In this section we consider the problem of modeling periodic behavior.

POINTS TO STRESS

1. Understanding the meanings of amplitude, period, and frequency, and determining them from the formula $y = a \sin \omega t$ or $y = a \cos \omega t$.
2. Understanding the physical meaning of a phase shift in formulas such as $y = a \sin (\omega(t - c))$ and $y = a \cos (\omega(t - c))$.
3. Damped harmonic motion ($y = ke^{-ct} \sin \omega t$, $y = ke^{-ct} \cos \omega t$).

Modeling Periodic Behavior

The trigonometric functions are ideally suited for modeling periodic behavior. A glance at the graphs of the sine and cosine functions, for instance, tells us that these functions themselves exhibit periodic behavior. Figure 1 shows the graph of $y = \sin t$. If we think of t as time, we see that as time goes on, $y = \sin t$ increases and decreases over and over again. Figure 2 shows that the motion of a vibrating mass on a spring is modeled very accurately by $y = \sin t$.

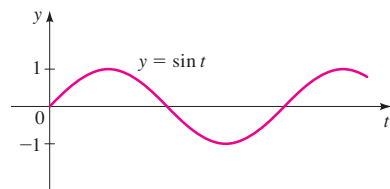


Figure 1
 $y = \sin t$

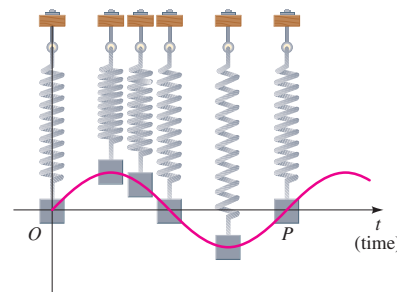


Figure 2
Motion of a vibrating spring is modeled by $y = \sin t$.

Notice that the mass returns to its original position over and over again. A **cycle** is one complete vibration of an object, so the mass in Figure 2 completes one cycle of its motion between O and P . Our observations about how the sine and cosine functions model periodic behavior are summarized in the following box.

Simple Harmonic Motion

If the equation describing the displacement y of an object at time t is

$$y = a \sin \omega t \quad \text{or} \quad y = a \cos \omega t$$

then the object is in **simple harmonic motion**. In this case,

amplitude = $|a|$ *Maximum displacement of the object*

period = $\frac{2\pi}{\omega}$ *Time required to complete one cycle*

frequency = $\frac{\omega}{2\pi}$ *Number of cycles per unit of time*

The main difference between the two equations describing simple harmonic motion is the starting point. At $t = 0$, we get

$$y = a \sin \omega \cdot 0 = 0$$

$$y = a \cos \omega \cdot 0 = a$$

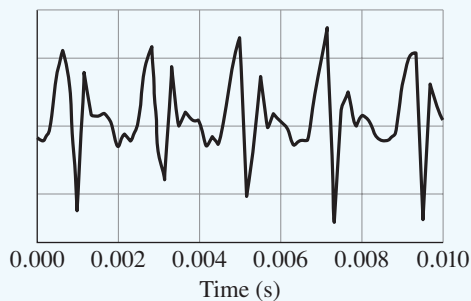
In the first case the motion “starts” with zero displacement, whereas in the second case the motion “starts” with the displacement at maximum (at the amplitude a).

IN-CLASS MATERIALS

Terms used in this section such as amplitude, period, and frequency can be used when describing any sort of periodic function.

IN-CLASS MATERIALS

Musical notes can be thought of as harmonic motion where the t -axis represents time and the y -axis represents air pressure. A sound wave is a periodic variation of air pressure. The amplitude corresponds to volume, and the frequency corresponds to the note; a higher frequency means a higher pitch. A pure sine or cosine wave sounds like a note played on a tuning fork—not a lot of character. The sound wave from a trumpet looks like this:



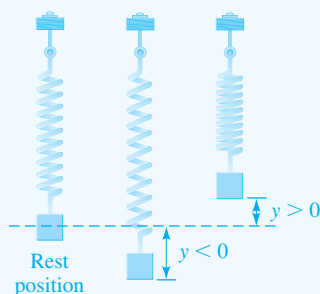
Dept. of Physics, Concordia College, Moorhead, MN

ALTERNATE EXAMPLE 1

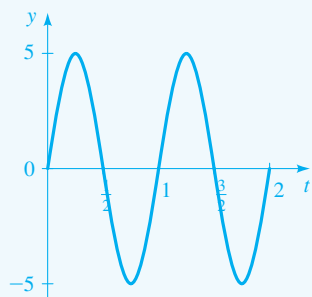
The displacement of a mass suspended by a spring is modeled by the function

$$y = 5 \sin 2\pi t$$

where y is measured in inches and t in seconds.



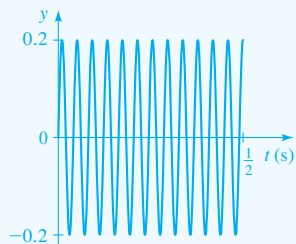
Sketch the graph of the displacement of the mass.

ANSWER**ALTERNATE EXAMPLE 2**

A tuba player plays the note G and sustains the sound for some time. For a pure G the variation in pressure from normal air pressure is given by

$$V(t) = 0.2 \sin 49\pi t$$

where V is measured in pounds per square inch and t in seconds. Sketch a graph of V .

ANSWER

The symbol ω is the lowercase Greek letter “omega,” and ν is the letter “nu.”

Notice that the functions

$$y = a \sin 2\pi\nu t \quad \text{and} \quad y = a \cos 2\pi\nu t$$

have frequency ν , because $2\pi\nu/(2\pi) = \nu$. Since we can immediately read the frequency from these equations, we often write equations of simple harmonic motion in this form.

Example 1 A Vibrating Spring

The displacement of a mass suspended by a spring is modeled by the function

$$y = 10 \sin 4\pi t$$

where y is measured in inches and t in seconds (see Figure 3).

- Find the amplitude, period, and frequency of the motion of the mass.
- Sketch the graph of the displacement of the mass.

Solution

- From the formulas for amplitude, period, and frequency, we get

$$\text{amplitude} = |a| = 10 \text{ in.}$$

$$\text{period} = \frac{2\pi}{\omega} = \frac{2\pi}{4\pi} = \frac{1}{2} \text{ s}$$

$$\text{frequency} = \frac{\omega}{2\pi} = \frac{4\pi}{2\pi} = 2 \text{ Hz}$$

- The graph of the displacement of the mass at time t is shown in Figure 4. ■

An important situation where simple harmonic motion occurs is in the production of sound. Sound is produced by a regular variation in air pressure from the normal pressure. If the pressure varies in simple harmonic motion, then a pure sound is produced. The tone of the sound depends on the frequency and the loudness depends on the amplitude.

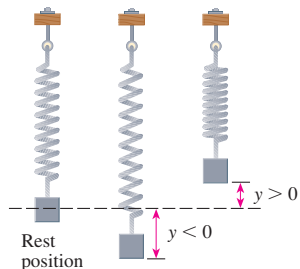
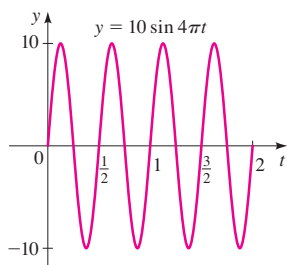
Example 2 Vibrations of a Musical Note

A tuba player plays the note E and sustains the sound for some time. For a pure E the variation in pressure from normal air pressure is given by

$$V(t) = 0.2 \sin 80\pi t$$

where V is measured in pounds per square inch and t in seconds.

- Find the amplitude, period, and frequency of V .
- Sketch a graph of V .
- If the tuba player increases the loudness of the note, how does the equation for V change?
- If the player is playing the note incorrectly and it is a little flat, how does the equation for V change?

**Figure 3****Figure 4****DRILL QUESTION**

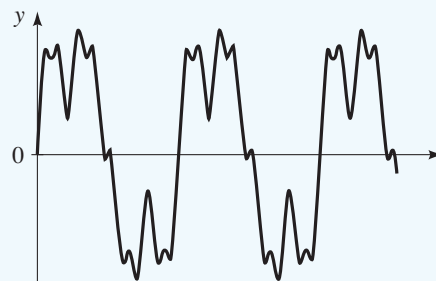
What is the frequency of the harmonic function $y = \sin(6\pi t)$?

Answer

3 Hz

IN-CLASS MATERIALS

Musical instruments produce a basic sound, plus a series of overtones. We can think of the basic sound as a sine wave with a (relatively) large amplitude, and then the overtones as waves whose frequencies are rational multiples of the basic frequency, and whose amplitudes are smaller. We add the sine curves together to get a sound wave that looks something like this:



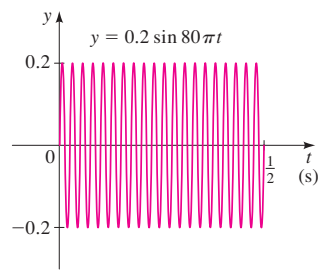


Figure 5

Solution

(a) From the formulas for amplitude, period, and frequency, we get

$$\text{amplitude} = |0.2| = 0.2$$

$$\text{period} = \frac{2\pi}{80\pi} = \frac{1}{40}$$

$$\text{frequency} = \frac{80\pi}{2\pi} = 40$$

(b) The graph of V is shown in Figure 5.

(c) If the player increases the loudness the amplitude increases. So the number 0.2 is replaced by a larger number.

(d) If the note is flat, then the frequency is decreased. Thus, the coefficient of t is less than 80π . ■

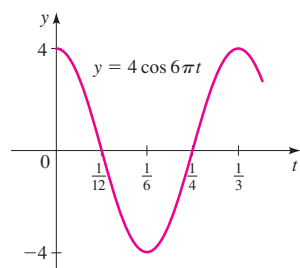
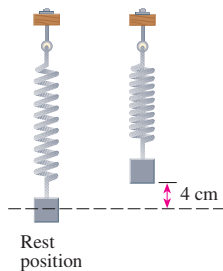


Figure 6

Example 3 Modeling a Vibrating Spring

A mass is suspended from a spring. The spring is compressed a distance of 4 cm and then released. It is observed that the mass returns to the compressed position after $\frac{1}{3}$ s.

(a) Find a function that models the displacement of the mass.

(b) Sketch the graph of the displacement of the mass.

Solution

(a) The motion of the mass is given by one of the equations for simple harmonic motion. The amplitude of the motion is 4 cm. Since this amplitude is reached at time $t = 0$, an appropriate function that models the displacement is of the form

$$y = a \cos \omega t$$

Since the period is $p = \frac{1}{3}$, we can find ω from the following equation:

$$\text{period} = \frac{2\pi}{\omega}$$

$$\frac{1}{3} = \frac{2\pi}{\omega} \quad \text{Period} = \frac{1}{3}$$

$$\omega = 6\pi \quad \text{Solve for } \omega$$

So, the motion of the mass is modeled by the function

$$y = 4 \cos 6\pi t$$

where y is the displacement from the rest position at time t . Notice that when $t = 0$, the displacement is $y = 4$, as we expect.

(b) The graph of the displacement of the mass at time t is shown in Figure 6. ■

SAMPLE QUESTION**Text Question**

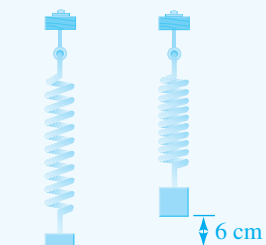
What is the relationship between the period of a harmonic function and its frequency?

Answer

They are reciprocals.

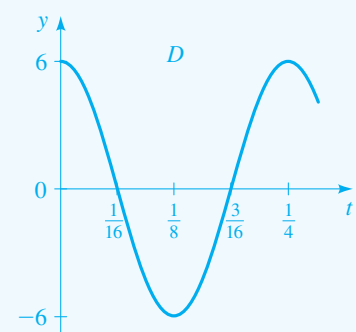
ALTERNATE EXAMPLE 3

A mass is suspended from a spring. The spring is compressed a distance of 6 cm and then released. It is observed that the mass returns to the compressed position after $\frac{1}{4}$ s.



Rest position

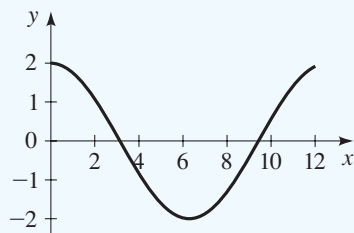
Sketch the graph of the displacement of the mass.

ANSWER**IN-CLASS MATERIALS**

An easy way to plot a trigonometric function given an amplitude and period (no phase shift) is to plot the zeros of one cycle, followed by the maximum and the minimum (between the zeros), and then draw the graph. This method also works with a phase shift, but is not as straightforward.

EXAMPLE

$$y = 4 \sin\left(\pi\left(x - \frac{1}{4}\right)\right)$$

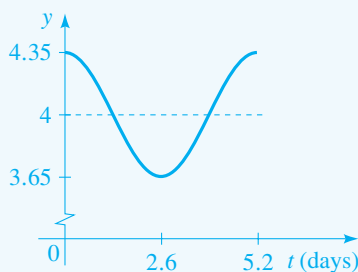


Period = 2, frequency = $\frac{1}{2}$,
amplitude = 4

ALTERNATE EXAMPLE 4

A variable star is one whose brightness alternately increases and decreases. For the variable star Eta Pi, the time between periods of maximum brightness is 5.2 days. The average brightness (or magnitude) of the star is 4.0, and its brightness varies by ± 0.35 magnitude.

Sketch a graph of the brightness of Eta Pi as a function of time.

ANSWER

In general, the sine or cosine functions representing harmonic motion may be shifted horizontally or vertically. In this case, the equations take the form

$$y = a \sin(\omega(t - c)) + b \quad \text{or} \quad y = a \cos(\omega(t - c)) + b$$

The vertical shift b indicates that the variation occurs around an average value b . The horizontal shift c indicates the position of the object at $t = 0$. (See Figure 7.)

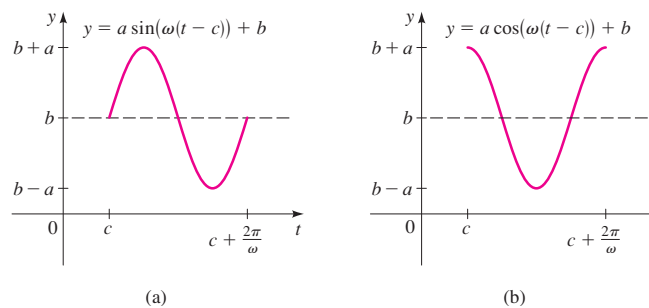


Figure 7

Example 4 Modeling the Brightness of a Variable Star

A variable star is one whose brightness alternately increases and decreases. For the variable star Delta Cephei, the time between periods of maximum brightness is 5.4 days. The average brightness (or magnitude) of the star is 4.0, and its brightness varies by ± 0.35 magnitude.

- Find a function that models the brightness of Delta Cephei as a function of time.
- Sketch a graph of the brightness of Delta Cephei as a function of time.

Solution

- Let's find a function in the form

$$y = a \cos(\omega(t - c)) + b$$

The amplitude is the maximum variation from average brightness, so the amplitude is $a = 0.35$ magnitude. We are given that the period is 5.4 days, so

$$\omega = \frac{2\pi}{5.4} \approx 1.164$$

Since the brightness varies from an average value of 4.0 magnitudes, the graph is shifted upward by $b = 4.0$. If we take $t = 0$ to be a time when the star is at maximum brightness, there is no horizontal shift, so $c = 0$ (because a cosine curve achieves its maximum at $t = 0$). Thus, the function we want is

$$y = 0.35 \cos(1.16t) + 4.0$$

where t is the number of days from a time when the star is at maximum brightness.

- The graph is sketched in Figure 8. ■

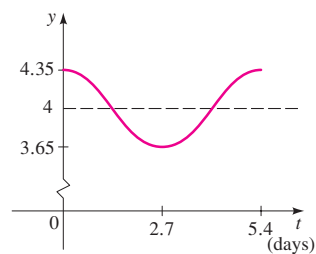
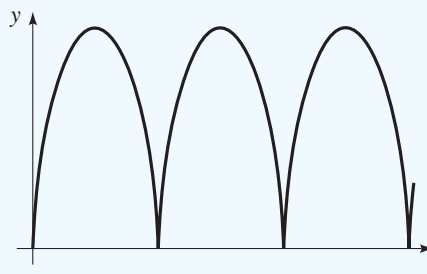


Figure 8

IN-CLASS MATERIALS

Note that not all periodic functions are trigonometric functions. For example, mark a point on a car tire and plot its distance from the ground versus time as the car moves. The resulting graph is called a cycloid and looks like this:



The number of hours of daylight varies throughout the course of a year. In the Northern Hemisphere, the longest day is June 21, and the shortest is December 21. The average length of daylight is 12 h, and the variation from this average depends on the latitude. (For example, Fairbanks, Alaska, experiences more than 20 h of daylight on the longest day and less than 4 h on the shortest day!) The graph in Figure 9 shows the number of hours of daylight at different times of the year for various latitudes. It's apparent from the graph that the variation in hours of daylight is simple harmonic.

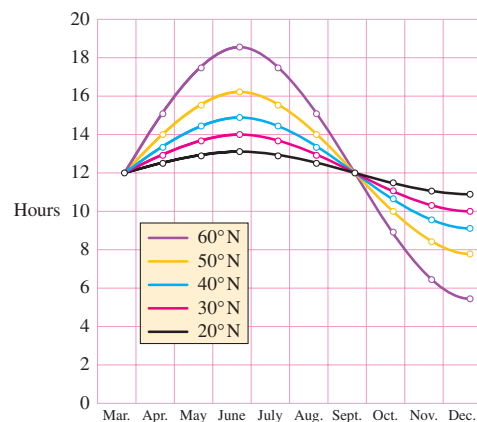


Figure 9

Graph of the length of daylight from March 21 through December 21 at various latitudes

Source: Lucia C. Harrison, *Daylight, Twilight, Darkness and Time* (New York: Silver, Burdett, 1935), page 40.

Example 5 Modeling the Number of Hours of Daylight



In Philadelphia (40°N latitude), the longest day of the year has 14 h 50 min of daylight and the shortest day has 9 h 10 min of daylight.

- Find a function L that models the length of daylight as a function of t , the number of days from January 1.
- An astronomer needs at least 11 hours of darkness for a long exposure astronomical photograph. On what days of the year are such long exposures possible?

Solution

- We need to find a function in the form

$$y = a \sin(\omega(t - c)) + b$$

whose graph is the 40°N latitude curve in Figure 9. From the information given, we see that the amplitude is

$$a = \frac{1}{2} \left(14\frac{5}{6} - 9\frac{1}{6} \right) \approx 2.83 \text{ h}$$

Since there are 365 days in a year, the period is 365, so

$$\omega = \frac{2\pi}{365} \approx 0.0172$$

ALTERNATE EXAMPLE 5a

In Winnipeg, Manitoba (50°N latitude), the longest day of the year has 16 h 7 min of daylight, and the shortest day of the year has 7 h 53 min of daylight.

Find a function L that models the length of daylight as a function of t , the number of days since January 1.

ANSWER

Amplitude: 4.12 h

$\omega \approx 0.0172$

$b = 12$

$c = 80$

$y = 4.12 \sin(0.0172(t - 80)) + 12$

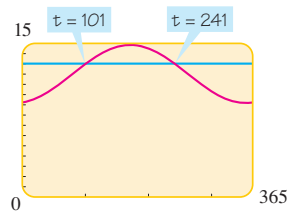


Figure 10

Since the average length of daylight is 12 h, the graph is shifted upward by 12, so $b = 12$. Since the curve attains the average value (12) on March 21, the 80th day of the year, the curve is shifted 80 units to the right. Thus, $c = 80$. So a function that models the number of hours of daylight is

$$y = 2.83 \sin(0.0172(t - 80)) + 12$$

where t is the number of days from January 1.

- (b) A day has 24 h, so 11 h of night correspond to 13 h of daylight. So we need to solve the inequality $y \leq 13$. To solve this inequality graphically, we graph $y = 2.83 \sin 0.0172(t - 80) + 12$ and $y = 13$ on the same graph. From the graph in Figure 10 we see that there are fewer than 13 h of daylight between day 1 (January 1) and day 101 (April 11) and from day 241 (August 29) to day 365 (December 31).

Another situation where simple harmonic motion occurs is in alternating current (AC) generators. Alternating current is produced when an armature rotates about its axis in a magnetic field.

Figure 11 represents a simple version of such a generator. As the wire passes through the magnetic field, a voltage E is generated in the wire. It can be shown that the voltage generated is given by

$$E(t) = E_0 \cos \omega t$$

where E_0 is the maximum voltage produced (which depends on the strength of the magnetic field) and $\omega/(2\pi)$ is the number of revolutions per second of the armature (the frequency).

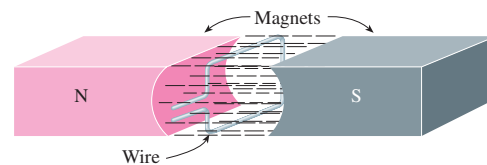


Figure 11

Why do we say that household current is 110 V when the maximum voltage produced is 155 V? From the symmetry of the cosine function, we see that the average voltage produced is zero. This average value would be the same for all AC generators and so gives no information about the voltage generated. To obtain a more informative measure of voltage, engineers use the **root-mean-square** (rms) method. It can be shown that the rms voltage is $1/\sqrt{2}$ times the maximum voltage. So, for household current the rms voltage is

$$155 \times \frac{1}{\sqrt{2}} \approx 110 \text{ V}$$

ALTERNATE EXAMPLE 6

An alternating current varies from +135 V to -135 V with a frequency of 70 Hz (cycles per second). Find an equation that describes this variation in voltage.

(**HINT:** Take $t = 0$ to be a time when the voltage is +135 V.)

The equation is

$$E(t) = \underline{\hspace{2cm}}$$

ANSWER

$$E(t) = 135 \cos 140\pi t$$

Example 6 Modeling Alternating Current

Ordinary 110-V household alternating current varies from +155 V to -155 V with a frequency of 60 Hz (cycles per second). Find an equation that describes this variation in voltage.

Solution The variation in voltage is simple harmonic. Since the frequency is 60 cycles per second, we have

$$\frac{\omega}{2\pi} = 60 \quad \text{or} \quad \omega = 120\pi$$

Let's take $t = 0$ to be a time when the voltage is +155 V. Then

$$E(t) = a \cos \omega t = 155 \cos 120\pi t$$

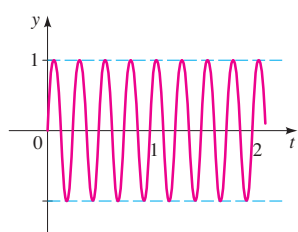
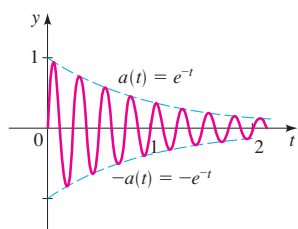
(a) Harmonic motion: $y = \sin 8\pi t$ (b) Damped harmonic motion:
 $y = e^{-t} \sin 8\pi t$

Figure 12

Hz is the abbreviation for hertz. One hertz is one cycle per second.

Damped Harmonic Motion

The spring in Figure 2 on page 443 is assumed to oscillate in a frictionless environment. In this hypothetical case, the amplitude of the oscillation will not change. In the presence of friction, however, the motion of the spring eventually “dies down”; that is, the amplitude of the motion decreases with time. Motion of this type is called *damped harmonic motion*.

Damped Harmonic Motion

If the equation describing the displacement y of an object at time t is

$$y = ke^{-ct} \sin \omega t \quad \text{or} \quad y = ke^{-ct} \cos \omega t \quad (c > 0)$$

then the object is in **damped harmonic motion**. The constant c is the **damping constant**, k is the initial amplitude, and $2\pi/\omega$ is the period.*

Damped harmonic motion is simply harmonic motion for which the amplitude is governed by the function $a(t) = ke^{-ct}$. Figure 12 shows the difference between harmonic motion and damped harmonic motion.

Example 7 Modeling Damped Harmonic Motion

Two mass-spring systems are experiencing damped harmonic motion, both at 0.5 cycles per second, and both with an initial maximum displacement of 10 cm. The first has a damping constant of 0.5 and the second has a damping constant of 0.1.

- Find functions of the form $g(t) = ke^{-ct} \cos \omega t$ to model the motion in each case.
- Graph the two functions you found in part (a). How do they differ?

Solution

- At time $t = 0$, the displacement is 10 cm. Thus $g(0) = ke^{-c \cdot 0} \cos(\omega \cdot 0) = k$, and so $k = 10$. Also, the frequency is $f = 0.5$ Hz, and since $\omega = 2\pi f$ (see page 443), we get $\omega = 2\pi(0.5) = \pi$. Using the given damping constants, we find that the motions of the two springs are given by the functions

$$g_1(t) = 10e^{-0.5t} \cos \pi t \quad \text{and} \quad g_2(t) = 10e^{-0.1t} \cos \pi t$$

- The functions g_1 and g_2 are graphed in Figure 13. From the graphs we see that in the first case (where the damping constant is larger) the motion dies down quickly, whereas in the second case, perceptible motion continues much longer.

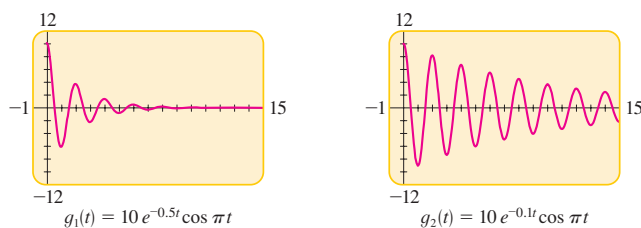


Figure 13

*In the case of damped harmonic motion, the term *quasi-period* is often used instead of *period* because the motion is not actually periodic—it diminishes with time. However, we will continue to use the term *period* to avoid confusion.

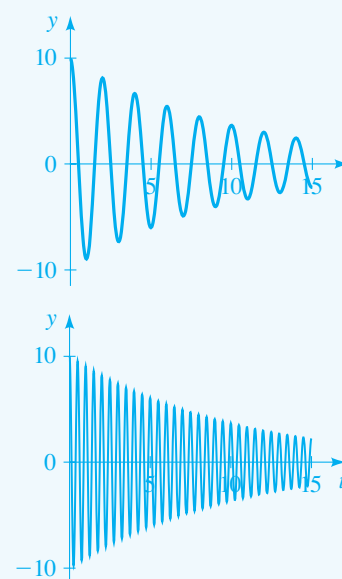
ALTERNATE EXAMPLE 7

Two mass-spring systems are experiencing damped harmonic motion with an initial maximum displacement of 10 cm. They both have damping constant 0.1. One is oscillating at 0.5 cycles per second, and the other is oscillating at 2 cycles per second.

- Find functions of the form $g(t) = ke^{-ct} \cos(\omega t)$ to model the motion in each case.
- Graph the two functions you found in part (a) and notice how they differ.

ANSWERS

- $g(t) = 10e^{-0.1t} \cos(\pi t)$ and $g(t) = 10e^{-0.1t} \cos(4\pi t)$
-



ALTERNATE EXAMPLE 8

A C string on a harp is pulled a distance of 0.25 cm from its rest position, then released and allowed to vibrate. The damping constant c for this string is determined to be 1.25. Find an equation that describes the motion of the point at which it was plucked. (The frequency of C is 256 Hz.)

ANSWER

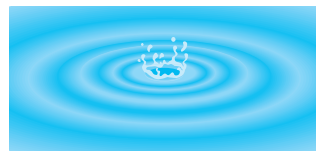
$$f(t) = 0.25e^{-1.25t} \cos(512\pi t)$$

ALTERNATE EXAMPLE 9

A stone is dropped in a calm lake, as in Example 9. After 30 seconds, the amplitude has been reduced by 80%. Find the damping constant c .

ANSWER

$$c = \ln(5)/30 \approx 0.0536$$



As the preceding example indicates, the larger the damping constant c , the quicker the oscillation dies down. When a guitar string is plucked and then allowed to vibrate freely, a point on that string undergoes damped harmonic motion. We hear the damping of the motion as the sound produced by the vibration of the string fades. How fast the damping of the string occurs (as measured by the size of the constant c) is a property of the size of the string and the material it is made of. Another example of damped harmonic motion is the motion that a shock absorber on a car undergoes when the car hits a bump in the road. In this case, the shock absorber is engineered to damp the motion as quickly as possible (large c) and to have the frequency as small as possible (small ω). On the other hand, the sound produced by a tuba player playing a note is undamped as long as the player can maintain the loudness of the note. The electromagnetic waves that produce light move in simple harmonic motion that is not damped.

Example 8 A Vibrating Violin String

The G-string on a violin is pulled a distance of 0.5 cm above its rest position, then released and allowed to vibrate. The damping constant c for this string is determined to be 1.4. Suppose that the note produced is a pure G (frequency = 200 Hz). Find an equation that describes the motion of the point at which the string was plucked.

Solution Let P be the point at which the string was plucked. We will find a function $f(t)$ that gives the distance at time t of the point P from its original rest position. Since the maximum displacement occurs at $t = 0$, we find an equation in the form

$$y = ke^{-ct} \cos \omega t$$

From this equation, we see that $f(0) = k$. But we know that the original displacement of the string is 0.5 cm. Thus, $k = 0.5$. Since the frequency of the vibration is 200, we have $\omega = 2\pi f = 2\pi(200) = 400\pi$. Finally, since we know that the damping constant is 1.4, we get

$$f(t) = 0.5e^{-1.4t} \cos 400\pi t \quad \blacksquare$$

Example 9 Ripples on a Pond

A stone is dropped in a calm lake, causing waves to form. The up-and-down motion of a point on the surface of the water is modeled by damped harmonic motion. At some time the amplitude of the wave is measured, and 20 s later it is found that the amplitude has dropped to $\frac{1}{10}$ of this value. Find the damping constant c .

Solution The amplitude is governed by the coefficient ke^{-ct} in the equations for damped harmonic motion. Thus, the amplitude at time t is ke^{-ct} , and 20 s later, it is $ke^{-c(t+20)}$. So, because the later value is $\frac{1}{10}$ the earlier value, we have

$$ke^{-c(t+20)} = \frac{1}{10}ke^{-ct}$$

We now solve this equation for c . Canceling k and using the Laws of Exponents, we get

$$e^{-ct} \cdot e^{-20c} = \frac{1}{10}e^{-ct}$$

$$e^{-20c} = \frac{1}{10}$$

Cancel e^{-ct}

$$e^{20c} = 10$$

Take reciprocals

Taking the natural logarithm of each side gives

$$20c = \ln(10)$$

$$c = \frac{1}{20} \ln(10) \approx \frac{1}{20}(2.30) \approx 0.12$$

Thus, the damping constant is $c \approx 0.12$. ■

5.5 Exercises

1–8 ■ The given function models the displacement of an object moving in simple harmonic motion.

- (a) Find the amplitude, period, and frequency of the motion.
 (b) Sketch a graph of the displacement of the object over one complete period.

1. $y = 2 \sin 3t$ 2. $y = 3 \cos \frac{1}{2}t$
 3. $y = -\cos 0.3t$ 4. $y = 2.4 \sin 3.6t$
 5. $y = -0.25 \cos\left(1.5t - \frac{\pi}{3}\right)$ 6. $y = -\frac{3}{2} \sin(0.2t + 1.4)$
 7. $y = 5 \cos\left(\frac{2}{3}t + \frac{3}{4}\right)$ 8. $y = 1.6 \sin(t - 1.8)$

9–12 ■ Find a function that models the simple harmonic motion having the given properties. Assume that the displacement is zero at time $t = 0$.

9. amplitude 10 cm, period 3 s
 10. amplitude 24 ft, period 2 min
 11. amplitude 6 in., frequency $5/\pi$ Hz
 12. amplitude 1.2 m, frequency 0.5 Hz

13–16 ■ Find a function that models the simple harmonic motion having the given properties. Assume that the displacement is at its maximum at time $t = 0$.

13. amplitude 60 ft, period 0.5 min
 14. amplitude 35 cm, period 8 s
 15. amplitude 2.4 m, frequency 750 Hz
 16. amplitude 6.25 in., frequency 60 Hz

17–24 ■ An initial amplitude k , damping constant c , and frequency f or period p are given. (Recall that frequency and period are related by the equation $f = 1/p$.)

- (a) Find a function that models the damped harmonic motion. Use a function of the form $y = ke^{-ct} \cos \omega t$ in Exercises 17–20, and of the form $y = ke^{-ct} \sin \omega t$ in Exercises 21–24.
 (b) Graph the function.
17. $k = 2$, $c = 1.5$, $f = 3$
 18. $k = 15$, $c = 0.25$, $f = 0.6$

19. $k = 100$, $c = 0.05$, $p = 4$
 20. $k = 0.75$, $c = 3$, $p = 3\pi$
 21. $k = 7$, $c = 10$, $p = \pi/6$
 22. $k = 1$, $c = 1$, $p = 1$
 23. $k = 0.3$, $c = 0.2$, $f = 20$
 24. $k = 12$, $c = 0.01$, $f = 8$

Applications

25. A Bobbing Cork A cork floating in a lake is bobbing in simple harmonic motion. Its displacement above the bottom of the lake is modeled by

$$y = 0.2 \cos 20\pi t + 8$$

where y is measured in meters and t is measured in minutes.

- (a) Find the frequency of the motion of the cork.
 (b) Sketch a graph of y .
 (c) Find the maximum displacement of the cork above the lake bottom.

26. FM Radio Signals The carrier wave for an FM radio signal is modeled by the function

$$y = a \sin(2\pi(9.15 \times 10^7)t)$$

where t is measured in seconds. Find the period and frequency of the carrier wave.

27. Predator Population Model In a predator/prey model (see page 432), the predator population is modeled by the function

$$y = 900 \cos 2t + 8000$$

where t is measured in years.

- (a) What is the maximum population?
 (b) Find the length of time between successive periods of maximum population.

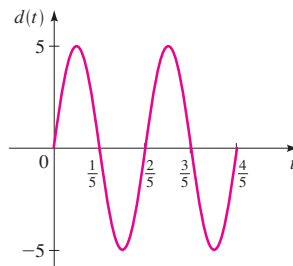
28. Blood Pressure Each time your heart beats, your blood pressure increases, then decreases as the heart rests between beats. A certain person's blood pressure is modeled by the function

$$p(t) = 115 + 25 \sin(160\pi t)$$

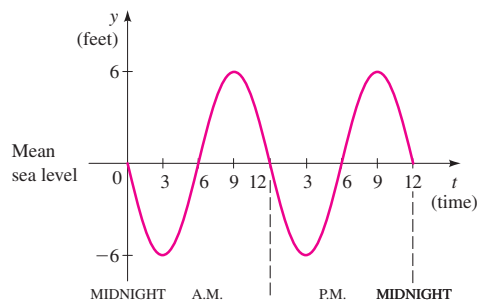
where $p(t)$ is the pressure in mmHg at time t , measured in minutes.

- Find the amplitude, period, and frequency of p .
- Sketch a graph of p .
- If a person is exercising, his heart beats faster. How does this affect the period and frequency of p ?

- 29. Spring–Mass System** A mass attached to a spring is moving up and down in simple harmonic motion. The graph gives its displacement $d(t)$ from equilibrium at time t . Express the function d in the form $d(t) = a \sin \omega t$.



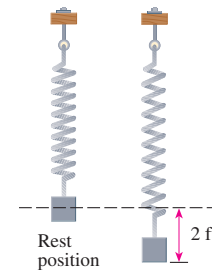
- 30. Tides** The graph shows the variation of the water level relative to mean sea level in Commencement Bay at Tacoma, Washington, for a particular 24-hour period. Assuming that this variation is modeled by simple harmonic motion, find an equation of the form $y = a \sin \omega t$ that describes the variation in water level as a function of the number of hours after midnight.



- 31. Tides** The Bay of Fundy in Nova Scotia has the highest tides in the world. In one 12-hour period the water starts at mean sea level, rises to 21 ft above, drops to 21 ft below, then returns to mean sea level. Assuming that the motion of the tides is simple harmonic, find an equation that describes the height of the tide in the Bay of Fundy above

mean sea level. Sketch a graph that shows the level of the tides over a 12-hour period.

- 32. Spring–Mass System** A mass suspended from a spring is pulled down a distance of 2 ft from its rest position, as shown in the figure. The mass is released at time $t = 0$ and allowed to oscillate. If the mass returns to this position after 1 s, find an equation that describes its motion.



- 33. Spring–Mass System** A mass is suspended on a spring. The spring is compressed so that the mass is located 5 cm above its rest position. The mass is released at time $t = 0$ and allowed to oscillate. It is observed that the mass reaches its lowest point $\frac{1}{2}$ s after it is released. Find an equation that describes the motion of the mass.

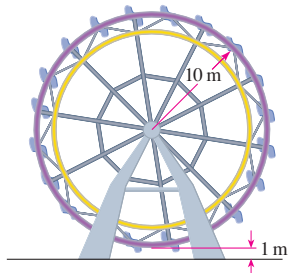
- 34. Spring–Mass System** The frequency of oscillation of an object suspended on a spring depends on the stiffness k of the spring (called the *spring constant*) and the mass m of the object. If the spring is compressed a distance a and then allowed to oscillate, its displacement is given by

$$f(t) = a \cos \sqrt{k/m} t$$

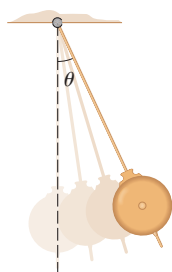
- A 10-g mass is suspended from a spring with stiffness $k = 3$. If the spring is compressed a distance 5 cm and then released, find the equation that describes the oscillation of the spring.
- Find a general formula for the frequency (in terms of k and m).
- How is the frequency affected if the mass is increased? Is the oscillation faster or slower?
- How is the frequency affected if a stiffer spring is used (larger k)? Is the oscillation faster or slower?

- 35. Ferris Wheel** A ferris wheel has a radius of 10 m, and the bottom of the wheel passes 1 m above the ground. If the ferris wheel makes one complete revolution every 20 s, find an

equation that gives the height above the ground of a person on the ferris wheel as a function of time.



- 36. Clock Pendulum** The pendulum in a grandfather clock makes one complete swing every 2 s. The maximum angle that the pendulum makes with respect to its rest position is 10° . We know from physical principles that the angle θ between the pendulum and its rest position changes in simple harmonic fashion. Find an equation that describes the size of the angle θ as a function of time. (Take $t = 0$ to be a time when the pendulum is vertical.)



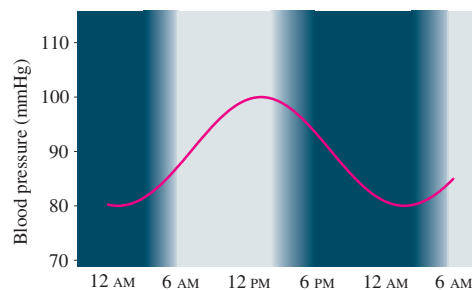
- 37. Variable Stars** The variable star Zeta Gemini has a period of 10 days. The average brightness of the star is 3.8 magnitudes, and the maximum variation from the average is 0.2 magnitude. Assuming that the variation in brightness is simple harmonic, find an equation that gives the brightness of the star as a function of time.
- 38. Variable Stars** Astronomers believe that the radius of a variable star increases and decreases with the brightness of the star. The variable star Delta Cephei (Example 4) has an average radius of 20 million miles and changes by a maximum of 1.5 million miles from this average during a single pulsation. Find an equation that describes the radius of this star as a function of time.
- 39. Electric Generator** The armature in an electric generator is rotating at the rate of 100 revolutions per

second (rps). If the maximum voltage produced is 310 V, find an equation that describes this variation in voltage. What is the rms voltage? (See Example 6 and the margin note adjacent to it.)

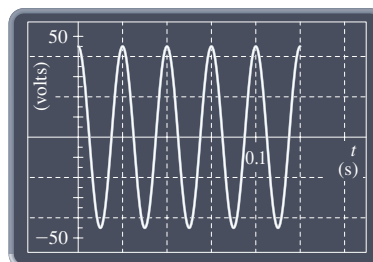
- 40. Biological Clocks** *Circadian rhythms* are biological processes that oscillate with a period of approximately 24 hours. That is, a circadian rhythm is an internal daily biological clock. Blood pressure appears to follow such a rhythm. For a certain individual the average resting blood pressure varies from a maximum of 100 mmHg at 2:00 P.M. to a minimum of 80 mmHg at 2:00 A.M. Find a sine function of the form

$$f(t) = a \sin(\omega(t - c)) + b$$

that models the blood pressure at time t , measured in hours from midnight.



- 41. Electric Generator** The graph shows an oscilloscope reading of the variation in voltage of an AC current produced by a simple generator.
- Find the maximum voltage produced.
 - Find the frequency (cycles per second) of the generator.
 - How many revolutions per second does the armature in the generator make?
 - Find a formula that describes the variation in voltage as a function of time.



- 42. Doppler Effect** When a car with its horn blowing drives by an observer, the pitch of the horn seems higher as it approaches and lower as it recedes (see the figure). This phenomenon is called the **Doppler effect**. If the sound source is moving at speed v relative to the observer and if the speed of sound is v_0 , then the perceived frequency f is related to the actual frequency f_0 as follows:

$$f = f_0 \left(\frac{v_0}{v_0 \pm v} \right)$$

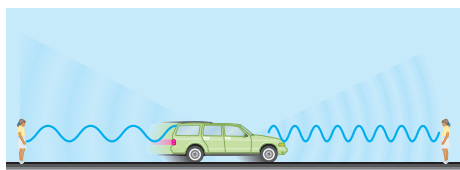
We choose the minus sign if the source is moving toward the observer and the plus sign if it is moving away.

Suppose that a car drives at 110 ft/s past a woman standing on the shoulder of a highway, blowing its horn, which has a frequency of 500 Hz. Assume that the speed of sound is 1130 ft/s. (This is the speed in dry air at 70° F.)

- (a) What are the frequencies of the sounds that the woman hears as the car approaches her and as it moves away from her?
 (b) Let A be the amplitude of the sound. Find functions of the form

$$y = A \sin \omega t$$

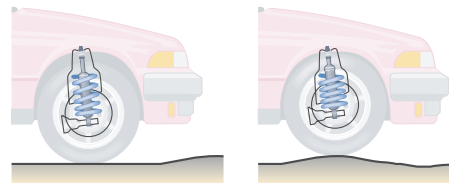
that model the perceived sound as the car approaches the woman and as it recedes.



- 43. Motion of a Building** A strong gust of wind strikes a tall building, causing it to sway back and forth in damped harmonic motion. The frequency of the oscillation is 0.5 cycle per second and the damping constant is $c = 0.9$. Find an equation that describes the motion of the building. (Assume

$k = 1$ and take $t = 0$ to be the instant when the gust of wind strikes the building.)

- 44. Shock Absorber** When a car hits a certain bump on the road, a shock absorber on the car is compressed a distance of 6 in., then released (see the figure). The shock absorber vibrates in damped harmonic motion with a frequency of 2 cycles per second. The damping constant for this particular shock absorber is 2.8.
- (a) Find an equation that describes the displacement of the shock absorber from its rest position as a function of time. Take $t = 0$ to be the instant that the shock absorber is released.
 (b) How long does it take for the amplitude of the vibration to decrease to 0.5 in?



- 45. Tuning Fork** A tuning fork is struck and oscillates in damped harmonic motion. The amplitude of the motion is measured, and 3 s later it is found that the amplitude has dropped to $\frac{1}{4}$ of this value. Find the damping constant c for this tuning fork.
- 46. Guitar String** A guitar string is pulled at point P a distance of 3 cm above its rest position. It is then released and vibrates in damped harmonic motion with a frequency of 165 cycles per second. After 2 s, it is observed that the amplitude of the vibration at point P is 0.6 cm.
- (a) Find the damping constant c .
 (b) Find an equation that describes the position of point P above its rest position as a function of time. Take $t = 0$ to be the instant that the string is released.

5 Review

Concept Check

1. (a) What is the unit circle?
 (b) Use a diagram to explain what is meant by the terminal point determined by a real number t .
 (c) What is the reference number \bar{t} associated with t ?
 (d) If t is a real number and $P(x, y)$ is the terminal point determined by t , write equations that define $\sin t$, $\cos t$, $\tan t$, $\cot t$, $\sec t$, and $\csc t$.

- (e) What are the domains of the six functions that you defined in part (d)?
- (f) Which trigonometric functions are positive in quadrants I, II, III, and IV?
2. (a) What is an even function?
(b) Which trigonometric functions are even?
(c) What is an odd function?
(d) Which trigonometric functions are odd?
3. (a) State the reciprocal identities.
(b) State the Pythagorean identities.
4. (a) What is a periodic function?
(b) What are the periods of the six trigonometric functions?
5. Graph the sine and cosine functions. How is the graph of cosine related to the graph of sine?
6. Write expressions for the amplitude, period, and phase shift of the sine curve $y = a \sin k(x - b)$ and the cosine curve $y = a \cos k(x - b)$.
7. (a) Graph the tangent and cotangent functions.
(b) State the periods of the tangent curve $y = a \tan kx$ and the cotangent curve $y = a \cot kx$.
8. (a) Graph the secant and cosecant functions.
(b) State the periods of the secant curve $y = a \sec kx$ and the cosecant curve $y = a \csc kx$.
9. (a) What is simple harmonic motion?
(b) What is damped harmonic motion?
(c) Give three real-life examples of simple harmonic motion and of damped harmonic motion.

Exercises

1–2 ■ A point $P(x, y)$ is given.

- (a) Show that P is on the unit circle.
(b) Suppose that P is the terminal point determined by t . Find $\sin t$, $\cos t$, and $\tan t$.

1. $P\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ 2. $P\left(\frac{3}{5}, -\frac{4}{5}\right)$

3–6 ■ A real number t is given.

- (a) Find the reference number for t .
(b) Find the terminal point $P(x, y)$ on the unit circle determined by t .
(c) Find the six trigonometric functions of t .

3. $t = \frac{2\pi}{3}$ 4. $t = \frac{5\pi}{3}$

5. $t = -\frac{11\pi}{4}$ 6. $t = -\frac{7\pi}{6}$

7–16 ■ Find the value of the trigonometric function. If possible, give the exact value; otherwise, use a calculator to find an approximate value correct to five decimal places.

7. (a) $\sin \frac{3\pi}{4}$ (b) $\cos \frac{3\pi}{4}$

8. (a) $\tan \frac{\pi}{3}$ (b) $\tan\left(-\frac{\pi}{3}\right)$

9. (a) $\sin 1.1$ (b) $\cos 1.1$

10. (a) $\cos \frac{\pi}{5}$ (b) $\cos\left(-\frac{\pi}{5}\right)$

11. (a) $\cos \frac{9\pi}{2}$ (b) $\sec \frac{9\pi}{2}$

12. (a) $\sin \frac{\pi}{7}$ (b) $\csc \frac{\pi}{7}$

13. (a) $\tan \frac{5\pi}{2}$ (b) $\cot \frac{5\pi}{2}$

14. (a) $\sin 2\pi$ (b) $\csc 2\pi$

15. (a) $\tan \frac{5\pi}{6}$ (b) $\cot \frac{5\pi}{6}$

16. (a) $\cos \frac{\pi}{3}$ (b) $\sin \frac{\pi}{6}$

17–20 ■ Use the fundamental identities to write the first expression in terms of the second.

17. $\frac{\tan t}{\cos t}$, $\sin t$ 18. $\tan^2 t \sec t$, $\cos t$

19. $\tan t$, $\sin t$; t in quadrant IV

20. $\sec t$, $\sin t$; t in quadrant II

21–24 ■ Find the values of the remaining trigonometric functions at t from the given information.

21. $\sin t = \frac{5}{13}$, $\cos t = -\frac{12}{13}$

22. $\sin t = -\frac{1}{2}$, $\cos t > 0$

23. $\cot t = -\frac{1}{2}$, $\csc t = \sqrt{5}/2$

24. $\cos t = -\frac{3}{5}$, $\tan t < 0$

25. If $\tan t = \frac{1}{4}$ and the terminal point for t is in quadrant III, find $\sec t + \cot t$.
26. If $\sin t = -\frac{8}{17}$ and the terminal point for t is in quadrant IV, find $\csc t + \sec t$.
27. If $\cos t = \frac{3}{5}$ and the terminal point for t is in quadrant I, find $\tan t + \sec t$.
28. If $\sec t = -5$ and the terminal point for t is in quadrant II, find $\sin^2 t + \cos^2 t$.

29–36 ■ A trigonometric function is given.

- (a) Find the amplitude, period, and phase shift of the function.
- (b) Sketch the graph.

29. $y = 10 \cos \frac{1}{2}x$

30. $y = 4 \sin 2\pi x$

31. $y = -\sin \frac{1}{2}x$

32. $y = 2 \sin \left(x - \frac{\pi}{4}\right)$

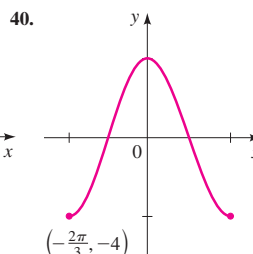
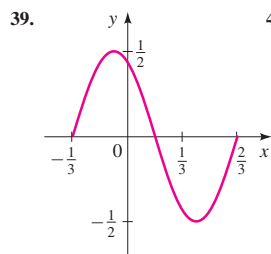
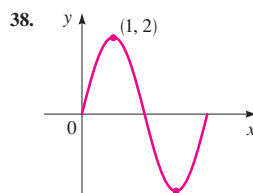
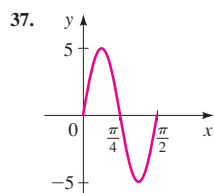
33. $y = 3 \sin(2x - 2)$

34. $y = \cos 2 \left(x - \frac{\pi}{2}\right)$

35. $y = -\cos \left(\frac{\pi}{2}x + \frac{\pi}{6}\right)$

36. $y = 10 \sin \left(2x - \frac{\pi}{2}\right)$

37–40 ■ The graph of one period of a function of the form $y = a \sin k(x - b)$ or $y = a \cos k(x - b)$ is shown. Determine the function.



41–48 ■ Find the period, and sketch the graph.

41. $y = 3 \tan x$

42. $y = \tan \pi x$

43. $y = 2 \cot \left(x - \frac{\pi}{2}\right)$

44. $y = \sec \left(\frac{1}{2}x - \frac{\pi}{2}\right)$

45. $y = 4 \csc(2x + \pi)$

46. $y = \tan \left(x + \frac{\pi}{6}\right)$

47. $y = \tan \left(\frac{1}{2}x - \frac{\pi}{8}\right)$

48. $y = -4 \sec 4\pi x$

49–54 ■ A function is given.

- (a) Use a graphing device to graph the function.
- (b) Determine from the graph whether the function is periodic and, if so, determine the period.
- (c) Determine from the graph whether the function is odd, even, or neither.

49. $y = |\cos x|$

50. $y = \sin(\cos x)$

51. $y = \cos(2^{0.1x})$

52. $y = 1 + 2^{\cos x}$

53. $y = |x| \cos 3x$

54. $y = \sqrt{x} \sin 3x \quad (x > 0)$

55–58 ■ Graph the three functions on a common screen. How are the graphs related?

55. $y = x, \quad y = -x, \quad y = x \sin x$

56. $y = 2^{-x}, \quad y = -2^{-x}, \quad y = 2^{-x} \cos 4\pi x$

57. $y = x, \quad y = \sin 4x, \quad y = x + \sin 4x$

58. $y = \sin^2 x, \quad y = \cos^2 x, \quad y = \sin^2 x + \cos^2 x$

59–60 ■ Find the maximum and minimum values of the function.

59. $y = \cos x + \sin 2x$

60. $y = \cos x + \sin^2 x$

61. Find the solutions of $\sin x = 0.3$ in the interval $[0, 2\pi]$.

62. Find the solutions of $\cos 3x = x$ in the interval $[0, \pi]$.

63. Let $f(x) = \frac{\sin^2 x}{x}$.

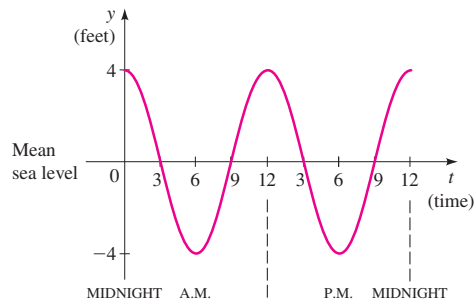
- (a) Is the function f even, odd, or neither?
- (b) Find the x -intercepts of the graph of f .
- (c) Graph f in an appropriate viewing rectangle.
- (d) Describe the behavior of the function as x becomes large.
- (e) Notice that $f(x)$ is not defined when $x = 0$. What happens as x approaches 0?

64. Let $y_1 = \cos(\sin x)$ and $y_2 = \sin(\cos x)$.

- (a) Graph y_1 and y_2 in the same viewing rectangle.
- (b) Determine the period of each of these functions from its graph.
- (c) Find an inequality between $\sin(\cos x)$ and $\cos(\sin x)$ that is valid for all x .
65. A point P moving in simple harmonic motion completes 8 cycles every second. If the amplitude of the motion is 50 cm, find an equation that describes the motion of P as a function of time. Assume the point P is at its maximum displacement when $t = 0$.
66. A mass suspended from a spring oscillates in simple harmonic motion at a frequency of 4 cycles per second. The

distance from the highest to the lowest point of the oscillation is 100 cm. Find an equation that describes the distance of the mass from its rest position as a function of time. Assume the mass is at its lowest point when $t = 0$.

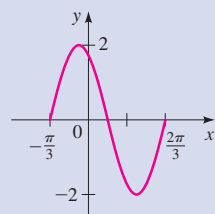
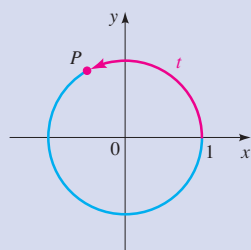
67. The graph shows the variation of the water level relative to mean sea level in the Long Beach harbor for a particular 24-hour period. Assuming that this variation is simple harmonic, find an equation of the form $y = a \cos \omega t$ that describes the variation in water level as a function of the number of hours after midnight.



68. The top floor of a building undergoes damped harmonic motion after a sudden brief earthquake. At time $t = 0$ the displacement is at a maximum, 16 cm from the normal position. The damping constant is $c = 0.72$ and the building vibrates at 1.4 cycles per second.

- (a) Find a function of the form $y = ke^{-ct} \cos \omega t$ to model the motion.
 (b) Graph the function you found in part (a).
 (c) What is the displacement at time $t = 10$ s?

5 Test



1. The point $P(x, y)$ is on the unit circle in quadrant IV. If $x = \sqrt{11}/6$, find y .
 2. The point P in the figure at the left has y -coordinate $\frac{4}{5}$. Find:
 - (a) $\sin t$
 - (b) $\cos t$
 - (c) $\tan t$
 - (d) $\sec t$
 3. Find the exact value.
 - (a) $\sin \frac{7\pi}{6}$
 - (b) $\cos \frac{13\pi}{4}$
 - (c) $\tan\left(-\frac{5\pi}{3}\right)$
 - (d) $\csc \frac{3\pi}{2}$
 4. Express $\tan t$ in terms of $\sin t$, if the terminal point determined by t is in quadrant II.
 5. If $\cos t = -\frac{8}{17}$ and if the terminal point determined by t is in quadrant III, find $\tan t \cot t + \csc t$.
- 6–7** ■ A trigonometric function is given.
- (a) Find the amplitude, period, and phase shift of the function.
 - (b) Sketch the graph.
6. $y = -5 \cos 4x$
 7. $y = 2 \sin\left(\frac{1}{2}x - \frac{\pi}{6}\right)$
- 8–9** ■ Find the period, and graph the function.
8. $y = -\csc 2x$
 9. $y = \tan\left(2x - \frac{\pi}{2}\right)$
10. The graph shown at left is one period of a function of the form $y = a \sin k(x - b)$. Determine the function.
11. Let $f(x) = \frac{\cos x}{1 + x^2}$.
- (a) Use a graphing device to graph f in an appropriate viewing rectangle.
 - (b) Determine from the graph if f is even, odd, or neither.
 - (c) Find the minimum and maximum values of f .
12. A mass suspended from a spring oscillates in simple harmonic motion. The mass completes 2 cycles every second and the distance between the highest point and the lowest point of the oscillation is 10 cm. Find an equation of the form $y = a \sin \omega t$ that gives the distance of the mass from its rest position as a function of time.
13. An object is moving up and down in damped harmonic motion. Its displacement at time $t = 0$ is 16 in; this is its maximum displacement. The damping constant is $c = 0.1$ and the frequency is 12 Hz.
- (a) Find a function that models this motion.
 - (b) Graph the function.

Focus on Modeling

Fitting Sinusoidal Curves to Data

In the *Focus on Modeling* that follows Chapter 2 (page 239), we learned how to construct linear models from data. Figure 1 shows some scatter plots of data; the first plot appears to be linear but the others are not. What do we do when the data we are studying are not linear? In this case, our model would be some other type of function that best fits the data. If the scatter plot indicates simple harmonic motion, then we might try to model the data with a sine or cosine function. The next example illustrates this process.

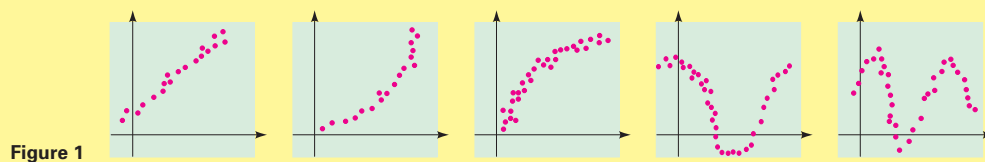


Figure 1



Example 1 Modeling the Height of a Tide

The water depth in a narrow channel varies with the tides. Table 1 shows the water depth over a 12-hour period.

- Make a scatter plot of the water depth data.
- Find a function that models the water depth with respect to time.
- If a boat needs at least 11 ft of water to cross the channel, during which times can it safely do so?

Solution

- A scatter plot of the data is shown in Figure 2.

Table 1

Time	Depth (ft)
12:00 A.M.	9.8
1:00 A.M.	11.4
2:00 A.M.	11.6
3:00 A.M.	11.2
4:00 A.M.	9.6
5:00 A.M.	8.5
6:00 A.M.	6.5
7:00 A.M.	5.7
8:00 A.M.	5.4
9:00 A.M.	6.0
10:00 A.M.	7.0
11:00 A.M.	8.6
12:00 P.M.	10.0

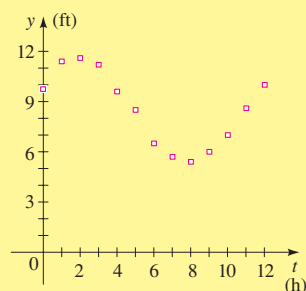


Figure 2

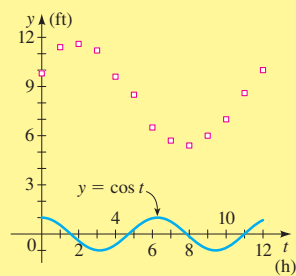
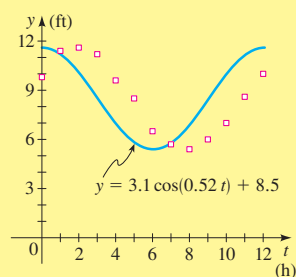
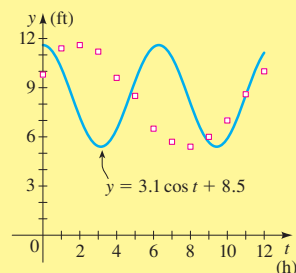
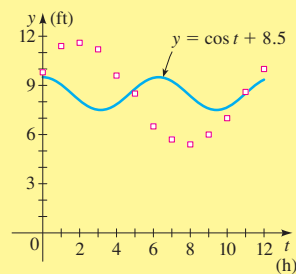


Figure 3



- (b) The data appear to lie on a cosine (or sine) curve. But if we graph $y = \cos t$ on the same graph as the scatter plot, the result in Figure 3 is not even close to the data—to fit the data we need to adjust the vertical shift, amplitude, period, and phase shift of the cosine curve. In other words, we need to find a function of the form

$$y = a \cos(\omega(t - c)) + b$$

We use the following steps, which are illustrated by the graphs in the margin.

■ Adjust the Vertical Shift

The vertical shift b is the average of the maximum and minimum values:

$$\begin{aligned} b &= \text{vertical shift} \\ &= \frac{1}{2} \cdot (\text{maximum value} + \text{minimum value}) \\ &= \frac{1}{2}(11.6 + 5.4) = 8.5 \end{aligned}$$

■ Adjust the Amplitude

The amplitude a is half of the difference between the maximum and minimum values:

$$\begin{aligned} a &= \text{amplitude} \\ &= \frac{1}{2} \cdot (\text{maximum value} - \text{minimum value}) \\ &= \frac{1}{2}(11.6 - 5.4) = 3.1 \end{aligned}$$

■ Adjust the Period

The time between consecutive maximum and minimum values is half of one period. Thus

$$\begin{aligned} \frac{2\pi}{\omega} &= \text{period} \\ &= 2 \cdot (\text{time of maximum value} - \text{time of minimum value}) \\ &= 2(8 - 2) = 12 \end{aligned}$$

Thus, $\omega = 2\pi/12 = 0.52$.

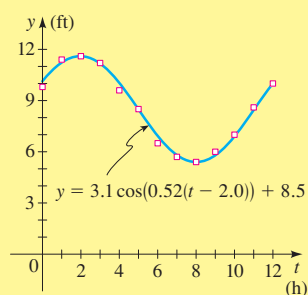


Figure 4

■ Adjust the Horizontal Shift

Since the maximum value of the data occurs at approximately $t = 2.0$, it represents a cosine curve shifted 2 h to the right. So

$$\begin{aligned} c &= \text{phase shift} \\ &= \text{time of maximum value} \\ &= 2.0 \end{aligned}$$

■ The Model

We have shown that a function that models the tides over the given time period is given by

$$y = 3.1 \cos(0.52(t - 2.0)) + 8.5$$

A graph of the function and the scatter plot are shown in Figure 4. It appears that the model we found is a good approximation to the data.

- (c) We need to solve the inequality $y \geq 11$. We solve this inequality graphically by graphing $y = 3.1 \cos(0.52(t - 2.0)) + 8.5$ and $y = 11$ on the same graph. From the graph in Figure 5 we see the water depth is higher than 11 ft between $t \approx 0.8$ and $t \approx 3.2$. This corresponds to the times 12:48 A.M. to 3:12 A.M. ■

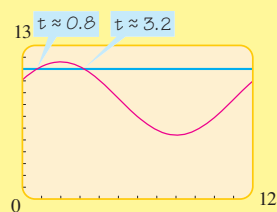


Figure 5

For the TI-83 and TI-86 the command **SinReg** (for sine regression) finds the sine curve that best fits the given data.

In Example 1 we used the scatter plot to guide us in finding a cosine curve that gives an approximate model of the data. Some graphing calculators are capable of finding a sine or cosine curve that best fits a given set of data points. The method these calculators use is similar to the method of finding a line of best fit, as explained on pages 239–240.

Example 2 Fitting a Sine Curve to Data

- Use a graphing device to find the sine curve that best fits the depth of water data in Table 1 on page 459.
- Compare your result to the model found in Example 1.


```

SinReg
y=a*sin(bx+c)+d
a=3.097877596
b=.5268322697
c=.5493035195
d=8.424021899

```

Output of the SinReg function on the TI-83.

Solution

- (a) Using the data in Table 1 and the `SinReg` command on the TI-83 calculator, we get a function of the form

$$y = a \sin(bt + c) + d$$

where

$$a = 3.1 \quad b = 0.53$$

$$c = 0.55 \quad d = 8.42$$

So, the sine function that best fits the data is

$$y = 3.1 \sin(0.53t + 0.55) + 8.42$$

- (b) To compare this with the function in Example 1, we change the sine function to a cosine function by using the reduction formula $\sin u = \cos(u - \pi/2)$.

$$y = 3.1 \sin(0.53t + 0.55) + 8.42$$

$$= 3.1 \cos\left(0.53t + 0.55 - \frac{\pi}{2}\right) + 8.42 \quad \text{Reduction formula}$$

$$= 3.1 \cos(0.53t - 1.02) + 8.42$$

$$= 3.1 \cos(0.53(t - 1.92)) + 8.42 \quad \text{Factor 0.53}$$

Comparing this with the function we obtained in Example 1, we see that there are small differences in the coefficients. In Figure 6 we graph a scatter plot of the data together with the sine function of best fit.

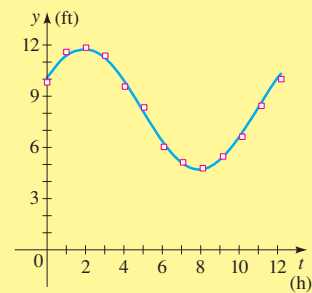



Figure 6

In Example 1 we estimated the values of the amplitude, period, and shifts from the data. In Example 2 the calculator computed the sine curve that best fits the data (that is, the curve that deviates least from the data as explained on page 240). The different ways of obtaining the model account for the differences in the functions.

Problems

1–4 ■ Modeling Periodic Data A set of data is given.

- Make a scatter plot of the data.
- Find a cosine function of the form $y = a \cos(\omega(t - c)) + b$ that models the data, as in Example 1.
- Graph the function you found in part (b) together with the scatter plot. How well does the curve fit the data?
-  Use a graphing calculator to find the sine function that best fits the data, as in Example 2.
- Compare the functions you found in parts (b) and (d). [Use the reduction formula $\sin u = \cos(u - \pi/2)$.]

1.

t	y
0	2.1
2	1.1
4	-0.8
6	-2.1
8	-1.3
10	0.6
12	1.9
14	1.5

2.

t	y
0	190
25	175
50	155
75	125
100	110
125	95
150	105
175	120
200	140
225	165
250	185
275	200
300	195
325	185
350	165


3.

t	y
0.1	21.1
0.2	23.6
0.3	24.5
0.4	21.7
0.5	17.5
0.6	12.0
0.7	5.6
0.8	2.2
0.9	1.0
1.0	3.5
1.1	7.6
1.2	13.2
1.3	18.4
1.4	23.0
1.5	25.1

4.


t	y
0.0	0.56
0.5	0.45
1.0	0.29
1.5	0.13
2.0	0.05
2.5	-0.10
3.0	0.02
3.5	0.12
4.0	0.26
4.5	0.43
5.0	0.54
5.5	0.63
6.0	0.59

5. Annual Temperature Change The table gives the average monthly temperature in Montgomery County, Maryland.

- Make a scatter plot of the data.
- Find a cosine curve that models the data (as in Example 1).
- Graph the function you found in part (b) together with the scatter plot.
-  Use a graphing calculator to find the sine curve that best fits the data (as in Example 2).


Month	Average temperature (°F)	Month	Average temperature (°F)
January	40.0	July	85.8
February	43.1	August	83.9
March	54.6	September	76.9
April	64.2	October	66.8
May	73.8	November	55.5
June	81.8	December	44.5

6. Circadian Rhythms Circadian rhythm (from the Latin *circa*—about, and *diem*—day) is the daily biological pattern by which body temperature, blood pressure, and other physiological variables change. The data in the table below show typical changes in human body temperature over a 24-hour period ($t = 0$ corresponds to midnight).

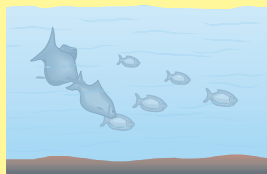
- Make a scatter plot of the data.
- Find a cosine curve that models the data (as in Example 1).
- Graph the function you found in part (b) together with the scatter plot.
-  Use a graphing calculator to find the sine curve that best fits the data (as in Example 2).

Time	Body temperature (°C)	Time	Body temperature (°C)
0	36.8	14	37.3
2	36.7	16	37.4
4	36.6	18	37.3
6	36.7	20	37.2
8	36.8	22	37.0
10	37.0	24	36.8
12	37.2		

7. Predator Population When two species interact in a predator/prey relationship (see page 432), the populations of both species tend to vary in a sinusoidal fashion. In a certain midwestern county, the main food source for barn owls consists of field mice and other small mammals. The table gives the population of barn owls in this county every July 1 over a 12-year period.

- Make a scatter plot of the data.
- Find a sine curve that models the data (as in Example 1).
- Graph the function you found in part (b) together with the scatter plot.
-  Use a graphing calculator to find the sine curve that best fits the data (as in Example 2). Compare to your answer from part (b).

Year	Owl population
0	50
1	62
2	73
3	80
4	71
5	60
6	51
7	43
8	29
9	20
10	28
11	41
12	49



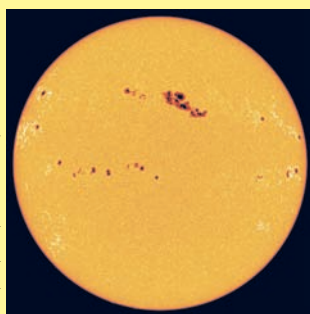
8. Salmon Survival For reasons not yet fully understood, the number of fingerling salmon that survive the trip from their riverbed spawning grounds to the open ocean varies approximately sinusoidally from year to year. The table shows the number of salmon that hatch in a certain British Columbia creek and then make their way to the Strait of Georgia. The data is given in thousands of fingerlings, over a period of 16 years.

- Make a scatter plot of the data.
- Find a sine curve that models the data (as in Example 1).
- Graph the function you found in part (b) together with the scatter plot.
- Use a graphing calculator to find the sine curve that best fits the data (as in Example 2). Compare to your answer from part (b).

Year	Salmon ($\times 1000$)	Year	Salmon ($\times 1000$)
1985	43	1993	56
1986	36	1994	63
1987	27	1995	57
1988	23	1996	50
1989	26	1997	44
1990	33	1998	38
1991	43	1999	30
1992	50	2000	22

9. Sunspot Activity Sunspots are relatively “cool” regions on the sun that appear as dark spots when observed through special solar filters. The number of sunspots varies in an 11-year cycle. The table gives the average daily sunspot count for the years 1975–2004.

- Make a scatter plot of the data.
- Find a cosine curve that models the data (as in Example 1).
- Graph the function you found in part (b) together with the scatter plot.
- Use a graphing calculator to find the sine curve that best fits the data (as in Example 2). Compare to your answer in part (b).



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Year	Sunspots	Year	Sunspots	Year	Sunspots
1975	16	1985	18	1995	18
1976	13	1986	13	1996	9
1977	28	1987	29	1997	21
1978	93	1988	100	1998	64
1979	155	1989	158	1999	93
1980	155	1990	143	2000	119
1981	140	1991	146	2001	111
1982	116	1992	94	2002	104
1983	67	1993	55	2003	64
1984	46	1994	30	2004	40