4

Exponential and Logarithmic Functions

4.1 Exponential Functions
4.2 Logarithmic Functions
4.3 Laws of Logarithms
4.4 Exponential and Logarithmic Equations
4.5 Modeling with Exponential and Logarithmic Functions

Chapter Overview

In this chapter we study a new class of functions called *exponential functions*. For example,

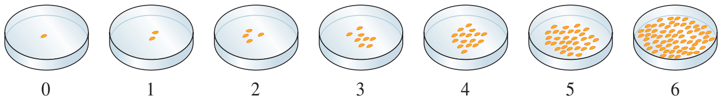
$$f(x) = 2^x$$

is an exponential function (with base 2). Notice how quickly the values of this function increase:

$$f(3) = 2^3 = 8$$
$$f(10) = 2^{10} = 1024$$
$$f(30) = 2^{30} = 1{,}073{,}741{,}824$$

Compare this with the function $g(x) = x^2$, where $g(30) = 30^2 = 900$. The point is, when the variable is in the exponent, even a small change in the variable can cause a dramatic change in the value of the function.

In spite of this incomprehensibly huge growth, exponential functions are appropriate for modeling population growth for all living things, from bacteria to elephants. To understand how a population grows, consider the case of a single bacterium, which divides every hour. After one hour we would have 2 bacteria, after two hours 2^2 or 4 bacteria, after three hours 2^3 or 8 bacteria, and so on. After x hours we would have 2^x bacteria. This leads us to model the bacteria population by the function $f(x) = 2^x$.



The principle governing population growth is the following: The larger the population, the greater the number of offspring. This same principle is present in many other real-life situations. For example, the larger your bank account, the more interest you get. So we also use exponential functions to find compound interest.

We use *logarithmic functions*, which are inverses of exponential functions, to help us answer such questions as, When will my investment grow to $100,000? In *Focus on Modeling* (page 386) we explore how to fit exponential and logarithmic models to data.

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SUGGESTED TIME AND EMPHASIS

$1\text{–}1\frac{1}{2}$ classes.

Essential material.

POINTS TO STRESS

1. The definition of an exponential function, including what it means to raise an irrational number to a power.
2. The geometry of exponential functions and their transformations.
3. The base *e*.
4. Periodically and continuously compounded interest.

4.1 Exponential Functions

So far, we have studied polynomial and rational functions. We now study one of the most important functions in mathematics, the *exponential function*. This function is used to model such natural processes as population growth and radioactive decay.

Exponential Functions

In Section 1.2 we defined a^x for $a > 0$ and x a rational number, but we have not yet defined irrational powers. So, what is meant by $5^{\sqrt{3}}$ or 2^{π}? To define a^x when x is irrational, we approximate x by rational numbers. For example, since

$$\sqrt{3} \approx 1.73205\ldots$$

is an irrational number, we successively approximate $a^{\sqrt{3}}$ by the following rational powers:

$$a^{1.7}, a^{1.73}, a^{1.732}, a^{1.7320}, a^{1.73205}, \ldots$$

Intuitively, we can see that these rational powers of a are getting closer and closer to $a^{\sqrt{3}}$. It can be shown using advanced mathematics that there is exactly one number that these powers approach. We define $a^{\sqrt{3}}$ to be this number.

For example, using a calculator we find

$$5^{\sqrt{3}} \approx 5^{1.732}$$
$$\approx 16.2411\ldots$$

The more decimal places of $\sqrt{3}$ we use in our calculation, the better our approximation of $5^{\sqrt{3}}$.

The Laws of Exponents are listed on page 14.

It can be proved that the *Laws of Exponents are still true when the exponents are real numbers*.

Exponential Functions

The **exponential function with base *a*** is defined for all real numbers x by

$$f(x) = a^x$$

where $a > 0$ and $a \neq 1$.

We assume $a \neq 1$ because the function $f(x) = 1^x = 1$ is just a constant function. Here are some examples of exponential functions:

$$f(x) = 2^x \qquad g(x) = 3^x \qquad h(x) = 10^x$$

Base 2 Base 3 Base 10

Example 1 Evaluating Exponential Functions

Let $f(x) = 3^x$ and evaluate the following:

(a) $f(2)$ (b) $f(-\frac{2}{3})$ (c) $f(\pi)$ (d) $f(\sqrt{2})$

Solution We use a calculator to obtain the values of f.

	Calculator keystrokes	Output
(a) $f(2) = 3^2 = 9$	$\boxed{3}\ \boxed{\wedge}\ \boxed{2}\ \boxed{\text{ENTER}}$	$\boxed{9}$
(b) $f(-\frac{2}{3}) = 3^{-2/3} \approx 0.4807$	$\boxed{3}\ \boxed{\wedge}\ \boxed{(}\ \boxed{(-)}\ \boxed{2}\ \boxed{\div}\ \boxed{3}\ \boxed{)}\ \boxed{\text{ENTER}}$	$\boxed{0.4807498}$
(c) $f(\pi) = 3^{\pi} \approx 31.544$	$\boxed{3}\ \boxed{\wedge}\ \boxed{\pi}\ \boxed{\text{ENTER}}$	$\boxed{31.5442807}$
(d) $f(\sqrt{2}) = 3^{\sqrt{2}} \approx 4.7288$	$\boxed{3}\ \boxed{\wedge}\ \boxed{\sqrt{\ }}\ \boxed{2}\ \boxed{\text{ENTER}}$	$\boxed{4.7288043}$

■

Graphs of Exponential Functions

We first graph exponential functions by plotting points. We will see that the graphs of such functions have an easily recognizable shape.

Example 2 Graphing Exponential Functions by Plotting Points

Draw the graph of each function.

(a) $f(x) = 3^x$ (b) $g(x) = \left(\dfrac{1}{3}\right)^x$

Solution We calculate values of $f(x)$ and $g(x)$ and plot points to sketch the graphs in Figure 1.

x	$f(x) = 3^x$	$g(x) = \left(\frac{1}{3}\right)^x$
-3	$\frac{1}{27}$	27
-2	$\frac{1}{9}$	9
-1	$\frac{1}{3}$	3
0	1	1
1	3	$\frac{1}{3}$
2	9	$\frac{1}{9}$
3	27	$\frac{1}{27}$

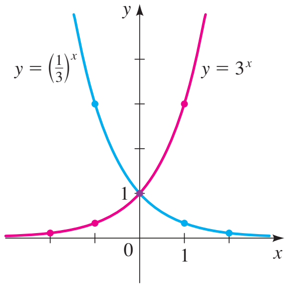


Figure 1

Notice that

$$g(x) = \left(\frac{1}{3}\right)^x = \frac{1}{3^x} = 3^{-x} = f(-x)$$

and so we could have obtained the graph of g from the graph of f by reflecting in the y-axis. ■

Reflecting graphs is explained in Section 2.4.

ALTERNATE EXAMPLE 1

Let $f(x) = 1.5^x$ and evaluate the following:

(a) $f(3)$
(b) $f(-1)$
(c) $f\left(\sqrt{2} - \sqrt{3}\right)$

ANSWERS

(a) 3.375
(b) $\frac{2}{3}$ or approximately 0.6667
(c) approximately 0.8791

ALTERNATE EXAMPLE 2

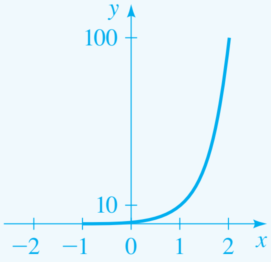
Draw the graph of each function by plotting points.

(a) $f(x) = 10^x$
(b) $g(x) = (1/10)^x$
(c) draw $f(x)$ where the x and y scales are the same

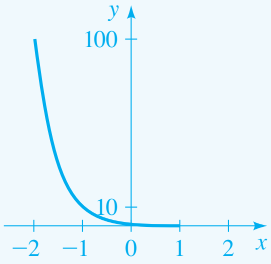
ANSWERS

(Note that the x and y scales are vastly different for parts (a) and (b).)

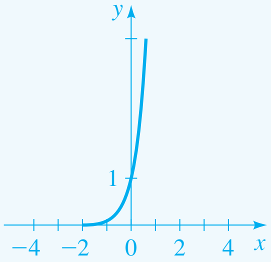
(a)



(b)



(c)



SAMPLE QUESTION
Text Question

Sketch a graph of $y = \left(\frac{1}{2}\right)^x$.

Answer

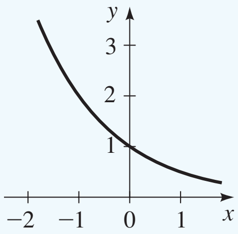


Figure 2 shows the graphs of the family of exponential functions $f(x) = a^x$ for various values of the base a. All of these graphs pass through the point $(0, 1)$ because $a^0 = 1$ for $a \neq 0$. You can see from Figure 2 that there are two kinds of exponential functions: If $0 < a < 1$, the exponential function decreases rapidly. If $a > 1$, the function increases rapidly (see the margin note).

To see just how quickly $f(x) = 2^x$ increases, let's perform the following thought experiment. Suppose we start with a piece of paper a thousandth of an inch thick, and we fold it in half 50 times. Each time we fold the paper, the thickness of the paper stack doubles, so the thickness of the resulting stack would be $2^{50}/1000$ inches. How thick do you think that is? It works out to be more than 17 million miles!

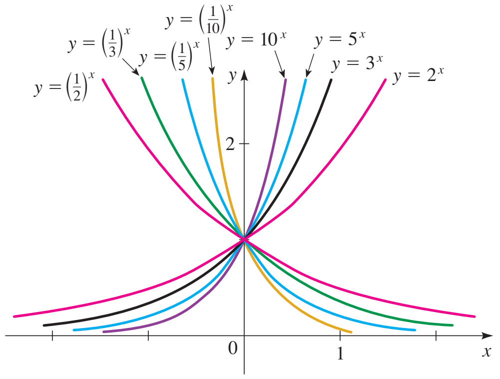


Figure 2
A family of exponential functions

The x-axis is a horizontal asymptote for the exponential function $f(x) = a^x$. This is because when $a > 1$, we have $a^x \to 0$ as $x \to -\infty$, and when $0 < a < 1$, we have $a^x \to 0$ as $x \to \infty$ (see Figure 2). Also, $a^x > 0$ for all $x \in \mathbb{R}$, so the function $f(x) = a^x$ has domain $\mathbb{R}$ and range $(0, \infty)$. These observations are summarized in the following box.

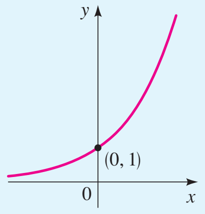
See Section 3.6, page 301, where the "arrow notation" used here is explained.

Graphs of Exponential Functions

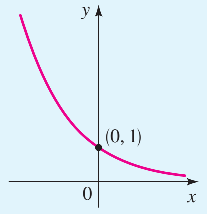
The exponential function

$$f(x) = a^x \qquad (a > 0,\ a \neq 1)$$

has domain $\mathbb{R}$ and range $(0, \infty)$. The line $y = 0$ (the x-axis) is a horizontal asymptote of f. The graph of f has one of the following shapes.



$f(x) = a^x$ for $a > 1$



$f(x) = a^x$ for $0 < a < 1$

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The **Gateway Arch** in St. Louis, Missouri, is shaped in the form of the graph of a combination of exponential functions (*not* a parabola, as it might first appear). Specifically, it is a **catenary**, which is the graph of an equation of the form

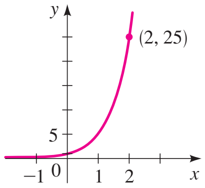
$$y = a(e^{bx} + e^{-bx})$$

(see Exercise 57). This shape was chosen because it is optimal for distributing the internal structural forces of the arch. Chains and cables suspended between two points (for example, the stretches of cable between pairs of telephone poles) hang in the shape of a catenary.

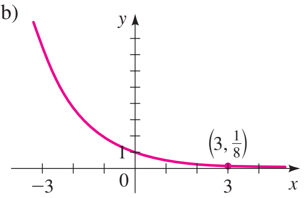
Example 3 Identifying Graphs of Exponential Functions

Find the exponential function $f(x) = a^x$ whose graph is given.

(a)



(b)



Solution

(a) Since $f(2) = a^2 = 25$, we see that the base is $a = 5$. So $f(x) = 5^x$.

(b) Since $f(3) = a^3 = \frac{1}{8}$, we see that the base is $a = \frac{1}{2}$. So $f(x) = \left(\frac{1}{2}\right)^x$. ■

In the next example we see how to graph certain functions, not by plotting points, but by taking the basic graphs of the exponential functions in Figure 2 and applying the shifting and reflecting transformations of Section 2.4.

Shifting and reflecting of graphs is explained in Section 2.4.

Example 4 Transformations of Exponential Functions

Use the graph of $f(x) = 2^x$ to sketch the graph of each function.

(a) $g(x) = 1 + 2^x$ (b) $h(x) = -2^x$ (c) $k(x) = 2^{x-1}$

Solution

(a) To obtain the graph of $g(x) = 1 + 2^x$, we start with the graph of $f(x) = 2^x$ and shift it upward 1 unit. Notice from Figure 3(a) that the line $y = 1$ is now a horizontal asymptote.

(b) Again we start with the graph of $f(x) = 2^x$, but here we reflect in the x-axis to get the graph of $h(x) = -2^x$ shown in Figure 3(b).

(c) This time we start with the graph of $f(x) = 2^x$ and shift it to the right by 1 unit, to get the graph of $k(x) = 2^{x-1}$ shown in Figure 3(c).

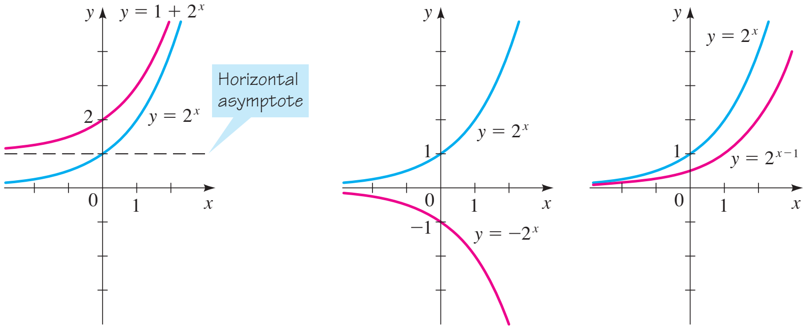
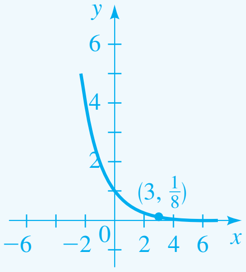


Figure 3 (a) (b) (c) ■

ALTERNATE EXAMPLE 3
Find the coordinates of the points with $x = -3, -2, -1, 0, 1, 2,$ and 3, on the curve of equation $f(x) = 4^x$.

ANSWER
$\left(-3, \frac{1}{64}\right), \left(-2, \frac{1}{16}\right), \left(-1, \frac{1}{4}\right),$ $(0, 1), (1, 4), (2, 16), (3, 64)$

ALTERNATE EXAMPLE 4
Find the exponential function $f(x) = a^x$ whose graph is given.



ANSWER
$f(x) = \left(\frac{1}{2}\right)^x$

IN-CLASS MATERIALS

Start to draw a graph of $y = 2^x$, using a carefully measured scale of 1 in. per unit on both axes. Point out that after 1 ft, the height would be over 100 yards (the length of a football field). After 2 ft, the height would be 264 miles, after 3 ft it would be 1,000,000 miles (four times the distance to the moon), after 3.5 ft it would be in the heart of the sun. If the graph extended 5 ft to the right, $x = 60$, then y would be over 1 light year up.

ALTERNATE EXAMPLE 5
In what direction and how many units is it necessary to move the graph of the equation $f(x) = 5^x$ to obtain the graph of the equation $g(x) = 5^x - 2$?

ANSWER
Downward, 2

Example 5 Comparing Exponential and Power Functions

Compare the rates of growth of the exponential function $f(x) = 2^x$ and the power function $g(x) = x^2$ by drawing the graphs of both functions in the following viewing rectangles.

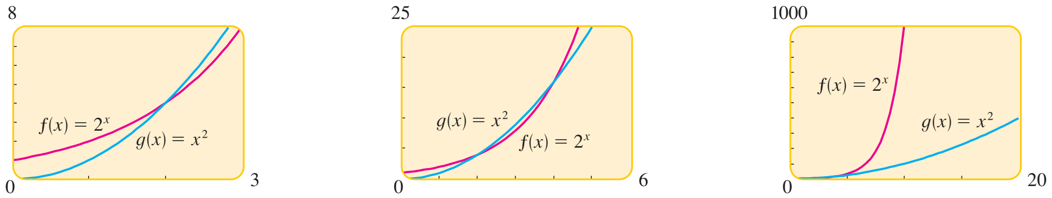
(a) $[0, 3]$ by $[0, 8]$
(b) $[0, 6]$ by $[0, 25]$
(c) $[0, 20]$ by $[0, 1000]$

Solution

(a) Figure 4(a) shows that the graph of $g(x) = x^2$ catches up with, and becomes higher than, the graph of $f(x) = 2^x$ at $x = 2$.

(b) The larger viewing rectangle in Figure 4(b) shows that the graph of $f(x) = 2^x$ overtakes that of $g(x) = x^2$ when $x = 4$.

(c) Figure 4(c) gives a more global view and shows that, when x is large, $f(x) = 2^x$ is much larger than $g(x) = x^2$.



(a) (b) (c)

Figure 4

■

The Natural Exponential Function

Any positive number can be used as the base for an exponential function, but some bases are used more frequently than others. We will see in the remaining sections of this chapter that the bases 2 and 10 are convenient for certain applications, but the most important base is the number denoted by the letter e.

The number e is defined as the value that $(1 + 1/n)^n$ approaches as n becomes large. (In calculus this idea is made more precise through the concept of a limit. See Exercise 55.) The table in the margin shows the values of the expression $(1 + 1/n)^n$ for increasingly large values of n. It appears that, correct to five decimal places, $e \approx 2.71828$; in fact, the approximate value to 20 decimal places is

$$e \approx 2.71828182845904523536$$

It can be shown that e is an irrational number, so we cannot write its exact value in decimal form.

Why use such a strange base for an exponential function? It may seem at first that a base such as 10 is easier to work with. We will see, however, that in certain applications the number e is the best possible base. In this section we study how e occurs in the description of compound interest.

n	$\left(1 + \frac{1}{n}\right)^n$
1	2.00000
5	2.48832
10	2.59374
100	2.70481
1000	2.71692
10,000	2.71815
100,000	2.71827
1,000,000	2.71828

The notation e was chosen by Leonhard Euler (see page 288), probably because it is the first letter of the word *exponential*.

IN-CLASS MATERIALS

Point out this contrast between exponential and linear functions: For equally spaced x-values, linear functions have constant *differences* in y-values, while pure exponential functions have constant *ratios* in y-values. Use this fact to show that the following table describes an exponential function, not a linear one.

x	y
−6.2	0.62000
−2.4	0.65100
1.4	0.68355
5.2	0.71773
9.0	0.75361
12.8	0.79129

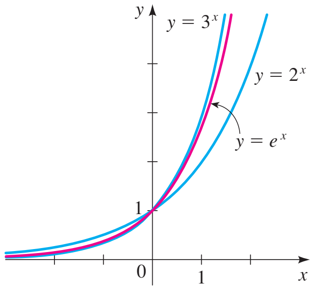


Figure 5

Graph of the natural exponential function

The Natural Exponential Function

The **natural exponential function** is the exponential function

$$f(x) = e^x$$

with base e. It is often referred to as *the* exponential function.

Since $2 < e < 3$, the graph of the natural exponential function lies between the graphs of $y = 2^x$ and $y = 3^x$, as shown in Figure 5.

Scientific calculators have a special key for the function $f(x) = e^x$. We use this key in the next example.

Example 6 Evaluating the Exponential Function

Evaluate each expression correct to five decimal places.

(a) e^3 (b) $2e^{-0.53}$ (c) $e^{4.8}$

Solution We use the $\boxed{e^x}$ key on a calculator to evaluate the exponential function.

(a) $e^3 \approx 20.08554$

(b) $2e^{-0.53} \approx 1.17721$

(c) $e^{4.8} \approx 121.51042$ ■

Example 7 Transformations of the Exponential Function

Sketch the graph of each function.

(a) $f(x) = e^{-x}$ (b) $g(x) = 3e^{0.5x}$

Solution

(a) We start with the graph of $y = e^x$ and reflect in the y-axis to obtain the graph of $y = e^{-x}$ as in Figure 6.

(b) We calculate several values, plot the resulting points, then connect the points with a smooth curve. The graph is shown in Figure 7.

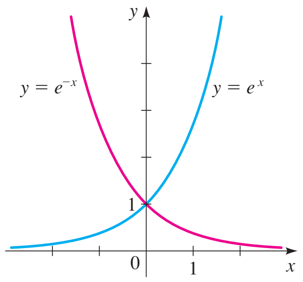


Figure 6

x	$f(x) = 3e^{0.5x}$
−3	0.67
−2	1.10
−1	1.82
0	3.00
1	4.95
2	8.15
3	13.45

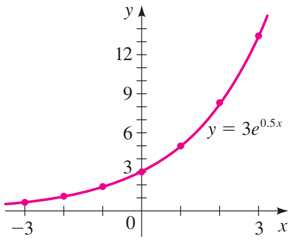


Figure 7 ■

ALTERNATE EXAMPLE 6

How many points of interception have the graphs of the functions $f(x) = 2^x$ and $g(x) = x^2$ drawing in the viewing rectangle [0, 3] by [0, 8]?

ANSWER

1

ALTERNATE EXAMPLE 7

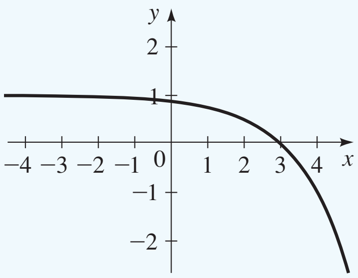
Evaluate the expression correct to five decimal places: $e^{3.7}$

ANSWER

40.4473

EXAMPLE

A shifted exponential curve:



$f(x) = -2^{x-3} + 1$

IN-CLASS MATERIALS

Estimate where $3^x > x^3$ and where $2^x > x^8$ using technology. Notice that exponential functions start by growing slower than polynomial functions, and then wind up growing much *faster*. For example, if one were to graph x^2 vs x using one inch per unit, then when $x = 60$, y would be only 100 yards, as opposed to a light year for $y = 2^x$. (The sun is only 8 light minutes from the earth.)

ALTERNATE EXAMPLE 8
For the graph of the function $g(x) = 4e^{0.4x}$, find the coordinates of the points with $x = -3, -2, -1, 0, 1, 2, 3$. Round each value to two decimal places.

ANSWER
$(-3, 1.2), (-2, 1.8), (-1, 2.68), (0, 4), (1, 5.97), (2, 8.9), (3, 13.28)$

Example 8 An Exponential Model for the Spread of a Virus

An infectious disease begins to spread in a small city of population 10,000. After t days, the number of persons who have succumbed to the virus is modeled by the function

$$v(t) = \frac{10{,}000}{5 + 1245e^{-0.97t}}$$

(a) How many infected people are there initially (at time $t = 0$)?

(b) Find the number of infected people after one day, two days, and five days.

(c) Graph the function v and describe its behavior.

Solution

(a) Since $v(0) = 10{,}000/(5 + 1245e^{0}) = 10{,}000/1250 = 8$, we conclude that 8 people initially have the disease.

(b) Using a calculator, we evaluate $v(1)$, $v(2)$, and $v(5)$, and then round off to obtain the following values.

Days	Infected people
1	21
2	54
5	678

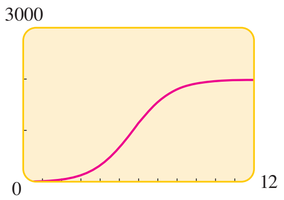


Figure 8

$v(t) = \dfrac{10{,}000}{5 + 1245e^{-0.97t}}$

(c) From the graph in Figure 8, we see that the number of infected people first rises slowly; then rises quickly between day 3 and day 8, and then levels off when about 2000 people are infected. ■

The graph in Figure 8 is called a *logistic curve* or a *logistic growth model*. Curves like it occur frequently in the study of population growth. (See Exercises 69–72.)

Compound Interest

Exponential functions occur in calculating compound interest. If an amount of money P, called the **principal**, is invested at an interest rate i per time period, then after one time period the interest is Pi, and the amount A of money is

$$A = P + Pi = P(1 + i)$$

If the interest is reinvested, then the new principal is $P(1 + i)$, and the amount after another time period is $A = P(1 + i)(1 + i) = P(1 + i)^2$. Similarly, after a third time period the amount is $A = P(1 + i)^3$. In general, after k periods the amount is

$$A = P(1 + i)^k$$

Notice that this is an exponential function with base $1 + i$.

If the annual interest rate is r and if interest is compounded n times per year, then in each time period the interest rate is $i = r/n$, and there are nt time periods in t years. This leads to the following formula for the amount after t years.

IN-CLASS MATERIALS

Have your students fill out the following table, using their calculators, to give them a feel for $y = e^x$.

x	2^x	3^x	e^x
-2	$\frac{1}{4} = 0.25$	$\frac{1}{9} \approx 0.111$	≈ 0.135
-1	$\frac{1}{2} = 0.5$	$\frac{1}{3} \approx 0.333$	≈ 0.368
0	1	1	1
1	2	3	≈ 2.718
2	4	9	≈ 7.389
3	8	27	≈ 20.086

Compound Interest

Compound interest is calculated by the formula

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

where $A(t)$ = amount after t years

P = principal

r = interest rate per year

n = number of times interest is compounded per year

t = number of years

r is often referred to as the nominal annual interest rate.

Example 9 Calculating Compound Interest

A sum of $1000 is invested at an interest rate of 12% per year. Find the amounts in the account after 3 years if interest is compounded annually, semiannually, quarterly, monthly, and daily.

Solution We use the compound interest formula with $P = \$1000$, $r = 0.12$, and $t = 3$.

Compounding	n	Amount after 3 years
Annual	1	$1000\left(1 + \frac{0.12}{1}\right)^{1(3)} = \$1404.93$
Semiannual	2	$1000\left(1 + \frac{0.12}{2}\right)^{2(3)} = \$1418.52$
Quarterly	4	$1000\left(1 + \frac{0.12}{4}\right)^{4(3)} = \$1425.76$
Monthly	12	$1000\left(1 + \frac{0.12}{12}\right)^{12(3)} = \$1430.77$
Daily	365	$1000\left(1 + \frac{0.12}{365}\right)^{365(3)} = \$1433.24$

We see from Example 9 that the interest paid increases as the number of compounding periods n increases. Let's see what happens as n increases indefinitely. If we let $m = n/r$, then

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt} = P\left[\left(1 + \frac{r}{n}\right)^{n/r}\right]^{rt} = P\left[\left(1 + \frac{1}{m}\right)^{m}\right]^{rt}$$

Recall that as m becomes large, the quantity $(1 + 1/m)^m$ approaches the number e. Thus, the amount approaches $A = Pe^{rt}$. This expression gives the amount when the interest is compounded at "every instant."

ALTERNATE EXAMPLE 9

An infectious disease begins to spread in a small city with a population of 8000. After t days, the number of persons who have succumbed to the virus is modeled by the function

$$v(t) = \frac{8000}{4 + 1596e^{-0.95t}}$$

How many infected people are there initially (at time $t = 0$)? Determine the number of infected people after one day, two days, and six days.

ANSWER

5, 13, 33, 857

EXAMPLE

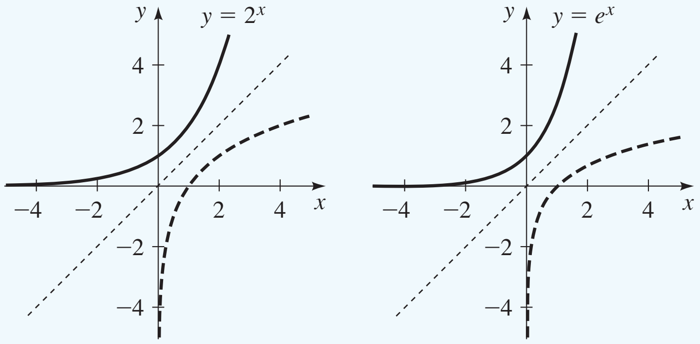
A comparison of compounding rates: $2000 is put in an IRA that earns 7%. Its worth after 10 years is given in the following table.

Compounded annually	$3934.30
Compounded semi-annually	$3979.58
Compounded monthly	$4019.32
Compounded weekly	$4025.61
Compounded daily	$4027.24
Compounded hourly	$4027.50
Compounded instantaneously	$4027.51

Note: Most calculators will not be able to determine the amount earned if the interest is compounded, say, every tenth of a second. Underflow errors in the microprocessor will make the computation difficult.

IN-CLASS MATERIALS

Anticipate the next section by having students sketch the graphs of the inverse functions of 2^x and e^x by reflecting them across the line $y = x$.



DRILL QUESTION

Find the total amount of money in an account after 2 years if \$100 is invested at an interest rate of 5.5% per year, compounded continuously.

Answer

\$111.63

ALTERNATE EXAMPLE 10

Find the amount after 4 years if \$1100 is invested at an interest rate of 11% per year, compounded continuously. Round to the nearest cent.

ANSWER

\$1707.98

Continuously Compounded Interest

Continuously compounded interest is calculated by the formula

$$A(t) = Pe^{rt}$$

where $A(t)$ = amount after t years
P = principal
r = interest rate per year
t = number of years

Example 10 Calculating Continuously Compounded Interest

Find the amount after 3 years if \$1000 is invested at an interest rate of 12% per year, compounded continuously.

Solution We use the formula for continuously compounded interest with $P = \$1000$, $r = 0.12$, and $t = 3$ to get

$$A(3) = 1000e^{(0.12)3} = 1000e^{0.36} = \$1433.33$$

Compare this amount with the amounts in Example 9. ■

4.1 Exercises

1–4 ■ Use a calculator to evaluate the function at the indicated values. Round your answers to three decimals.

1. $f(x) = 4^x;\quad f(0.5), f(\sqrt{2}), f(\pi), f(\frac{1}{3})$

2. $f(x) = 3^{x+1};\quad f(-1.5), f(\sqrt{3}), f(e), f(-\frac{5}{4})$

3. $g(x) = (\frac{2}{3})^{x-1};\quad g(1.3), g(\sqrt{5}), g(2\pi), g(-\frac{1}{2})$

4. $g(x) = (\frac{3}{4})^{2x};\quad g(0.7), g(\sqrt{7}/2), g(1/\pi), g(\frac{2}{3})$

5–10 ■ Sketch the graph of the function by making a table of values. Use a calculator if necessary.

5. $f(x) = 2^x$

6. $g(x) = 8^x$

7. $f(x) = (\frac{1}{3})^x$

8. $h(x) = (1.1)^x$

9. $g(x) = 3e^x$

10. $h(x) = 2e^{-0.5x}$

11–14 ■ Graph both functions on one set of axes.

11. $f(x) = 2^x$ and $g(x) = 2^{-x}$

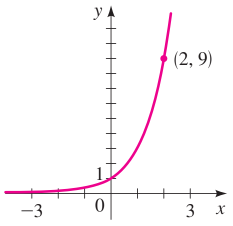
12. $f(x) = 3^{-x}$ and $g(x) = (\frac{1}{3})^x$

13. $f(x) = 4^x$ and $g(x) = 7^x$

14. $f(x) = (\frac{2}{3})^x$ and $g(x) = (\frac{4}{3})^x$

15–18 ■ Find the exponential function $f(x) = a^x$ whose graph is given.

15.



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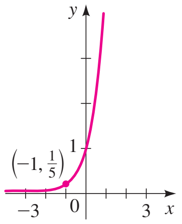

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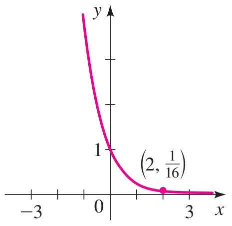

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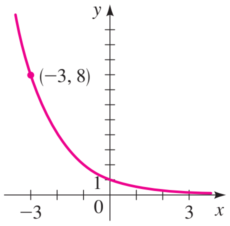


19–24 ■ Match the exponential function with one of the graphs labeled I–VI.

19. $f(x) = 5^x$

20. $f(x) = -5^x$

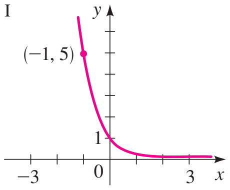
21. $f(x) = 5^{-x}$

22. $f(x) = 5^x + 3$

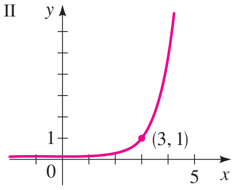
23. $f(x) = 5^{x-3}$

24. $f(x) = 5^{x+1} - 4$

I



II



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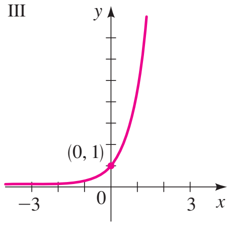
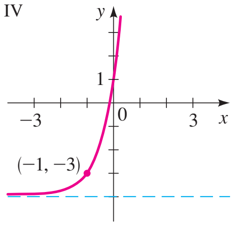
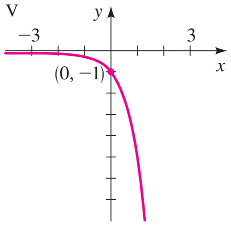


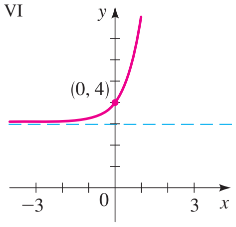
IV



V



VI



25–38 ■ Graph the function, not by plotting points, but by starting from the graphs in Figures 2 and 5. State the domain, range, and asymptote.

25. $f(x) = -3^x$

26. $f(x) = 10^{-x}$

27. $g(x) = 2^x - 3$

28. $g(x) = 2^{x-3}$

29. $h(x) = 4 + \left(\frac{1}{2}\right)^x$

30. $h(x) = 6 - 3^x$

31. $f(x) = 10^{x+3}$

32. $f(x) = -\left(\frac{1}{5}\right)^x$

33. $f(x) = -e^x$

34. $y = 1 - e^x$

35. $y = e^{-x} - 1$

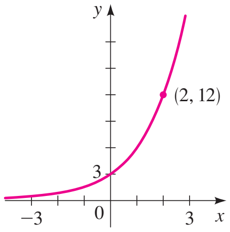
36. $f(x) = -e^{-x}$

37. $f(x) = e^{x-2}$

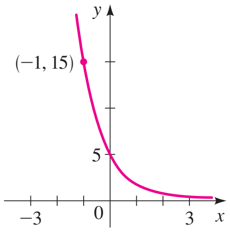
38. $y = e^{x-3} + 4$

39–40 ■ Find the function of the form $f(x) = Ca^x$ whose graph is given.

39.



40.



41. (a) Sketch the graphs of $f(x) = 2^x$ and $g(x) = 3(2^x)$.

(b) How are the graphs related?

42. (a) Sketch the graphs of $f(x) = 9^{x/2}$ and $g(x) = 3^x$.

(b) Use the Laws of Exponents to explain the relationship between these graphs.

43. If $f(x) = 10^x$, show that

$$\frac{f(x+h) - f(x)}{h} = 10^x\left(\frac{10^h - 1}{h}\right)$$

44. Compare the functions $f(x) = x^3$ and $g(x) = 3^x$ by evaluating both of them for $x = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 15,$ and 20. Then draw the graphs of f and g on the same set of axes.

45. The *hyperbolic cosine function* is defined by

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

Sketch the graphs of the functions $y = \frac{1}{2}e^x$ and $y = \frac{1}{2}e^{-x}$ on the same axes and use graphical addition (see Section 2.7) to sketch the graph of $y = \cosh(x)$.

46. The *hyperbolic sine function* is defined by

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

Sketch the graph of this function using graphical addition as in Exercise 45.

47–50 ■ Use the definitions in Exercises 45 and 46 to prove the identity.

47. $\cosh(-x) = \cosh(x)$

48. $\sinh(-x) = -\sinh(x)$

49. $[\cosh(x)]^2 - [\sinh(x)]^2 = 1$

50. $\sinh(x + y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y)$

51. **(a)** Compare the rates of growth of the functions $f(x) = 2^x$ and $g(x) = x^5$ by drawing the graphs of both functions in the following viewing rectangles.

(i) $[0, 5]$ by $[0, 20]$

(ii) $[0, 25]$ by $[0, 10^7]$

(iii) $[0, 50]$ by $[0, 10^8]$

(b) Find the solutions of the equation $2^x = x^5$, correct to one decimal place.

52. **(a)** Compare the rates of growth of the functions $f(x) = 3^x$ and $g(x) = x^4$ by drawing the graphs of both functions in the following viewing rectangles:

(i) $[-4, 4]$ by $[0, 20]$ (ii) $[0, 10]$ by $[0, 5000]$

(iii) $[0, 20]$ by $[0, 10^5]$

(b) Find the solutions of the equation $3^x = x^4$, correct to two decimal places.

53–54 ■ Draw graphs of the given family of functions for $c = 0.25, 0.5, 1, 2, 4$. How are the graphs related?

53. $f(x) = c2^x$ **54.** $f(x) = 2^{cx}$

55. Illustrate the definition of the number e by graphing the curve $y = (1 + 1/x)^x$ and the line $y = e$ on the same screen using the viewing rectangle $[0, 40]$ by $[0, 4]$.

56. Investigate the behavior of the function

$$f(x) = \left(1 - \frac{1}{x}\right)^x$$

as $x \to \infty$ by graphing f and the line $y = 1/e$ on the same screen using the viewing rectangle $[0, 20]$ by $[0, 1]$.

57. **(a)** Draw the graphs of the family of functions

$$f(x) = \frac{a}{2}(e^{x/a} + e^{-x/a})$$

for $a = 0.5, 1, 1.5$, and 2.

(b) How does a larger value of a affect the graph?

58–59 ■ Graph the function and comment on vertical and horizontal asymptotes.

58. $y = 2^{1/x}$ **59.** $y = \dfrac{e^x}{x}$

60–61 ■ Find the local maximum and minimum values of the function and the value of x at which each occurs. State each answer correct to two decimal places.

60. $g(x) = x^x \quad (x > 0)$ **61.** $g(x) = e^x + e^{-3x}$

62–63 ■ Find, correct to two decimal places, (a) the intervals on which the function is increasing or decreasing, and (b) the range of the function.

62. $y = 10^{x-x^2}$ **63.** $y = xe^{-x}$

Applications

64. Medical Drugs When a certain medical drug is administered to a patient, the number of milligrams remaining in the patient's bloodstream after t hours is modeled by

$$D(t) = 50e^{-0.2t}$$

How many milligrams of the drug remain in the patient's bloodstream after 3 hours?

65. Radioactive Decay A radioactive substance decays in such a way that the amount of mass remaining after t days is given by the function

$$m(t) = 13e^{-0.015t}$$

where $m(t)$ is measured in kilograms.

(a) Find the mass at time $t = 0$.

(b) How much of the mass remains after 45 days?

66. Radioactive Decay Radioactive iodine is used by doctors as a tracer in diagnosing certain thyroid gland disorders. This type of iodine decays in such a way that the mass remaining after t days is given by the function

$$m(t) = 6e^{-0.087t}$$

where $m(t)$ is measured in grams.

(a) Find the mass at time $t = 0$.

(b) How much of the mass remains after 20 days?

67. Sky Diving A sky diver jumps from a reasonable height above the ground. The air resistance she experiences is proportional to her velocity, and the constant of proportionality is 0.2. It can be shown that the downward velocity of the sky diver at time t is given by

$$v(t) = 80(1 - e^{-0.2t})$$

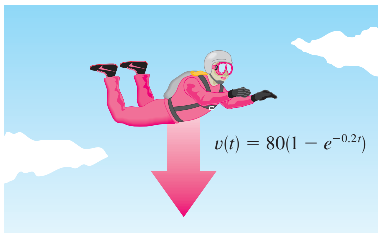
where t is measured in seconds and $v(t)$ is measured in feet per second (ft/s).

(a) Find the initial velocity of the sky diver.

(b) Find the velocity after 5 s and after 10 s.

(c) Draw a graph of the velocity function $v(t)$.

(d) The maximum velocity of a falling object with wind resistance is called its *terminal velocity*. From the graph in part (c) find the terminal velocity of this sky diver.



68. Mixtures and Concentrations A 50-gallon barrel is filled completely with pure water. Salt water with a concentration of 0.3 lb/gal is then pumped into the barrel, and the resulting mixture overflows at the same rate. The amount of salt in the barrel at time t is given by

$$Q(t) = 15(1 - e^{-0.04t})$$

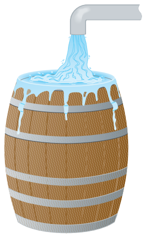
where t is measured in minutes and $Q(t)$ is measured in pounds.

(a) How much salt is in the barrel after 5 min?

(b) How much salt is in the barrel after 10 min?

(c) Draw a graph of the function $Q(t)$.

(d) Use the graph in part (c) to determine the value that the amount of salt in the barrel approaches as t becomes large. Is this what you would expect?



$Q(t) = 15(1 - e^{-0.04t})$

69. Logistic Growth Animal populations are not capable of unrestricted growth because of limited habitat and food supplies. Under such conditions the population follows a *logistic growth model*

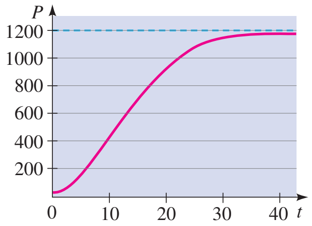
$$P(t) = \frac{d}{1 + ke^{-ct}}$$

where c, d, and k are positive constants. For a certain fish population in a small pond $d = 1200$, $k = 11$, $c = 0.2$, and t is measured in years. The fish were introduced into the pond at time $t = 0$.

(a) How many fish were originally put in the pond?

(b) Find the population after 10, 20, and 30 years.

(c) Evaluate $P(t)$ for large values of t. What value does the population approach as $t \to \infty$? Does the graph shown confirm your calculations?



70. Bird Population The population of a certain species of bird is limited by the type of habitat required for nesting. The population behaves according to the logistic growth model

$$n(t) = \frac{5600}{0.5 + 27.5e^{-0.044t}}$$

where t is measured in years.

(a) Find the initial bird population.

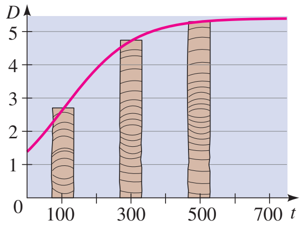
(b) Draw a graph of the function $n(t)$.

(c) What size does the population approach as time goes on?

71. Tree Diameter For a certain type of tree the diameter D (in feet) depends on the tree's age t (in years) according to the logistic growth model

$$D(t) = \frac{5.4}{1 + 2.9e^{-0.01t}}$$

Find the diameter of a 20-year-old tree.



72. Rabbit Population Assume that a population of rabbits behaves according to the logistic growth model

$$n(t) = \frac{300}{0.05 + \left(\dfrac{300}{n_0} - 0.05\right)e^{-0.55t}}$$

where n_0 is the initial rabbit population.

(a) If the initial population is 50 rabbits, what will the population be after 12 years?

(b) Draw graphs of the function $n(t)$ for $n_0 = 50$, 500, 2000, 8000, and 12,000 in the viewing rectangle $[0, 15]$ by $[0, 12{,}000]$.

(c) From the graphs in part (b), observe that, regardless of the initial population, the rabbit population seems to approach a certain number as time goes on. What is that number? (This is the number of rabbits that the island can support.)

73–74 ■ Compound Interest An investment of \$5000 is deposited into an account in which interest is compounded monthly. Complete the table by filling in the amounts to which the investment grows at the indicated times or interest rates.

73. $r = 4\%$

Time (years)	Amount
1	
2	
3	
4	
5	
6	

74. $t = 5$ years

Rate per year	Amount
1%	
2%	
3%	
4%	
5%	
6%	

75. Compound Interest If \$10,000 is invested at an interest rate of 10% per year, compounded semiannually, find the value of the investment after the given number of years.

(a) 5 years

(b) 10 years

(c) 15 years

76. Compound Interest If \$4000 is borrowed at a rate of 16% interest per year, compounded quarterly, find the amount due at the end of the given number of years.

(a) 4 years

(b) 6 years

(c) 8 years

77. Compound Interest If \$3000 is invested at an interest rate of 9% per year, find the amount of the investment at the end of 5 years for the following compounding methods.

(a) Annual

(b) Semiannual

(c) Monthly

(d) Weekly

(e) Daily

(f) Hourly

(g) Continuously

78. Compound Interest If \$4000 is invested in an account for which interest is compounded quarterly, find the amount of the investment at the end of 5 years for the following interest rates.

(a) 6% **(b)** $6\frac{1}{2}\%$

(c) 7% **(d)** 8%

79. Compound Interest Which of the given interest rates and compounding periods would provide the best investment?

(i) $8\frac{1}{2}\%$ per year, compounded semiannually

(ii) $8\frac{1}{4}\%$ per year, compounded quarterly

(iii) 8% per year, compounded continuously

80. Compound Interest Which of the given interest rates and compounding periods would provide the better investment?

(i) $9\frac{1}{4}\%$ per year, compounded semiannually

(ii) 9% per year, compounded continuously

81. Present Value The **present value** of a sum of money is the amount that must be invested now, at a given rate of interest, to produce the desired sum at a later date.

(a) Find the present value of \$10,000 if interest is paid at a rate of 9% per year, compounded semiannually, for 3 years.

(b) Find the present value of \$100,000 if interest is paid at a rate of 8% per year, compounded monthly, for 5 years.

82. Investment A sum of \$5000 is invested at an interest rate of 9% per year, compounded semiannually.

(a) Find the value $A(t)$ of the investment after t years.

(b) Draw a graph of $A(t)$.

(c) Use the graph of $A(t)$ to determine when this investment will amount to \$25,000.

Discovery • Discussion

83. Growth of an Exponential Function Suppose you are offered a job that lasts one month, and you are to be very well paid. Which of the following methods of payment is more profitable for you?

(a) One million dollars at the end of the month

(b) Two cents on the first day of the month, 4 cents on the second day, 8 cents on the third day, and, in general, 2^n cents on the nth day

84. The Height of the Graph of an Exponential Function Your mathematics instructor asks you to sketch a graph of the exponential function

$$f(x) = 2^x$$

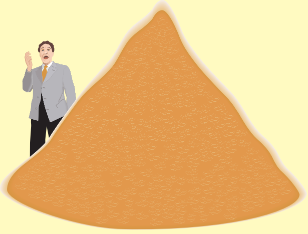
for x between 0 and 40, using a scale of 10 units to one inch. What are the dimensions of the sheet of paper you will need to sketch this graph?

DISCOVERY PROJECT

Exponential Explosion

To help us grasp just how explosive exponential growth is, let's try a thought experiment.

Suppose you put a penny in your piggy bank today, two pennies tomorrow, four pennies the next day, and so on, doubling the number of pennies you add to the bank each day (see the table). How many pennies will you put in your piggy bank on day 30? The answer is 2^{30} pennies. That's simple, but can you guess how many dollars that is? 2^{30} pennies is more than 10 million dollars!



Day	Pennies
0	1
1	2
2	4
3	8
4	16
⋮	⋮
n	2^n
⋮	⋮

As you can see, the exponential function $f(x) = 2^x$ grows extremely fast. This is the principle behind atomic explosions. An atom splits releasing two neutrons, which cause two atoms to split, each releasing two neutrons, causing four atoms to split, and so on. At the nth stage 2^n atoms split—an exponential explosion!

Populations also grow exponentially. Let's see what this means for a type of bacteria that splits every minute. Suppose that at 12:00 noon a single bacterium colonizes a discarded food can. The bacterium and his descendants are all happy, but they fear the time when the can is completely full of bacteria—doomsday.

1. How many bacteria are in the can at 12:05? At 12:10?

2. The can is completely full of bacteria at 1:00 P.M. At what time was the can only half full of bacteria?

3. When the can is exactly half full, the president of the bacteria colony reassures his constituents that doomsday is far away—after all, there is as much room left in the can as has been used in the entire previous history of the colony. Is the president correct? How much time is left before doomsday?

4. When the can is one-quarter full, how much time remains till doomsday?

5. A wise bacterium decides to start a new colony in another can and slow down splitting time to 2 minutes. How much time does this new colony have?

SUGGESTED TIME AND EMPHASIS

1 class.
Essential material.

POINTS TO STRESS

1. Definition of the logarithm function as the inverse of the exponential function, from both a numeric and geometric perspective.
2. Properties of the logarithm function, emphasizing the natural and common logarithms.

4.2 Logarithmic Functions

In this section we study the inverse of exponential functions.

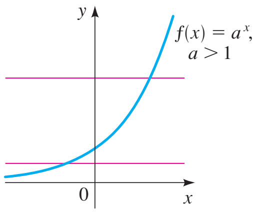


Figure 1
$f(x) = a^x$ is one-to-one

Logarithmic Functions

Every exponential function $f(x) = a^x$, with $a > 0$ and $a \neq 1$, is a one-to-one function by the Horizontal Line Test (see Figure 1 for the case $a > 1$) and therefore has an inverse function. The inverse function f^{-1} is called the *logarithmic function with base a* and is denoted by $\log_a$. Recall from Section 2.8 that f^{-1} is defined by

$$f^{-1}(x) = y \quad \Leftrightarrow \quad f(y) = x$$

This leads to the following definition of the logarithmic function.

Definition of the Logarithmic Function

Let a be a positive number with $a \neq 1$. The **logarithmic function with base** a, denoted by $\mathbf{\log_a}$, is defined by

$$\log_a x = y \quad \Leftrightarrow \quad a^y = x$$

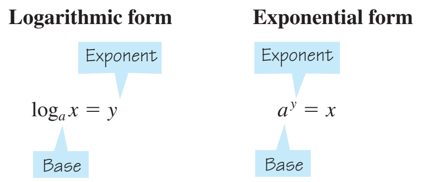
So, $\log_a x$ is the *exponent* to which the base a must be raised to give x.

We read $\log_a x = y$ as "log base a of x is y."

By tradition, the name of the logarithmic function is $\log_a$, not just a single letter. Also, we usually omit the parentheses in the function notation and write

$$\log_a(x) = \log_a x$$

When we use the definition of logarithms to switch back and forth between the **logarithmic form** $\log_a x = y$ and the **exponential form** $a^y = x$, it's helpful to notice that, in both forms, the base is the same:



ALTERNATE EXAMPLE 1

Write the following equation in exponential form: $\log_3 9 = 2$

ANSWER

$3^2 = 9$

Example 1 Logarithmic and Exponential Forms

The logarithmic and exponential forms are equivalent equations—if one is true, then so is the other. So, we can switch from one form to the other as in the following illustrations.

Logarithmic form	Exponential form
$\log_{10} 100{,}000 = 5$	$10^5 = 100{,}000$
$\log_2 8 = 3$	$2^3 = 8$
$\log_2\left(\frac{1}{8}\right) = -3$	$2^{-3} = \frac{1}{8}$
$\log_5 s = r$	$5^r = s$

■

It's important to understand that $\log_a x$ is an *exponent*. For example, the numbers in the right column of the table in the margin are the logarithms (base 10) of the

SAMPLE QUESTION

Text Question

It is a fact that $10^{\pi} \approx 1385.46$. Is it possible to approximate log 1385.46 without the use of a calculator? If so, then approximate this number. If not, why not?

Answer

$\log_{10} 1385.46 \approx \pi$

x	$\log_{10} x$
10^4	4
10^3	3
10^2	2
10	1
1	0
10^{-1}	-1
10^{-2}	-2
10^{-3}	-3
10^{-4}	-4

numbers in the left column. This is the case for all bases, as the following example illustrates.

Example 2 Evaluating Logarithms

(a) $\log_{10} 1000 = 3$ because $10^3 = 1000$

(b) $\log_2 32 = 5$ because $2^5 = 32$

(c) $\log_{10} 0.1 = -1$ because $10^{-1} = 0.1$

(d) $\log_{16} 4 = \frac{1}{2}$ because $16^{1/2} = 4$ ■

Inverse Function Property:

$f^{-1}(f(x)) = x$

$f(f^{-1}(x)) = x$

When we apply the Inverse Function Property described on page 227 to $f(x) = a^x$ and $f^{-1}(x) = \log_a x$, we get

$$\log_a(a^x) = x \qquad x \in \mathbb{R}$$

$$a^{\log_a x} = x \qquad x > 0$$

We list these and other properties of logarithms discussed in this section.

Properties of Logarithms

Property	Reason
1. $\log_a 1 = 0$	We must raise a to the power 0 to get 1.
2. $\log_a a = 1$	We must raise a to the power 1 to get a.
3. $\log_a a^x = x$	We must raise a to the power x to get a^x.
4. $a^{\log_a x} = x$	$\log_a x$ is the power to which a must be raised to get x.

Example 3 Applying Properties of Logarithms

We illustrate the properties of logarithms when the base is 5.

$\log_5 1 = 0$ Property 1 $\qquad \log_5 5 = 1$ Property 2

$\log_5 5^8 = 8$ Property 3 $\qquad 5^{\log_5 12} = 12$ Property 4 ■

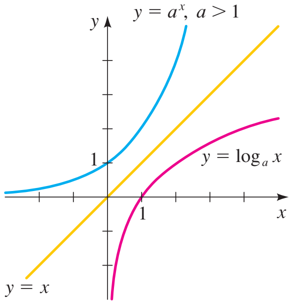


Figure 2

Graph of the logarithmic function $f(x) = \log_a x$

Graphs of Logarithmic Functions

Recall that if a one-to-one function f has domain A and range B, then its inverse function f^{-1} has domain B and range A. Since the exponential function $f(x) = a^x$ with $a \neq 1$ has domain $\mathbb{R}$ and range $(0, \infty)$, we conclude that its inverse function, $f^{-1}(x) = \log_a x$, has domain $(0, \infty)$ and range $\mathbb{R}$.

The graph of $f^{-1}(x) = \log_a x$ is obtained by reflecting the graph of $f(x) = a^x$ in the line $y = x$. Figure 2 shows the case $a > 1$. The fact that $y = a^x$ (for $a > 1$) is a very rapidly increasing function for $x > 0$ implies that $y = \log_a x$ is a very slowly increasing function for $x > 1$ (see Exercise 84).

Since $\log_a 1 = 0$, the x-intercept of the function $y = \log_a x$ is 1. The y-axis is a vertical asymptote of $y = \log_a x$ because $\log_a x \to -\infty$ as $x \to 0^+$.

Arrow notation is explained on page 301.

ALTERNATE EXAMPLE 2

Evaluate the following logarithm: $\log_{10} 100{,}000$

ANSWER

5

ALTERNATE EXAMPLE 3a

In what direction and how many units is it necessary to move the graph of the function $g(x) = \log_5 x$ to obtain the graph of the function $f(x) = -4 + \log_5 x$?

ANSWER

Downward, 4

ALTERNATE EXAMPLE 3b

In what direction and how many units is it necessary to move the graph of the function $h(x) = \log_{10} x$ to obtain the graph of the function $g(x) = \log_{10}(x - 2)$?

ANSWER

Right, 2

DRILL QUESTION

Compute $\log_4 \frac{1}{64}$.

Answer

-3

Mathematics in the Modern World

Bettmann/Corbis

Hulton/Deutch Collection/Corbis

Law Enforcement

Mathematics aids law enforcement in numerous and surprising ways, from the reconstruction of bullet trajectories, to determining the time of death, to calculating the probability that a DNA sample is from a particular person. One interesting use is in the search for missing persons. If a person has been missing for several years, that person may look quite different from their most recent available photograph. This is particularly true if the missing person is a child. Have you ever wondered what you will look like 5, 10, or 15 years from now?

Researchers have found that different parts of the body grow at different rates. For example, you have no doubt noticed that a baby's head is much larger relative to its body than an adult's. As another example, the ratio of arm length to height is $\frac{1}{3}$ in a child but about $\frac{2}{5}$ in an adult. By collecting data and analyzing the graphs, researchers are able to determine the functions that model growth. As in all growth phenomena, exponential and logarithmic functions play a crucial role. For instance, the formula that relates arm length l to height h is $l = ae^{kh}$ where a and k are constants. By studying various physical characteristics of a person, mathematical biologists model each characteristic by a function that describes how it changes over time. Models of facial characteristics can

(*continued*)

Example 4 Graphing a Logarithmic Function by Plotting Points

Sketch the graph of $f(x) = \log_2 x$.

Solution To make a table of values, we choose the x-values to be powers of 2 so that we can easily find their logarithms. We plot these points and connect them with a smooth curve as in Figure 3.

x	$\log_2 x$
2^3	3
2^2	2
2	1
1	0
2^{-1}	-1
2^{-2}	-2
2^{-3}	-3
2^{-4}	-4

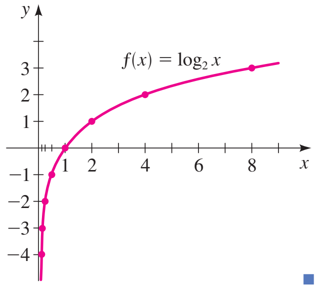


Figure 3

Figure 4 shows the graphs of the family of logarithmic functions with bases 2, 3, 5, and 10. These graphs are drawn by reflecting the graphs of $y = 2^x$, $y = 3^x$, $y = 5^x$, and $y = 10^x$ (see Figure 2 in Section 4.1) in the line $y = x$. We can also plot points as an aid to sketching these graphs, as illustrated in Example 4.

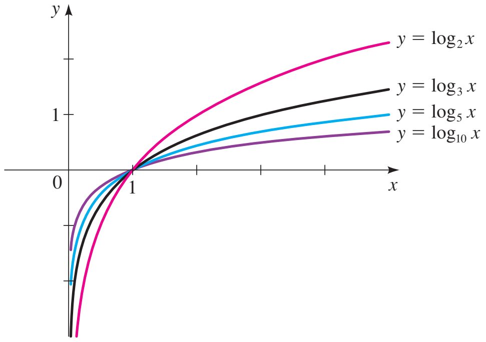


Figure 4
A family of logarithmic functions

In the next two examples we graph logarithmic functions by starting with the basic graphs in Figure 4 and using the transformations of Section 2.4.

ALTERNATE EXAMPLE 4

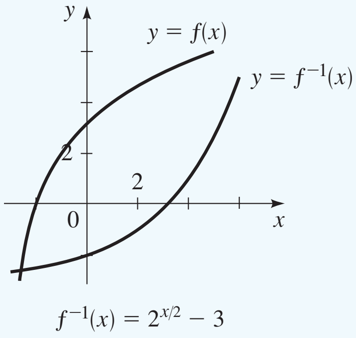
Evaluate $\log_5 5^{14}$ and $5^{\log_5 15}$.

ANSWER

14, 15

IN-CLASS MATERIALS

Sketch a graph of $f(x) = 2\log_2 (x + 3)$. Sketch the inverse function, then find an algebraic formula for the inverse. Foreshadow the next section by showing that the graph of $f(x)$ is the same as that of $g(x) = \log_2 \left((x + 3)^2\right)$.



IN-CLASS MATERIALS

When the logarithm function is graphed on a calculator, it appears to have a horizontal asymptote. Point out that the graph is misleading in that way. Start a graph of $y = \log_{10} x$ on the blackboard, noting the domain and the vertical asymptote. Using the scale of 1 in. = 1 unit (on the x-axis) and 1 ft = 1 unit (on the y-axis), plot some points:

x	0.1	1	2	3	4	5	6	7	8	9	10
$\log_{10} x$	-1	0	0.30	0.47	0.60	0.70	0.78	0.85	0.90	0.95	1

Now ask how many inches we would have to go out to get up to $y = 2$ ft. (Answer: 100 in., or $8\frac{1}{3}$ ft.) If the blackboard is large enough, plot the point (100, 2). Then ask how far we would have to go to get up to $y = 5$ ft. (Answer: 1.57 miles.) Note how it turns out to take close to a mile and a half to go from $y = 4$ to $y = 5$, and that (if you graphed it out) it would look a lot like there is a horizontal asymptote. Find the distance from your classroom to a city or landmark in another state, and ask the class to estimate the log of that distance, using the same scale.

be programmed into a computer to give a picture of how a person's appearance changes over time. These pictures aid law enforcement agencies in locating missing persons.

Example 5 Reflecting Graphs of Logarithmic Functions

Sketch the graph of each function.

(a) $g(x) = -\log_2 x$ (b) $h(x) = \log_2(-x)$

Solution

(a) We start with the graph of $f(x) = \log_2 x$ and reflect in the x-axis to get the graph of $g(x) = -\log_2 x$ in Figure 5(a).

(b) We start with the graph of $f(x) = \log_2 x$ and reflect in the y-axis to get the graph of $h(x) = \log_2(-x)$ in Figure 5(b).

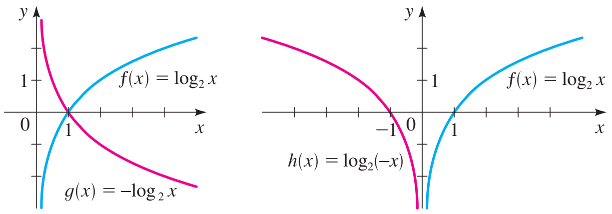


Figure 5 (a) (b) ■

Example 6 Shifting Graphs of Logarithmic Functions

Find the domain of each function, and sketch the graph.

(a) $g(x) = 2 + \log_5 x$ (b) $h(x) = \log_{10}(x - 3)$

Solution

(a) The graph of g is obtained from the graph of $f(x) = \log_5 x$ (Figure 4) by shifting upward 2 units (see Figure 6). The domain of f is $(0, \infty)$.

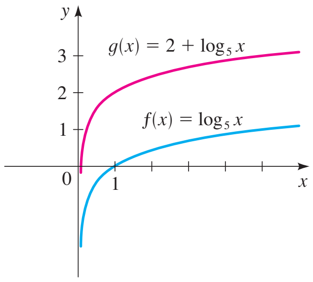


Figure 6

(b) The graph of h is obtained from the graph of $f(x) = \log_{10} x$ (Figure 4) by shifting to the right 3 units (see Figure 7 on the next page). The line $x = 3$ is a vertical asymptote. Since $\log_{10} x$ is defined only when $x > 0$, the domain

ALTERNATE EXAMPLE 5

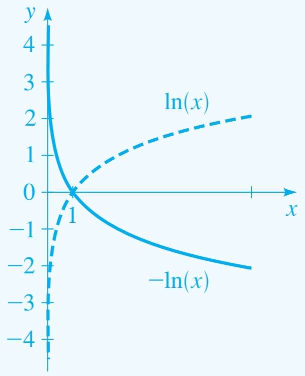
Sketch the graph of each function:

(a) $g(x) = -\ln(x)$

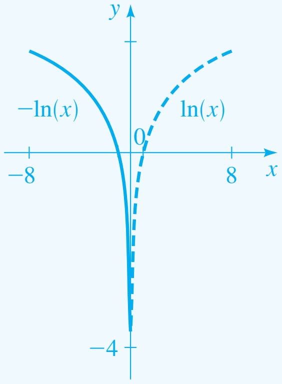
(b) $g(x) = \ln(-x)$

ANSWERS

(a)



(b)



ALTERNATE EXAMPLE 6

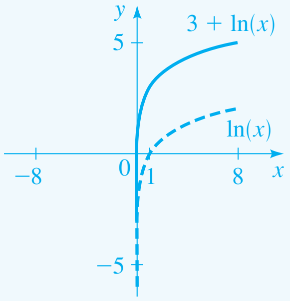
Sketch the graph of each function:

(a) $g(x) = 3 + \ln(x)$

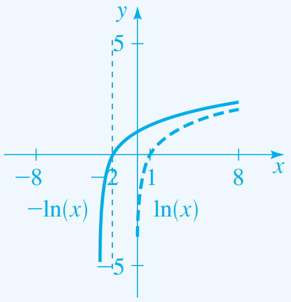
(b) $g(x) = \ln(3 + x)$

ANSWERS

(a)

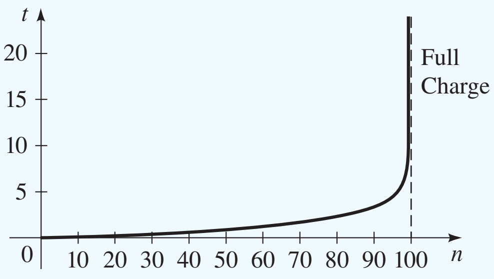


(b)



IN-CLASS MATERIALS

Ask your students if they have ever had to deal with recharging a battery, for example the battery to a music device or a laptop. Assume that the battery is dead, and it takes an hour to charge it up halfway. Ask them how much it will be charged in two hours. (The answer is 75% charged.) They may have noticed that, when charging older laptop batteries, the power meter rarely says "100% charged." It turns out that the time it takes to charge the battery to n% is given by $t = -k \ln\left(1 - \frac{n}{100}\right)$; in our example $k = 1.4427$. Have students compute how long it would take the battery to get a 97% charge, a 98% charge, and a 99% charge. (Remind them that it took only an hour to go from 0% to 50%.) Graph t versus n to demonstrate that the battery will never be fully charged.



of $h(x) = \log_{10}(x - 3)$ is

$$\{x \mid x - 3 > 0\} = \{x \mid x > 3\} = (3, \infty)$$

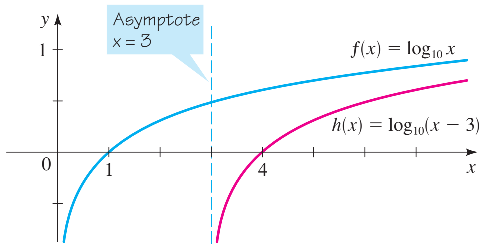


Figure 7

■

Common Logarithms

We now study logarithms with base 10.

Common Logarithm

The logarithm with base 10 is called the **common logarithm** and is denoted by omitting the base:

$$\log x = \log_{10} x$$

From the definition of logarithms we can easily find that

$$\log 10 = 1 \qquad \text{and} \qquad \log 100 = 2$$

But how do we find log 50? We need to find the exponent y such that $10^y = 50$. Clearly, 1 is too small and 2 is too large. So

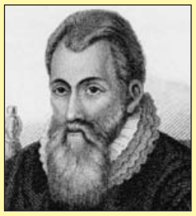
$$1 < \log 50 < 2$$

To get a better approximation, we can experiment to find a power of 10 closer to 50. Fortunately, scientific calculators are equipped with a LOG key that directly gives values of common logarithms.

Example 7 Evaluating Common Logarithms

Use a calculator to find appropriate values of $f(x) = \log x$ and use the values to sketch the graph.

Solution We make a table of values, using a calculator to evaluate the function at those values of x that are not powers of 10. We plot those points and connect them by a smooth curve as in Figure 8.



John Napier (1550–1617) was a Scottish landowner for whom mathematics was a hobby. We know him today because of his key invention—logarithms, which he published in 1614 under the title *A Description of the Marvelous Rule of Logarithms*. In Napier's time, logarithms were used exclusively for simplifying complicated calculations. For example, to multiply two large numbers we would write them as powers of 10. The exponents are simply the logarithms of the numbers. For instance,

$$\begin{aligned} 4532 \times 57783 &\approx 10^{3.65629} \times 10^{4.76180} \\ &= 10^{8.41809} \\ &\approx 261{,}872{,}564 \end{aligned}$$

The idea is that multiplying powers of 10 is easy (we simply add their exponents). Napier produced extensive tables giving the logarithms (or exponents) of numbers. Since the advent of calculators and computers, logarithms are no longer used for this purpose. The logarithmic functions, however, have found many applications, some of which are described in this chapter.

Napier wrote on many topics. One of his most colorful works is a book entitled *A Plaine Discovery of the Whole Revelation of Saint John*, in which he predicted that the world would end in the year 1700.

ALTERNATE EXAMPLE 7

Find appropriate values of $f(x) = \log x$, if $x = 0.1, 1, 4$, and 10. Use a calculator to evaluate the function for those values of x that are not powers of 10.

ANSWER

$(0.1, -1), (1, 0), (4, 0.602), (10, 1)$

IN-CLASS MATERIALS

Show students semilog graph paper (available from the web, university bookstores, or your friendly neighborhood physics teacher). Point out how the distance between the y-axis lines is based on the logarithm of the y-coordinate, not on the y-coordinate itself. Have them graph $y = 2^x$ on semilog graph paper.

x	$\log x$
0.01	-2
0.1	-1
0.5	-0.301
1	0
4	0.602
5	0.699
10	1

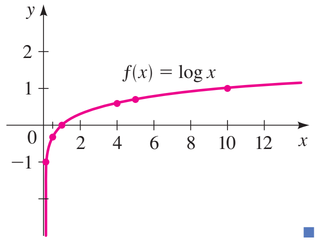
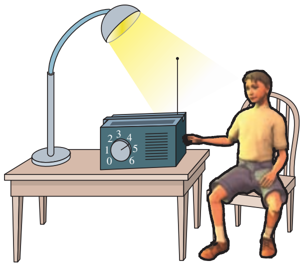


Figure 8 ■



Human response to sound and light intensity is logarithmic.

Scientists model human response to stimuli (such as sound, light, or pressure) using logarithmic functions. For example, the intensity of a sound must be increased manyfold before we "feel" that the loudness has simply doubled. The psychologist Gustav Fechner formulated the law as

$$S = k\log\left(\frac{I}{I_0}\right)$$

where S is the subjective intensity of the stimulus, I is the physical intensity of the stimulus, I_0 stands for the threshold physical intensity, and k is a constant that is different for each sensory stimulus.

We study the decibel scale in more detail in Section 4.5.

Example 8 Common Logarithms and Sound

The perception of the loudness B (in decibels, dB) of a sound with physical intensity I (in W/m^2) is given by

$$B = 10\log\left(\frac{I}{I_0}\right)$$

where I_0 is the physical intensity of a barely audible sound. Find the decibel level (loudness) of a sound whose physical intensity I is 100 times that of I_0.

Solution We find the decibel level B by using the fact that $I = 100I_0$.

$$\begin{aligned} B &= 10\log\left(\frac{I}{I_0}\right) && \text{Definition of } B \\ &= 10\log\left(\frac{100I_0}{I_0}\right) && I = 100I_0 \\ &= 10\log 100 && \text{Cancel } I_0 \\ &= 10\cdot 2 = 20 && \text{Definition of log} \end{aligned}$$

The loudness of the sound is 20 dB. ■

Natural Logarithms

Of all possible bases a for logarithms, it turns out that the most convenient choice for the purposes of calculus is the number e, which we defined in Section 4.1.

ALTERNATE EXAMPLE 8

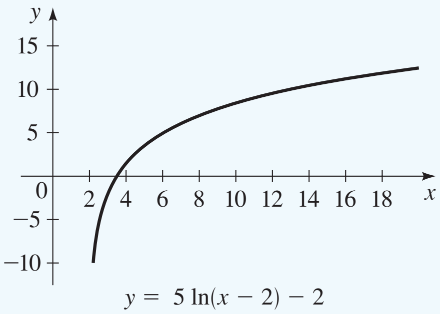
Find the decibel level of a sound whose physical intensity I is 1000 times that of I_0.

ANSWER

$10\log\left(\frac{1000I_0}{I_0}\right) = 30$. Compare this result with the example in the text—an increase of 10 in the decibel level reflected the intensity going from $100I_0$ to $1000I_0$.

EXAMPLE

A shifted logarithmic curve:



The notation ln is an abbreviation for the Latin name *logarithmus naturalis*.

Natural Logarithm

The logarithm with base e is called the **natural logarithm** and is denoted by **ln**:

$$\ln x = \log_e x$$

The natural logarithmic function $y = \ln x$ is the inverse function of the exponential function $y = e^x$. Both functions are graphed in Figure 9. By the definition of inverse functions we have

$$\boxed{\ln x = y \quad \Leftrightarrow \quad e^y = x}$$

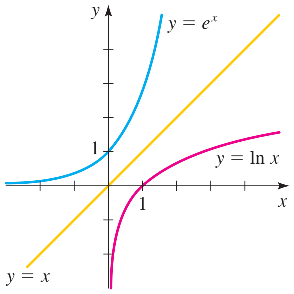


Figure 9
Graph of the natural logarithmic function

If we substitute $a = e$ and write "ln" for "$\log_e$" in the properties of logarithms mentioned earlier, we obtain the following properties of natural logarithms.

Properties of Natural Logarithms

Property	Reason
1. $\ln 1 = 0$	We must raise e to the power 0 to get 1.
2. $\ln e = 1$	We must raise e to the power 1 to get e.
3. $\ln e^x = x$	We must raise e to the power x to get e^x.
4. $e^{\ln x} = x$	$\ln x$ is the power to which e must be raised to get x.

Calculators are equipped with an LN key that directly gives the values of natural logarithms.

Example 9 Evaluating the Natural Logarithm Function

(a) $\ln e^8 = 8$	*Definition of natural logarithm*
(b) $\ln\left(\dfrac{1}{e^2}\right) = \ln e^{-2} = -2$	*Definition of natural logarithm*
(c) $\ln 5 \approx 1.609$	*Use LN key on calculator* ■

ALTERNATE EXAMPLE 9

Evaluate the following:

(a) $\ln(e^5)$

(b) $\ln\left(\sqrt[3]{e^7}\right)$

(c) $\ln(2.72)$

ANSWERS

(a) 5

(b) $\frac{7}{3}$

(c) $\ln(2.72) \approx 1.0006$

Example 10 Finding the Domain of a Logarithmic Function

Find the domain of the function $f(x) = \ln(4 - x^2)$.

Solution As with any logarithmic function, $\ln x$ is defined when $x > 0$. Thus, the domain of f is

$$\{x \mid 4 - x^2 > 0\} = \{x \mid x^2 < 4\} = \{x \mid |x| < 2\}$$
$$= \{x \mid -2 < x < 2\} = (-2, 2)$$

■

Example 11 Drawing the Graph of a Logarithmic Function

Draw the graph of the function $y = x\ln(4 - x^2)$ and use it to find the asymptotes and local maximum and minimum values.

Solution As in Example 10 the domain of this function is the interval $(-2, 2)$, so we choose the viewing rectangle $[-3, 3]$ by $[-3, 3]$. The graph is shown in Figure 10, and from it we see that the lines $x = -2$ and $x = 2$ are vertical asymptotes.

The function has a local maximum point to the right of $x = 1$ and a local minimum point to the left of $x = -1$. By zooming in and tracing along the graph with the cursor, we find that the local maximum value is approximately 1.13 and this occurs when $x \approx 1.15$. Similarly (or by noticing that the function is odd), we find that the local minimum value is about -1.13, and it occurs when $x \approx -1.15$. ■

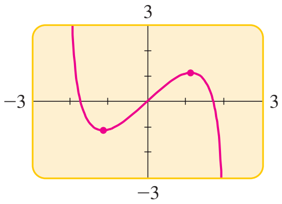


Figure 10
$y = x\ln(4 - x^2)$

4.2 Exercises

1–2 ■ Complete the table by finding the appropriate logarithmic or exponential form of the equation, as in Example 1.

1.

Logarithmic form	Exponential form
$\log_8 8 = 1$	
$\log_8 64 = 2$	
	$8^{2/3} = 4$
	$8^3 = 512$
$\log_8\left(\frac{1}{8}\right) = -1$	
	$8^{-2} = \frac{1}{64}$

2.

Logarithmic form	Exponential form
	$4^3 = 64$
$\log_4 2 = \frac{1}{2}$	
	$4^{3/2} = 8$
$\log_4\left(\frac{1}{16}\right) = -2$	
$\log_4\left(\frac{1}{2}\right) = -\frac{1}{2}$	
	$4^{-5/2} = \frac{1}{32}$

3–8 ■ Express the equation in exponential form.

3. **(a)** $\log_5 25 = 2$ **(b)** $\log_5 1 = 0$

4. **(a)** $\log_{10} 0.1 = -1$ **(b)** $\log_8 512 = 3$

5. **(a)** $\log_8 2 = \frac{1}{3}$ **(b)** $\log_2\left(\frac{1}{8}\right) = -3$

6. **(a)** $\log_3 81 = 4$ **(b)** $\log_8 4 = \frac{2}{3}$

7. **(a)** $\ln 5 = x$ **(b)** $\ln y = 5$

8. **(a)** $\ln(x + 1) = 2$ **(b)** $\ln(x - 1) = 4$

9–14 ■ Express the equation in logarithmic form.

9. **(a)** $5^3 = 125$ **(b)** $10^{-4} = 0.0001$

10. **(a)** $10^3 = 1000$ **(b)** $81^{1/2} = 9$

11. **(a)** $8^{-1} = \frac{1}{8}$ **(b)** $2^{-3} = \frac{1}{8}$

12. **(a)** $4^{-3/2} = 0.125$ **(b)** $7^3 = 343$

13. **(a)** $e^x = 2$ **(b)** $e^3 = y$

14. **(a)** $e^{x+1} = 0.5$ **(b)** $e^{0.5x} = t$

15–24 ■ Evaluate the expression.

15. **(a)** $\log_3 3$ **(b)** $\log_3 1$ **(c)** $\log_3 3^2$

16. **(a)** $\log_5 5^4$ **(b)** $\log_4 64$ **(c)** $\log_9 9$

ALTERNATE EXAMPLE 10

Find the domain of the function $f(x) = \ln(x^2 - 9)$.

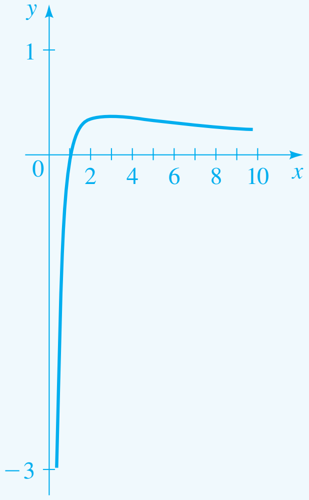
ANSWER

$(-\infty, -3) \cup (3, \infty)$

ALTERNATE EXAMPLE 11

Draw the graph of $y = \ln(x)/x$ and use it to find the asymptotes and local extreme values.

There is a vertical asymptote at $x = 0$, and a horizontal one at $y = 0$. There is a local maximum at about $x = 2.718$, and its value is about 0.3679. (Using calculus, we can find the exact values: $x = e$, $y = 1/e$.)



17. **(a)** $\log_6 36$ **(b)** $\log_9 81$ **(c)** $\log_7 7^{10}$

18. **(a)** $\log_2 32$ **(b)** $\log_8 8^{17}$ **(c)** $\log_6 1$

19. **(a)** $\log_3\left(\frac{1}{27}\right)$ **(b)** $\log_{10}\sqrt{10}$ **(c)** $\log_5 0.2$

20. **(a)** $\log_5 125$ **(b)** $\log_{49} 7$ **(c)** $\log_9\sqrt{3}$

21. **(a)** $2^{\log_2 37}$ **(b)** $3^{\log_3 8}$ **(c)** $e^{\ln\sqrt{5}}$

22. **(a)** $e^{\ln \pi}$ **(b)** $10^{\log 5}$ **(c)** $10^{\log 87}$

23. **(a)** $\log_8 0.25$ **(b)** $\ln e^4$ **(c)** $\ln(1/e)$

24. **(a)** $\log_4\sqrt{2}$ **(b)** $\log_4\left(\frac{1}{2}\right)$ **(c)** $\log_4 8$

25–32 ■ Use the definition of the logarithmic function to find x.

25. **(a)** $\log_2 x = 5$ **(b)** $\log_2 16 = x$

26. **(a)** $\log_5 x = 4$ **(b)** $\log_{10} 0.1 = x$

27. **(a)** $\log_3 243 = x$ **(b)** $\log_3 x = 3$

28. **(a)** $\log_4 2 = x$ **(b)** $\log_4 x = 2$

29. **(a)** $\log_{10} x = 2$ **(b)** $\log_5 x = 2$

30. **(a)** $\log_x 1000 = 3$ **(b)** $\log_x 25 = 2$

31. **(a)** $\log_x 16 = 4$ **(b)** $\log_x 8 = \frac{3}{2}$

32. **(a)** $\log_x 6 = \frac{1}{2}$ **(b)** $\log_x 3 = \frac{1}{3}$

33–36 ■ Use a calculator to evaluate the expression, correct to four decimal places.

33. **(a)** $\log 2$ **(b)** $\log 35.2$ **(c)** $\log\left(\frac{2}{3}\right)$

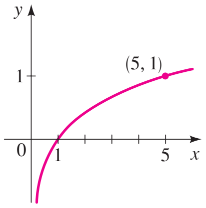
34. **(a)** $\log 50$ **(b)** $\log\sqrt{2}$ **(c)** $\log(3\sqrt{2})$

35. **(a)** $\ln 5$ **(b)** $\ln 25.3$ **(c)** $\ln(1 + \sqrt{3})$

36. **(a)** $\ln 27$ **(b)** $\ln 7.39$ **(c)** $\ln 54.6$

37–40 ■ Find the function of the form $y = \log_a x$ whose graph is given.

37.



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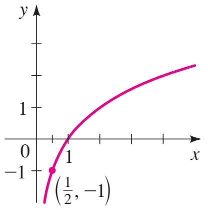

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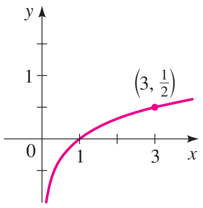

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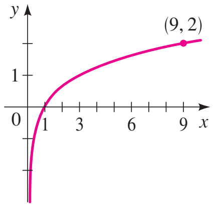


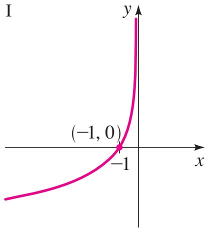
41–46 ■ Match the logarithmic function with one of the graphs labeled I–VI.

41. $f(x) = -\ln x$ **42.** $f(x) = \ln(x - 2)$

43. $f(x) = 2 + \ln x$ **44.** $f(x) = \ln(-x)$

45. $f(x) = \ln(2 - x)$ **46.** $f(x) = -\ln(-x)$

I



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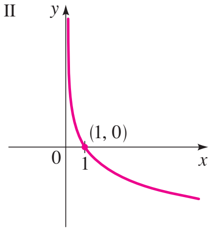

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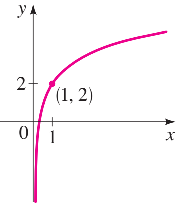
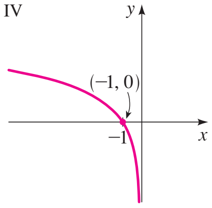
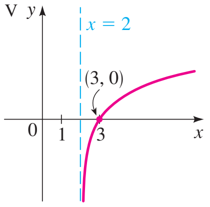


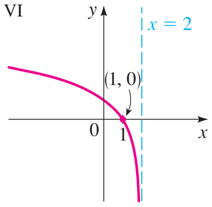
IV



V



VI



47. Draw the graph of $y = 4^x$, then use it to draw the graph of $y = \log_4 x$.

48. Draw the graph of $y = 3^x$, then use it to draw the graph of $y = \log_3 x$.

49–58 ■ Graph the function, not by plotting points, but by starting from the graphs in Figures 4 and 9. State the domain, range, and asymptote.

49. $f(x) = \log_2(x - 4)$ **50.** $f(x) = -\log_{10} x$

51. $g(x) = \log_5(-x)$ **52.** $g(x) = \ln(x + 2)$

53. $y = 2 + \log_3 x$ **54.** $y = \log_3(x - 1) - 2$

55. $y = 1 - \log_{10} x$ **56.** $y = 1 + \ln(-x)$

57. $y = |\ln x|$ **58.** $y = \ln|x|$

59–64 ■ Find the domain of the function.

59. $f(x) = \log_{10}(x + 3)$

60. $f(x) = \log_5(8 - 2x)$

61. $g(x) = \log_3(x^2 - 1)$

62. $g(x) = \ln(x - x^2)$

63. $h(x) = \ln x + \ln(2 - x)$

64. $h(x) = \sqrt{x - 2} - \log_5(10 - x)$

65–70 ■ Draw the graph of the function in a suitable viewing rectangle and use it to find the domain, the asymptotes, and the local maximum and minimum values.

65. $y = \log_{10}(1 - x^2)$

66. $y = \ln(x^2 - x)$

67. $y = x + \ln x$

68. $y = x(\ln x)^2$

69. $y = \dfrac{\ln x}{x}$

70. $y = x \log_{10}(x + 10)$

71. Compare the rates of growth of the functions $f(x) = \ln x$ and $g(x) = \sqrt{x}$ by drawing their graphs on a common screen using the viewing rectangle $[-1, 30]$ by $[-1, 6]$.

72. **(a)** By drawing the graphs of the functions

$$f(x) = 1 + \ln(1 + x) \qquad \text{and} \qquad g(x) = \sqrt{x}$$

in a suitable viewing rectangle, show that even when a logarithmic function starts out higher than a root function, it is ultimately overtaken by the root function.

(b) Find, correct to two decimal places, the solutions of the equation $\sqrt{x} = 1 + \ln(1 + x)$.

73–74 ■ A family of functions is given.

(a) Draw graphs of the family for $c = 1, 2, 3,$ and 4.

(b) How are the graphs in part (a) related?

73. $f(x) = \log(cx)$

74. $f(x) = c \log x$

75–76 ■ A function $f(x)$ is given.

(a) Find the domain of the function f.

(b) Find the inverse function of f.

75. $f(x) = \log_2(\log_{10} x)$

76. $f(x) = \ln(\ln(\ln x))$

77. **(a)** Find the inverse of the function $f(x) = \dfrac{2^x}{1 + 2^x}$.

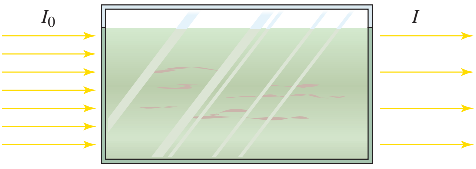
(b) What is the domain of the inverse function?

Applications

78. Absorption of Light A spectrophotometer measures the concentration of a sample dissolved in water by shining a light through it and recording the amount of light that emerges. In other words, if we know the amount of light absorbed, we can calculate the concentration of the sample. For a certain substance, the concentration (in moles/liter) is found using the formula

$$C = -2500 \ln\left(\frac{I}{I_0}\right)$$

where I_0 is the intensity of the incident light and I is the intensity of light that emerges. Find the concentration of the substance if the intensity I is 70% of I_0.



79. Carbon Dating The age of an ancient artifact can be determined by the amount of radioactive carbon-14 remaining in it. If D_0 is the original amount of carbon-14 and D is the amount remaining, then the artifact's age A (in years) is given by

$$A = -8267 \ln\left(\frac{D}{D_0}\right)$$

Find the age of an object if the amount D of carbon-14 that remains in the object is 73% of the original amount D_0.

80. Bacteria Colony A certain strain of bacteria divides every three hours. If a colony is started with 50 bacteria, then the time t (in hours) required for the colony to grow to N bacteria is given by

$$t = 3\frac{\log(N/50)}{\log 2}$$

Find the time required for the colony to grow to a million bacteria.

81. Investment The time required to double the amount of an investment at an interest rate r compounded continuously is given by

$$t = \frac{\ln 2}{r}$$

Find the time required to double an investment at 6%, 7%, and 8%.

82. Charging a Battery The rate at which a battery charges is slower the closer the battery is to its maximum charge C_0. The time (in hours) required to charge a fully discharged battery to a charge C is given by

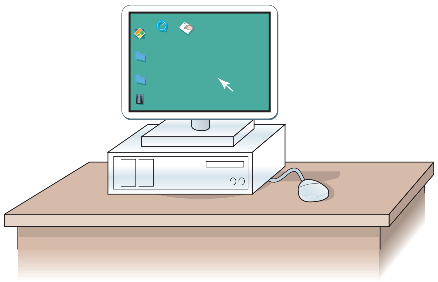
$$t = -k \ln\left(1 - \frac{C}{C_0}\right)$$

where k is a positive constant that depends on the battery. For a certain battery, $k = 0.25$. If this battery is fully discharged, how long will it take to charge to 90% of its maximum charge C_0?

83. Difficulty of a Task The difficulty in "acquiring a target" (such as using your mouse to click on an icon on your computer screen) depends on the distance to the target and the size of the target. According to Fitts's Law, the index of difficulty (ID) is given by

$$\text{ID} = \frac{\log(2A/W)}{\log 2}$$

where W is the width of the target and A is the distance to the center of the target. Compare the difficulty of clicking on an icon that is 5 mm wide to one that is 10 mm wide. In each case, assume the mouse is 100 mm from the icon.



Discovery • Discussion

84. The Height of the Graph of a Logarithmic Function Suppose that the graph of $y = 2^x$ is drawn on a coordinate plane where the unit of measurement is an inch.

(a) Show that at a distance 2 ft to the right of the origin the height of the graph is about 265 mi.

(b) If the graph of $y = \log_2 x$ is drawn on the same set of axes, how far to the right of the origin do we have to go before the height of the curve reaches 2 ft?

85. The Googolplex A **googol** is 10^{100}, and a **googolplex** is 10^{googol}. Find

$$\log(\log(\text{googol})) \qquad \text{and} \qquad \log(\log(\log(\text{googolplex})))$$

86. Comparing Logarithms Which is larger, $\log_4 17$ or $\log_5 24$? Explain your reasoning.

87. The Number of Digits in an Integer Compare log 1000 to the number of digits in 1000. Do the same for 10,000. How many digits does any number between 1000 and 10,000 have? Between what two values must the common logarithm of such a number lie? Use your observations to explain why the number of digits in any positive integer x is $[\![\log x]\!] + 1$. (The symbol $[\![n]\!]$ is the greatest integer function defined in Section 2.2.) How many digits does the number 2^{100} have?

SUGGESTED TIME AND EMPHASIS

1–2 classes.

Essential material.

4.3 Laws of Logarithms

In this section we study properties of logarithms. These properties give logarithmic functions a wide range of applications, as we will see in Section 4.5.

Laws of Logarithms

Since logarithms are exponents, the Laws of Exponents give rise to the Laws of Logarithms.

Laws of Logarithms

Let a be a positive number, with $a \neq 1$. Let A, B, and C be any real numbers with $A > 0$ and $B > 0$.

Law	Description
1. $\log_a(AB) = \log_a A + \log_a B$	The logarithm of a product of numbers is the sum of the logarithms of the numbers.
2. $\log_a\left(\frac{A}{B}\right) = \log_a A - \log_a B$	The logarithm of a quotient of numbers is the difference of the logarithms of the numbers.
3. $\log_a(A^C) = C \log_a A$	The logarithm of a power of a number is the exponent times the logarithm of the number.

POINT TO STRESS

The laws of logarithms, including the change of base formula.

■ **Proof** We make use of the property $\log_a a^x = x$ from Section 4.2.

Law 1. Let $\log_a A = u$ and $\log_a B = v$. When written in exponential form, these equations become

$$a^u = A \qquad \text{and} \qquad a^v = B$$

Thus

$$\begin{aligned}\log_a(AB) &= \log_a(a^u a^v) = \log_a(a^{u+v})\\ &= u + v = \log_a A + \log_a B\end{aligned}$$

Law 2. Using Law 1, we have

$$\log_a A = \log_a\left[\left(\frac{A}{B}\right)B\right] = \log_a\left(\frac{A}{B}\right) + \log_a B$$

so

$$\log_a\left(\frac{A}{B}\right) = \log_a A - \log_a B$$

Law 3. Let $\log_a A = u$. Then $a^u = A$, so

$$\log_a(A^C) = \log_a(a^u)^C = \log_a(a^{uC}) = uC = C\log_a A$$ ■

Example 1 Using the Laws of Logarithms to Evaluate Expressions

Evaluate each expression.

(a) $\log_4 2 + \log_4 32$ (b) $\log_2 80 - \log_2 5$ (c) $-\frac{1}{3}\log 8$

Solution

(a) $\log_4 2 + \log_4 32 = \log_4(2 \cdot 32)$ Law 1

$= \log_4 64 = 3$ Because $64 = 4^3$

(b) $\log_2 80 - \log_2 5 = \log_2\left(\frac{80}{5}\right)$ Law 2

$= \log_2 16 = 4$ Because $16 = 2^4$

(c) $-\frac{1}{3}\log 8 = \log 8^{-1/3}$ Law 3

$= \log\left(\frac{1}{2}\right)$ Property of negative exponents

≈ -0.301 Calculator ■

Expanding and Combining Logarithmic Expressions

The laws of logarithms allow us to write the logarithm of a product or a quotient as the sum or difference of logarithms. This process, called *expanding* a logarithmic expression, is illustrated in the next example.

Example 2 Expanding Logarithmic Expressions

Use the Laws of Logarithms to expand each expression.

(a) $\log_2(6x)$ (b) $\log_5(x^3y^6)$ (c) $\ln\left(\frac{ab}{\sqrt[3]{c}}\right)$

Solution

(a) $\log_2(6x) = \log_2 6 + \log_2 x$ Law 1

SAMPLE QUESTION
Text Question

Given that $\log_2 3 \approx 1.58496$, approximately what is $\log_2 9$?

Answer

$\log_2 9 \approx 3.16992$

ALTERNATE EXAMPLE 1

Find the solution of the equation $3^{x+3} = 5$. If necessary, correct the result to six decimal places.

ANSWER

-1.535026

ALTERNATE EXAMPLE 2

Use the laws of logarithms to expand each expression:

(a) $\log_3(9x)$

(b) $\log_{10}(x^3y^4)$

(c) $\ln\left(\frac{a\sqrt{b}}{c^8}\right)$

ANSWERS

(a) $\log_3 9 + \log_3 x = 2 + \log_3 x$

(b) $3\log_{10} x + 4\log_{10} y$

(c) $\ln a + \frac{1}{2}\ln b - 8\ln c$

(b) $\log_5(x^3y^6) = \log_5 x^3 + \log_5 y^6$ Law 1

$= 3\log_5 x + 6\log_5 y$ Law 3

(c) $\ln\left(\dfrac{ab}{\sqrt[3]{c}}\right) = \ln(ab) - \ln\sqrt[3]{c}$ Law 2

$= \ln a + \ln b - \ln c^{1/3}$ Law 1

$= \ln a + \ln b - \frac{1}{3}\ln c$ Law 3 ■

The laws of logarithms also allow us to reverse the process of expanding done in Example 2. That is, we can write sums and differences of logarithms as a single logarithm. This process, called *combining* logarithmic expressions, is illustrated in the next example.

Example 3 Combining Logarithmic Expressions

Combine $3\log x + \frac{1}{2}\log(x + 1)$ into a single logarithm.

Solution

$$3\log x + \tfrac{1}{2}\log(x+1) = \log x^3 + \log(x+1)^{1/2} \quad \text{Law 3}$$

$$= \log(x^3(x+1)^{1/2}) \quad \text{Law 1}$$ ■

Example 4 Combining Logarithmic Expressions

Combine $3\ln s + \frac{1}{2}\ln t - 4\ln(t^2 + 1)$ into a single logarithm.

Solution

$$3\ln s + \tfrac{1}{2}\ln t - 4\ln(t^2+1) = \ln s^3 + \ln t^{1/2} - \ln(t^2+1)^4 \quad \text{Law 3}$$

$$= \ln(s^3t^{1/2}) - \ln(t^2+1)^4 \quad \text{Law 1}$$

$$= \ln\left(\frac{s^3\sqrt{t}}{(t^2+1)^4}\right) \quad \text{Law 2}$$ ■

WARNING Although the Laws of Logarithms tell us how to compute the logarithm of a product or a quotient, *there is no corresponding rule for the logarithm of a sum or a difference*. For instance,

$$\log_a(x+y) \xcancel{=} \log_a x + \log_a y$$

In fact, we know that the right side is equal to $\log_a(xy)$. Also, don't improperly simplify quotients or powers of logarithms. For instance,

$$\frac{\log 6}{\log 2} \xcancel{=} \log\left(\frac{6}{2}\right) \quad \text{and} \quad (\log_2 x)^3 \xcancel{=} 3\log_2 x$$

Logarithmic functions are used to model a variety of situations involving human behavior. One such behavior is how quickly we forget things we have learned. For example, if you learn algebra at a certain performance level (say 90% on a test) and then don't use algebra for a while, how much will you retain after a week, a month, or a year? Hermann Ebbinghaus (1850–1909) studied this phenomenon and formulated the law described in the next example.

ALTERNATE EXAMPLE 3

Combine $3\ln(x) + \frac{3}{2}\ln(x + 3)$ into a single logarithm.

ANSWER

$\ln\left(x^3\sqrt{(x+3)^3}\right)$

ALTERNATE EXAMPLE 4

Combine $\log(x) - 3\log(y) + \log(x + y)$ into a single logarithm.

ANSWER

$\log\left(\dfrac{x(x+y)}{y^3}\right)$

DRILL QUESTION

Express $5\log(x+2) - \dfrac{1}{3}\log x$ as a single logarithm.

Answer

$\log\left(\dfrac{(x+2)^5}{\sqrt[3]{x}}\right)$

EXAMPLE

A set of logarithms that can be combined:

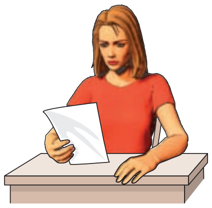
$a\ln(b + c) + \dfrac{3}{4}(\ln a - \ln b) =$

$\ln\left((b+c)^a\sqrt[4]{\left(\dfrac{a}{b}\right)^3}\right)$

IN-CLASS MATERIALS

Make sure your students do not neglect the warning after Example 4. Perhaps have them write out all the rules of logarithms they have learned so far, organized as shown in the figure at right. Perhaps if your students memorize the non-rules, they will be less likely to indulge in algebraic mischief under exam pressure.

The Equivalence Definition	$y = \log_m x$ is equivalent to $x = m^y$
The Conversion Rule	$\log_m(\text{expression}) = \dfrac{\log_n(\text{expression})}{\log_n(m)}$
The Combining Rules	$\log_m(m^{\text{expression}}) = \text{expression}$ $m^{\log_m(\text{expression})} = \text{expression}$
The Arithmetic Rules	$\log_m 1 = 0$ $\log_m(ab) = \log_m a + \log_m b$ $\log_m\left(\dfrac{a}{b}\right) = \log_m a - \log_m b$ $\log_m(a^b) = b\log_m a$
The Non-Rules	$\log_m(a+b) = \log_m(a+b)$ $\dfrac{\log_m a}{\log_m b} = \dfrac{\log_m a}{\log_m b}$ $(\log_m a)^b = (\log_m a)^b$



Forgetting what we've learned depends logarithmically on how long ago we learned it.

Example 5 The Law of Forgetting

Ebbinghaus' Law of Forgetting states that if a task is learned at a performance level P_0, then after a time interval t the performance level P satisfies

$$\log P = \log P_0 - c\log(t+1)$$

where c is a constant that depends on the type of task and t is measured in months.

(a) Solve for P.

(b) If your score on a history test is 90, what score would you expect to get on a similar test after two months? After a year? (Assume $c = 0.2$.)

Solution

(a) We first combine the right-hand side.

$$\log P = \log P_0 - c\log(t+1) \qquad \text{Given equation}$$

$$\log P = \log P_0 - \log(t+1)^c \qquad \text{Law 3}$$

$$\log P = \log\frac{P_0}{(t+1)^c} \qquad \text{Law 2}$$

$$P = \frac{P_0}{(t+1)^c} \qquad \text{Because log is one-to-one}$$

(b) Here $P_0 = 90$, $c = 0.2$, and t is measured in months.

In two months: $t = 2$ and $P = \dfrac{90}{(2+1)^{0.2}} \approx 72$

In one year: $t = 12$ and $P = \dfrac{90}{(12+1)^{0.2}} \approx 54$

Your expected scores after two months and one year are 72 and 54, respectively. ■

Change of Base

For some purposes, we find it useful to change from logarithms in one base to logarithms in another base. Suppose we are given $\log_a x$ and want to find $\log_b x$. Let

$$y = \log_b x$$

We write this in exponential form and take the logarithm, with base a, of each side.

$$b^y = x \qquad \text{Exponential form}$$

$$\log_a(b^y) = \log_a x \qquad \text{Take } \log_a \text{ of each side}$$

$$y\log_a b = \log_a x \qquad \text{Law 3}$$

$$y = \frac{\log_a x}{\log_a b} \qquad \text{Divide by } \log_a b$$

This proves the following formula.

ALTERNATE EXAMPLE 5

A student memorizes the Latin terms for 50 animals in 9th grade. One year later, she still knows 40 of them. According to Ebbinghaus' Law of Forgetting, how many will she know two years after that—when she is in 12th grade?

ANSWER

First we need to find c: $\log 40 = \log 50 - c\log(13)$. So $c \approx 0.087$.

Next, we use our value of c in the Ebbinghaus' equation: $\log P = \log 50 - 0.087\log(37)$. So P is between 36 and 37 Latin terms.

EXAMPLE

A set of logarithms that can be expanded:

Given $\ln 2 \approx 0.693$ and $\ln 3 \approx 1.099$, compute $\ln \frac{8}{9}\sqrt[5]{6}$.

ANSWER

$$\frac{8}{9}\sqrt[5]{6} = \frac{2^3}{3^2} = (2\cdot 3)^{1/5}$$

$$\ln\left(\frac{8}{9}\sqrt[5]{6}\right) = 3\ln 2 - 2\ln 3 + \frac{1}{5}(\ln 2 + \ln 3)$$

$$\approx 0.240569$$

IN-CLASS MATERIALS

Note, without necessarily emphasizing, the importance of domains when applying the log rules. For example, $\ln\left(\frac{-5}{-6}\right)$ is not equal to $\ln(-5) - \ln(-6)$.

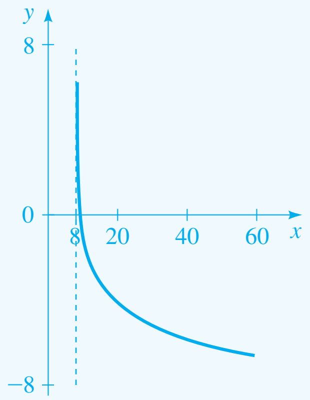
ALTERNATE EXAMPLE 6
Use the change of base formula and common or natural logarithms to evaluate each logarithm, correct to 5 decimal places.

(a) $\log_2 268, 435, 456$
(b) $\log_8 1000$

ANSWERS
(a) 28
(b) 3.32193

ALTERNATE EXAMPLE 7
Use a graphing calculator to graph $f(x) = -3 \log_6 (x - 8)$.

ANSWER
Using the identity $\log_6 (x - 8) = \dfrac{\ln (x - 8)}{\ln (6)}$ we obtain the graph:



We may write the Change of Base Formula as

$$\log_b x = \left(\frac{1}{\log_a b}\right)\log_a x$$

So, $\log_b x$ is just a constant multiple of $\log_a x$; the constant is $\dfrac{1}{\log_a b}$.

Change of Base Formula

$$\log_b x = \frac{\log_a x}{\log_a b}$$

In particular, if we put $x = a$, then $\log_a a = 1$ and this formula becomes

$$\log_b a = \frac{1}{\log_a b}$$

We can now evaluate a logarithm to *any* base by using the Change of Base Formula to express the logarithm in terms of common logarithms or natural logarithms and then using a calculator.

Example 6 Evaluating Logarithms with the Change of Base Formula

Use the Change of Base Formula and common or natural logarithms to evaluate each logarithm, correct to five decimal places.

(a) $\log_8 5$ (b) $\log_9 20$

Solution

(a) We use the Change of Base Formula with $b = 8$ and $a = 10$:

$$\log_8 5 = \frac{\log_{10} 5}{\log_{10} 8} \approx 0.77398$$

(b) We use the Change of Base Formula with $b = 9$ and $a = e$:

$$\log_9 20 = \frac{\ln 20}{\ln 9} \approx 1.36342$$

■

We get the same answer whether we use $\log_{10}$ or ln:

$$\log_8 5 = \frac{\ln 5}{\ln 8} \approx 0.77398$$

Example 7 Using the Change of Base Formula to Graph a Logarithmic Function

Use a graphing calculator to graph $f(x) = \log_6 x$.

Solution Calculators don't have a key for $\log_6$, so we use the Change of Base Formula to write

$$f(x) = \log_6 x = \frac{\ln x}{\ln 6}$$

Since calculators do have an LN key, we can enter this new form of the function and graph it. The graph is shown in Figure 1. ■

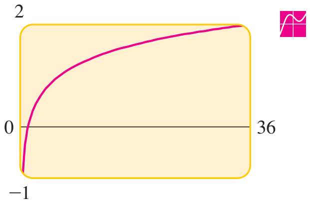


Figure 1
$f(x) = \log_6 x = \dfrac{\ln x}{\ln 6}$

4.3 Exercises

1–12 ■ Evaluate the expression.

1. $\log_3 \sqrt{27}$
2. $\log_2 160 - \log_2 5$
3. $\log 4 + \log 25$
4. $\log \dfrac{1}{\sqrt{1000}}$
5. $\log_4 192 - \log_4 3$
6. $\log_{12} 9 + \log_{12} 16$
7. $\log_2 6 - \log_2 15 + \log_2 20$
8. $\log_3 100 - \log_3 18 - \log_3 50$

IN-CLASS MATERIALS

Mention how logarithms were used to do calculations before the widespread availability of calculators. For example, every scientist had a table like this one:

x	$\ln x$	x	$\ln x$
4.4	1.48160	9.8	2.28238
4.5	1.50408	9.9	2.29253
4.6	1.52606	10.0	2.30259
4.7	1.54756	10.1	2.31254
4.8	1.56862	10.2	2.32239

Now let's say a scientist wanted to find $\sqrt[3]{100}$. He would write

$$\begin{aligned}\sqrt[3]{100} &= 10^{2/3}\\ \ln(\sqrt[3]{100}) &= \ln(10^{2/3})\\ &= \frac{2}{3}\ln(10)\end{aligned}$$

Then he would look up ln (10) from the table to get

$$\begin{aligned}\ln(\sqrt[3]{100}) &\approx \frac{2}{3}(2.30259)\\ &\approx 1.53506\end{aligned}$$

And to find $\sqrt[3]{100}$, he would try to find 1.53506 in the right-hand column of the table. The result is that $\sqrt[3]{100}$ is between 4.6 and 4.7. In practice, the distance between table entries was much closer than 0.1. For a quicker, less accurate estimate, a slide rule was used.

9. $\log_4 16^{100}$

10. $\log_2 8^{33}$

11. $\log(\log 10^{10,000})$

12. $\ln(\ln e^{e^{200}})$

13–38 ■ Use the Laws of Logarithms to expand the expression.

13. $\log_2(2x)$

14. $\log_3(5y)$

15. $\log_2(x(x - 1))$

16. $\log_5 \dfrac{x}{2}$

17. $\log 6^{10}$

18. $\ln \sqrt{z}$

19. $\log_2(AB^2)$

20. $\log_6 \sqrt[4]{17}$

21. $\log_3(x\sqrt{y})$

22. $\log_2(xy)^{10}$

23. $\log_5 \sqrt[3]{x^2 + 1}$

24. $\log_a\left(\dfrac{x^2}{yz^3}\right)$

25. $\ln \sqrt{ab}$

26. $\ln \sqrt[3]{3r^2 s}$

27. $\log\left(\dfrac{x^3 y^4}{z^6}\right)$

28. $\log\left(\dfrac{a^2}{b^4\sqrt{c}}\right)$

29. $\log_2\left(\dfrac{x(x^2 + 1)}{\sqrt{x^2 - 1}}\right)$

30. $\log_5 \sqrt{\dfrac{x - 1}{x + 1}}$

31. $\ln\left(x\sqrt{\dfrac{y}{z}}\right)$

32. $\ln \dfrac{3x^2}{(x + 1)^{10}}$

33. $\log \sqrt[4]{x^2 + y^2}$

34. $\log\left(\dfrac{x}{\sqrt[3]{1 - x}}\right)$

35. $\log \sqrt{\dfrac{x^2 + 4}{(x^2 + 1)(x^3 - 7)^2}}$

36. $\log \sqrt{x\sqrt{y\sqrt{z}}}$

37. $\ln\left(\dfrac{x^3\sqrt{x - 1}}{3x + 4}\right)$

38. $\log\left(\dfrac{10^x}{x(x^2 + 1)(x^4 + 2)}\right)$

39–48 ■ Use the Laws of Logarithms to combine the expression.

39. $\log_3 5 + 5 \log_3 2$

40. $\log 12 + \frac{1}{2} \log 7 - \log 2$

41. $\log_2 A + \log_2 B - 2 \log_2 C$

42. $\log_5(x^2 - 1) - \log_5(x - 1)$

43. $4 \log x - \frac{1}{3} \log(x^2 + 1) + 2 \log(x - 1)$

44. $\ln(a + b) + \ln(a - b) - 2 \ln c$

45. $\ln 5 + 2 \ln x + 3 \ln(x^2 + 5)$

46. $2(\log_5 x + 2 \log_5 y - 3 \log_5 z)$

47. $\frac{1}{3} \log(2x + 1) + \frac{1}{2}[\log(x - 4) - \log(x^4 - x^2 - 1)]$

48. $\log_a b + c \log_a d - r \log_a s$

49–56 ■ Use the Change of Base Formula and a calculator to evaluate the logarithm, correct to six decimal places. Use either natural or common logarithms.

49. $\log_2 5$

50. $\log_5 2$

51. $\log_3 16$

52. $\log_6 92$

53. $\log_7 2.61$

54. $\log_6 532$

55. $\log_4 125$

56. $\log_{12} 2.5$

57. Use the Change of Base Formula to show that

$$\log_3 x = \frac{\ln x}{\ln 3}$$

Then use this fact to draw the graph of the function $f(x) = \log_3 x$.

58. Draw graphs of the family of functions $y = \log_a x$ for $a = 2, e, 5$, and 10 on the same screen, using the viewing rectangle $[0, 5]$ by $[-3, 3]$. How are these graphs related?

59. Use the Change of Base Formula to show that

$$\log e = \frac{1}{\ln 10}$$

60. Simplify: $(\log_2 5)(\log_5 7)$

61. Show that $-\ln(x - \sqrt{x^2 - 1}) = \ln(x + \sqrt{x^2 - 1})$.

Applications

62. Forgetting Use the Ebbinghaus Forgetting Law (Example 5) to estimate a student's score on a biology test two years after he got a score of 80 on a test covering the same material. Assume $c = 0.3$ and t is measured in months.

63. Wealth Distribution Vilfredo Pareto (1848–1923) observed that most of the wealth of a country is owned by a few members of the population. **Pareto's Principle** is

$$\log P = \log c - k \log W$$

where W is the wealth level (how much money a person has) and P is the number of people in the population having that much money.

(a) Solve the equation for P.

(b) Assume $k = 2.1$, $c = 8000$, and W is measured in millions of dollars. Use part (a) to find the number of people who have \$2 million or more. How many people have \$10 million or more?

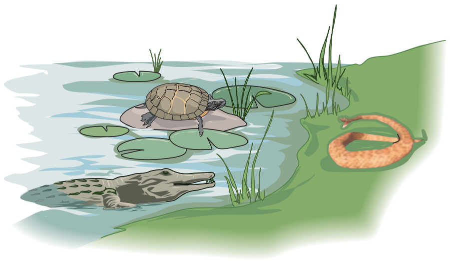
64. Biodiversity Some biologists model the number of species S in a fixed area A (such as an island) by the Species-Area relationship

$$\log S = \log c + k \log A$$

where c and k are positive constants that depend on the type of species and habitat.

(a) Solve the equation for S.

(b) Use part (a) to show that if $k = 3$ then doubling the area increases the number of species eightfold.



65. Magnitude of Stars The magnitude M of a star is a measure of how bright a star appears to the human eye. It is defined by

$$M = -2.5 \log\left(\frac{B}{B_0}\right)$$

where B is the actual brightness of the star and B_0 is a constant.

(a) Expand the right-hand side of the equation.

(b) Use part (a) to show that the brighter a star, the less its magnitude.

(c) Betelgeuse is about 100 times brighter than Albiero. Use part (a) to show that Betelgeuse is 5 magnitudes less than Albiero.

Discovery • Discussion

66. True or False? Discuss each equation and determine whether it is true for all possible values of the variables. (Ignore values of the variables for which any term is undefined.)

(a) $\log\left(\frac{x}{y}\right) = \frac{\log x}{\log y}$

(b) $\log_2(x - y) = \log_2 x - \log_2 y$

(c) $\log_5\left(\frac{a}{b^2}\right) = \log_5 a - 2\log_5 b$

(d) $\log 2^z = z \log 2$

(e) $(\log P)(\log Q) = \log P + \log Q$

(f) $\frac{\log a}{\log b} = \log a - \log b$

(g) $(\log_2 7)^x = x \log_2 7$

(h) $\log_a a^a = a$

(i) $\log(x - y) = \frac{\log x}{\log y}$

(j) $-\ln\left(\frac{1}{A}\right) = \ln A$

67. Find the Error What is wrong with the following argument?

$$\begin{aligned} \log 0.1 &< 2 \log 0.1 \\ &= \log(0.1)^2 \\ &= \log 0.01 \\ \log 0.1 &< \log 0.01 \\ 0.1 &< 0.01 \end{aligned}$$

68. Shifting, Shrinking, and Stretching Graphs of Functions Let $f(x) = x^2$. Show that $f(2x) = 4f(x)$, and explain how this shows that shrinking the graph of f horizontally has the same effect as stretching it vertically. Then use the identities $e^{2+x} = e^2e^x$ and $\ln(2x) = \ln 2 + \ln x$ to show that for $g(x) = e^x$, a horizontal shift is the same as a vertical stretch and for $h(x) = \ln x$, a horizontal shrinking is the same as a vertical shift.

4.4 Exponential and Logarithmic Equations

In this section we solve equations that involve exponential or logarithmic functions. The techniques we develop here will be used in the next section for solving applied problems.

Exponential Equations

An *exponential equation* is one in which the variable occurs in the exponent. For example,

$$2^x = 7$$

The variable x presents a difficulty because it is in the exponent. To deal with this

SUGGESTED TIME AND EMPHASIS

$\frac{1}{2}$–1 class.

Essential material. Can be combined with Section 4.5.

POINT TO STRESS

Solving equations involving exponential and logarithmic functions, algebraically and graphically.

difficulty, we take the logarithm of each side and then use the Laws of Logarithms to "bring down x" from the exponent.

$$
\begin{aligned}
2^x &= 7 && \text{Given equation}\\
\ln 2^x &= \ln 7 && \text{Take ln of each side}\\
x \ln 2 &= \ln 7 && \text{Law 3 (bring down exponent)}\\
x &= \frac{\ln 7}{\ln 2} && \text{Solve for x}\\
&\approx 2.807 && \text{Calculator}
\end{aligned}
$$

Recall that Law 3 of the Laws of Logarithms says that $\log_a A^C = C \log_a A$.

The method we used to solve $2^x = 7$ is typical of how we solve exponential equations in general.

Guidelines for Solving Exponential Equations

1. Isolate the exponential expression on one side of the equation.
2. Take the logarithm of each side, then use the Laws of Logarithms to "bring down the exponent."
3. Solve for the variable.

Example 1 Solving an Exponential Equation

Find the solution of the equation $3^{x+2} = 7$, correct to six decimal places.

Solution We take the common logarithm of each side and use Law 3.

$$
\begin{aligned}
3^{x+2} &= 7 && \text{Given equation}\\
\log(3^{x+2}) &= \log 7 && \text{Take log of each side}\\
(x + 2)\log 3 &= \log 7 && \text{Law 3 (bring down exponent)}\\
x + 2 &= \frac{\log 7}{\log 3} && \text{Divide by log 3}\\
x &= \frac{\log 7}{\log 3} - 2 && \text{Subtract 2}\\
&\approx -0.228756 && \text{Calculator}
\end{aligned}
$$

■

We could have used natural logarithms instead of common logarithms. In fact, using the same steps, we get

$$x = \frac{\ln 7}{\ln 3} - 2 \approx -0.228756$$

Check Your Answer Substituting $x = -0.228756$ into the original equation and using a calculator, we get

$$3^{(-0.228756)+2} \approx 7 \quad ✓$$

SAMPLE QUESTION
Text Question

Solve the equation $4 + 3\log(2x) = 16$.

Answer

$x = 5000$

ALTERNATE EXAMPLE 1

Find the solution of the equation $3^{x+3} = 5$. If necessary, correct the result to six decimal places.

ANSWER

-1.535026

ALTERNATE EXAMPLE 2
Solve the equation $8\,e^{2x} = 24$. Please round the answer to the nearest thousandth.

ANSWER
0.549

EXAMPLE
Solve $(\ln x - 2)^3 - 4(\ln x - 2) = 0$.

ANSWER
$x = 1, e^2, e^4$

ALTERNATE EXAMPLE 3
Solve the equation $e^{5-2x} = 3$. Please round the answer to the nearest thousandth.

ANSWER
1.951

Example 2 Solving an Exponential Equation

Solve the equation $8e^{2x} = 20$.

Solution We first divide by 8 in order to isolate the exponential term on one side of the equation.

$$\begin{aligned} 8e^{2x} &= 20 && \text{Given equation} \\ e^{2x} &= \tfrac{20}{8} && \text{Divide by 8} \\ \ln e^{2x} &= \ln 2.5 && \text{Take ln of each side} \\ 2x &= \ln 2.5 && \text{Property of ln} \\ x &= \frac{\ln 2.5}{2} && \text{Divide by 2} \\ &\approx 0.458 && \text{Calculator} \end{aligned}$$

■

Check Your Answer Substituting $x = 0.458$ into the original equation and using a calculator, we get

$$8e^{2(0.458)} \approx 20 \quad ✓$$

Radiocarbon dating is a method archeologists use to determine the age of ancient objects. The carbon dioxide in the atmosphere always contains a fixed fraction of radioactive carbon, carbon-14 (^{14}C), with a half-life of about 5730 years. Plants absorb carbon dioxide from the atmosphere, which then makes its way to animals through the food chain. Thus, all living creatures contain the same fixed proportions of ^{14}C to nonradioactive ^{12}C as the atmosphere.

After an organism dies, it stops assimilating ^{14}C, and the amount of ^{14}C in it begins to decay exponentially. We can then determine the time elapsed since the death of the organism by measuring the amount of ^{14}C left in it.

For example, if a donkey bone contains 73% as much ^{14}C as a living donkey and it died t years ago, then by the formula for radioactive decay (Section 4.5),

$$0.73 = (1.00)e^{-(t \ln 2)/5730}$$

We solve this exponential equation to find $t \approx 2600$, so the bone is about 2600 years old.

Example 3 Solving an Exponential Equation Algebraically and Graphically

Solve the equation $e^{3-2x} = 4$ algebraically and graphically.

Solution 1: Algebraic

Since the base of the exponential term is e, we use natural logarithms to solve this equation.

$$\begin{aligned} e^{3-2x} &= 4 && \text{Given equation} \\ \ln(e^{3-2x}) &= \ln 4 && \text{Take ln of each side} \\ 3 - 2x &= \ln 4 && \text{Property of ln} \\ 2x &= 3 - \ln 4 && \\ x &= \tfrac{1}{2}(3 - \ln 4) \approx 0.807 && \end{aligned}$$

You should check that this answer satisfies the original equation.

Solution 2: Graphical

We graph the equations $y = e^{3-2x}$ and $y = 4$ in the same viewing rectangle as in Figure 1. The solutions occur where the graphs intersect. Zooming in on the point of intersection of the two graphs, we see that $x \approx 0.81$. ■

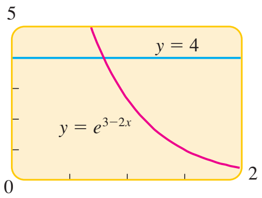


Figure 1

Example 4 An Exponential Equation of Quadratic Type

Solve the equation $e^{2x} - e^{x} - 6 = 0$.

Solution To isolate the exponential term, we factor.

$$\begin{aligned} e^{2x} - e^{x} - 6 &= 0 && \text{Given equation} \\ (e^{x})^2 - e^{x} - 6 &= 0 && \text{Law of Exponents} \\ (e^{x} - 3)(e^{x} + 2) &= 0 && \text{Factor (a quadratic in } e^{x}\text{)} \\ e^{x} - 3 = 0 \quad \text{or} \quad e^{x} + 2 &= 0 && \text{Zero-Product Property} \\ e^{x} = 3 \qquad\qquad e^{x} &= -2 \end{aligned}$$

The equation $e^{x} = 3$ leads to $x = \ln 3$. But the equation $e^{x} = -2$ has no solution because $e^{x} > 0$ for all x. Thus, $x = \ln 3 \approx 1.0986$ is the only solution. You should check that this answer satisfies the original equation. ■

If we let $w = e^{x}$, we get the quadratic equation

$$w^2 - w - 6 = 0$$

which factors as

$$(w - 3)(w + 2) = 0$$

ALTERNATE EXAMPLE 4

Solve the equation $e^{2x} - e^{x} - 2 = 0$. Please round the answer to four decimal places.

ANSWER

0.6931

Example 5 Solving an Exponential Equation

Solve the equation $3xe^{x} + x^2e^{x} = 0$.

Solution First we factor the left side of the equation.

$$\begin{aligned} 3xe^{x} + x^2e^{x} &= 0 && \text{Given equation} \\ x(3 + x)e^{x} &= 0 && \text{Factor out common factors} \\ x(3 + x) &= 0 && \text{Divide by } e^{x} \text{ (because } e^{x} \neq 0\text{)} \\ x = 0 \quad \text{or} \quad 3 + x &= 0 && \text{Zero-Product Property} \end{aligned}$$

Thus, the solutions are $x = 0$ and $x = -3$. ■

Check Your Answers

$x = 0$:

$3(0)e^{0} + 0^2e^{0} = 0$ ✓

$x = -3$:

$3(-3)e^{-3} + (-3)^2e^{-3}$

$= -9e^{-3} + 9e^{-3} = 0$ ✓

ALTERNATE EXAMPLE 5

Solve the equation $5x^4 e^{x} - x^5 e^{x} = 0$.

ANSWER

$x = 0, x = 5$

Logarithmic Equations

A *logarithmic equation* is one in which a logarithm of the variable occurs. For example,

$$\log_2(x + 2) = 5$$

To solve for x, we write the equation in exponential form.

$$\begin{aligned} x + 2 &= 2^5 && \text{Exponential form} \\ x &= 32 - 2 = 30 && \text{Solve for } x \end{aligned}$$

Another way of looking at the first step is to raise the base, 2, to each side of the equation.

$$\begin{aligned} 2^{\log_2(x+2)} &= 2^5 && \text{Raise 2 to each side} \\ x + 2 &= 2^5 && \text{Property of logarithms} \\ x &= 32 - 2 = 30 && \text{Solve for } x \end{aligned}$$

The method used to solve this simple problem is typical. We summarize the steps as follows.

IN-CLASS MATERIALS

Review the concept of inverse functions. Have students find the inverse of functions such as $f(x) = 2^{x^3+1}$ and $f(x) = \ln(x - 5) + e^3$.

Guidelines for Solving Logarithmic Equations

1. Isolate the logarithmic term on one side of the equation; you may first need to combine the logarithmic terms.
2. Write the equation in exponential form (or raise the base to each side of the equation).
3. Solve for the variable.

ALTERNATE EXAMPLE 6a
Solve the equation $\ln x = 3$. Round the answer to the nearest integer.

ANSWER
20

ALTERNATE EXAMPLE 6b
Solve the equation $\log_2(22 - x) = 3$.

ANSWER
$x = 14$

Example 6 Solving Logarithmic Equations

Solve each equation for x.

(a) $\ln x = 8$ (b) $\log_2(25 - x) = 3$

Solution

(a)

$\ln x = 8$	*Given equation*
$x = e^8$	*Exponential form*

Therefore, $x = e^8 \approx 2981$.

We can also solve this problem another way:

$\ln x = 8$	*Given equation*
$e^{\ln x} = e^8$	*Raise e to each side*
$x = e^8$	*Property of ln*

(b) The first step is to rewrite the equation in exponential form.

$\log_2(25 - x) = 3$	*Given equation*
$25 - x = 2^3$	*Exponential form (or raise 2 to each side)*
$25 - x = 8$	
$x = 25 - 8 = 17$	

■

Check Your Answer

If $x = 17$, we get

$\log_2(25 - 17) = \log_2 8 = 3$ ✓

ALTERNATE EXAMPLE 7
Solve the equation $5 + 4\log(6x) = 25$. Round non-integer answers to the nearest hundredth.

ANSWER
$x = 16666.67$

Example 7 Solving a Logarithmic Equation

Solve the equation $4 + 3\log(2x) = 16$.

Solution We first isolate the logarithmic term. This allows us to write the equation in exponential form.

$4 + 3\log(2x) = 16$	*Given equation*
$3\log(2x) = 12$	*Subtract 4*
$\log(2x) = 4$	*Divide by 3*
$2x = 10^4$	*Exponential form (or raise 10 to each side)*
$x = 5000$	*Divide by 2*

■

Check Your Answer

If $x = 5000$, we get

$$\begin{aligned} 4 + 3\log 2(5000) &= 4 + 3\log 10{,}000 \\ &= 4 + 3(4) \\ &= 16 \end{aligned}$$ ✓

Example 8 Solving a Logarithmic Equation Algebraically and Graphically

Solve the equation $\log(x+2) + \log(x-1) = 1$ algebraically and graphically.

Solution 1: Algebraic

We first combine the logarithmic terms using the Laws of Logarithms.

$$\begin{aligned} \log[(x+2)(x-1)] &= 1 && \text{Law 1} \\ (x+2)(x-1) &= 10 && \text{Exponential form (or raise 10 to each side)} \\ x^2 + x - 2 &= 10 && \text{Expand left side} \\ x^2 + x - 12 &= 0 && \text{Subtract 10} \\ (x+4)(x-3) &= 0 && \text{Factor} \\ x = -4 \quad \text{or} \quad x &= 3 \end{aligned}$$

We check these potential solutions in the original equation and find that $x = -4$ is not a solution (because logarithms of negative numbers are undefined), but $x = 3$ is a solution. (See *Check Your Answers*.)

Check Your Answers

$x = -4$:
$\log(-4+2) + \log(-4-1)$
$= \log(-2) + \log(-5)$
undefined ✗

$x = 3$:
$\log(3+2) + \log(3-1)$
$= \log 5 + \log 2 = \log(5 \cdot 2)$
$= \log 10 = 1$ ✓

Solution 2: Graphical

We first move all terms to one side of the equation:

$$\log(x+2) + \log(x-1) - 1 = 0$$

Then we graph

$$y = \log(x+2) + \log(x-1) - 1$$

as in Figure 2. The solutions are the x-intercepts of the graph. Thus, the only solution is $x \approx 3$.

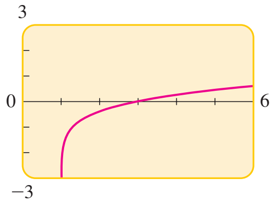


Figure 2 ■

ALTERNATE EXAMPLE 8
Solve the equation $\log(x+8) + \log(x-1) = 1$.

ANSWER
$x = 2$

EXAMPLE
Solve $\log(x^2 - 1) - \log(x+1) = 3$.

ANSWER
$x = 1001$

Example 9 Solving a Logarithmic Equation Graphically

Solve the equation $x^2 = 2\ln(x+2)$.

Solution We first move all terms to one side of the equation

$$x^2 - 2\ln(x+2) = 0$$

Then we graph

$$y = x^2 - 2\ln(x+2)$$

In Example 9, it's not possible to isolate x algebraically, so we must solve the equation graphically.

ALTERNATE EXAMPLE 9
Solve the equation $x^2 = 15\ln(x+5)$ to the nearest integer.

ANSWER
$-3, 6$

IN-CLASS MATERIALS

At this point students know the algebraic rules for working with exponential and logarithmic functions. Stress that if these rules do not suffice to solve an equation, there is a good chance that they cannot find an exact solution. Give students an equation such as $x^2 = 2\ln(x+2)$ (Example 9) and have them try to solve it algebraically. The correct answer is that students cannot do so, but you will find that many make up rules and somehow wind up with a solution.

as in Figure 3. The solutions are the x-intercepts of the graph. Zooming in on the x-intercepts, we see that there are two solutions:

$$x \approx -0.71 \qquad \text{and} \qquad x \approx 1.60$$

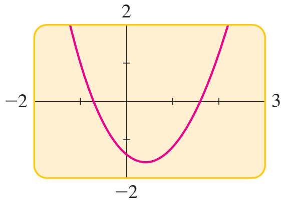


Figure 3

■

The intensity of light in a lake diminishes with depth.

Logarithmic equations are used in determining the amount of light that reaches various depths in a lake. (This information helps biologists determine the types of life a lake can support.) As light passes through water (or other transparent materials such as glass or plastic), some of the light is absorbed. It's easy to see that the murkier the water the more light is absorbed. The exact relationship between light absorption and the distance light travels in a material is described in the next example.

Example 10 Transparency of a Lake

If I_0 and I denote the intensity of light before and after going through a material and x is the distance (in feet) the light travels in the material, then according to the **Beer-Lambert Law**

$$-\frac{1}{k}\ln\left(\frac{I}{I_0}\right) = x$$

where k is a constant depending on the type of material.

(a) Solve the equation for I.

(b) For a certain lake $k = 0.025$ and the light intensity is $I_0 = 14$ lumens (lm). Find the light intensity at a depth of 20 ft.

Solution

(a) We first isolate the logarithmic term.

$$-\frac{1}{k}\ln\left(\frac{I}{I_0}\right) = x \qquad \text{Given equation}$$

$$\ln\left(\frac{I}{I_0}\right) = -kx \qquad \text{Multiply by } -k$$

$$\frac{I}{I_0} = e^{-kx} \qquad \text{Exponential form}$$

$$I = I_0e^{-kx} \qquad \text{Multiply by } I_0$$

(b) We find I using the formula from part (a).

$$I = I_0e^{-kx} \qquad \text{From part (a)}$$

$$= 14e^{(-0.025)(20)} \qquad I_0 = 14,\ k = 0.025,\ x = 20$$

$$\approx 8.49 \qquad \text{Calculator}$$

The light intensity at a depth of 20 ft is about 8.5 lm. ■

ALTERNATE EXAMPLE 10

For a certain lake, the light intensity goes from 20 lumens at the surface to 18 lumens at a depth of 5 feet.

(a) Find the value of k.

(b) What will the intensity be at a depth of 40 feet?

ANSWERS

(a) $-\frac{1}{k}\ln\left(\frac{18}{20}\right) = 5$. Therefore $k \approx 0.0211$.

(b) $-\frac{1}{0.0211}\ln\left(\frac{I}{20}\right) = 40$. Solving for I gives $I \approx 8.5997426$ lumens.

Compound Interest

Recall the formulas for interest that we found in Section 4.1. If a principal P is invested at an interest rate r for a period of t years, then the amount A of the investment is given by

$$\begin{aligned} A &= P(1 + r) && \text{Simple interest (for one year)} \\ A(t) &= P\left(1 + \frac{r}{n}\right)^{nt} && \text{Interest compounded } n \text{ times per year} \\ A(t) &= Pe^{rt} && \text{Interest compounded continuously} \end{aligned}$$

We can use logarithms to determine the time it takes for the principal to increase to a given amount.

Example 11 Finding the Term for an Investment to Double

A sum of \$5000 is invested at an interest rate of 5% per year. Find the time required for the money to double if the interest is compounded according to the following method.

(a) Semiannual (b) Continuous

Solution

(a) We use the formula for compound interest with $P = \$5000$, $A(t) = \$10{,}000$, $r = 0.05$, $n = 2$, and solve the resulting exponential equation for t.

$$\begin{aligned} 5000\left(1 + \frac{0.05}{2}\right)^{2t} &= 10{,}000 && P\left(1+\frac{r}{n}\right)^{nt} = A \\ (1.025)^{2t} &= 2 && \text{Divide by 5000} \\ \log 1.025^{2t} &= \log 2 && \text{Take log of each side} \\ 2t \log 1.025 &= \log 2 && \text{Law 3 (bring down the exponent)} \\ t &= \frac{\log 2}{2 \log 1.025} && \text{Divide by } 2 \log 1.025 \\ t &\approx 14.04 && \text{Calculator} \end{aligned}$$

The money will double in 14.04 years.

(b) We use the formula for continuously compounded interest with $P = \$5000$, $A(t) = \$10{,}000$, $r = 0.05$, and solve the resulting exponential equation for t.

$$\begin{aligned} 5000e^{0.05t} &= 10{,}000 && Pe^{rt} = A \\ e^{0.05t} &= 2 && \text{Divide by 5000} \\ \ln e^{0.05t} &= \ln 2 && \text{Take ln of each side} \\ 0.05t &= \ln 2 && \text{Property of ln} \\ t &= \frac{\ln 2}{0.05} && \text{Divide by 0.05} \\ t &\approx 13.86 && \text{Calculator} \end{aligned}$$

The money will double in 13.86 years. ■

ALTERNATE EXAMPLE 11

A sum of \$2000 is invested at an interest rate of 3.25% per year. Find the time for the money to double if the interest is compounded according to the following method:

(a) Semiannual
(b) Monthly
(c) Continuous

ANSWERS

(a) Approximately 21.500 years
(b) Approximately 21.356 years
(c) Approximately 21.328 years

IN-CLASS MATERIALS

Build up some tough problems from simple ones. For example, first have students solve $x^2 + 2x - 15 = 0$. Then have them solve $(e^x - 1)^2 + 2(e^x - 1) - 15 = 0$. One can even belabor the point with $\ln((e^x - 1)^2 + 2(e^x - 1) - 14) = 0$.

ALTERNATE EXAMPLE 12

A sum of \$4000 is invested at an interest rate of 9% per year. Find the time required for the money to double if the interest is compounded continuously.

ANSWER

7.7

Example 12 Time Required to Grow an Investment

A sum of \$1000 is invested at an interest rate of 4% per year. Find the time required for the amount to grow to \$4000 if interest is compounded continuously.

Solution We use the formula for continuously compounded interest with $P = \$1000$, $A(t) = \$4000$, $r = 0.04$, and solve the resulting exponential equation for t.

$$
\begin{aligned}
1000e^{0.04t} &= 4000 && \text{$Pe^{rt} = A$}\\
e^{0.04t} &= 4 && \text{Divide by 1000}\\
0.04t &= \ln 4 && \text{Take ln of each side}\\
t &= \frac{\ln 4}{0.04} && \text{Divide by 0.04}\\
t &\approx 34.66 && \text{Calculator}
\end{aligned}
$$

The amount will be \$4000 in about 34 years and 8 months. ■

If an investment earns compound interest, then the **annual percentage yield** (APY) is the *simple* interest rate that yields the same amount at the end of one year.

ALTERNATE EXAMPLE 13

Find the annual percentage yield for an investment that earns interest at a rate of 3% per year, compounded daily.

ANSWER

3.045%

Example 13 Calculating the Annual Percentage Yield

Find the annual percentage yield for an investment that earns interest at a rate of 6% per year, compounded daily.

Solution After one year, a principal P will grow to the amount

$$A = P\left(1 + \frac{0.06}{365}\right)^{365} = P(1.06183)$$

The formula for simple interest is

$$A = P(1 + r)$$

Comparing, we see that $1 + r = 1.06183$, so $r = 0.06183$. Thus the annual percentage yield is 6.183%. ■

4.4 Exercises

1–26 ■ Find the solution of the exponential equation, correct to four decimal places.

1. $10^x = 25$

2. $10^{-x} = 4$

3. $e^{-2x} = 7$

4. $e^{3x} = 12$

5. $2^{1-x} = 3$

6. $3^{2x-1} = 5$

7. $3e^x = 10$

8. $2e^{12x} = 17$

9. $e^{1-4x} = 2$

10. $4(1 + 10^{5x}) = 9$

11. $4 + 3^{5x} = 8$

12. $2^{3x} = 34$

13. $8^{0.4x} = 5$

14. $3^{x/14} = 0.1$

15. $5^{-x/100} = 2$

16. $e^{3-5x} = 16$

17. $e^{2x+1} = 200$

18. $\left(\frac{1}{4}\right)^x = 75$

19. $5^x = 4^{x+1}$

20. $10^{1-x} = 6^x$

21. $2^{3x+1} = 3^{x-2}$

22. $7^{x/2} = 5^{1-x}$

23. $\dfrac{50}{1 + e^{-x}} = 4$

24. $\dfrac{10}{1 + e^{-x}} = 2$

25. $100(1.04)^{2t} = 300$

26. $(1.00625)^{12t} = 2$

DRILL QUESTION

If I invest \$2000 at an annual interest rate of 3%, compounded continuously, how long will it take the investment to double?

Answer

$$\frac{\ln 2}{0.03} \approx 23 \text{ years}$$

27–34 ■ Solve the equation.

27. $x^2 2^x - 2^x = 0$

28. $x^2 10^x - x10^x = 2(10^x)$

29. $4x^3 e^{-3x} - 3x^4 e^{-3x} = 0$

30. $x^2 e^x + xe^x - e^x = 0$

31. $e^{2x} - 3e^x + 2 = 0$

32. $e^{2x} - e^x - 6 = 0$

33. $e^{4x} + 4e^{2x} - 21 = 0$

34. $e^x - 12e^{-x} - 1 = 0$

35–50 ■ Solve the logarithmic equation for x.

35. $\ln x = 10$

36. $\ln(2 + x) = 1$

37. $\log x = -2$

38. $\log(x - 4) = 3$

39. $\log(3x + 5) = 2$

40. $\log_3(2 - x) = 3$

41. $2 - \ln(3 - x) = 0$

42. $\log_2(x^2 - x - 2) = 2$

43. $\log_2 3 + \log_2 x = \log_2 5 + \log_2(x - 2)$

44. $2 \log x = \log 2 + \log(3x - 4)$

45. $\log x + \log(x - 1) = \log(4x)$

46. $\log_5 x + \log_5(x + 1) = \log_5 20$

47. $\log_5(x + 1) - \log_5(x - 1) = 2$

48. $\log x + \log(x - 3) = 1$

49. $\log_9(x - 5) + \log_9(x + 3) = 1$

50. $\ln(x - 1) + \ln(x + 2) = 1$

51. For what value of x is the following true?

$$\log(x + 3) = \log x + \log 3$$

52. For what value of x is it true that $(\log x)^3 = 3\log x$?

53. Solve for x: $2^{2/\log_5 x} = \frac{1}{16}$

54. Solve for x: $\log_2(\log_3 x) = 4$

55–62 ■ Use a graphing device to find all solutions of the equation, correct to two decimal places.

55. $\ln x = 3 - x$

56. $\log x = x^2 - 2$

57. $x^3 - x = \log(x + 1)$

58. $x = \ln(4 - x^2)$

59. $e^x = -x$

60. $2^{-x} = x - 1$

61. $4^{-x} = \sqrt{x}$

62. $e^{x^2} - 2 = x^3 - x$

63–66 ■ Solve the inequality.

63. $\log(x - 2) + \log(9 - x) < 1$

64. $3 \le \log_2 x \le 4$

65. $2 < 10^x < 5$

66. $x^2 e^x - 2e^x < 0$

Applications

67. Compound Interest A man invests $5000 in an account that pays 8.5% interest per year, compounded quarterly.

(a) Find the amount after 3 years.

(b) How long will it take for the investment to double?

68. Compound Interest A man invests $6500 in an account that pays 6% interest per year, compounded continuously.

(a) What is the amount after 2 years?

(b) How long will it take for the amount to be $8000?

69. Compound Interest Find the time required for an investment of $5000 to grow to $8000 at an interest rate of 7.5% per year, compounded quarterly.

70. Compound Interest Nancy wants to invest $4000 in saving certificates that bear an interest rate of 9.75% per year, compounded semiannually. How long a time period should she choose in order to save an amount of $5000?

71. Doubling an Investment How long will it take for an investment of $1000 to double in value if the interest rate is 8.5% per year, compounded continuously?

72. Interest Rate A sum of $1000 was invested for 4 years, and the interest was compounded semiannually. If this sum amounted to $1435.77 in the given time, what was the interest rate?

73. Annual Percentage Yield Find the annual percentage yield for an investment that earns 8% per year, compounded monthly.

74. Annual Percentage Yield Find the annual percentage yield for an investment that earns $5\frac{1}{2}$% per year, compounded continuously.

75. Radioactive Decay A 15-g sample of radioactive iodine decays in such a way that the mass remaining after t days is given by $m(t) = 15e^{-0.087t}$ where $m(t)$ is measured in grams. After how many days is there only 5 g remaining?

76. Skydiving The velocity of a sky diver t seconds after jumping is given by $v(t) = 80(1 - e^{-0.2t})$. After how many seconds is the velocity 70 ft/s?

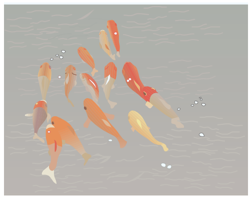
77. Fish Population A small lake is stocked with a certain species of fish. The fish population is modeled by the function

$$P = \frac{10}{1 + 4e^{-0.8t}}$$

where P is the number of fish in thousands and t is measured in years since the lake was stocked.

(a) Find the fish population after 3 years.

(b) After how many years will the fish population reach 5000 fish?



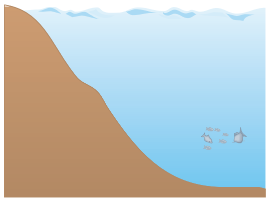
78. Transparency of a Lake Environmental scientists measure the intensity of light at various depths in a lake to find the "transparency" of the water. Certain levels of transparency are required for the biodiversity of the submerged macrophyte population. In a certain lake the intensity of light at depth x is given by

$$I = 10e^{-0.008x}$$

where I is measured in lumens and x in feet.

(a) Find the intensity I at a depth of 30 ft.

(b) At what depth has the light intensity dropped to $I = 5$?



79. Atmospheric Pressure Atmospheric pressure P (in kilopascals, kPa) at altitude h (in kilometers, km) is governed by the formula

$$\ln\left(\frac{P}{P_0}\right) = -\frac{h}{k}$$

where $k = 7$ and $P_0 = 100$ kPa are constants.

(a) Solve the equation for P.

(b) Use part (a) to find the pressure P at an altitude of 4 km.

80. Cooling an Engine Suppose you're driving your car on a cold winter day (20 °F outside) and the engine overheats (at about 220 °F). When you park, the engine begins to cool down. The temperature T of the engine t minutes after you park satisfies the equation

$$\ln\left(\frac{T - 20}{200}\right) = -0.11t$$

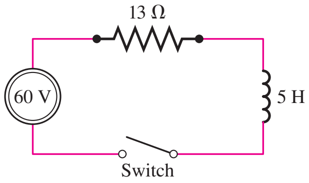
(a) Solve the equation for T.

(b) Use part (a) to find the temperature of the engine after 20 min ($t = 20$).

81. Electric Circuits An electric circuit contains a battery that produces a voltage of 60 volts (V), a resistor with a resistance of 13 ohms (Ω), and an inductor with an inductance of 5 henrys (H), as shown in the figure. Using calculus, it can be shown that the current $I = I(t)$ (in amperes, A) t seconds after the switch is closed is $I = \frac{60}{13}(1 - e^{-13t/5})$.

(a) Use this equation to express the time t as a function of the current I.

(b) After how many seconds is the current 2 A?



82. Learning Curve A *learning curve* is a graph of a function $P(t)$ that measures the performance of someone learning a skill as a function of the training time t. At first, the rate of learning is rapid. Then, as performance increases and approaches a maximal value M, the rate of learning decreases. It has been found that the function

$$P(t) = M - Ce^{-kt}$$

where k and C are positive constants and $C < M$ is a reasonable model for learning.

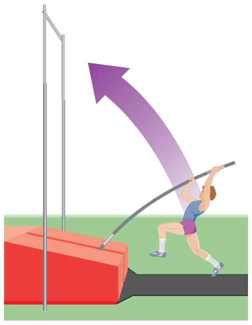
(a) Express the learning time t as a function of the performance level P.

(b) For a pole-vaulter in training, the learning curve is given by

$$P(t) = 20 - 14e^{-0.024t}$$

where $P(t)$ is the height he is able to pole-vault after t months. After how many months of training is he able to vault 12 ft?

(c) Draw a graph of the learning curve in part (b).



Discovery • Discussion

83. Estimating a Solution Without actually solving the equation, find two whole numbers between which the solution of $9^x = 20$ must lie. Do the same for $9^x = 100$. Explain how you reached your conclusions.

84. A Surprising Equation Take logarithms to show that the equation

$$x^{1/\log x} = 5$$

has no solution. For what values of k does the equation

$$x^{1/\log x} = k$$

have a solution? What does this tell us about the graph of the function $f(x) = x^{1/\log x}$? Confirm your answer using a graphing device.

85. Disguised Equations Each of these equations can be transformed into an equation of linear or quadratic type by applying the hint. Solve each equation.

(a) $(x - 1)^{\log(x-1)} = 100(x - 1)$ [Take log of each side.]

(b) $\log_2 x + \log_4 x + \log_8 x = 11$ [Change all logs to base 2.]

(c) $4^x - 2^{x+1} = 3$ [Write as a quadratic in 2^x.]

4.5 Modeling with Exponential and Logarithmic Functions

SUGGESTED TIME AND EMPHASIS

$\frac{1}{2}$–1 class.

Recommended material. Can be combined with Section 4.5.

Many processes that occur in nature, such as population growth, radioactive decay, heat diffusion, and numerous others, can be modeled using exponential functions. Logarithmic functions are used in models for the loudness of sounds, the intensity of earthquakes, and many other phenomena. In this section we study exponential and logarithmic models.

Exponential Models of Population Growth

Biologists have observed that the population of a species doubles its size in a fixed period of time. For example, under ideal conditions a certain population of bacteria doubles in size every 3 hours. If the culture is started with 1000 bacteria, then after 3 hours there will be 2000 bacteria, after another 3 hours there will be 4000, and so on. If we let $n = n(t)$ be the number of bacteria after t hours, then

$$\begin{aligned}
n(0) &= 1000 \\
n(3) &= 1000 \cdot 2 \\
n(6) &= (1000 \cdot 2) \cdot 2 = 1000 \cdot 2^2 \\
n(9) &= (1000 \cdot 2^2) \cdot 2 = 1000 \cdot 2^3 \\
n(12) &= (1000 \cdot 2^3) \cdot 2 = 1000 \cdot 2^4
\end{aligned}$$

From this pattern it appears that the number of bacteria after t hours is modeled by the function

$$n(t) = 1000 \cdot 2^{t/3}$$

In general, suppose that the initial size of a population is n_0 and the doubling period is a. Then the size of the population at time t is modeled by

$$n(t) = n_0 2^{ct}$$

where $c = 1/a$. If we knew the tripling time b, then the formula would be $n(t) = n_0 3^{ct}$ where $c = 1/b$. These formulas indicate that the growth of the bacteria is modeled by

POINTS TO STRESS

1. Translating verbal descriptions of problems into mathematical models, and solving the problems using the models.
2. Certain standard types of problems such as those dealing with exponential growth and decay and logarithmic scales.

an exponential function. But what base should we use? The answer is e, because then it can be shown (using calculus) that the population is modeled by

$$n(t) = n_0 e^{rt}$$

where r is the *relative rate of growth of population, expressed as a proportion of the population at any time*. For instance, if $r = 0.02$, then at any time t the growth rate is 2% of the population at time t.

Notice that the formula for population growth is the same as that for continuously compounded interest. In fact, the same principle is at work in both cases: The growth of a population (or an investment) per time period is proportional to the size of the population (or the amount of the investment). A population of 1,000,000 will increase more in one year than a population of 1000; in exactly the same way, an investment of \$1,000,000 will increase more in one year than an investment of \$1000.

Exponential Growth Model

A population that experiences **exponential growth** increases according to the model

$$n(t) = n_0 e^{rt}$$

where $n(t)$ = population at time t

n_0 = initial size of the population

r = relative rate of growth (expressed as a proportion of the population)

t = time

In the following examples we assume that the populations grow exponentially.

Example 1 Predicting the Size of a Population

The initial bacterium count in a culture is 500. A biologist later makes a sample count of bacteria in the culture and finds that the relative rate of growth is 40% per hour.

(a) Find a function that models the number of bacteria after t hours.

(b) What is the estimated count after 10 hours?

(c) Sketch the graph of the function $n(t)$.

Solution

(a) We use the exponential growth model with $n_0 = 500$ and $r = 0.4$ to get

$$n(t) = 500e^{0.4t}$$

where t is measured in hours.

(b) Using the function in part (a), we find that the bacterium count after 10 hours is

$$n(10) = 500e^{0.4(10)} = 500e^{4} \approx 27{,}300$$

(c) The graph is shown in Figure 1. ■

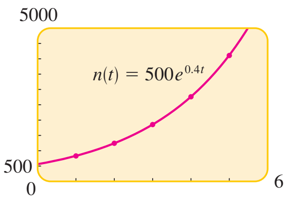


Figure 1

ALTERNATE EXAMPLE 1
The initial bacterium count in a culture is 400. A biologist later makes a sample count of bacteria in the culture and finds that the relative rate of growth is 40% per hour. Using the exponential growth model, find a function that models the number of bacteria after t hours and use it to estimate (to the nearest hundred) the bacterium count after 10 hours.

ANSWER
$n(t) = 400e^{0.4t}$; 21,800

SAMPLE QUESTION
Text Question

Recall that Newton's Law of Cooling is given by

$$T(t) = T_s + D_0 e^{-kt}$$

Which of the constants in this law correspond to surrounding temperature? Which represents the initial difference between the object and its surroundings? How do you know?

Answer

T_s, D_0. There is a horizontal asymptote at T_s, which would have to correspond to the surrounding temperature, because things cool off to the surrounding temperature. Their initial difference is D_0, because when $t = 0$ we know that $T = T_s + D_0$.

Example 2 Comparing Different Rates of Population Growth

In 2000 the population of the world was 6.1 billion and the relative rate of growth was 1.4% per year. It is claimed that a rate of 1.0% per year would make a significant difference in the total population in just a few decades. Test this claim by estimating the population of the world in the year 2050 using a relative rate of growth of (a) 1.4% per year and (b) 1.0% per year.

Graph the population functions for the next 100 years for the two relative growth rates in the same viewing rectangle.

Solution

(a) By the exponential growth model, we have

$$n(t) = 6.1e^{0.014t}$$

where $n(t)$ is measured in billions and t is measured in years since 2000. Because the year 2050 is 50 years after 2000, we find

$$n(50) = 6.1e^{0.014(50)} = 6.1e^{0.7} \approx 12.3$$

The estimated population in the year 2050 is about 12.3 billion.

(b) We use the function

$$n(t) = 6.1e^{0.010t}$$

and find $$n(50) = 6.1e^{0.010(50)} = 6.1e^{0.50} \approx 10.1$$

The estimated population in the year 2050 is about 10.1 billion.

The graphs in Figure 2 show that a small change in the relative rate of growth will, over time, make a large difference in population size. ■

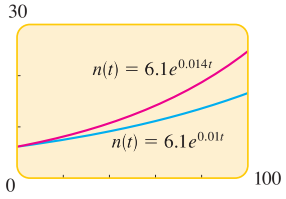
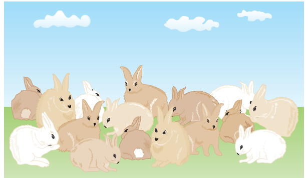


Figure 2

Example 3 Finding the Initial Population



A certain breed of rabbit was introduced onto a small island about 8 years ago. The current rabbit population on the island is estimated to be 4100, with a relative growth rate of 55% per year.

(a) What was the initial size of the rabbit population?

(b) Estimate the population 12 years from now.

Solution

(a) From the exponential growth model, we have

$$n(t) = n_0 e^{0.55t}$$

and we know that the population at time $t = 8$ is $n(8) = 4100$. We substitute what we know into the equation and solve for n_0:

$$4100 = n_0 e^{0.55(8)}$$

$$n_0 = \frac{4100}{e^{0.55(8)}} \approx \frac{4100}{81.45} \approx 50$$

Thus, we estimate that 50 rabbits were introduced onto the island.

ALTERNATE EXAMPLE 2

In 2000 the population of the world was 6.3 billion and the relative rate of growth was 1.4% per year. It is claimed that a rate of 1.0% per year would make a significant difference in the total population in just a few decades. Test this claim by estimating to the nearest tenth of billion the population of the world in the year 2040 using a relative rate of growth of

(a) 1.4% per year

(b) 1% per year

ANSWERS

(a) 11.0

(b) 9.4

ALTERNATE EXAMPLE 3

A certain breed of rabbit was introduced onto a small island about 9 years ago. The current rabbit population on the island is estimated to be 4700, with a relative growth rate of 55% per year.

(a) What was the initial size of the rabbit population?

(b) Estimate the population 11 years from now.

Round both answers to the nearest integer.

ANSWERS

(a) 33

(b) 1,975,847

DRILL QUESTIONS

Recall that the pH of a substance is given by $-\log [H^+]$, where H^+ is the concentration of hydrogen ions measured in moles per liter. Also recall that solutions with a pH of 7 are neutral, those with pH > 7 are basic, and those with pH < 7 are acidic.

(a) If a sample was measured to have hydrogen concentration of $[H^+] = 4 \times 10^{-8}$ M, what would the pH be?

(b) What is the hydrogen ion concentration in a neutral substance?

Answers

(a) 7.4

(b) 10^{-7}

(b) Now that we know n_0, we can write a formula for population growth:

$$n(t) = 50e^{0.55t}$$

Twelve years from now, $t = 20$ and

$$n(20) = 50e^{0.55(20)} \approx 2{,}993{,}707$$

We estimate that the rabbit population on the island 12 years from now will be about 3 million. ■

Another way to solve part (b) is to let t be the number of years from now. In this case, $n_0 = 4100$ (the current population), and the population 12 years from now will be

$$n(12) = 4100e^{0.55(12)} \approx 3 \text{ million}$$

Can the rabbit population in Example 3(b) actually reach such a high number? In reality, as the island becomes overpopulated with rabbits, the rabbit population growth will be slowed due to food shortage and other factors. One model that takes into account such factors is the *logistic growth model* described in the *Focus on Modeling*, page 392.

Standing Room Only

The population of the world was about 6.1 billion in 2000, and was increasing at 1.4% per year. Assuming that each person occupies an average of 4 ft^2 of the surface of the earth, the exponential model for population growth projects that by the year 2801 there will be standing room only! (The total land surface area of the world is about 1.8×10^{15} ft^2.)

Example 4 World Population Projections

The population of the world in 2000 was 6.1 billion, and the estimated relative growth rate was 1.4% per year. If the population continues to grow at this rate, when will it reach 122 billion?

Solution We use the population growth function with $n_0 = 6.1$ billion, $r = 0.014$, and $n(t) = 122$ billion. This leads to an exponential equation, which we solve for t.

$$\begin{aligned}
6.1e^{0.014t} &= 122 && n_0 e^{rt} = n(t)\\
e^{0.014t} &= 20 && \text{Divide by 6.1}\\
\ln e^{0.014t} &= \ln 20 && \text{Take ln of each side}\\
0.014t &= \ln 20 && \text{Property of ln}\\
t &= \frac{\ln 20}{0.014} && \text{Divide by 0.014}\\
t &\approx 213.98 && \text{Calculator}
\end{aligned}$$

Thus, the population will reach 122 billion in approximately 214 years, that is, in the year 2000 + 214 = 2214. ■

Example 5 The Number of Bacteria in a Culture

A culture starts with 10,000 bacteria, and the number doubles every 40 min.

(a) Find a function that models the number of bacteria at time t.

(b) Find the number of bacteria after one hour.

(c) After how many minutes will there be 50,000 bacteria?

(d) Sketch a graph of the number of bacteria at time t.

Solution

(a) To find the function that models this population growth, we need to find the rate r. To do this, we use the formula for population growth with $n_0 = 10{,}000$, $t = 40$, and $n(t) = 20{,}000$, and then solve for r.

ALTERNATE EXAMPLE 5

A culture starts with 5000 bacteria. If this population doubles every 30 minutes, find the number of bacteria after one hour.

ANSWER

20,000

IN-CLASS MATERIALS

Show that the expression $y = e^{kt}$ can be written as $y = at$ and vice versa. Add that e^{kt+c} is equivalent to Ae^{kt}.

$$\begin{aligned}
10{,}000e^{r(40)} &= 20{,}000 && \text{$n_0e^{rt} = n(t)$}\\
e^{40r} &= 2 && \text{Divide by 10,000}\\
\ln e^{40r} &= \ln 2 && \text{Take ln of each side}\\
40r &= \ln 2 && \text{Property of ln}\\
r &= \frac{\ln 2}{40} && \text{Divide by 40}\\
r &\approx 0.01733 && \text{Calculator}
\end{aligned}$$

Now that we know $r \approx 0.01733$, we can write the function for the population growth:

$$n(t) = 10{,}000e^{0.01733t}$$

(b) Using the function we found in part (a) with $t = 60$ min (one hour), we get

$$n(60) = 10{,}000e^{0.01733(60)} \approx 28{,}287$$

Thus, the number of bacteria after one hour is approximately 28,000.

(c) We use the function we found in part (a) with $n(t) = 50{,}000$ and solve the resulting exponential equation for t.

$$\begin{aligned}
10{,}000e^{0.01733t} &= 50{,}000 && \text{$n_0e^{rt} = n(t)$}\\
e^{0.01733t} &= 5 && \text{Divide by 10,000}\\
\ln e^{0.01733t} &= \ln 5 && \text{Take ln of each side}\\
0.01733t &= \ln 5 && \text{Property of ln}\\
t &= \frac{\ln 5}{0.01733} && \text{Divide by 0.01733}\\
t &\approx 92.9 && \text{Calculator}
\end{aligned}$$

The bacterium count will reach 50,000 in approximately 93 min.

(d) The graph of the function $n(t) = 10{,}000e^{0.01733t}$ is shown in Figure 3. ■

Radioactive Decay

Radioactive substances decay by spontaneously emitting radiation. The rate of decay is directly proportional to the mass of the substance. This is analogous to population growth, except that the mass of radioactive material *decreases*. It can be shown that the mass $m(t)$ remaining at time t is modeled by the function

$$m(t) = m_0e^{-rt}$$

where r is the rate of decay expressed as a proportion of the mass and m_0 is the initial mass. Physicists express the rate of decay in terms of **half-life**, the time required for half the mass to decay. We can obtain the rate r from this as follows. If h is the

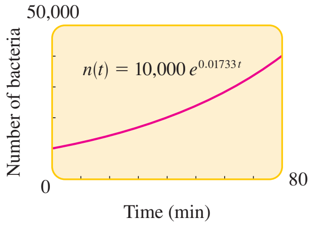


Figure 3

The half-lives of **radioactive elements** vary from very long to very short. Here are some examples.

Element	Half-life
Thorium-232	14.5 billion years
Uranium-235	4.5 billion years
Thorium-230	80,000 years
Plutonium-239	24,360 years
Carbon-14	5,730 years
Radium-226	1,600 years
Cesium-137	30 years
Strontium-90	28 years
Polonium-210	140 days
Thorium-234	25 days
Iodine-135	8 days
Radon-222	3.8 days
Lead-211	3.6 minutes
Krypton-91	10 seconds

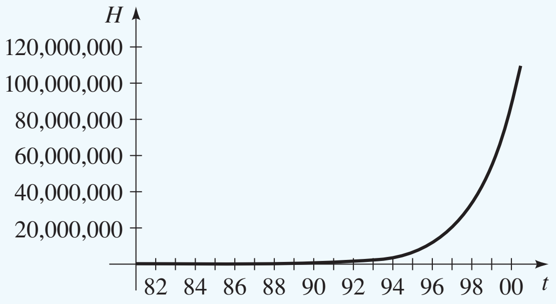
IN-CLASS MATERIALS

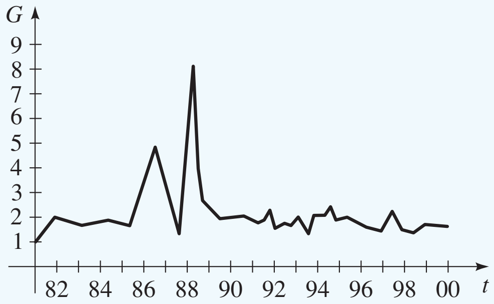
One way to measure the growth of the Internet is to measure the number of Internet hosts. The following data show the number of Internet hosts over time. Try to determine with students if this is exponential growth. (Note: Do not show the students the third column right away. Let them come up with the idea of finding growth rates between data points.)

Month	Hosts	Growth
Aug 1981	213	—
May 1982	235	1.1404
Aug 1983	562	2.0065
Oct 1984	1024	1.6701
Oct 1985	1961	1.9150
Feb 1986	2308	1.6217
Nov 1986	5089	2.8782
Dec 1987	28,174	4.8615
Jul 1988	33,000	1.3112
Oct 1988	56,000	8.1509
Jan 1989	80,000	4.1168
Jul 1989	130,000	2.6620
Oct 1989	159,000	2.2231
Oct 1990	313,000	1.9686
Jan 1991	376,000	2.0824
Jul 1991	535,000	2.0364
Oct 1991	617,000	1.7608
Jan 1992	727,000	1.9172
Apr 1992	890,000	2.2511
Jul 1992	992,000	1.5453
Oct 1992	1,136,000	1.7122
Jan 1993	1,313,000	1.7762
Apr 1993	1,486,000	1.6520
Jul 1993	1,776,000	2.0443
Oct 1993	2,056,000	1.7875
Jan 1994	2,217,000	1.3487
Jul 1994	3,212,000	2.1120
Oct 1994	3,864,000	2.0818
Jan 1995	4,852,000	2.4618
Jul 1995	6,642,000	1.8739
Jan 1996	9,472,000	2.0357
Jul 1996	12,881,000	1.8525
Jan 1997	16,146,000	1.5654
Jul 1997	19,540,000	1.4692
Jan 1998	29,670,000	2.2900
Jul 1998	36,739,000	1.5387
Jan 1999	43,230,000	1.3809
Jul 1999	56,218,000	1.6985
Jan 2000	72,398,092	1.6516
Jul 2000	93,047,785	1.6541
Jan 2001	109,574,429	1.3831

Answer

We can graph the data, and get a curve that looks like exponential growth. We can also graph growth rate and see (except for two spikes in the late 1980s) a more-or-less constant growth rate.





half-life, then a mass of 1 unit becomes $\frac{1}{2}$ unit when $t = h$. Substituting this into the model, we get

$$\begin{aligned} \tfrac{1}{2} &= 1 \cdot e^{-rh} && \textsf{\small $m(t) = m_0e^{-rt}$} \\ \ln\left(\tfrac{1}{2}\right) &= -rh && \textsf{Take ln of each side} \\ r &= -\frac{1}{h}\ln(2^{-1}) && \textsf{Solve for } r \\ r &= \frac{\ln 2}{h} && \textsf{ln } 2^{-1} = -\textsf{ln } 2 \textsf{ by Law 3} \end{aligned}$$

This last equation allows us to find the rate r from the half-life h.

Radioactive Decay Model

If m_0 is the initial mass of a radioactive substance with half-life h, then the mass remaining at time t is modeled by the function

$$m(t) = m_0e^{-rt}$$

where $r = \dfrac{\ln 2}{h}$.

Example 6 Radioactive Decay

Polonium-210 (^{210}Po) has a half-life of 140 days. Suppose a sample of this substance has a mass of 300 mg.

(a) Find a function that models the amount of the sample remaining at time t.

(b) Find the mass remaining after one year.

(c) How long will it take for the sample to decay to a mass of 200 mg?

(d) Draw a graph of the sample mass as a function of time.

Solution

(a) Using the model for radioactive decay with $m_0 = 300$ and $r = (\ln 2/140) \approx 0.00495$, we have

$$m(t) = 300e^{-0.00495t}$$

(b) We use the function we found in part (a) with $t = 365$ (one year).

$$m(365) = 300e^{-0.00495(365)} \approx 49.256$$

Thus, approximately 49 mg of ^{210}Po remains after one year.

(c) We use the function we found in part (a) with $m(t) = 200$ and solve the resulting exponential equation for t.

$$\begin{aligned} 300e^{-0.00495t} &= 200 && m(t) = m_0e^{-rt} \\ e^{-0.00495t} &= \tfrac{2}{3} && \textsf{Divided by 300} \\ \ln e^{-0.00495t} &= \ln \tfrac{2}{3} && \textsf{Take ln of each side} \end{aligned}$$

Radioactive Waste

Harmful radioactive isotopes are produced whenever a nuclear reaction occurs, whether as the result of an atomic bomb test, a nuclear accident such as the one at Chernobyl in 1986, or the uneventful production of electricity at a nuclear power plant.

One radioactive material produced in atomic bombs is the isotope strontium-90 (^{90}Sr), with a half-life of 28 years. This is deposited like calcium in human bone tissue, where it can cause leukemia and other cancers. However, in the decades since atmospheric testing of nuclear weapons was halted, ^{90}Sr levels in the environment have fallen to a level that no longer poses a threat to health.

Nuclear power plants produce radioactive plutonium-239 (^{239}Pu), which has a half-life of 24,360 years. Because of its long half-life, ^{239}Pu could pose a threat to the environment for thousands of years. So, great care must be taken to dispose of it properly. The difficulty of ensuring the safety of the disposed radioactive waste is one reason that nuclear power plants remain controversial.

Joel W. Rogers/Corbis

ALTERNATE EXAMPLE 6

Polonium-210 (^{210}Po) has a half-life of 140 days. Suppose a sample of this substance has a mass of 100 mg. Find the mass remaining after one year (365 days).

ANSWER

16 mg

IN-CLASS MATERIALS

In 1985 there were 15,948 diagnosed cases of AIDS in the United States. In 1990 there were 156,024. Scientists said that if there was no research done, the disease would grow exponentially. Compute the number of cases this model predicts for the year 2000. The actual number was 774,467. Discuss possible flaws in the model with students, and point out the dangers of extrapolation.

$$-0.00495t = \ln\tfrac{2}{3} \qquad \text{Property of ln}$$

$$t = -\frac{\ln\frac{2}{3}}{0.00495} \qquad \text{Divide by } -0.00495$$

$$t \approx 81.9 \qquad \text{Calculator}$$

The time required for the sample to decay to 200 mg is about 82 days.

(d) A graph of the function $m(t) = 300e^{-0.00495t}$ is shown in Figure 4. ■

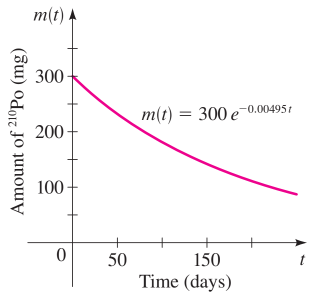


Figure 4

Newton's Law of Cooling

Newton's Law of Cooling states that the rate of cooling of an object is proportional to the temperature difference between the object and its surroundings, provided that the temperature difference is not too large. Using calculus, the following model can be deduced from this law.

Newton's Law of Cooling

If D_0 is the initial temperature difference between an object and its surroundings, and if its surroundings have temperature T_s, then the temperature of the object at time t is modeled by the function

$$T(t) = T_s + D_0e^{-kt}$$

where k is a positive constant that depends on the type of object.

Example 7 Newton's Law of Cooling

A cup of coffee has a temperature of 200°F and is placed in a room that has a temperature of 70°F. After 10 min the temperature of the coffee is 150°F.

(a) Find a function that models the temperature of the coffee at time t.

(b) Find the temperature of the coffee after 15 min.

(c) When will the coffee have cooled to 100°F?

(d) Illustrate by drawing a graph of the temperature function.

Solution

(a) The temperature of the room is $T_s = 70$°F, and the initial temperature difference is

$$D_0 = 200 - 70 = 130°\text{F}$$

So, by Newton's Law of Cooling, the temperature after t minutes is modeled by the function

$$T(t) = 70 + 130e^{-kt}$$

We need to find the constant k associated with this cup of coffee. To do this, we use the fact that when $t = 10$, the temperature is $T(10) = 150$.

ALTERNATE EXAMPLE 7

A cup of coffee has a temperature of 210°F and is placed in a room that has a temperature of 85°F. After 20 minutes the temperature of the coffee is 140°F. Find the temperature of the coffee after 25 minutes.

ANSWER

130°F

So we have

$$
\begin{aligned}
70 + 130e^{-10k} &= 150 && \text{$T_s + D_0e^{-kt} = T(t)$} \\
130e^{-10k} &= 80 && \text{Subtract 70} \\
e^{-10k} &= \tfrac{8}{13} && \text{Divide by 130} \\
-10k &= \ln \tfrac{8}{13} && \text{Take ln of each side} \\
k &= -\tfrac{1}{10} \ln \tfrac{8}{13} && \text{Divide by -10} \\
k &\approx 0.04855 && \text{Caculator}
\end{aligned}
$$

Substituting this value of k into the expression for $T(t)$, we get

$$T(t) = 70 + 130e^{-0.04855t}$$

(b) We use the function we found in part (a) with $t = 15$.

$$T(15) = 70 + 130e^{-0.04855(15)} \approx 133\,°\text{F}$$

(c) We use the function we found in part (a) with $T(t) = 100$ and solve the resulting exponential equation for t.

$$
\begin{aligned}
70 + 130e^{-0.04855t} &= 100 && \text{$T_s + D_0e^{-kt} = T(t)$} \\
130e^{-0.04855t} &= 30 && \text{Subtract 70} \\
e^{-0.04855t} &= \tfrac{3}{13} && \text{Divide by 130} \\
-0.04855t &= \ln \tfrac{3}{13} && \text{Take ln of each side} \\
t &= \frac{\ln \frac{3}{13}}{-0.04855} && \text{Divide by -0.04855} \\
t &\approx 30.2 && \text{Calculator}
\end{aligned}
$$

The coffee will have cooled to 100°F after about half an hour.

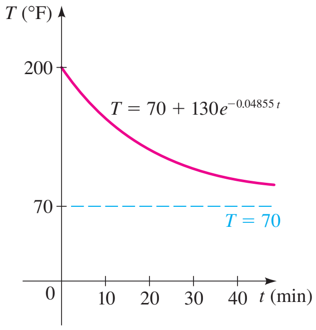


Figure 5
Temperature of coffee after t minutes

(d) The graph of the temperature function is sketched in Figure 5. Notice that the line $t = 70$ is a horizontal asymptote. (Why?) ■

Logarithmic Scales

When a physical quantity varies over a very large range, it is often convenient to take its logarithm in order to have a more manageable set of numbers. We discuss three such situations: the pH scale, which measures acidity; the Richter scale, which measures the intensity of earthquakes; and the decibel scale, which measures the loudness of sounds. Other quantities that are measured on logarithmic scales are light intensity, information capacity, and radiation.

THE pH SCALE Chemists measured the acidity of a solution by giving its hydrogen ion concentration until Sorensen, in 1909, proposed a more convenient measure. He defined

$$\boxed{\text{pH} = -\log[\text{H}^+]}$$

IN-CLASS MATERIALS

Discuss the logistic growth model $P = \dfrac{M}{1 + Ae^{-kt}}$. Have students graph a few of these curves with different values of M and k. This model assumes that an environment has a carrying capacity M. It assumes that when a population is much less than M, a population's growth will look like exponential growth, but that when the population approaches M, the population growth gets very slow, asymptotically approaching M. If the population starts out greater than M, then it will decay, exponentially, to M.

where $[H^+]$ is the concentration of hydrogen ions measured in moles per liter (M). He did this to avoid very small numbers and negative exponents. For instance,

$$\text{if} \quad [H^+] = 10^{-4}\,\text{M}, \qquad \text{then} \qquad \text{pH} = -\log_{10}(10^{-4}) = -(-4) = 4$$

Solutions with a pH of 7 are defined as *neutral*, those with pH < 7 are *acidic*, and those with pH > 7 are *basic*. Notice that when the pH increases by one unit, $[H^+]$ decreases by a factor of 10.

pH for Some Common Substances

Substance	pH
Milk of Magnesia	10.5
Seawater	8.0–8.4
Human blood	7.3–7.5
Crackers	7.0–8.5
Hominy (lye)	6.9–7.9
Cow's milk	6.4–6.8
Spinach	5.1–5.7
Tomatoes	4.1–4.4
Oranges	3.0–4.0
Apples	2.9–3.3
Limes	1.3–2.0
Battery acid	1.0

Example 8 pH Scale and Hydrogen Ion Concentration

(a) The hydrogen ion concentration of a sample of human blood was measured to be $[H^+] = 3.16 \times 10^{-8}$ M. Find the pH and classify the blood as acidic or basic.

(b) The most acidic rainfall ever measured occurred in Scotland in 1974; its pH was 2.4. Find the hydrogen ion concentration.

Solution

(a) A calculator gives

$$\text{pH} = -\log[H^+] = -\log(3.16 \times 10^{-8}) \approx 7.5$$

Since this is greater than 7, the blood is basic.

(b) To find the hydrogen ion concentration, we need to solve for $[H^+]$ in the logarithmic equation

$$\log[H^+] = -\text{pH}$$

So, we write it in exponential form.

$$[H^+] = 10^{-\text{pH}}$$

In this case, pH $= 2.4$, so

$$[H^+] = 10^{-2.4} \approx 4.0 \times 10^{-3}\ \text{M}$$

■

ALTERNATE EXAMPLE 8

The hydrogen ion concentration in a sample of human blood was measured to be $[H^+] = 5.41 \times 10^{-8}$ M. Find the pH and classify the blood as acidic or basic.

ANSWER

pH $= 7.3$, basic

THE RICHTER SCALE In 1935 the American geologist Charles Richter (1900–1984) defined the magnitude M of an earthquake to be

$$\boxed{M = \log \frac{I}{S}}$$

where I is the intensity of the earthquake (measured by the amplitude of a seismograph reading taken 100 km from the epicenter of the earthquake) and S is the intensity of a "standard" earthquake (whose amplitude is 1 micron $= 10^{-4}$ cm). The magnitude of a standard earthquake is

$$M = \log \frac{S}{S} = \log 1 = 0$$

Richter studied many earthquakes that occurred between 1900 and 1950. The largest had magnitude 8.9 on the Richter scale, and the smallest had magnitude 0. This corresponds to a ratio of intensities of 800,000,000, so the Richter scale provides more

Largest Earthquakes

Location	Date	Magnitude
Chile	1960	9.5
Alaska	1964	9.2
Alaska	1957	9.1
Kamchatka	1952	9.0
Sumatra	2004	9.0
Ecuador	1906	8.8
Alaska	1965	8.7
Tibet	1950	8.6
Kamchatka	1923	8.5
Indonesia	1938	8.5
Kuril Islands	1963	8.5

manageable numbers to work with. For instance, an earthquake of magnitude 6 is ten times stronger than an earthquake of magnitude 5.

Example 9 Magnitude of Earthquakes

The 1906 earthquake in San Francisco had an estimated magnitude of 8.3 on the Richter scale. In the same year a powerful earthquake occurred on the Colombia-Ecuador border and was four times as intense. What was the magnitude of the Colombia-Ecuador earthquake on the Richter scale?

Solution If I is the intensity of the San Francisco earthquake, then from the definition of magnitude we have

$$M = \log \frac{I}{S} = 8.3$$

The intensity of the Colombia-Ecuador earthquake was $4I$, so its magnitude was

$$M = \log \frac{4I}{S} = \log 4 + \log \frac{I}{S} = \log 4 + 8.3 \approx 8.9$$ ■

Example 10 Intensity of Earthquakes

The 1989 Loma Prieta earthquake that shook San Francisco had a magnitude of 7.1 on the Richter scale. How many times more intense was the 1906 earthquake (see Example 9) than the 1989 event?

Solution If I_1 and I_2 are the intensities of the 1906 and 1989 earthquakes, then we are required to find I_1/I_2. To relate this to the definition of magnitude, we divide numerator and denominator by S.

$$\begin{aligned} \log \frac{I_1}{I_2} &= \log \frac{I_1/S}{I_2/S} && \text{Divide numerator and denominator by } S \\ &= \log \frac{I_1}{S} - \log \frac{I_2}{S} && \text{Law 2 of logarithms} \\ &= 8.3 - 7.1 = 1.2 && \text{Definition of earthquake magnitude} \end{aligned}$$

Therefore

$$\frac{I_1}{I_2} = 10^{\log(I_1/I_2)} = 10^{1.2} \approx 16$$

The 1906 earthquake was about 16 times as intense as the 1989 earthquake. ■


Roger Ressmeyer/Corbis

THE DECIBEL SCALE The ear is sensitive to an extremely wide range of sound intensities. We take as a reference intensity $I_0 = 10^{-12}$ W/m^2 (watts per square meter) at a frequency of 1000 hertz, which measures a sound that is just barely audible (the threshold of hearing). The psychological sensation of loudness varies with the logarithm of the intensity (the Weber-Fechner Law) and so the **intensity level** B, measured in decibels (dB), is defined as

$$\boxed{B = 10 \log \frac{I}{I_0}}$$

ALTERNATE EXAMPLE 9

An earthquake in the United States has an estimated magnitude of 3.7 on the Richter scale. In the same year an earthquake in Ecuador was four times as intense. What was the magnitude of the Ecuador earthquake on the Richter scale?

ANSWER

4.3

ALTERNATE EXAMPLE 10

A San Francisco earthquake has a magnitude of 7.5 on the Richter scale. Five years before, San Francisco had one with a magnitude of 8.1 on the Richter scale. How many times more intense was the past earthquake compared to the present one?

ANSWER

4

IN-CLASS MATERIALS

Go over Examples 9 and 10, the Richter scale. Ask students the open-ended question, “How much worse is an earthquake that measures 7 on the Richter scale than an earthquake that measures 6?” and discuss the issue.

The **intensity levels of sounds** that we can hear vary from very loud to very soft. Here are some examples of the decibel levels of commonly heard sounds.

Source of sound	B (dB)
Jet takeoff	140
Jackhammer	130
Rock concert	120
Subway	100
Heavy traffic	80
Ordinary traffic	70
Normal conversation	50
Whisper	30
Rustling leaves	10–20
Threshold of hearing	0

The intensity level of the barely audible reference sound is

$$B = 10 \log \frac{I_0}{I_0} = 10 \log 1 = 0 \text{ dB}$$

Example 11 Sound Intensity of a Jet Takeoff

Find the decibel intensity level of a jet engine during takeoff if the intensity was measured at 100 W/m^2.

Solution From the definition of intensity level we see that

$$B = 10 \log \frac{I}{I_0} = 10 \log \frac{10^2}{10^{-12}} = 10 \log 10^{14} = 140 \text{ dB}$$

Thus, the intensity level is 140 dB. ■

The table in the margin lists decibel intensity levels for some common sounds ranging from the threshold of human hearing to the jet takeoff of Example 11. The threshold of pain is about 120 dB.

ALTERNATE EXAMPLE 11

Find the decibel intensity level of a jet engine during takeoff if the intensity was measured at 10,000 W/m^2.

ANSWER

160

4.5 Exercises

1–13 ■ These exercises use the population growth model.

1. **Bacteria Culture** The number of bacteria in a culture is modeled by the function

$$n(t) = 500e^{0.45t}$$

where t is measured in hours.

(a) What is the initial number of bacteria?

(b) What is the relative rate of growth of this bacterium population? Express your answer as a percentage.

(c) How many bacteria are in the culture after 3 hours?

(d) After how many hours will the number of bacteria reach 10,000?

2. **Fish Population** The number of a certain species of fish is modeled by the function

$$n(t) = 12e^{0.012t}$$

where t is measured in years and $n(t)$ is measured in millions.

(a) What is the relative rate of growth of the fish population? Express your answer as a percentage.

(b) What will the fish population be after 5 years?

(c) After how many years will the number of fish reach 30 million?

(d) Sketch a graph of the fish population function $n(t)$.

3. **Fox Population** The fox population in a certain region has a relative growth rate of 8% per year. It is estimated that the population in 2000 was 18,000.

(a) Find a function that models the population t years after 2000.

(b) Use the function from part (a) to estimate the fox population in the year 2008.

(c) Sketch a graph of the fox population function for the years 2000–2008.

4. **Population of a Country** The population of a country has a relative growth rate of 3% per year. The government is trying to reduce the growth rate to 2%. The population in 1995 was approximately 110 million. Find the projected population for the year 2020 for the following conditions.

(a) The relative growth rate remains at 3% per year.

(b) The relative growth rate is reduced to 2% per year.

5. **Population of a City** The population of a certain city was 112,000 in 1998, and the observed relative growth rate is 4% per year.

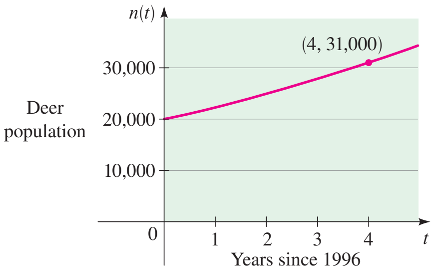
(a) Find a function that models the population after t years.

(b) Find the projected population in the year 2004.

(c) In what year will the population reach 200,000?

6. **Frog Population** The frog population in a small pond grows exponentially. The current population is 85 frogs, and the relative growth rate is 18% per year.
   - **(a)** Find a function that models the population after t years.
   - **(b)** Find the projected population after 3 years.
   - **(c)** Find the number of years required for the frog population to reach 600.

7. **Deer Population** The graph shows the deer population in a Pennsylvania county between 1996 and 2000. Assume that the population grows exponentially.
   - **(a)** What was the deer population in 1996?
   - **(b)** Find a function that models the deer population t years after 1996.
   - **(c)** What is the projected deer population in 2004?
   - **(d)** In what year will the deer population reach 100,000?



8. **Bacteria Culture** A culture contains 1500 bacteria initially and doubles every 30 min.
   - **(a)** Find a function that models the number of bacteria $n(t)$ after t minutes.
   - **(b)** Find the number of bacteria after 2 hours.
   - **(c)** After how many minutes will the culture contain 4000 bacteria?

9. **Bacteria Culture** A culture starts with 8600 bacteria. After one hour the count is 10,000.
   - **(a)** Find a function that models the number of bacteria $n(t)$ after t hours.
   - **(b)** Find the number of bacteria after 2 hours.
   - **(c)** After how many hours will the number of bacteria double?

10. **Bacteria Culture** The count in a culture of bacteria was 400 after 2 hours and 25,600 after 6 hours.
    - **(a)** What is the relative rate of growth of the bacteria population? Express your answer as a percentage.
    - **(b)** What was the initial size of the culture?
    - **(c)** Find a function that models the number of bacteria $n(t)$ after t hours.
    - **(d)** Find the number of bacteria after 4.5 hours.
    - **(e)** When will the number of bacteria be 50,000?

11. **World Population** The population of the world was 5.7 billion in 1995 and the observed relative growth rate was 2% per year.
    - **(a)** By what year will the population have doubled?
    - **(b)** By what year will the population have tripled?

12. **Population of California** The population of California was 10,586,223 in 1950 and 23,668,562 in 1980. Assume the population grows exponentially.
    - **(a)** Find a function that models the population t years after 1950.
    - **(b)** Find the time required for the population to double.
    - **(c)** Use the function from part (a) to predict the population of California in the year 2000. Look up California's actual population in 2000, and compare.

13. **Infectious Bacteria** An infectious strain of bacteria increases in number at a relative growth rate of 200% per hour. When a certain critical number of bacteria are present in the bloodstream, a person becomes ill. If a single bacterium infects a person, the critical level is reached in 24 hours. How long will it take for the critical level to be reached if the same person is infected with 10 bacteria?

14–22 ■ These exercises use the radioactive decay model.

14. **Radioactive Radium** The half-life of radium-226 is 1600 years. Suppose we have a 22-mg sample.
    - **(a)** Find a function that models the mass remaining after t years.
    - **(b)** How much of the sample will remain after 4000 years?
    - **(c)** After how long will only 18 mg of the sample remain?

15. **Radioactive Cesium** The half-life of cesium-137 is 30 years. Suppose we have a 10-g sample.
    - **(a)** Find a function that models the mass remaining after t years.
    - **(b)** How much of the sample will remain after 80 years?
    - **(c)** After how long will only 2 g of the sample remain?

16. **Radioactive Thorium** The mass $m(t)$ remaining after t days from a 40-g sample of thorium-234 is given by

$$m(t) = 40e^{-0.0277t}$$

    - **(a)** How much of the sample will remain after 60 days?
    - **(b)** After how long will only 10 g of the sample remain?
    - **(c)** Find the half-life of thorium-234.

17. **Radioactive Strontium** The half-life of strontium-90 is 28 years. How long will it take a 50-mg sample to decay to a mass of 32 mg?

18. Radioactive Radium Radium-221 has a half-life of 30 s. How long will it take for 95% of a sample to decay?

19. Finding Half-life If 250 mg of a radioactive element decays to 200 mg in 48 hours, find the half-life of the element.

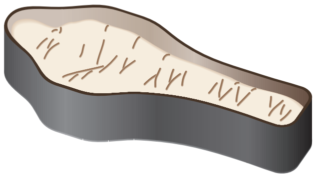
20. Radioactive Radon After 3 days a sample of radon-222 has decayed to 58% of its original amount.

(a) What is the half-life of radon-222?

(b) How long will it take the sample to decay to 20% of its original amount?

21. Carbon-14 Dating A wooden artifact from an ancient tomb contains 65% of the carbon-14 that is present in living trees. How long ago was the artifact made? (The half-life of carbon-14 is 5730 years.)

22. Carbon-14 Dating The burial cloth of an Egyptian mummy is estimated to contain 59% of the carbon-14 it contained originally. How long ago was the mummy buried? (The half-life of carbon-14 is 5730 years.)



23–26 ■ These exercises use Newton's Law of Cooling.

23. Cooling Soup A hot bowl of soup is served at a dinner party. It starts to cool according to Newton's Law of Cooling so that its temperature at time t is given by

$$T(t) = 65 + 145e^{-0.05t}$$

where t is measured in minutes and T is measured in °F.

(a) What is the initial temperature of the soup?

(b) What is the temperature after 10 min?

(c) After how long will the temperature be 100°F?

24. Time of Death Newton's Law of Cooling is used in homicide investigations to determine the time of death. The normal body temperature is 98.6°F. Immediately following death, the body begins to cool. It has been determined experimentally that the constant in Newton's Law of Cooling is approximately $k = 0.1947$, assuming time is measured in hours. Suppose that the temperature of the surroundings is 60°F.

(a) Find a function $T(t)$ that models the temperature t hours after death.

(b) If the temperature of the body is now 72°F, how long ago was the time of death?

25. Cooling Turkey A roasted turkey is taken from an oven when its temperature has reached 185°F and is placed on a table in a room where the temperature is 75°F.

(a) If the temperature of the turkey is 150°F after half an hour, what is its temperature after 45 min?

(b) When will the turkey cool to 100°F?

26. Boiling Water A kettle full of water is brought to a boil in a room with temperature 20°C. After 15 min the temperature of the water has decreased from 100°C to 75°C. Find the temperature after another 10 min. Illustrate by graphing the temperature function.

27–41 ■ These exercises deal with logarithmic scales.

27. Finding pH The hydrogen ion concentration of a sample of each substance is given. Calculate the pH of the substance.

(a) Lemon juice: $[H^+] = 5.0 \times 10^{-3}$ M

(b) Tomato juice: $[H^+] = 3.2 \times 10^{-4}$ M

(c) Seawater: $[H^+] = 5.0 \times 10^{-9}$ M

28. Finding pH An unknown substance has a hydrogen ion concentration of $[H^+] = 3.1 \times 10^{-8}$ M. Find the pH and classify the substance as acidic or basic.

29. Ion Concentration The pH reading of a sample of each substance is given. Calculate the hydrogen ion concentration of the substance.

(a) Vinegar: pH = 3.0

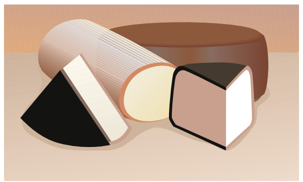
(b) Milk: pH = 6.5

30. Ion Concentration The pH reading of a glass of liquid is given. Find the hydrogen ion concentration of the liquid.

(a) Beer: pH = 4.6

(b) Water: pH = 7.3

31. Finding pH The hydrogen ion concentrations in cheeses range from 4.0×10^{-7} M to 1.6×10^{-5} M. Find the corresponding range of pH readings.



32. Ion Concentration in Wine The pH readings for wines vary from 2.8 to 3.8. Find the corresponding range of hydrogen ion concentrations.

33. Earthquake Magnitudes If one earthquake is 20 times as intense as another, how much larger is its magnitude on the Richter scale?

34. Earthquake Magnitudes The 1906 earthquake in San Francisco had a magnitude of 8.3 on the Richter scale. At the same time in Japan an earthquake with magnitude 4.9

caused only minor damage. How many times more intense was the San Francisco earthquake than the Japanese earthquake?

35. Earthquake Magnitudes The Alaska earthquake of 1964 had a magnitude of 8.6 on the Richter scale. How many times more intense was this than the 1906 San Francisco earthquake? (See Exercise 34.)

36. Earthquake Magnitudes The Northridge, California, earthquake of 1994 had a magnitude of 6.8 on the Richter scale. A year later, a 7.2-magnitude earthquake struck Kobe, Japan. How many times more intense was the Kobe earthquake than the Northridge earthquake?

37. Earthquake Magnitudes The 1985 Mexico City earthquake had a magnitude of 8.1 on the Richter scale. The 1976 earthquake in Tangshan, China, was 1.26 times as intense. What was the magnitude of the Tangshan earthquake?

38. Traffic Noise The intensity of the sound of traffic at a busy intersection was measured at 2.0×10^{-5} W/m^2. Find the intensity level in decibels.

39. Subway Noise The intensity of the sound of a subway train was measured at 98 dB. Find the intensity in W/m^2.

40. Comparing Decibel Levels The noise from a power mower was measured at 106 dB. The noise level at a rock concert was measured at 120 dB. Find the ratio of the intensity of the rock music to that of the power mower.

41. Inverse Square Law for Sound A law of physics states that the intensity of sound is inversely proportional to the square of the distance d from the source: $I = k/d^2$.

(a) Use this model and the equation

$$B = 10 \log \frac{I}{I_0}$$

(described in this section) to show that the decibel levels B_1 and B_2 at distances d_1 and d_2 from a sound source are related by the equation

$$B_2 = B_1 + 20 \log \frac{d_1}{d_2}$$

(b) The intensity level at a rock concert is 120 dB at a distance 2 m from the speakers. Find the intensity level at a distance of 10 m.

4 Review

Concept Check

1. **(a)** Write an equation that defines the exponential function with base a.

(b) What is the domain of this function?

(c) What is the range of this function?

(d) Sketch the general shape of the graph of the exponential function for each case.

(i) $a > 1$ (ii) $0 < a < 1$

2. If x is large, which function grows faster, $y = 2^x$ or $y = x^2$?

3. **(a)** How is the number e defined?

(b) What is the natural exponential function?

4. **(a)** How is the logarithmic function $y = \log_a x$ defined?

(b) What is the domain of this function?

(c) What is the range of this function?

(d) Sketch the general shape of the graph of the function $y = \log_a x$ if $a > 1$.

(e) What is the natural logarithm?

(f) What is the common logarithm?

5. State the three Laws of Logarithms.

6. State the Change of Base Formula.

7. **(a)** How do you solve an exponential equation?

(b) How do you solve a logarithmic equation?

8. Suppose an amount P is invested at an interest rate r and A is the amount after t years.

(a) Write an expression for A if the interest is compounded n times per year.

(b) Write an expression for A if the interest is compounded continuously.

9. If the initial size of a population is n_0 and the population grows exponentially with relative growth rate r, write an expression for the population $n(t)$ at time t.

10. **(a)** What is the half-life of a radioactive substance?

(b) If a radioactive substance has initial mass m_0 and half-life h, write an expression for the mass $m(t)$ remaining at time t.

11. What does Newton's Law of Cooling say?

12. What do the pH scale, the Richter scale, and the decibel scale have in common? What do they measure?

Exercises

1–12 ■ Sketch the graph of the function. State the domain, range, and asymptote.

1. $f(x) = 2^{-x+1}$

2. $f(x) = 3^{x-2}$

3. $g(x) = 3 + 2^x$

4. $g(x) = 5^{-x} - 5$

5. $f(x) = \log_3(x - 1)$

6. $g(x) = \log(-x)$

7. $f(x) = 2 - \log_2 x$

8. $f(x) = 3 + \log_5(x + 4)$

9. $F(x) = e^x - 1$

10. $G(x) = \frac{1}{2}e^{x-1}$

11. $g(x) = 2 \ln x$

12. $g(x) = \ln(x^2)$

13–16 ■ Find the domain of the function.

13. $f(x) = 10^{x^2} + \log(1 - 2x)$

14. $g(x) = \ln(2 + x - x^2)$

15. $h(x) = \ln(x^2 - 4)$

16. $k(x) = \ln|x|$

17–20 ■ Write the equation in exponential form.

17. $\log_2 1024 = 10$

18. $\log_6 37 = x$

19. $\log x = y$

20. $\ln c = 17$

21–24 ■ Write the equation in logarithmic form.

21. $2^6 = 64$

22. $49^{-1/2} = \frac{1}{7}$

23. $10^x = 74$

24. $e^k = m$

25–40 ■ Evaluate the expression without using a calculator.

25. $\log_2 128$

26. $\log_8 1$

27. $10^{\log 45}$

28. $\log 0.000001$

29. $\ln(e^6)$

30. $\log_4 8$

31. $\log_3\left(\frac{1}{27}\right)$

32. $2^{\log_2 13}$

33. $\log_5 \sqrt{5}$

34. $e^{2\ln 7}$

35. $\log 25 + \log 4$

36. $\log_3 \sqrt{243}$

37. $\log_2 16^{23}$

38. $\log_5 250 - \log_5 2$

39. $\log_8 6 - \log_8 3 + \log_8 2$

40. $\log \log 10^{100}$

41–46 ■ Expand the logarithmic expression.

41. $\log(AB^2C^3)$

42. $\log_2(x\sqrt{x^2 + 1})$

43. $\ln\sqrt{\dfrac{x^2 - 1}{x^2 + 1}}$

44. $\log\left(\dfrac{4x^3}{y^2(x - 1)^5}\right)$

45. $\log_5\left(\dfrac{x^2(1 - 5x)^{3/2}}{\sqrt{x^3 - x}}\right)$

46. $\ln\left(\dfrac{\sqrt[3]{x^4 + 12}}{(x + 16)\sqrt{x - 3}}\right)$

47–52 ■ Combine into a single logarithm.

47. $\log 6 + 4 \log 2$

48. $\log x + \log(x^2y) + 3 \log y$

49. $\frac{3}{2}\log_2(x - y) - 2\log_2(x^2 + y^2)$

50. $\log_5 2 + \log_5(x + 1) - \frac{1}{3}\log_5(3x + 7)$

51. $\log(x - 2) + \log(x + 2) - \frac{1}{2}\log(x^2 + 4)$

52. $\frac{1}{2}[\ln(x - 4) + 5\ln(x^2 + 4x)]$

53–62 ■ Solve the equation. Find the exact solution if possible; otherwise approximate to two decimals.

53. $\log_2(1 - x) = 4$

54. $2^{3x-5} = 7$

55. $5^{5-3x} = 26$

56. $\ln(2x - 3) = 14$

57. $e^{3x/4} = 10$

58. $2^{1-x} = 3^{2x+5}$

59. $\log x + \log(x + 1) = \log 12$

60. $\log_8(x + 5) - \log_8(x - 2) = 1$

61. $x^2e^{2x} + 2xe^{2x} = 8e^{2x}$

62. $2^{3^x} = 5$

63–66 ■ Use a calculator to find the solution of the equation, correct to six decimal places.

63. $5^{-2x/3} = 0.63$

64. $2^{3x-5} = 7$

65. $5^{2x+1} = 3^{4x-1}$

66. $e^{-15k} = 10{,}000$

67–70 ■ Draw a graph of the function and use it to determine the asymptotes and the local maximum and minimum values.

67. $y = e^{x/(x+2)}$

68. $y = 2x^2 - \ln x$

69. $y = \log(x^3 - x)$

70. $y = 10^x - 5^x$

71–72 ■ Find the solutions of the equation, correct to two decimal places.

71. $3 \log x = 6 - 2x$

72. $4 - x^2 = e^{-2x}$

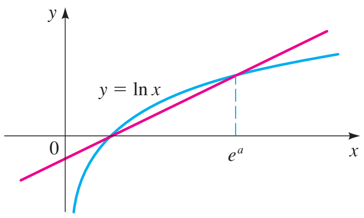
73–74 ■ Solve the inequality graphically.

73. $\ln x > x - 2$

74. $e^x < 4x^2$

75. Use a graph of $f(x) = e^x - 3e^{-x} - 4x$ to find, approximately, the intervals on which f is increasing and on which f is decreasing.

76. Find an equation of the line shown in the figure.



77. Evaluate $\log_4 15$, correct to six decimal places.

78. Solve the inequality: $0.2 \leq \log x < 2$

79. Which is larger, $\log_4 258$ or $\log_5 620$?

80. Find the inverse of the function $f(x) = 2^{3^x}$ and state its domain and range.

81. If $12,000 is invested at an interest rate of 10% per year, find the amount of the investment at the end of 3 years for each compounding method.

(a) Semiannual **(b)** Monthly

(c) Daily **(d)** Continuous

82. A sum of $5000 is invested at an interest rate of $8\frac{1}{2}\%$ per year, compounded semiannually.

(a) Find the amount of the investment after $1\frac{1}{2}$ years.

(b) After what period of time will the investment amount to $7000?

83. The stray-cat population in a small town grows exponentially. In 1999, the town had 30 stray cats and the relative growth rate was 15% per year.

(a) Find a function that models the stray-cat population $n(t)$ after t years.

(b) Find the projected population after 4 years.

(c) Find the number of years required for the stray-cat population to reach 500.

84. A culture contains 10,000 bacteria initially. After an hour the bacteria count is 25,000.

(a) Find the doubling period.

(b) Find the number of bacteria after 3 hours.

85. Uranium-234 has a half-life of 2.7×10^5 years.

(a) Find the amount remaining from a 10-mg sample after a thousand years.

(b) How long will it take this sample to decompose until its mass is 7 mg?

86. A sample of bismuth-210 decayed to 33% of its original mass after 8 days.

(a) Find the half-life of this element.

(b) Find the mass remaining after 12 days.

87. The half-life of radium-226 is 1590 years.

(a) If a sample has a mass of 150 mg, find a function that models the mass that remains after t years.

(b) Find the mass that will remain after 1000 years.

(c) After how many years will only 50 mg remain?

88. The half-life of palladium-100 is 4 days. After 20 days a sample has been reduced to a mass of 0.375 g.

(a) What was the initial mass of the sample?

(b) Find a function that models the mass remaining after t days.

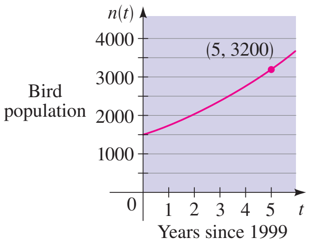
(c) What is the mass after 3 days?

(d) After how many days will only 0.15 g remain?

89. The graph shows the population of a rare species of bird, where t represents years since 1999 and $n(t)$ is measured in thousands.

(a) Find a function that models the bird population at time t in the form $n(t) = n_0 e^{rt}$.

(b) What is the bird population expected to be in the year 2010?



90. A car engine runs at a temperature of 190°F. When the engine is turned off, it cools according to Newton's Law of Cooling with constant $k = 0.0341$, where the time is measured in minutes. Find the time needed for the engine to cool to 90°F if the surrounding temperature is 60°F.

91. The hydrogen ion concentration of fresh egg whites was measured as

$$[\mathrm{H}^+] = 1.3 \times 10^{-8}\ \mathrm{M}$$

Find the pH, and classify the substance as acidic or basic.

92. The pH of lime juice is 1.9. Find the hydrogen ion concentration.

93. If one earthquake has magnitude 6.5 on the Richter scale, what is the magnitude of another quake that is 35 times as intense?

94. The drilling of a jackhammer was measured at 132 dB. The sound of whispering was measured at 28 dB. Find the ratio of the intensity of the drilling to that of the whispering.

4 Test

1. Graph the functions $y = 2^x$ and $y = \log_2 x$ on the same axes.

2. Sketch the graph of the function $f(x) = \log(x + 1)$ and state the domain, range, and asymptote.

3. Evaluate each logarithmic expression.

(a) $\log_3 \sqrt{27}$ (b) $\log_2 80 - \log_2 10$

(c) $\log_8 4$ (d) $\log_6 4 + \log_6 9$

4. Use the Laws of Logarithms to expand the expression.

$$\log \sqrt[3]{\frac{x + 2}{x^4(x^2 + 4)}}$$

5. Combine into a single logarithm: $\ln x - 2\ln(x^2 + 1) + \frac{1}{2}\ln(3 - x^4)$

6. Find the solution of the equation, correct to two decimal places.

(a) $2^{x-1} = 10$ (b) $5\ln(3 - x) = 4$

(c) $10^{x+3} = 6^{2x}$ (d) $\log_2(x + 2) + \log_2(x - 1) = 2$

7. The initial size of a culture of bacteria is 1000. After one hour the bacteria count is 8000.

(a) Find a function that models the population after t hours.

(b) Find the population after 1.5 hours.

(c) When will the population reach 15,000?

(d) Sketch the graph of the population function.

8. Suppose that \$12,000 is invested in a savings account paying 5.6% interest per year.

(a) Write the formula for the amount in the account after t years if interest is compounded monthly.

(b) Find the amount in the account after 3 years if interest is compounded daily.

(c) How long will it take for the amount in the account to grow to \$20,000 if interest is compounded semiannually?

9. Let $f(x) = \dfrac{e^x}{x^3}$.

(a) Graph f in an appropriate viewing rectangle.

(b) State the asymptotes of f.

(c) Find, correct to two decimal places, the local minimum value of f and the value of x at which it occurs.

(d) Find the range of f.

(e) Solve the equation $\dfrac{e^x}{x^3} = 2x + 1$. State each solution correct to two decimal places.

Focus on Modeling

Fitting Exponential and Power Curves to Data

In *Focus on Modeling* (page 320) we learned that the shape of a scatter plot helps us choose the type of curve to use in modeling data. The first plot in Figure 1 fairly screams for a line to be fitted through it, and the second one points to a cubic polynomial. For the third plot it is tempting to fit a second-degree polynomial. But what if an exponential curve fits better? How do we decide this? In this section we learn how to fit exponential and power curves to data and how to decide which type of curve fits the data better. We also learn that for scatter plots like those in the last two plots in Figure 1, the data can be modeled by logarithmic or logistic functions.

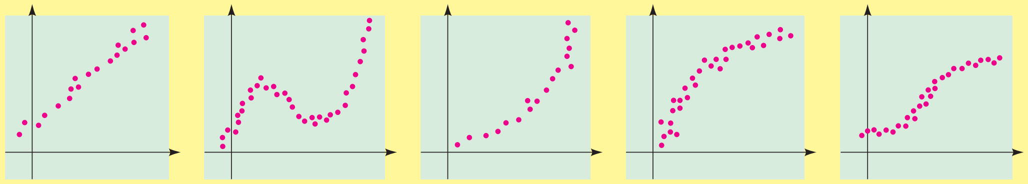


Figure 1

Modeling with Exponential Functions

If a scatter plot shows that the data increases rapidly, we might want to model the data using an *exponential model*, that is, a function of the form

$$\boxed{f(x) = Ce^{kx}}$$

where C and k are constants. In the first example we model world population by an exponential model. Recall from Section 4.5 that population tends to increase exponentially.

Table 1 World population

Year (t)	World population (P in millions)
1900	1650
1910	1750
1920	1860
1930	2070
1940	2300
1950	2520
1960	3020
1970	3700
1980	4450
1990	5300
2000	6060

Example 1 An Exponential Model for World Population

Table 1 gives the population of the world in the 20th century.

(a) Draw a scatter plot and note that a linear model is not appropriate.

(b) Find an exponential function that models population growth.

(c) Draw a graph of the function you found together with the scatter plot. How well does the model fit the data?

(d) Use the model you found to predict world population in the year 2020.

Solution

(a) The scatter plot is shown in Figure 2. The plotted points do not appear to lie


Chabruken / The Image Bank/Getty Images

The population of the world increases exponentially

along a straight line, so a linear model is not appropriate.

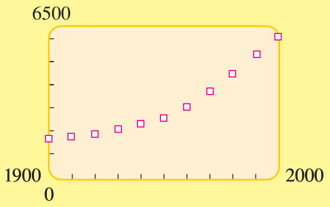


Figure 2
Scatter plot of world population

(b) Using a graphing calculator and the `ExpReg` command (see Figure 3(a)), we get the exponential model

$$P(t) = (0.0082543) \cdot (1.0137186)^t$$

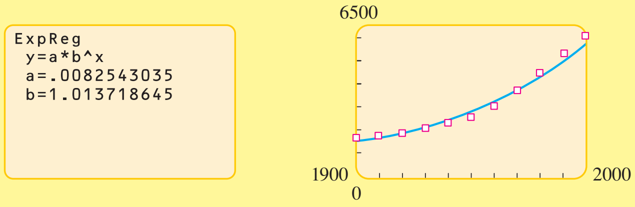
This is a model of the form $y = Cb^t$. To convert this to the form $y = Ce^{kt}$, we use the properties of exponentials and logarithms as follows:

$$\begin{aligned} 1.0137186^t &= e^{\ln 1.0137186^t} && A = e^{\ln A} \\ &= e^{t \ln 1.0137186} && \ln A^B = B \ln A \\ &= e^{0.013625t} && \ln 1.0137186 \approx 0.013625 \end{aligned}$$

Thus, we can write the model as

$$P(t) = 0.0082543e^{0.013625t}$$

(c) From the graph in Figure 3(b), we see that the model appears to fit the data fairly well. The period of relatively slow population growth is explained by the depression of the 1930s and the two world wars.



```
ExpReg
 y=a*b^x
 a=.0082543035
 b=1.013718645
```

Figure 3
Exponential model for world population

(a)

(b)

(d) The model predicts that the world population in 2020 will be

$$\begin{aligned} P(2020) &= 0.0082543e^{(0.013625)(2020)} \\ &\approx 7{,}405{,}400{,}000 \end{aligned}$$

■

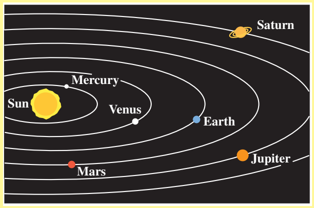
Modeling with Power Functions

If the scatter plot of the data we are studying resembles the graph of $y = ax^2$, $y = ax^{1.32}$, or some other power function, then we seek a *power model*, that is, a function of the form

$$\boxed{f(x) = ax^n}$$

where a is a positive constant and n is any real number.

In the next example we seek a power model for some astronomical data. In astronomy, distance in the solar system is often measured in astronomical units. An *astronomical unit* (AU) is the mean distance from the earth to the sun. The *period* of a planet is the time it takes the planet to make a complete revolution around the sun (measured in earth years). In this example we derive the remarkable relationship, first discovered by Johannes Kepler (see page 780), between the mean distance of a planet from the sun and its period.



Example 2 A Power Model for Planetary Periods

Table 2 Distances and periods of the planets

Planet	d	T
Mercury	0.387	0.241
Venus	0.723	0.615
Earth	1.000	1.000
Mars	1.523	1.881
Jupiter	5.203	11.861
Saturn	9.541	29.457
Uranus	19.190	84.008
Neptune	30.086	164.784
Pluto	39.507	248.350

Table 2 gives the mean distance d of each planet from the sun in astronomical units and its period T in years.

(a) Sketch a scatter plot. Is a linear model appropriate?

(b) Find a power function that models the data.

(c) Draw a graph of the function you found and the scatter plot on the same graph. How well does the model fit the data?

(d) Use the model you found to find the period of an asteroid whose mean distance from the sun is 5 AU.

Solution

(a) The scatter plot shown in Figure 4 indicates that the plotted points do not lie along a straight line, so a linear model is not appropriate.

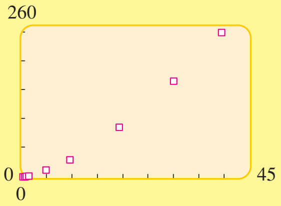


Figure 4
Scatter plot of planetary data

(b) Using a graphing calculator and the `PwrReg` command (see Figure 5(a)), we get the power model

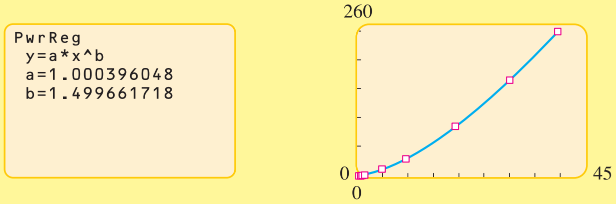
$$T = 1.000396d^{1.49966}$$

If we round both the coefficient and the exponent to three significant figures, we can write the model as

$$T = d^{1.5}$$

This is the relationship discovered by Kepler (see page 780). Sir Isaac Newton later used his Law of Gravity to derive this relationship theoretically, thereby providing strong scientific evidence that the Law of Gravity must be true.

(c) The graph is shown in Figure 5(b). The model appears to fit the data very well.



```
PwrReg
 y=a*x^b
 a=1.000396048
 b=1.499661718
```

Figure 5
Power model for planetary data

(a) (b)

(d) In this case, $d = 5$ AU and so our model gives

$$T = 1.00039 \cdot 5^{1.49966} \approx 11.22$$

The period of the asteroid is about 11.2 years. ■

Linearizing Data

We have used the shape of a scatter plot to decide which type of model to use—linear, exponential, or power. This works well if the data points lie on a straight line. But it's difficult to distinguish a scatter plot that is exponential from one that requires a power model. So, to help decide which model to use, we can *linearize* the data, that is, apply a function that "straightens" the scatter plot. The inverse of the linearizing function is then an appropriate model. We now describe how to linearize data that can be modeled by exponential or power functions.

■ Linearizing exponential data

If we suspect that the data points (x, y) lie on an exponential curve $y = Ce^{kx}$, then the points

$$(x, \ln y)$$

should lie on a straight line. We can see this from the following calculations:

$$\begin{aligned} \ln y &= \ln Ce^{kx} && \text{Assume } y = Ce^{kx} \text{ and take ln} \\ &= \ln e^{kx} + \ln C && \text{Property of ln} \\ &= kx + \ln C && \text{Property of ln} \end{aligned}$$

To see that $\ln y$ is a linear function of x, let $Y = \ln y$ and $A = \ln C$; then

$$Y = kx + A$$

Table 3 World population data

t	Population P (in millions)	$\ln P$
1900	1650	21.224
1910	1750	21.283
1920	1860	21.344
1930	2070	21.451
1940	2300	21.556
1950	2520	21.648
1960	3020	21.829
1970	3700	22.032
1980	4450	22.216
1990	5300	22.391
2000	6060	22.525

We apply this technique to the world population data (t, P) to obtain the points $(t, \ln P)$ in Table 3. The scatter plot in Figure 6 shows that the linearized data lie approximately on a straight line, so an exponential model should be appropriate.

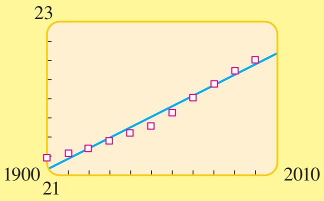


Figure 6

■ Linearizing power data

If we suspect that the data points (x, y) lie on a power curve $y = ax^n$, then the points

$$(\ln x, \ln y)$$

should be on a straight line. We can see this from the following calculations:

$$\begin{aligned} \ln y &= \ln ax^n && \text{Assume } y = ax^n \text{ and take ln} \\ &= \ln a + \ln x^n && \text{Property of ln} \\ &= \ln a + n \ln x && \text{Property of ln} \end{aligned}$$

To see that $\ln y$ is a linear function of $\ln x$, let $Y = \ln y$, $X = \ln x$, and $A = \ln a$; then

$$Y = nX + A$$

We apply this technique to the planetary data (d, T) in Table 2, to obtain the points $(\ln d, \ln T)$ in Table 4. The scatter plot in Figure 7 shows that the data lie on a straight line, so a power model seems appropriate.

Table 4 Log-log table

$\ln d$	$\ln T$
−0.94933	−1.4230
−0.32435	−0.48613
0	0
0.42068	0.6318
1.6492	2.4733
2.2556	3.3829
2.9544	4.4309
3.4041	5.1046
3.6765	5.5148

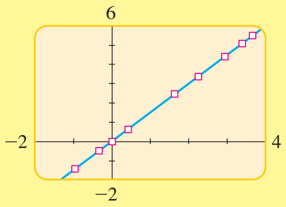


Figure 7
Log-log plot of data in Table 4

An Exponential or Power Model?

Suppose that a scatter plot of the data points (x, y) shows a rapid increase. Should we use an exponential function or a power function to model the data? To help us decide, we draw two scatter plots—one for the points $(x, \ln y)$ and the other for the points $(\ln x, \ln y)$. If the first scatter plot appears to lie along a line, then an exponential model is appropriate. If the second plot appears to lie along a line, then a power model is appropriate.

Example 3 An Exponential or Power Model?

Data points (x, y) are shown in Table 5.

(a) Draw a scatter plot of the data.

(b) Draw scatter plots of $(x, \ln y)$ and $(\ln x, \ln y)$.

(c) Is an exponential function or a power function appropriate for modeling this data?

(d) Find an appropriate function to model the data.

Table 5

x	y
1	2
2	6
3	14
4	22
5	34
6	46
7	64
8	80
9	102
10	130

Solution

(a) The scatter plot of the data is shown in Figure 8.

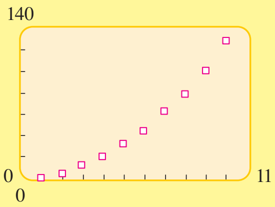


Figure 8

(b) We use the values from Table 6 to graph the scatter plots in Figures 9 and 10.

Table 6

x	$\ln x$	$\ln y$
1	0	0.7
2	0.7	1.8
3	1.1	2.6
4	1.4	3.1
5	1.6	3.5
6	1.8	3.8
7	1.9	4.2
8	2.1	4.4
9	2.2	4.6
10	2.3	4.9

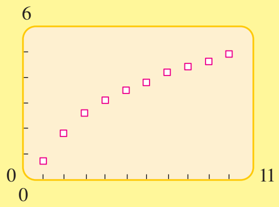


Figure 9

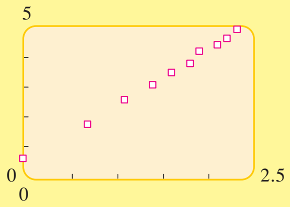


Figure 10

(c) The scatter plot of $(x, \ln y)$ in Figure 9 does not appear to be linear, so an exponential model is not appropriate. On the other hand, the scatter plot of $(\ln x, \ln y)$ in Figure 10 is very nearly linear, so a power model is appropriate.

(d) Using the `PwrReg` command on a graphing calculator, we find that the power function that best fits the data point is

$$y = 1.85x^{1.82}$$

The graph of this function and the original data points are shown in Figure 11.

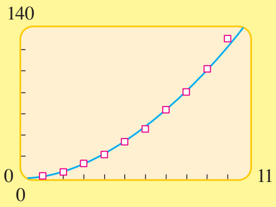


Figure 11

Before graphing calculators and statistical software became common, exponential and power models for data were often constructed by first finding a linear model for the linearized data. Then the model for the actual data was found by taking exponentials. For instance, if we find that $\ln y = A \ln x + B$, then by taking exponentials we get the model $y = e^B \cdot e^{A \ln x}$, or $y = Cx^A$ (where $C = e^B$). Special graphing paper called "log paper" or "log-log paper" was used to facilitate this process.

Modeling with Logistic Functions

A logistic growth model is a function of the form

$$f(t) = \frac{c}{1 + ae^{-bt}}$$

where a, b, and c are positive constants. Logistic functions are used to model populations where the growth is constrained by available resources. (See Exercises 69–72 of Section 4.1.)

Example 4 Stocking a Pond with Catfish

Much of the fish sold in supermarkets today is raised on commercial fish farms, not caught in the wild. A pond on one such farm is initially stocked with 1000 catfish, and the fish population is then sampled at 15-week intervals to estimate its size. The population data are given in Table 7.

Table 7

Week	Catfish
0	1000
15	1500
30	3300
45	4400
60	6100
75	6900
90	7100
105	7800
120	7900

(a) Find an appropriate model for the data.

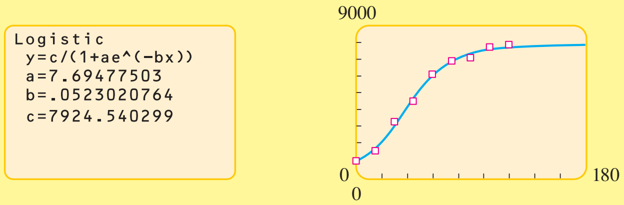
(b) Make a scatter plot of the data and graph the model you found in part (a) on the scatter plot.

(c) How does the model predict that the fish population will change with time?

Solution

(a) Since the catfish population is restricted by its habitat (the pond), a logistic model is appropriate. Using the `Logistic` command on a calculator (see Figure 12(a)), we find the following model for the catfish population $P(t)$:

$$P(t) = \frac{7925}{1 + 7.7e^{-0.052t}}$$



```
Logistic
 y=c/(1+ae^(-bx))
 a=7.69477503
 b=.0523020764
 c=7924.540299
```

Figure 12 (a) (b) Catfish population $y = P(t)$

(b) The scatter plot and the logistic curve are shown in Figure 12(b).

(c) From the graph of P in Figure 12(b), we see that the catfish population increases rapidly until about $t = 80$ weeks. Then growth slows down, and at about $t = 120$ weeks the population levels off and remains more or less constant at slightly over 7900. ■

The behavior exhibited by the catfish population in Example 4 is typical of logistic growth. After a rapid growth phase, the population approaches a constant level called the **carrying capacity** of the environment. This occurs because as $t \to \infty$, we have $e^{-bt} \to 0$ (see Section 4.1), and so

$$P(t) = \frac{c}{1 + ae^{-bt}} \quad \to \quad \frac{c}{1 + 0} = c$$

Thus, the carrying capacity is c.

Problems

1. U.S. Population The U.S. Constitution requires a census every 10 years. The census data for 1790–2000 is given in the table.

(a) Make a scatter plot of the data.

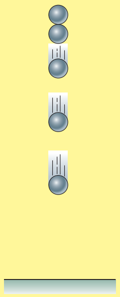
(b) Use a calculator to find an exponential model for the data.

(c) Use your model to predict the population at the 2010 census.

(d) Use your model to estimate the population in 1965.

(e) Compare your answers from parts (c) and (d) to the values in the table. Do you think an exponential model is appropriate for these data?

Year	Population (in millions)	Year	Population (in millions)	Year	Population (in millions)
1790	3.9	1870	38.6	1950	151.3
1800	5.3	1880	50.2	1960	179.3
1810	7.2	1890	63.0	1970	203.3
1820	9.6	1900	76.2	1980	226.5
1830	12.9	1910	92.2	1990	248.7
1840	17.1	1920	106.0	2000	281.4
1850	23.2	1930	123.2		
1860	31.4	1940	132.2		



Time (s)	Distance (m)
0.1	0.048
0.2	0.197
0.3	0.441
0.4	0.882
0.5	1.227
0.6	1.765
0.7	2.401
0.8	3.136
0.9	3.969
1.0	4.902

2. A Falling Ball In a physics experiment a lead ball is dropped from a height of 5 m. The students record the distance the ball has fallen every one-tenth of a second. (This can be done using a camera and a strobe light.)

(a) Make a scatter plot of the data.

(b) Use a calculator to find a power model.

(c) Use your model to predict how far a dropped ball would fall in 3 s.

3. Health-care Expenditures The U.S. health-care expenditures for 1970–2001 are given in the table on the next page, and a scatter plot of the data is shown in the figure.

(a) Does the scatter plot shown suggest an exponential model?

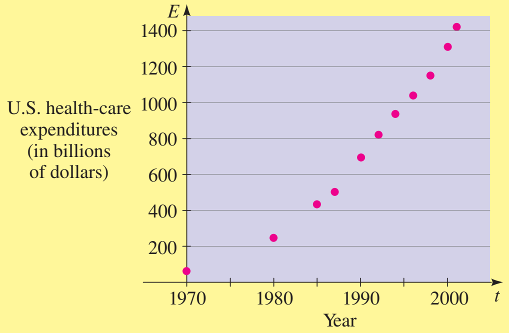
(b) Make a table of the values $(t, \ln E)$ and a scatter plot. Does the scatter plot appear to be linear?

Year	Health expenditures (in billions of dollars)
1970	74.3
1980	251.1
1985	434.5
1987	506.2
1990	696.6
1992	820.3
1994	937.2
1996	1039.4
1998	1150.0
2000	1310.0
2001	1424.5

(c) Find the regression line for the data in part (b).

(d) Use the results of part (c) to find an exponential model for the growth of health-care expenditures.

(e) Use your model to predict the total health-care expenditures in 2009.



Time (h)	Amount of ^{131}I (g)
0	4.80
8	4.66
16	4.51
24	4.39
32	4.29
40	4.14
48	4.04

4. Half-life of Radioactive Iodine A student is trying to determine the half-life of radioactive iodine-131. He measures the amount of iodine-131 in a sample solution every 8 hours. His data are shown in the table in the margin.

(a) Make a scatter plot of the data.

(b) Use a calculator to find an exponential model.

(c) Use your model to find the half-life of iodine-131.

5. The Beer-Lambert Law As sunlight passes through the waters of lakes and oceans, the light is absorbed and the deeper it penetrates, the more its intensity diminishes. The light intensity I at depth x is given by the Beer-Lambert Law:

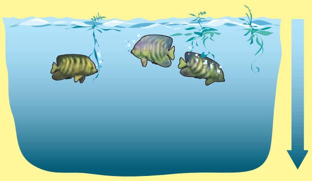
$$I = I_0 e^{-kx}$$

where I_0 is the light intensity at the surface and k is a constant that depends on the murkiness of the water (see page 364). A biologist uses a photometer to investigate light penetration in a northern lake, obtaining the data in the table.

(a) Use a graphing calculator to find an exponential function of the form given by the Beer-Lambert Law to model these data. What is the light intensity I_0 at the surface on this day, and what is the "murkiness" constant k for this lake? [*Hint:* If your calculator gives you a function of the form $I = ab^x$, convert this to the form you want using the identities $b^x = e^{\ln(b^x)} = e^{x \ln b}$. See Example 1(b).]

(b) Make a scatter plot of the data and graph the function that you found in part (a) on your scatter plot.

(c) If the light intensity drops below 0.15 lumens (lm), a certain species of algae can't survive because photosynthesis is impossible. Use your model from part (a) to determine the depth below which there is insufficient light to support this algae.



Light intensity decreases exponentially with depth.

Depth (ft)	Light intensity (lm)	Depth (ft)	Light intensity (lm)
5	13.0	25	1.8
10	7.6	30	1.1
15	4.5	35	0.5
20	2.7	40	0.3

6. Experimenting with "Forgetting" Curves Every one of us is all too familiar with the phenomenon of forgetting. Facts that we clearly understood at the time we first learned them sometimes fade from our memory by the time the final exam rolls around. Psychologists have proposed several ways to model this process. One such model is Ebbinghaus' Forgetting Curve, described on page 355. Other models use exponential or logarithmic functions. To develop her own model, a psychologist performs an experiment on a group of volunteers by asking them to memorize a list of 100 related words. She then tests how many of these words they can recall after various periods of time. The average results for the group are shown in the table.

(a) Use a graphing calculator to find a *power* function of the form $y = at^b$ that models the average number of words y that the volunteers remember after t hours. Then find an *exponential* function of the form $y = ab^t$ to model the data.

(b) Make a scatter plot of the data and graph both the functions that you found in part (a) on your scatter plot.

(c) Which of the two functions seems to provide the better model?

Time	Words recalled
15 min	64.3
1 h	45.1
8 h	37.3
1 day	32.8
2 days	26.9
3 days	25.6
5 days	22.9

7. Lead Emissions The table below gives U.S. lead emissions into the environment in millions of metric tons for 1970–1992.

(a) Find an exponential model for these data.

(b) Find a fourth-degree polynomial model for these data.

(c) Which of these curves gives a better model for the data? Use graphs of the two models to decide.

(d) Use each model to estimate the lead emissions in 1972 and 1982.

Year	Lead emissions
1970	199.1
1975	143.8
1980	68.0
1985	18.3
1988	5.9
1989	5.5
1990	5.1
1991	4.5
1992	4.7

8. Auto Exhaust Emissions A study by the U.S. Office of Science and Technology in 1972 estimated the cost of reducing automobile emissions by certain percentages. Find an exponential model that captures the "diminishing returns" trend of these data shown in the table below.

Reduction in emissions (%)	Cost per car ($)
50	45
55	55
60	62
65	70
70	80
75	90
80	100
85	200
90	375
95	600

9. Exponential or Power Model? Data points (x, y) are shown in the table.

(a) Draw a scatter plot of the data.

(b) Draw scatter plots of $(x, \ln y)$ and $(\ln x, \ln y)$.

(c) Which is more appropriate for modeling this data—an exponential function or a power function?

(d) Find an appropriate function to model the data.

x	y
2	0.08
4	0.12
6	0.18
8	0.25
10	0.36
12	0.52
14	0.73
16	1.06

10. Exponential or Power Model? Data points (x, y) are shown in the table in the margin.

(a) Draw a scatter plot of the data.

(b) Draw scatter plots of $(x, \ln y)$ and $(\ln x, \ln y)$.

(c) Which is more appropriate for modeling this data—an exponential function or a power function?

(d) Find an appropriate function to model the data.

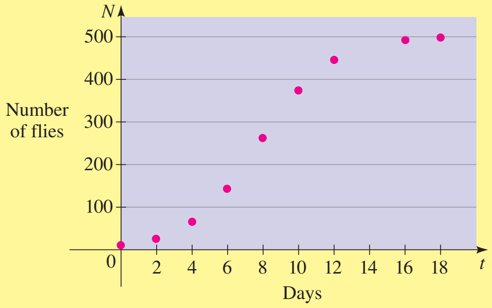
x	y
10	29
20	82
30	151
40	235
50	330
60	430
70	546
80	669
90	797

11. Logistic Population Growth The table and scatter plot give the population of black flies in a closed laboratory container over an 18-day period.

(a) Use the `Logistic` command on your calculator to find a logistic model for these data.

(b) Use the model to estimate the time when there were 400 flies in the container.

Time (days)	Number of flies
0	10
2	25
4	66
6	144
8	262
10	374
12	446
16	492
18	498



12. Logarithmic Models A **logarithmic model** is a function of the form

$$y = a + b \ln x$$

Many relationships between variables in the real world can be modeled by this type of function. The table and the scatter plot show the coal production (in metric tons) from a small mine in northern British Columbia.

(a) Use the `LnReg` command on your calculator to find a logarithmic model for these production figures.

(b) Use the model to predict coal production from this mine in 2010.

Year	Metric tons of coal
1950	882
1960	889
1970	894
1980	899
1990	905
2000	909