3 Polynomial and Rational Functions

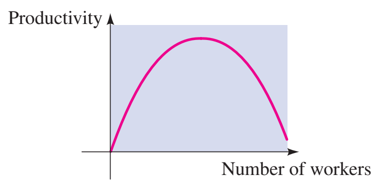
Chapter Overview

Functions defined by polynomial expressions are called polynomial functions. For example,

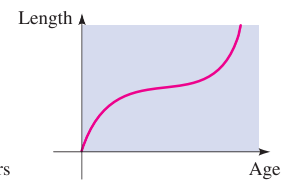
$$P(x) = 2x^3 - x + 1$$

is a polynomial function. Polynomial functions are easy to evaluate because they are defined using only addition, subtraction, and multiplication. This property makes them the most useful functions in mathematics.

The graphs of polynomial functions can increase and decrease several times. For this reason they are useful in modeling many real-world situations. For example, a factory owner notices that if she increases the number of workers, productivity increases, but if there are too many workers, productivity begins to decrease. This situation is modeled by a polynomial function of degree 2 (a quadratic polynomial). In many animal species the young experience an initial growth spurt, followed by a period of slow growth, followed by another growth spurt. This phenomenon is modeled by a polynomial function of degree 3 (a cubic polynomial).

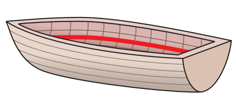


Productivity is modeled by a polynomial of degree 2.



Growth is modeled by a polynomial of degree 3.

The graphs of polynomial functions are beautiful, smooth curves that are used in design processes. For example, boat makers put together portions of the graphs of different cubic functions (called cubic splines) to design the natural curves for the hull of a boat.



J. L. Amos/SuperStock

In this chapter we also study rational functions, which are quotients of polynomial functions. We will see that rational functions also have many useful applications.

SUGGESTED TIME AND EMPHASIS

$\frac{1}{2}$–1 class.

Essential material.

3.1 Polynomial Functions and Their Graphs

Before we work with polynomial functions, we must agree on some terminology.

Polynomial Functions

A **polynomial function of degree *n*** is a function of the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

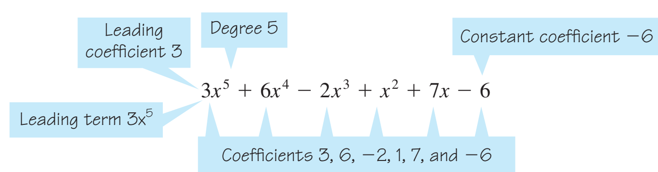
where n is a nonnegative integer and $a_n \neq 0$.

The numbers $a_0, a_1, a_2, \ldots, a_n$ are called the **coefficients** of the polynomial.

The number a_0 is the **constant coefficient** or **constant term**.

The number a_n, the coefficient of the highest power, is the **leading coefficient**, and the term $a_n x^n$ is the **leading term**.

We often refer to polynomial functions simply as *polynomials*. The following polynomial has degree 5, leading coefficient 3, and constant term −6.



Here are some more examples of polynomials.

$$P(x) = 3 \qquad \text{Degree 0}$$

$$Q(x) = 4x - 7 \qquad \text{Degree 1}$$

$$R(x) = x^2 + x \qquad \text{Degree 2}$$

$$S(x) = 2x^3 - 6x^2 - 10 \qquad \text{Degree 3}$$

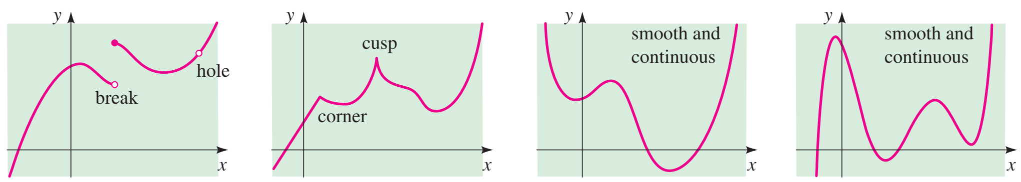
If a polynomial consists of just a single term, then it is called a **monomial**. For example, $P(x) = x^3$ and $Q(x) = -6x^5$ are monomials.

POINTS TO STRESS

1. The terminology and notation associated with polynomial functions.
2. Characteristics of polynomial graphs: smoothness, continuity, end behavior, and boundaries on the number of local maxima and minima.
3. Graphing polynomials using the zeros (taking into account multiplicity) and end behavior.

Graphs of Polynomials

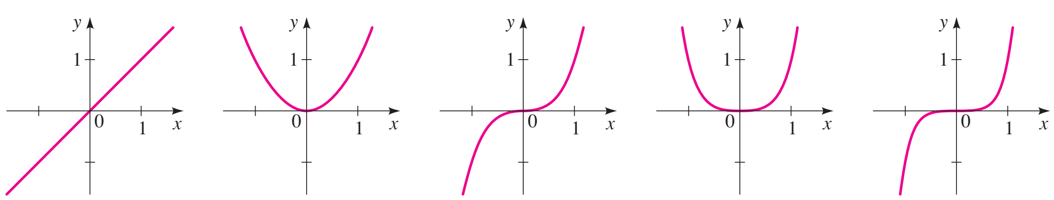
The graphs of polynomials of degree 0 or 1 are lines (Section 1.10), and the graphs of polynomials of degree 2 are parabolas (Section 2.5). The greater the degree of the polynomial, the more complicated its graph can be. However, the graph of a polynomial function is always a smooth curve; that is, it has no breaks or corners (see Figure 1). The proof of this fact requires calculus.



Not the graph of a polynomial function | Not the graph of a polynomial function | Graph of a polynomial function | Graph of a polynomial function

Figure 1

The simplest polynomial functions are the monomials $P(x) = x^n$, whose graphs are shown in Figure 2. As the figure suggests, the graph of $P(x) = x^n$ has the same general shape as $y = x^2$ when n is even, and the same general shape as $y = x^3$ when n is odd. However, as the degree n becomes larger, the graphs become flatter around the origin and steeper elsewhere.



(a) $y = x$ (b) $y = x^2$ (c) $y = x^3$ (d) $y = x^4$ (e) $y = x^5$

Figure 2
Graphs of monomials

Example 1 Transformations of Monomials

Sketch the graphs of the following functions.

(a) $P(x) = -x^3$ (b) $Q(x) = (x - 2)^4$

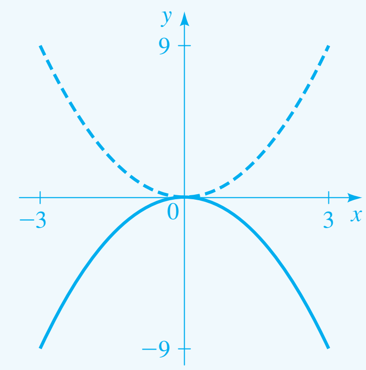
(c) $R(x) = -2x^5 + 4$

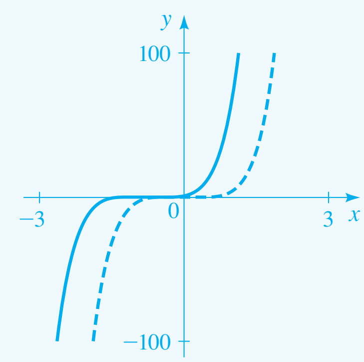
Solution We use the graphs in Figure 2 and transform them using the techniques of Section 2.4.

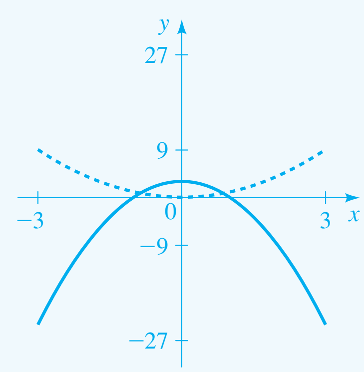
(a) The graph of $P(x) = -x^3$ is the reflection of the graph of $y = x^3$ in the x-axis, as shown in Figure 3(a) on the following page.

ALTERNATE EXAMPLE 1
Sketch the graphs of the following functions.

(a) $f(x) = -x^2$
(b) $g(x) = (x + 1)^5$
(c) $h(x) = -3x^2 + 3$







SAMPLE QUESTIONS
Text Questions

Which of the following are polynomial functions?

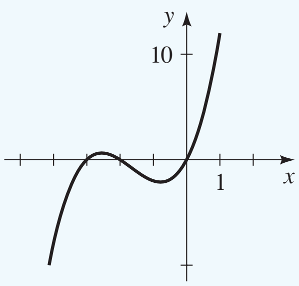
(a) $f(x) = -x^3 + 2x + 4$
(b) $f(x) = \left(\sqrt{x}\right)^3 - 2\left(\sqrt{x}\right)^2 + 5\left(\sqrt{x}\right) - 1$
(c) $f(x) = (x - 2)(x - 1)(x + 4)^2$
(d) $f(x) = \dfrac{x^2 + 2}{x^2 - 2}$

Answers
(a) and **(c)**

DRILL QUESTION

Sketch the graph of the polynomial $f(x) = x^3 + 5x^2 + 6x$.

Answer

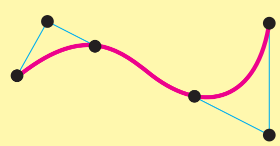


Mathematics in the Modern World

Splines

A spline is a long strip of wood that is curved while held fixed at certain points. In the old days shipbuilders used splines to create the curved shape of a boat's hull. Splines are also used to make the curves of a piano, a violin, or the spout of a teapot.

Mathematicians discovered that the shapes of splines can be obtained by piecing together parts of polynomials. For example, the graph of a cubic polynomial can be made to fit specified points by adjusting the coefficients of the polynomial (see Example 10, page 261). Curves obtained in this way are called cubic splines. In modern computer design programs, such as Adobe Illustrator or Microsoft Paint, a curve can be drawn by fixing two points, then using the mouse to drag one or more anchor points. Moving the anchor points amounts to adjusting the coefficients of a cubic polynomial.



(b) The graph of $Q(x) = (x - 2)^4$ is the graph of $y = x^4$ shifted to the right 2 units, as shown in Figure 3(b).

(c) We begin with the graph of $y = x^5$. The graph of $y = -2x^5$ is obtained by stretching the graph vertically and reflecting it in the x-axis (see the dashed blue graph in Figure 3(c)). Finally, the graph of $R(x) = -2x^5 + 4$ is obtained by shifting upward 4 units (see the red graph in Figure 3(c)).

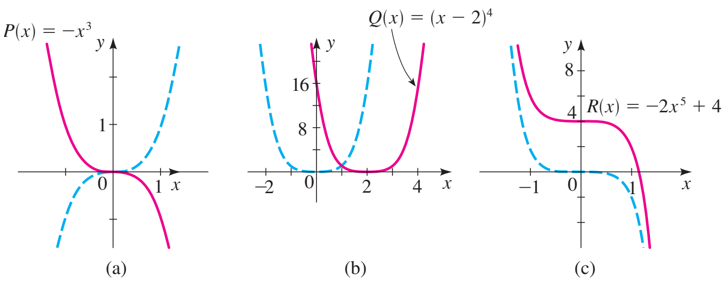


Figure 3

End Behavior and the Leading Term

The **end behavior** of a polynomial is a description of what happens as x becomes large in the positive or negative direction. To describe end behavior, we use the following notation:

$x \to \infty$	means	"x becomes large in the positive direction"

$x \to -\infty$	means	"x becomes large in the negative direction"

For example, the monomial $y = x^2$ in Figure 2(b) has the following end behavior:

$$y \to \infty \quad \text{as} \quad x \to \infty \qquad \text{and} \qquad y \to \infty \quad \text{as} \quad x \to -\infty$$

The monomial $y = x^3$ in Figure 2(c) has the end behavior

$$y \to \infty \quad \text{as} \quad x \to \infty \qquad \text{and} \qquad y \to -\infty \quad \text{as} \quad x \to -\infty$$

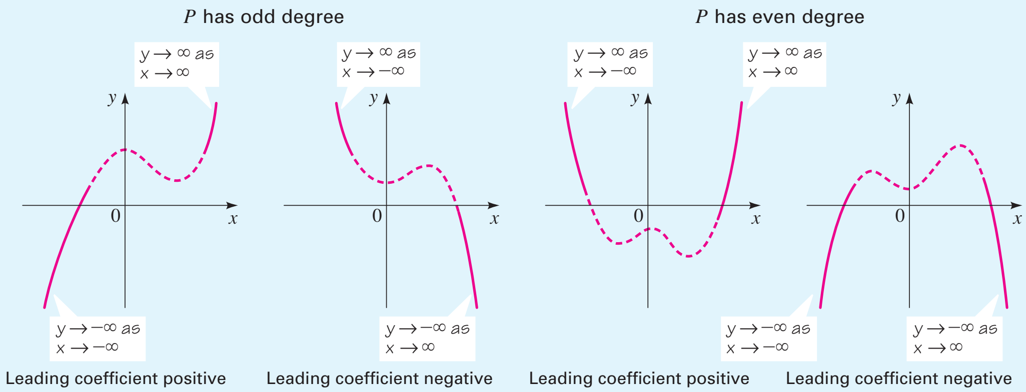
For any polynomial, *the end behavior is determined by the term that contains the highest power of* x, because when x is large, the other terms are relatively insignificant in size. The following box shows the four possible types of end behavior, based on the highest power and the sign of its coefficient.

IN-CLASS MATERIALS

Point out that while the students can, at this point, sketch the graph of a polynomial function like $f(x) = (x - 1)(x - 2)^2(x - 3)$ relatively quickly, they still cannot find the precise coordinates of the two local minima, nor could they tell how fast the function is increasing as x gets large. Point out that "for now" they have a good method of getting a general idea of the shape of a polynomial, and more precision will come with calculus.

End Behavior of Polynomials

The end behavior of the polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is determined by the degree n and the sign of the leading coefficient a_n, as indicated in the following graphs.



Example 2 End Behavior of a Polynomial

Determine the end behavior of the polynomial

$$P(x) = -2x^4 + 5x^3 + 4x - 7$$

Solution The polynomial P has degree 4 and leading coefficient -2. Thus, P has *even* degree and *negative* leading coefficient, so it has the following end behavior:

$$y \to -\infty \quad \text{as} \quad x \to \infty \qquad \text{and} \qquad y \to -\infty \quad \text{as} \quad x \to -\infty$$

The graph in Figure 4 illustrates the end behavior of P.

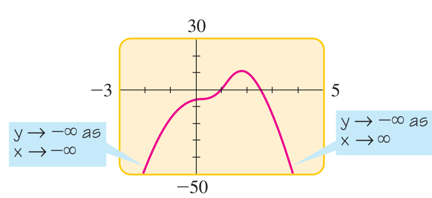


Figure 4
$P(x) = -2x^4 + 5x^3 + 4x - 7$

■

Example 3 End Behavior of a Polynomial

(a) Determine the end behavior of the polynomial $P(x) = 3x^5 - 5x^3 + 2x$.

(b) Confirm that P and its leading term $Q(x) = 3x^5$ have the same end behavior by graphing them together.

ALTERNATE EXAMPLE 2
Determine the end behavior of the polynomial
$f(x) = -3x^3 + 20x^2 + 60x + 2$.

ANSWER
$y \to -\infty$ as $x \to \infty$, $y \to \infty$ as $x \to -\infty$

ALTERNATE EXAMPLE 3
Determine the end behavior of the graph of the function
$y = 8x^3 - 7x^2 + 3x + 7$.

ANSWER
$y \to -\infty$ as $x \to -\infty$, and $y \to \infty$ as $x \to \infty$

IN-CLASS MATERIALS

This is one application of the Intermediate Value Theorem for Polynomials: Consider $f(x) = 90x^3 + 100x^2 + 10x + 1$ and $g(x) = 91x^3 - 60x^2$. Have students graph each on their calculator, if they can find a good window. It will be tough. After giving them some time, put some graphs on the board.

It certainly looks like these two curves never cross. One way to prove that they do would be to actually find the crossing point—to solve $f(x) - g(x) = 0$. But a much quicker way is to use the intermediate value property. Let $h(x) = f(x) - g(x)$. We know $g(x)$ will cross $f(x)$ when $h(x) = 0$. Now $h(0)$ is positive, and $h(1000)$ is negative. We don't *need* to go hunting for the value of x that makes $h(x) = 0$; we can simply invoke the intermediate value property to prove that such an x does exist.

Solution

(a) Since P has odd degree and positive leading coefficient, it has the following end behavior:

$$y \to \infty \quad \text{as} \quad x \to \infty \qquad \text{and} \qquad y \to -\infty \quad \text{as} \quad x \to -\infty$$

(b) Figure 5 shows the graphs of P and Q in progressively larger viewing rectangles. The larger the viewing rectangle, the more the graphs look alike. This confirms that they have the same end behavior. ■

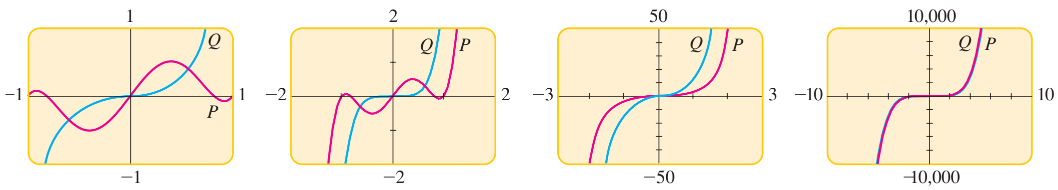


Figure 5
$P(x) = 3x^5 - 5x^3 + 2x$
$Q(x) = 3x^5$

To see algebraically why P and Q in Example 3 have the same end behavior, factor P as follows and compare with Q.

$$P(x) = 3x^5\left(1 - \frac{5}{3x^2} + \frac{2}{3x^4}\right) \qquad\qquad Q(x) = 3x^5$$

When x is large, the terms $5/3x^2$ and $2/3x^4$ are close to 0 (see Exercise 79 on page 12). So for large x, we have

$$P(x) \approx 3x^5(1 - 0 - 0) = 3x^5 = Q(x)$$

x	$P(x)$	$Q(x)$
15	2,261,280	2,278,125
30	72,765,060	72,900,000
50	936,875,100	937,500,000

So, when x is large, P and Q have approximately the same values. We can also see this numerically by making a table like the one in the margin.

By the same reasoning we can show that the end behavior of *any* polynomial is determined by its leading term.

Using Zeros to Graph Polynomials

If P is a polynomial function, then c is called a **zero** of P if $P(c) = 0$. In other words, the zeros of P are the solutions of the polynomial equation $P(x) = 0$. Note that if $P(c) = 0$, then the graph of P has an x-intercept at $x = c$, so the x-intercepts of the graph are the zeros of the function.

Real Zeros of Polynomials

If P is a polynomial and c is a real number, then the following are equivalent.

1. c is a zero of P.
2. $x = c$ is a solution of the equation $P(x) = 0$.
3. $x - c$ is a factor of $P(x)$.
4. $x = c$ is an x-intercept of the graph of P.

IN-CLASS MATERIALS

Explore, using technology, the concept of families of functions. Take, for example, the easy-to-graph curve $y = x^3 - x$. Add in a constant: $y = x^3 - x + 1$, $y = x^3 - x + 2$, $y = x^3 - x - 1$. Using material from Chapter 3, students should be able to predict what these graphs look like. Now add in a quadratic term: $y = x^3 + \frac{1}{2}x^2 - x$, $y = x^3 + x^2 - x$, $y = x^3 + 8x^2 - x$, $y = x^3 + -x^2 - x$. By graphing these curves on the same axes, have students attempt to put into words the effect that an x^2 term has on this cubic function.

To find the zeros of a polynomial P, we factor and then use the Zero-Product Property (see page 47). For example, to find the zeros of $P(x) = x^2 + x - 6$, we factor P to get

$$P(x) = (x - 2)(x + 3)$$

From this factored form we easily see that

1. 2 is a zero of P.
2. $x = 2$ is a solution of the equation $x^2 + x - 6 = 0$.
3. $x - 2$ is a factor of $x^2 + x - 6$.
4. $x = 2$ is an x-intercept of the graph of P.

The same facts are true for the other zero, -3.

The following theorem has many important consequences. (See, for instance, the Discovery Project on page 283.) Here we use it to help us graph polynomial functions.

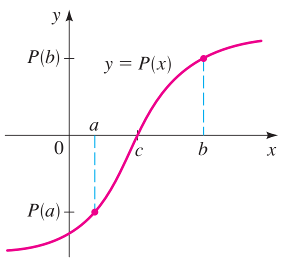


Figure 6

Intermediate Value Theorem for Polynomials

If P is a polynomial function and $P(a)$ and $P(b)$ have opposite signs, then there exists at least one value c between a and b for which $P(c) = 0$.

We will not prove this theorem, but Figure 6 shows why it is intuitively plausible.

One important consequence of this theorem is that between any two successive zeros, the values of a polynomial are either all positive or all negative. That is, between two successive zeros the graph of a polynomial lies *entirely above* or *entirely below* the x-axis. To see why, suppose c_1 and c_2 are successive zeros of P. If P has both positive and negative values between c_1 and c_2, then by the Intermediate Value Theorem P must have another zero between c_1 and c_2. But that's not possible because c_1 and c_2 are successive zeros. This observation allows us to use the following guidelines to graph polynomial functions.

Guidelines for Graphing Polynomial Functions

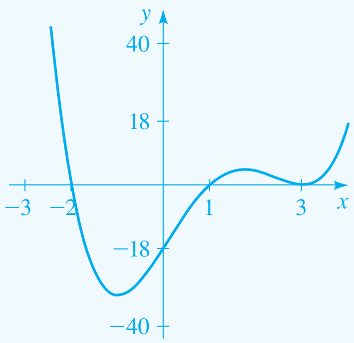
1. **Zeros.** Factor the polynomial to find all its real zeros; these are the x-intercepts of the graph.
2. **Test Points.** Make a table of values for the polynomial. Include test points to determine whether the graph of the polynomial lies above or below the x-axis on the intervals determined by the zeros. Include the y-intercept in the table.
3. **End Behavior.** Determine the end behavior of the polynomial.
4. **Graph.** Plot the intercepts and other points you found in the table. Sketch a smooth curve that passes through these points and exhibits the required end behavior.

IN-CLASS MATERIALS

When discussing local extrema, make sure that students understand that just because a fifth-degree polynomial (for example) *can* have four local extrema, doesn't mean it *must* have four local extrema. Have students graph $f(x) = x^5$ as a quick example, and then $f(x) = x^5 - x^3$ as an example of a fifth-degree polynomial with two local extrema. Have students try to come up with a proof that there can't be a fifth-degree polynomial with exactly one or three local extrema.

ALTERNATE EXAMPLE 4
Sketch the graph of the polynomial function $P(x) = (x - 1)(x + 2)(x - 3)^2$.

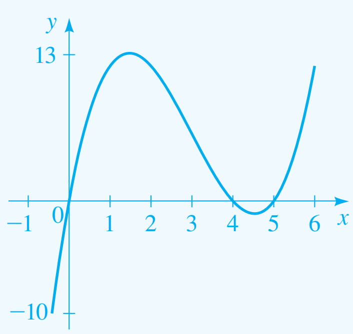
ANSWER
The zeros are -2, 1, and 3. We use test points $x = -3, 0, 2$, and 4. We obtain the graph:



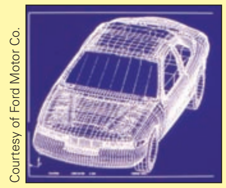
ALTERNATE EXAMPLE 5
Let $P(x) = x^3 - 9x^2 + 20x$.
(a) Find the zeros of P.
(b) Sketch the graph of P.

ANSWERS
(a) $P(x) = x(x - 4)(x - 5)$ so the zeros are $x = 0, x = 4, x = 5$.
(b) End term behavior: $y \to \infty$ as $x \to \infty$, $y \to -\infty$ as $x \to -\infty$.

We use test points -1, 3, 4.5, and 6 to obtain the graph



Mathematics in the Modern World



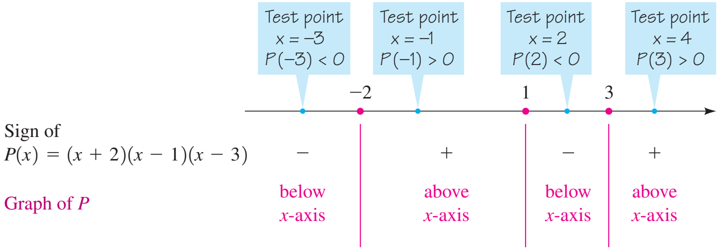
Automotive Design

Computer-aided design (CAD) has completely changed the way car companies design and manufacture cars. Before the 1980s automotive engineers would build a full-scale "nuts and bolts" model of a proposed new car; this was really the only way to tell whether the design was feasible. Today automotive engineers build a mathematical model, one that exists only in the memory of a computer. The model incorporates all the main design features of the car. Certain polynomial curves, called *splines*, are used in shaping the body of the car. The resulting "mathematical car" can be tested for structural stability, handling, aerodynamics, suspension response, and more. All this testing is done before a prototype is built. As you can imagine, CAD saves car manufacturers millions of dollars each year. More importantly, CAD gives automotive engineers far more flexibility in design; desired changes can be created and tested within seconds. With the help of computer graphics, designers can see how good the "mathematical car" looks before they build the real one. Moreover, the mathematical car can be viewed from any perspective; it can be moved, rotated, or seen from the inside. These manipulations of the car on the computer monitor translate mathematically into solving large systems of linear equations.

Example 4 Using Zeros to Graph a Polynomial Function

Sketch the graph of the polynomial function $P(x) = (x + 2)(x - 1)(x - 3)$.

Solution The zeros are $x = -2$, 1, and 3. These determine the intervals $(-\infty, -2)$, $(-2, 1)$, $(1, 3)$, and $(3, \infty)$. Using test points in these intervals, we get the information in the following sign diagram (see Section 1.7).



Plotting a few additional points and connecting them with a smooth curve helps us complete the graph in Figure 7.

	x	$P(x)$
Test point →	−3	−24
	−2	0
Test point →	−1	8
	0	6
	1	0
Test point →	2	−4
	3	0
Test point →	4	18

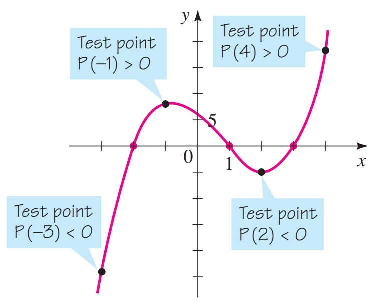


Figure 7
$P(x) = (x + 2)(x - 1)(x - 3)$ ■

Example 5 Finding Zeros and Graphing a Polynomial Function

Let $P(x) = x^3 - 2x^2 - 3x$.

(a) Find the zeros of P. (b) Sketch the graph of P.

Solution

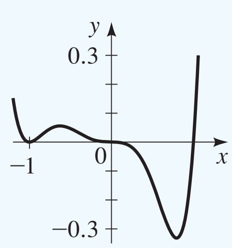
(a) To find the zeros, we factor completely.

$$\begin{aligned} P(x) &= x^3 - 2x^2 - 3x \\ &= x(x^2 - 2x - 3) && \text{Factor } x \\ &= x(x - 3)(x + 1) && \text{Factor quadratic} \end{aligned}$$

Thus, the zeros are $x = 0$, $x = 3$, and $x = -1$.

EXAMPLE
A polynomial function with zeros of various multiplicities:

$$f(x) = x^6 + x^5 - x^4 - x^3 = (x - 1)x^3(x + 1)^2$$



(b) The x-intercepts are $x = 0$, $x = 3$, and $x = -1$. The y-intercept is $P(0) = 0$. We make a table of values of $P(x)$, making sure we choose test points between (and to the right and left of) successive zeros.

Since P is of odd degree and its leading coefficient is positive, it has the following end behavior:

$$y \to \infty \quad \text{as} \quad x \to \infty \qquad \text{and} \qquad y \to -\infty \quad \text{as} \quad x \to -\infty$$

We plot the points in the table and connect them by a smooth curve to complete the graph, as shown in Figure 8.

	x	$P(x)$
Test point →	-2	-10
	-1	0
Test point →	$-\frac{1}{2}$	$\frac{7}{8}$
	0	0
Test point →	1	-4
	2	-6
	3	0
Test point →	4	20

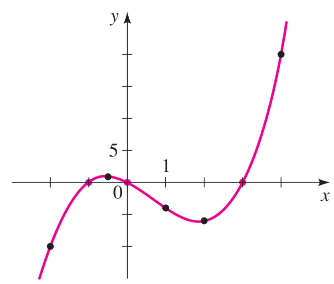


Figure 8
$P(x) = x^3 - 2x^2 - 3x$

Example 6 Finding Zeros and Graphing a Polynomial Function

Let $P(x) = -2x^4 - x^3 + 3x^2$.

(a) Find the zeros of P. (b) Sketch the graph of P.

Solution

(a) To find the zeros, we factor completely.

$$\begin{aligned} P(x) &= -2x^4 - x^3 + 3x^2 \\ &= -x^2(2x^2 + x - 3) && \text{Factor } -x^2 \\ &= -x^2(2x + 3)(x - 1) && \text{Factor quadratic} \end{aligned}$$

Thus, the zeros are $x = 0$, $x = -\frac{3}{2}$, and $x = 1$.

(b) The x-intercepts are $x = 0$, $x = -\frac{3}{2}$, and $x = 1$. The y-intercept is $P(0) = 0$. We make a table of values of $P(x)$, making sure we choose test points between (and to the right and left of) successive zeros.

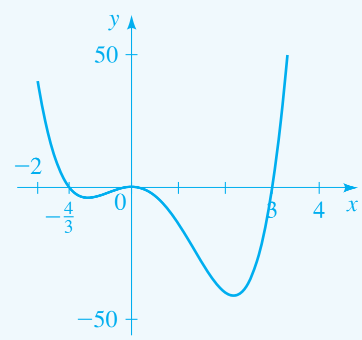
Since P is of even degree and its leading coefficient is negative, it has the following end behavior:

$$y \to -\infty \quad \text{as} \quad x \to \infty \qquad \text{and} \qquad y \to -\infty \quad \text{as} \quad x \to -\infty$$

ALTERNATE EXAMPLE 6
Let $P(x) = 3x^4 - 5x^3 - 12x^2$.
(a) Find the zeros of P.
(b) Sketch the graph of P.

ANSWERS
(a) $P(x) = x^2(x - 3)(3x + 4)$. The zeros are $x = 0$, $x = 3$, $x = -4/3$.
(b) End behavior: $y \to \infty$ as $x \to \infty$, $y \to -\infty$ as $x \to -\infty$

Graph:



EXAMPLES
Two sixth-degree polynomial functions that look very similar, but have different numbers of extrema: $f(x) = x^6 - 3x^3$ has one local minimum and a flat spot at $x = 0$. $f(x) = x^6 - 3.0x^3 - 0.015x^4 + 0.09x$ has two local minima and one local maximum—an obvious local minimum at $x \approx 1.145$, and two very subtle extrema at $x \approx \pm 0.1$.

We plot the points from the table and connect the points by a smooth curve to complete the graph in Figure 9.

Table of values are most easily calculated using a programmable calculator or a graphing calculator.

x	$P(x)$
-2	-12
-1.5	0
-1	2
-0.5	0.75
0	0
0.5	0.5
1	0
1.5	-6.75

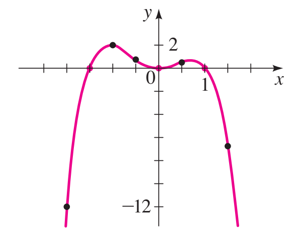


Figure 9
$P(x) = -2x^4 - x^3 + 3x^2$ ■

Example 7 Finding Zeros and Graphing a Polynomial Function

Let $P(x) = x^3 - 2x^2 + 4x + 8$.

(a) Find the zeros of P. (b) Sketch the graph of P.

Solution

(a) To find the zeros, we factor completely.

$$\begin{aligned} P(x) &= x^3 - 2x^2 - 4x + 8 \\ &= x^2(x-2) - 4(x-2) && \text{Group and factor} \\ &= (x^2-4)(x-2) && \text{Factor } x-2 \\ &= (x+2)(x-2)(x-2) && \text{Difference of squares} \\ &= (x+2)(x-2)^2 && \text{Simplify} \end{aligned}$$

Thus, the zeros are $x = -2$ and $x = 2$.

(b) The x-intercepts are $x = -2$ and $x = 2$. The y-intercept is $P(0) = 8$. The table gives additional values of $P(x)$.

Since P is of odd degree and its leading coefficient is positive, it has the following end behavior:

$$y \to \infty \quad \text{as} \quad x \to \infty \qquad \text{and} \qquad y \to -\infty \quad \text{as} \quad x \to -\infty$$

We connect the points by a smooth curve to complete the graph in Figure 10.

x	$P(x)$
-3	-25
-2	0
-1	9
0	8
1	3
2	0
3	5

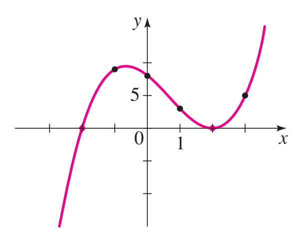
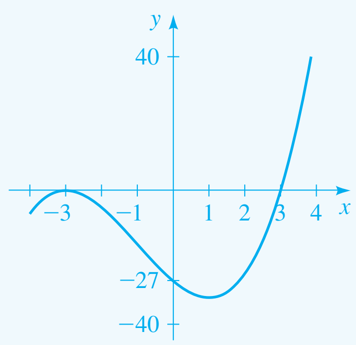


Figure 10
$P(x) = x^3 - 2x^2 - 4x + 8$ ■

ALTERNATE EXAMPLE 7
Let $P(x) = x^3 + 3x^2 - 9x - 27$.
(a) Find the zeros of P.
(b) Sketch the graph of P.

ANSWERS
(a) $P(x) = (x + 3)^2(x - 3)$. The zeros are $x = -3$, $x = 3$.
(b) $y \to \infty$ as $x \to \infty$, $y \to -\infty$ as $x \to -\infty$



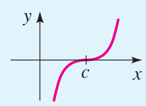
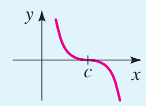
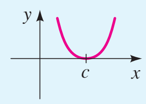
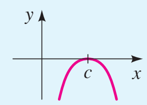
Shape of the Graph Near a Zero

Although $x = 2$ is a zero of the polynomial in Example 7, the graph does not cross the x-axis at the x-intercept 2. This is because the factor $(x - 2)^2$ corresponding to that zero is raised to an even power, so it doesn't change sign as we test points on either side of 2. In the same way, the graph does not cross the x-axis at $x = 0$ in Example 6.

In general, if c is a zero of P and the corresponding factor $x - c$ occurs exactly m times in the factorization of P then we say that c is a **zero of multiplicity *m***. By considering test points on either side of the x-intercept c, we conclude that the graph crosses the x-axis at c if the multiplicity m is odd and does not cross the x-axis if m is even. Moreover, it can be shown using calculus that near $x = c$ the graph has the same general shape as $A(x - c)^m$.

Shape of the Graph Near a Zero of Multiplicity m

Suppose that c is a zero of P of multiplicity m. Then the shape of the graph of P near c is as follows.

Multiplicity of c	Shape of the graph of P near the x-intercept c		
m odd, $m > 1$		OR	
m even, $m > 1$		OR	

Example 8 Graphing a Polynomial Function Using Its Zeros

Graph the polynomial $P(x) = x^4(x - 2)^3(x + 1)^2$.

Solution The zeros of P are -1, 0, and 2, with multiplicities 2, 4, and 3, respectively.

0 is a zero of multiplicity 4.

2 is a zero of multiplicity 3.

-1 is a zero of multiplicity 2.

$$P(x) = x^4(x - 2)^3(x + 1)^2$$

The zero 2 has *odd* multiplicity, so the graph crosses the x-axis at the x-intercept 2. But the zeros 0 and -1 have *even* multiplicity, so the graph does not cross the x-axis at the x-intercepts 0 and -1.

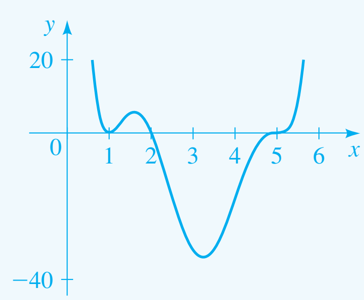
Since P is a polynomial of degree 9 and has positive leading coefficient, it has the following end behavior:

$$y \to \infty \quad \text{as} \quad x \to \infty \qquad \text{and} \qquad y \to -\infty \quad \text{as} \quad x \to -\infty$$

ALTERNATE EXAMPLE 8
Graph the polynomial $P(x) = (x - 1)^2(x - 2)(x - 5)^3$.

ANSWER
The zeros of P are 1, 2, and 5 with multiplicities 2, 1, and 3 respectively.

Since P is a polynomial of degree 6 with positive leading coefficient, the end behavior is $y \to \infty$ as $x \to \infty$, $y \to \infty$ as $x \to -\infty$.



With this information and a table of values, we sketch the graph in Figure 11.

x	$P(x)$
−1.3	−9.2
−1	0
−0.5	−3.9
0	0
1	−4
2	0
2.3	8.2

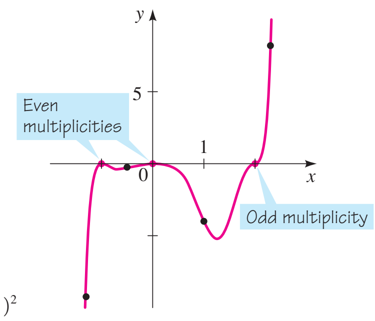


Figure 11
$P(x) = x^4(x-2)^3(x+1)^2$

Local Maxima and Minima of Polynomials

Recall from Section 2.5 that if the point $(a, f(a))$ is the highest point on the graph of f within some viewing rectangle, then $f(a)$ is a local maximum value of f, and if $(b, f(b))$ is the lowest point on the graph of f within a viewing rectangle, then $f(b)$ is a local minimum value (see Figure 12). We say that such a point $(a, f(a))$ is a **local maximum point** on the graph and that $(b, f(b))$ is a **local minimum point**. The set of all local maximum and minimum points on the graph of a function is called its **local extrema**.

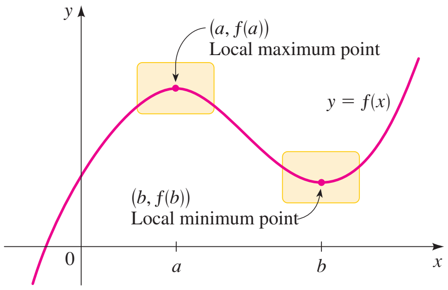


Figure 12

For a polynomial function the number of local extrema must be less than the degree, as the following principle indicates. (A proof of this principle requires calculus.)

Local Extrema of Polynomials

If $P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ is a polynomial of degree n, then the graph of P has at most $n - 1$ local extrema.

A polynomial of degree n may in fact have less than $n - 1$ local extrema. For example, $P(x) = x^5$ (graphed in Figure 2) has *no* local extrema, even though it is of degree 5. The preceding principle tells us only that a polynomial of degree n can have no more than $n - 1$ local extrema.

Example 9 The Number of Local Extrema

Determine how many local extrema each polynomial has.

(a) $P_1(x) = x^4 + x^3 - 16x^2 - 4x + 48$

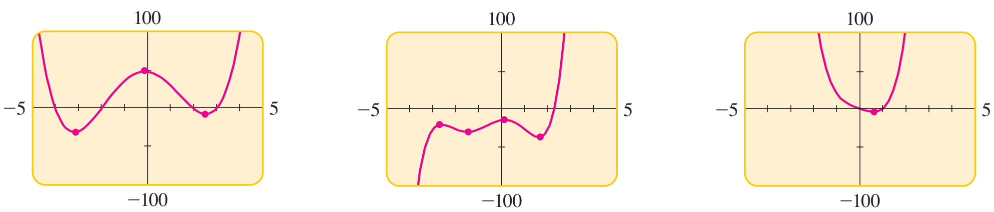
(b) $P_2(x) = x^5 + 3x^4 - 5x^3 - 15x^2 + 4x - 15$ (c) $P_3(x) = 7x^4 + 3x^2 - 10x$

Solution The graphs are shown in Figure 13.

(a) P_1 has two local minimum points and one local maximum point, for a total of three local extrema.

(b) P_2 has two local minimum points and two local maximum points, for a total of four local extrema.

(c) P_3 has just one local extremum, a local minimum.



(a) $P_1(x) = x^4 + x^3 - 16x^2 - 4x + 48$ (b) $P_2(x) = x^5 + 3x^4 - 5x^3 - 15x^2 + 4x - 15$ (c) $P_3(x) = 7x^4 + 3x^2 - 10x$

Figure 13 ■

With a graphing calculator we can quickly draw the graphs of many functions at once, on the same viewing screen. This allows us to see how changing a value in the definition of the functions affects the shape of its graph. In the next example we apply this principle to a family of third-degree polynomials.

Example 10 A Family of Polynomials

Sketch the family of polynomials $P(x) = x^3 - cx^2$ for $c = 0, 1, 2,$ and 3. How does changing the value of c affect the graph?

Solution The polynomials

$$P_0(x) = x^3 \qquad P_1(x) = x^3 - x^2$$
$$P_2(x) = x^3 - 2x^2 \qquad P_3(x) = x^3 - 3x^2$$

are graphed in Figure 14. We see that increasing the value of c causes the graph to develop an increasingly deep "valley" to the right of the y-axis, creating a local maximum at the origin and a local minimum at a point in quadrant IV. This local minimum moves lower and farther to the right as c increases. To see why this happens, factor $P(x) = x^2(x - c)$. The polynomial P has zeros at 0 and c, and the larger c gets, the farther to the right the minimum between 0 and c will be. ■

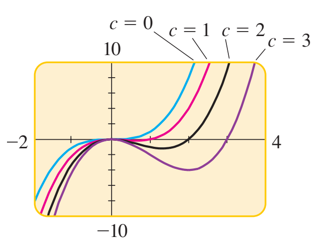
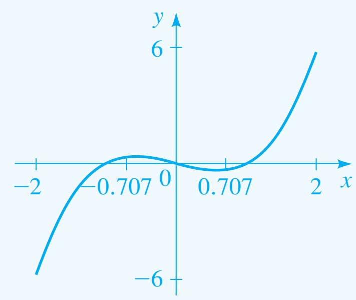
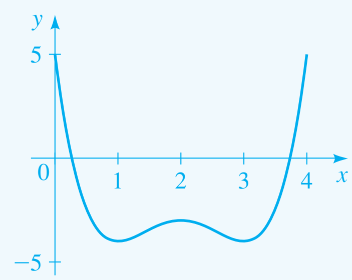


Figure 14
A family of polynomials $P(x) = x^3 - cx^2$

ALTERNATE EXAMPLE 9
Determine how many local extrema each polynomial has.
(a) $P_1(x) = x^3 - x$
(b) $P_2(x) = x^4 - 8x^3 + 22x^2 - 24x + 5$

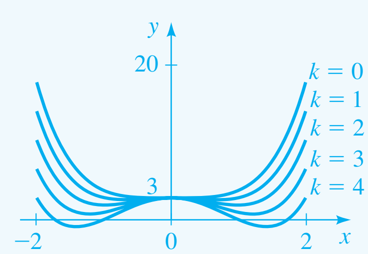
ANSWERS
(a) P_1 has one local minimum point and one local maximum point for a total of two local extrema.
(b) P_2 has two local minimum points and one local maximum point for a total of three local extrema.





ALTERNATE EXAMPLE 10
Sketch the family of polynomials $P(x) = x^4 - kx^2 + 3$ for $k = 0, 1, 2, 3, 4$. How does changing the value of k affect the graph?

ANSWER
The polynomials are graphed below. We see that increasing the value of k causes the two local minima to dip lower and lower.



3.1 Exercises

1–4 ■ Sketch the graph of each function by transforming the graph of an appropriate function of the form $y = x^n$ from Figure 2. Indicate all x- and y-intercepts on each graph.

1. (a) $P(x) = x^2 - 4$ **(b)** $Q(x) = (x - 4)^2$
(c) $R(x) = 2x^2 - 2$ **(d)** $S(x) = 2(x - 2)^2$

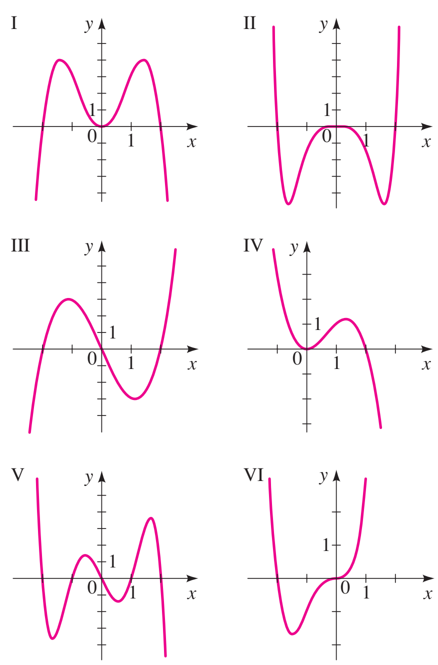
2. (a) $P(x) = x^4 - 16$ **(b)** $Q(x) = (x + 2)^4$
(c) $R(x) = (x + 2)^4 - 16$ **(d)** $S(x) = -2(x + 2)^4$

3. (a) $P(x) = x^3 - 8$ **(b)** $Q(x) = -x^3 + 27$
(c) $R(x) = -(x + 2)^3$ **(d)** $S(x) = \frac{1}{2}(x - 1)^3 + 4$

4. (a) $P(x) = (x + 3)^5$ **(b)** $Q(x) = 2(x + 3)^5 - 64$
(c) $R(x) = -\frac{1}{2}(x - 2)^5$ **(d)** $S(x) = -\frac{1}{2}(x - 2)^5 + 16$

5–10 ■ Match the polynomial function with one of the graphs I–VI. Give reasons for your choice.

5. $P(x) = x(x^2 - 4)$
6. $Q(x) = -x^2(x^2 - 4)$
7. $R(x) = -x^5 + 5x^3 - 4x$
8. $S(x) = \frac{1}{2}x^6 - 2x^4$
9. $T(x) = x^4 + 2x^3$
10. $U(x) = -x^3 + 2x^2$



11–22 ■ Sketch the graph of the polynomial function. Make sure your graph shows all intercepts and exhibits the proper end behavior.

11. $P(x) = (x - 1)(x + 2)$
12. $P(x) = (x - 1)(x + 1)(x - 2)$
13. $P(x) = x(x - 3)(x + 2)$
14. $P(x) = (2x - 1)(x + 1)(x + 3)$
15. $P(x) = (x - 3)(x + 2)(3x - 2)$
16. $P(x) = \frac{1}{5}x(x - 5)^2$
17. $P(x) = (x - 1)^2(x - 3)$
18. $P(x) = \frac{1}{4}(x + 1)^3(x - 3)$
19. $P(x) = \frac{1}{12}(x + 2)^2(x - 3)^2$
20. $P(x) = (x - 1)^2(x + 2)^3$
21. $P(x) = x^3(x + 2)(x - 3)^2$
22. $P(x) = (x - 3)^2(x + 1)^2$

23–36 ■ Factor the polynomial and use the factored form to find the zeros. Then sketch the graph.

23. $P(x) = x^3 - x^2 - 6x$
24. $P(x) = x^3 + 2x^2 - 8x$
25. $P(x) = -x^3 + x^2 + 12x$
26. $P(x) = -2x^3 - x^2 + x$
27. $P(x) = x^4 - 3x^3 + 2x^2$
28. $P(x) = x^5 - 9x^3$
29. $P(x) = x^3 + x^2 - x - 1$
30. $P(x) = x^3 + 3x^2 - 4x - 12$
31. $P(x) = 2x^3 - x^2 - 18x + 9$
32. $P(x) = \frac{1}{8}(2x^4 + 3x^3 - 16x - 24)^2$
33. $P(x) = x^4 - 2x^3 - 8x + 16$
34. $P(x) = x^4 - 2x^3 + 8x - 16$
35. $P(x) = x^4 - 3x^2 - 4$
36. $P(x) = x^6 - 2x^3 + 1$

37–42 ■ Determine the end behavior of P. Compare the graphs of P and Q on large and small viewing rectangles, as in Example 3(b).

37. $P(x) = 3x^3 - x^2 + 5x + 1$; $Q(x) = 3x^3$
38. $P(x) = -\frac{1}{8}x^3 + \frac{1}{4}x^2 + 12x$; $Q(x) = -\frac{1}{8}x^3$

39. $P(x) = x^4 - 7x^2 + 5x + 5$; $Q(x) = x^4$

40. $P(x) = -x^5 + 2x^2 + x$; $Q(x) = -x^5$

41. $P(x) = x^{11} - 9x^9$; $Q(x) = x^{11}$

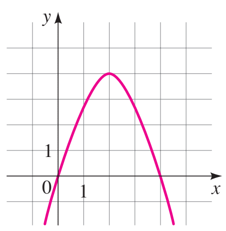
42. $P(x) = 2x^2 - x^{12}$; $Q(x) = -x^{12}$

43–46 ■ The graph of a polynomial function is given. From the graph, find

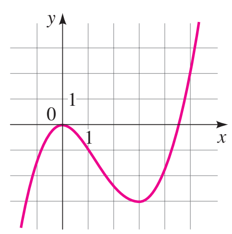
(a) the x- and y-intercepts

(b) the coordinates of all local extrema

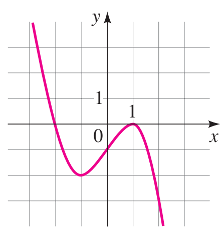
43. $P(x) = -x^2 + 4x$



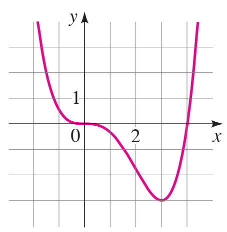
44. $P(x) = \frac{2}{9}x^3 - x^2$



45. $P(x) = -\frac{1}{2}x^3 + \frac{3}{2}x - 1$



46. $P(x) = \frac{1}{9}x^4 - \frac{4}{9}x^3$



47–54 ■ Graph the polynomial in the given viewing rectangle. Find the coordinates of all local extrema. State each answer correct to two decimal places.

47. $y = -x^2 + 8x$, $[-4, 12]$ by $[-50, 30]$

48. $y = x^3 - 3x^2$, $[-2, 5]$ by $[-10, 10]$

49. $y = x^3 - 12x + 9$, $[-5, 5]$ by $[-30, 30]$

50. $y = 2x^3 - 3x^2 - 12x - 32$, $[-5, 5]$ by $[-60, 30]$

51. $y = x^4 + 4x^3$, $[-5, 5]$ by $[-30, 30]$

52. $y = x^4 - 18x^2 + 32$, $[-5, 5]$ by $[-100, 100]$

53. $y = 3x^5 - 5x^3 + 3$, $[-3, 3]$ by $[-5, 10]$

54. $y = x^5 - 5x^2 + 6$, $[-3, 3]$ by $[-5, 10]$

55–64 ■ Graph the polynomial and determine how many local maxima and minima it has.

55. $y = -2x^2 + 3x + 5$

56. $y = x^3 + 12x$

57. $y = x^3 - x^2 - x$

58. $y = 6x^3 + 3x + 1$

59. $y = x^4 - 5x^2 + 4$

60. $y = 1.2x^5 + 3.75x^4 - 7x^3 - 15x^2 + 18x$

61. $y = (x - 2)^5 + 32$

62. $y = (x^2 - 2)^3$

63. $y = x^8 - 3x^4 + x$

64. $y = \frac{1}{3}x^7 - 17x^2 + 7$

65–70 ■ Graph the family of polynomials in the same viewing rectangle, using the given values of c. Explain how changing the value of c affects the graph.

65. $P(x) = cx^3$; $c = 1, 2, 5, \frac{1}{2}$

66. $P(x) = (x - c)^4$; $c = -1, 0, 1, 2$

67. $P(x) = x^4 + c$; $c = -1, 0, 1, 2$

68. $P(x) = x^3 + cx$; $c = 2, 0, -2, -4$

69. $P(x) = x^4 - cx$; $c = 0, 1, 8, 27$

70. $P(x) = x^c$; $c = 1, 3, 5, 7$

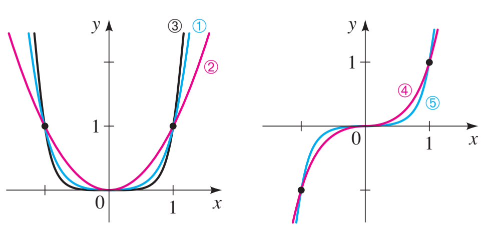
71. **(a)** On the same coordinate axes, sketch graphs (as accurately as possible) of the functions

$$y = x^3 - 2x^2 - x + 2 \quad \text{and} \quad y = -x^2 + 5x + 2$$

(b) Based on your sketch in part (a), at how many points do the two graphs appear to intersect?

(c) Find the coordinates of all intersection points.

72. Portions of the graphs of $y = x^2$, $y = x^3$, $y = x^4$, $y = x^5$, and $y = x^6$ are plotted in the figures. Determine which function belongs to each graph.



73. Recall that a function f is *odd* if $f(-x) = -f(x)$ or *even* if $f(-x) = f(x)$ for all real x.

(a) Show that a polynomial $P(x)$ that contains only odd powers of x is an odd function.

(b) Show that a polynomial $P(x)$ that contains only even powers of x is an even function.

(c) Show that if a polynomial $P(x)$ contains both odd and even powers of x, then it is neither an odd nor an even function.

(d) Express the function

$$P(x) = x^5 + 6x^3 - x^2 - 2x + 5$$

as the sum of an odd function and an even function.

74. **(a)** Graph the function $P(x) = (x - 1)(x - 3)(x - 4)$ and find all local extrema, correct to the nearest tenth.

(b) Graph the function

$$Q(x) = (x - 1)(x - 3)(x - 4) + 5$$

and use your answers to part (a) to find all local extrema, correct to the nearest tenth.

75. **(a)** Graph the function $P(x) = (x - 2)(x - 4)(x - 5)$ and determine how many local extrema it has.

(b) If $a < b < c$, explain why the function

$$P(x) = (x - a)(x - b)(x - c)$$

must have two local extrema.

76. **(a)** How many x-intercepts and how many local extrema does the polynomial $P(x) = x^3 - 4x$ have?

(b) How many x-intercepts and how many local extrema does the polynomial $Q(x) = x^3 + 4x$ have?

(c) If $a > 0$, how many x-intercepts and how many local extrema does each of the polynomials $P(x) = x^3 - ax$ and $Q(x) = x^3 + ax$ have? Explain your answer.

Applications

77. Market Research A market analyst working for a small-appliance manufacturer finds that if the firm produces and sells x blenders annually, the total profit (in dollars) is

$$P(x) = 8x + 0.3x^2 - 0.0013x^3 - 372$$

Graph the function P in an appropriate viewing rectangle and use the graph to answer the following questions.

(a) When just a few blenders are manufactured, the firm loses money (profit is negative). (For example, $P(10) = -263.3$, so the firm loses \$263.30 if it produces and sells only 10 blenders.) How many blenders must the firm produce to break even?

(b) Does profit increase indefinitely as more blenders are produced and sold? If not, what is the largest possible profit the firm could have?

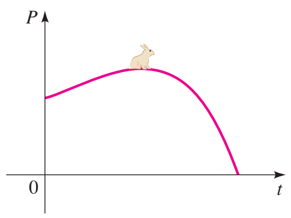
78. Population Change The rabbit population on a small island is observed to be given by the function

$$P(t) = 120t - 0.4t^4 + 1000$$

where t is the time (in months) since observations of the island began.

(a) When is the maximum population attained, and what is that maximum population?

(b) When does the rabbit population disappear from the island?

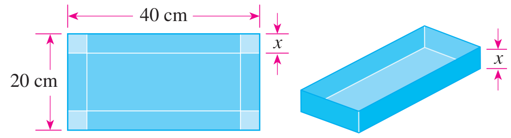


79. Volume of a Box An open box is to be constructed from a piece of cardboard 20 cm by 40 cm by cutting squares of side length x from each corner and folding up the sides, as shown in the figure.

(a) Express the volume V of the box as a function of x.

(b) What is the domain of V? (Use the fact that length and volume must be positive.)

(c) Draw a graph of the function V and use it to estimate the maximum volume for such a box.

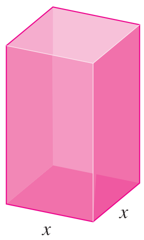


80. Volume of a Box A cardboard box has a square base, with each edge of the base having length x inches, as shown in the figure. The total length of all 12 edges of the box is 144 in.

(a) Show that the volume of the box is given by the function $V(x) = 2x^2(18 - x)$.

(b) What is the domain of V? (Use the fact that length and volume must be positive.)

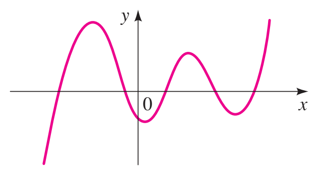
(c) Draw a graph of the function V and use it to estimate the maximum volume for such a box.



Discovery • Discussion

81. Graphs of Large Powers Graph the functions $y = x^2$, $y = x^3$, $y = x^4$, and $y = x^5$, for $-1 \le x \le 1$, on the same coordinate axes. What do you think the graph of $y = x^{100}$ would look like on this same interval? What about $y = x^{101}$? Make a table of values to confirm your answers.

82. Maximum Number of Local Extrema What is the smallest possible degree that the polynomial whose graph is shown can have? Explain.



83. Possible Number of Local Extrema Is it possible for a third-degree polynomial to have exactly one local extremum? Can a fourth-degree polynomial have exactly two local extrema? How many local extrema can polynomials of third, fourth, fifth, and sixth degree have? (Think about the end behavior of such polynomials.) Now give an example of a polynomial that has six local extrema.

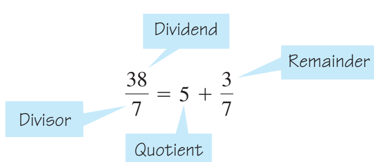
84. Impossible Situation? Is it possible for a polynomial to have two local maxima and no local minimum? Explain.

3.2 Dividing Polynomials

So far in this chapter we have been studying polynomial functions *graphically*. In this section we begin to study polynomials *algebraically*. Most of our work will be concerned with factoring polynomials, and to factor, we need to know how to divide polynomials.

Long Division of Polynomials

Dividing polynomials is much like the familiar process of dividing numbers. When we divide 38 by 7, the quotient is 5 and the remainder is 3. We write



To divide polynomials, we use long division, as in the next example.

SUGGESTED TIME AND EMPHASIS

$\frac{1}{2}$–1 class.

Essential material. Can be combined with Section 3.3.

POINTS TO STRESS

1. The division algorithm for polynomials.
2. Synthetic division.
3. The Remainder and Factor Theorems.

ALTERNATE EXAMPLE 1
Divide $8x^3 - 3x^2 + 2x - 1$ by $x + 2$.

ANSWER
Dividend $= 8x^3 - 3x^2 + 2x - 1$
Divisor $= x + 2$
Quotient $= 8x^2 - 19x + 40$
Remainder $= -81$

Example 1 Long Division of Polynomials

Divide $6x^2 - 26x + 12$ by $x - 4$.

Solution The *dividend* is $6x^2 - 26x + 12$ and the *divisor* is $x - 4$. We begin by arranging them as follows:

$$x - 4\overline{)6x^2 - 26x + 12}$$

Next we divide the leading term in the dividend by the leading term in the divisor to get the first term of the quotient: $6x^2/x = 6x$. Then we multiply the divisor by $6x$ and subtract the result from the dividend.

$$\begin{array}{r} 6x \\ x - 4\overline{)6x^2 - 26x + 12} \\ \underline{6x^2 - 24x} \\ -2x + 12 \end{array}$$

Divide leading terms: $\frac{6x^2}{x} = 6x$

Multiply: $6x(x - 4) = 6x^2 - 24x$

Subtract and "bring down" 12

We repeat the process using the last line $-2x + 12$ as the dividend.

$$\begin{array}{r} 6x - 2 \\ x - 4\overline{)6x^2 - 26x + 12} \\ \underline{6x^2 - 24x} \\ -2x + 12 \\ \underline{-2x + 8} \\ 4 \end{array}$$

Divide leading terms: $\frac{-2x}{x} = -2$

Multiply: $-2(x - 4) = -2x + 8$

Subtract

The division process ends when the last line is of lesser degree than the divisor. The last line then contains the *remainder*, and the top line contains the *quotient*. The result of the division can be interpreted in either of two ways.

$$\frac{6x^2 - 26x + 12}{x - 4} = 6x - 2 + \frac{4}{x - 4}$$

or

$$6x^2 - 26x + 12 = (x - 4)(6x - 2) + 4$$

Dividend — Divisor — Quotient — Remainder

We summarize the long division process in the following theorem.

Division Algorithm

If $P(x)$ and $D(x)$ are polynomials, with $D(x) \neq 0$, then there exist unique polynomials $Q(x)$ and $R(x)$, where $R(x)$ is either 0 or of degree less than the degree of $D(x)$, such that

$$P(x) = D(x) \cdot Q(x) + R(x)$$

Dividend — Divisor — Quotient — Remainder

The polynomials $P(x)$ and $D(x)$ are called the **dividend** and **divisor**, respectively, $Q(x)$ is the **quotient**, and $R(x)$ is the **remainder**.

To write the division algorithm another way, divide through by $D(x)$:

$$\frac{P(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}$$

SAMPLE QUESTION
Text Question

It is a fact that $x^3 + 2x^2 - 3x + 1 = (x + 2)(x^2 + 1) + (-4x - 1)$. Fill in the blanks:

______ Remainder: ______

$$x^2 + 1\overline{)x^3 + 2x^2 - 3x + 1}$$

Answer

$x + 2$, $-4x - 1$

Example 2 Long Division of Polynomials

Let $P(x) = 8x^4 + 6x^2 - 3x + 1$ and $D(x) = 2x^2 - x + 2$. Find polynomials $Q(x)$ and $R(x)$ such that $P(x) = D(x) \cdot Q(x) + R(x)$.

Solution We use long division after first inserting the term $0x^3$ into the dividend to ensure that the columns line up correctly.

$$\begin{array}{rl}
 & 4x^2 + 2x \\
2x^2 - x + 2 \,\big)\!\! & 8x^4 + 0x^3 + 6x^2 - 3x + 1 \\
 & \underline{8x^4 - 4x^3 + 8x^2} \\
 & 4x^3 - 2x^2 - 3x \\
 & \underline{4x^3 - 2x^2 + 4x} \\
 & -7x + 1
\end{array}$$

Multiply divisor by $4x^2$
Subtract
Multiply divisor by $2x$
Subtract

The process is complete at this point because $-7x + 1$ is of lesser degree than the divisor $2x^2 - x + 2$. From the above long division we see that $Q(x) = 4x^2 + 2x$ and $R(x) = -7x + 1$, so

$$8x^4 + 6x^2 - 3x + 1 = (2x^2 - x + 2)(4x^2 + 2x) + (-7x + 1)$$ ■

Synthetic Division

Synthetic division is a quick method of dividing polynomials; it can be used when the divisor is of the form $x - c$. In synthetic division we write only the essential parts of the long division. Compare the following long and synthetic divisions, in which we divide $2x^3 - 7x^2 + 5$ by $x - 3$. (We'll explain how to perform the synthetic division in Example 3.)

Long Division

$$\begin{array}{rl}
 & 2x^2 - x - 3 \\
x - 3 \,\big)\!\! & 2x^3 - 7x^2 + 0x + 5 \\
 & \underline{2x^3 - 6x^2} \\
 & -x^2 + 0x \\
 & \underline{-x^2 + 3x} \\
 & -3x + 5 \\
 & \underline{-3x + 9} \\
 & -4
\end{array}$$

Quotient (= $2x^2 - x - 3$); Remainder (= -4)

Synthetic Division

$$\begin{array}{r|rrrr}
3 & 2 & -7 & 0 & 5 \\
 & & 6 & -3 & -9 \\
\hline
 & 2 & -1 & -3 & -4
\end{array}$$

Quotient (= $2\ \ -1\ \ -3$); Remainder (= -4)

Note that in synthetic division we abbreviate $2x^3 - 7x^2 + 5$ by writing only the coefficients: 2 −7 0 5, and instead of $x - 3$, we simply write 3. (Writing 3 instead of −3 allows us to add instead of subtract, but this changes the sign of all the numbers that appear in the gold boxes.)

The next example shows how synthetic division is performed.

ALTERNATE EXAMPLE 2
Let $P(x) = 6x^4 + 2x^3 - x + 2$. Let $D(x) = x^2 - 2x + 2$. Find polynomials $Q(x)$ and $R(x)$ such that $P(x) = D(x) \cdot Q(x) + R(x)$.

ANSWER
$Q(x) = 6x^2 + 14x + 16$,
$R(x) = 3x - 30$

IN-CLASS MATERIALS

At this time, the teaching of long division in elementary schools is inconsistent. It will save time, in the long run, to do an integer long division problem for students, cautioning them to remind themselves of every step in the process, because you are going to be extending it to polynomials. For example, divide 31,673 by 5 using the long division algorithm. Then show how the answer can be written as

$$\frac{31{,}673}{5} = 6334 + \frac{3}{5} \text{ or } 6334 \text{ R } 3.$$

They will not be used to writing the result this way:

$$31673 = 5(6334) + 3$$

It is important that they understand the form

$$\text{dividend} = \text{divisor} \cdot \text{quotient} + \text{remainder}$$

because that is the form in which the division algorithm is presented, both in this course and any future math course involving generalized division. If students don't seem to understand (or start moving their lips as if beginning the process of rote memorization) it may even be worth the time to write out a very simple example, such as $35 = 3(11) + 2$, just so students are very clear how this is a trivial restatement of $\frac{35}{3} = 11$ R 2.

ALTERNATE EXAMPLE 3
Use synthetic division to divide $5x^3 - 2x^2 + x - 10$ by $x - 3$.

ANSWER
Quotient: $5x^2 + 13x + 40$; remainder: 110

DRILL QUESTION
Divide $x^3 + 2x^2 - 3x + 1$ by $x + 2$.

Answer

$$\frac{x^3 + 2x^2 - 3x + 1}{x + 2} = x^2 - 3 + \frac{7}{x+2}$$ or "$x^2 - 3$, remainder 7"

Example 3 Synthetic Division

Use synthetic division to divide $2x^3 - 7x^2 + 5$ by $x - 3$.

Solution We begin by writing the appropriate coefficients to represent the divisor and the dividend.

Divisor $x - 3$ →

3	2	−7	0	5

← Dividend $2x^3 - 7x^2 + 0x + 5$

We bring down the 2, multiply $3 \cdot 2 = 6$, and write the result in the middle row. Then we add:

3	2	−7	0	5	
		6			Multiply: $3 \cdot 2 = 6$
	2	−1			Add: $-7 + 6 = -1$

We repeat this process of multiplying and then adding until the table is complete.

3	2	−7	0	5	
		6	−3		Multiply: $3(-1) = -3$
	2	−1	−3		Add: $0 + (-3) = -3$

3	2	−7	0	5	
		6	−3	−9	Multiply: $3(-3) = 9$
	2	−1	−3	−4	Add: $5 + (-9) = -4$

Quotient $2x^2 - x - 3$ Remainder −4

From the last line of the synthetic division, we see that the quotient is $2x^2 - x - 3$ and the remainder is −4. Thus

$$2x^3 - 7x^2 + 5 = (x - 3)(2x^2 - x - 3) - 4$$ ■

The Remainder and Factor Theorems

The next theorem shows how synthetic division can be used to evaluate polynomials easily.

Remainder Theorem

If the polynomial $P(x)$ is divided by $x - c$, then the remainder is the value $P(c)$.

EXAMPLES
Fourth-degree polynomial functions with zeros at $x = -3$, 1, and 2:

$$f(x) = (x + 3)^2(x - 1)(x - 2) = x^4 + 3x^3 - 7x^2 - 15x + 18$$
$$f(x) = (x + 3)(x - 1)^2(x - 2) = x^4 - x^3 - 7x^2 + 13x - 6$$
$$f(x) = (x + 3)(x - 1)(x - 2)^2 = x^4 - 2x^3 - 7x^2 + 20x - 12$$

■ **Proof** If the divisor in the Division Algorithm is of the form $x - c$ for some real number c, then the remainder must be a constant (since the degree of the remainder is less than the degree of the divisor). If we call this constant r, then

$$P(x) = (x - c) \cdot Q(x) + r$$

Setting $x = c$ in this equation, we get $P(c) = (c - c) \cdot Q(x) + r = 0 + r = r$, that is, $P(c)$ is the remainder r. ■

Example 4 Using the Remainder Theorem to Find the Value of a Polynomial

Let $P(x) = 3x^5 + 5x^4 - 4x^3 + 7x + 3$.

(a) Find the quotient and remainder when $P(x)$ is divided by $x + 2$.

(b) Use the Remainder Theorem to find $P(-2)$.

Solution

(a) Since $x + 2 = x - (-2)$, the synthetic division for this problem takes the following form.

-2	3	5	-4	0	7	3
		-6	2	4	-8	2
	3	-1	-2	4	-1	5

Remainder is 5, so $P(-2) = 5$.

The quotient is $3x^4 - x^3 - 2x^2 + 4x - 1$ and the remainder is 5.

(b) By the Remainder Theorem, $P(-2)$ is the remainder when $P(x)$ is divided by $x - (-2) = x + 2$. From part (a) the remainder is 5, so $P(-2) = 5$. ■

The next theorem says that *zeros* of polynomials correspond to *factors*; we used this fact in Section 3.1 to graph polynomials.

Factor Theorem

c is a zero of P if and only if $x - c$ is a factor of $P(x)$.

■ **Proof** If $P(x)$ factors as $P(x) = (x - c) \cdot Q(x)$, then

$$P(c) = (c - c) \cdot Q(c) = 0 \cdot Q(c) = 0$$

Conversely, if $P(c) = 0$, then by the Remainder Theorem

$$P(x) = (x - c) \cdot Q(x) + 0 = (x - c) \cdot Q(x)$$

so $x - c$ is a factor of $P(x)$. ■

Example 5 Factoring a Polynomial Using the Factor Theorem

Let $P(x) = x^3 - 7x + 6$. Show that $P(1) = 0$, and use this fact to factor $P(x)$ completely.

1	1	0	-7	6
		1	1	-6
	1	1	-6	0

Solution Substituting, we see that $P(1) = 1^3 - 7 \cdot 1 + 6 = 0$. By the Factor Theorem, this means that $x - 1$ is a factor of $P(x)$. Using synthetic or long division

ALTERNATE EXAMPLE 4

Let $P(x) = 8x^5 - 2x^4 + 10x^3 + x^2 - 20x + 10$.

(a) Find the quotient and remainder when $P(x)$ is divided by $x + 2$.

(b) Use the Remainder Theorem to find $P(-2)$.

ANSWERS

(a) Quotient: $8x^4 - 18x^3 + 46x^2 - 91x + 162$; remainder: -314

(b) $P(-2) = -314$

EXAMPLE

An example to use in demonstrating the remainder and factor theorems: $f(x) = x^3 - x^2 - 14x + 24$ has zeros $x = -4$, 2, and 3; $f(0) = 24$, $f(1) = 10$, and $f(-1) = 36$.

ALTERNATE EXAMPLE 5

Let $P(x) = x^3 + 21x^2 - 157x + 135$. Use the fact that $P(1) = 0$ to factor $P(x)$ completely.

ANSWER

$P(1) = 0$ implies that $x - 1$ is a factor of $P(x)$. We divide by $x - 1$ to get a quotient of $x^2 + 22x - 135$ (and a remainder of zero). Now the quadratic formula, or factoring, gives the other factors of $(x - 5)$ and $(x + 27)$.

IN-CLASS MATERIALS

After doing a routine example, such as $\dfrac{x^4 + 3x^3 - x^2 - x + 3}{x^2 + 2x - 1} = x^2 + x - 2 + \dfrac{1 + 4x}{x^2 + 2x - 1}$, verify the answer by having students go through the multiplication. In other words, write $(x^2 + 2x - 1)(x^2 + x - 2) + (4x + 1)$ and multiply it out to verify that the result is $x^4 + 3x^3 - x^2 - x + 3$.

$$\begin{array}{r} x^2 + x - 6 \\ x - 1 \overline{) x^3 + 0x^2 - 7x + 6} \\ \underline{x^3 - x^2} \qquad\qquad \\ x^2 - 7x \quad\; \\ \underline{x^2 - x} \quad\; \\ -6x + 6 \\ \underline{-6x + 6} \\ 0 \end{array}$$

(shown in the margin), we see that

$$\begin{aligned} P(x) &= x^3 - 7x + 6 \\ &= (x - 1)(x^2 + x - 6) && \text{See margin} \\ &= (x - 1)(x - 2)(x + 3) && \text{Factor quadratic } x^2 + x - 6 \end{aligned}$$

■

Example 6 Finding a Polynomial with Specified Zeros

Find a polynomial of degree 4 that has zeros −3, 0, 1, and 5.

Solution By the Factor Theorem, $x - (-3)$, $x - 0$, $x - 1$, and $x - 5$ must all be factors of the desired polynomial, so let

$$P(x) = (x + 3)(x - 0)(x - 1)(x - 5) = x^4 - 3x^3 - 13x^2 + 15x$$

Since $P(x)$ is of degree 4 it is a solution of the problem. Any other solution of the problem must be a constant multiple of $P(x)$, since only multiplication by a constant does not change the degree. ■

The polynomial P of Example 6 is graphed in Figure 1. Note that the zeros of P correspond to the x-intercepts of the graph.

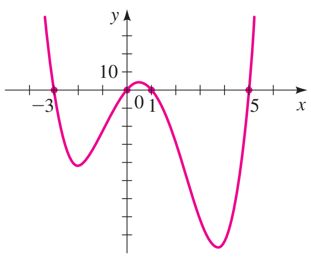


Figure 1
$P(x) = (x + 3)x(x - 1)(x - 5)$ has zeros −3, 0, 1, and 5.

3.2 Exercises

1–6 ■ Two polynomials P and D are given. Use either synthetic or long division to divide $P(x)$ by $D(x)$, and express P in the form $P(x) = D(x) \cdot Q(x) + R(x)$.

1. $P(x) = 3x^2 + 5x - 4, \quad D(x) = x + 3$
2. $P(x) = x^3 + 4x^2 - 6x + 1, \quad D(x) = x - 1$
3. $P(x) = 2x^3 - 3x^2 - 2x, \quad D(x) = 2x - 3$
4. $P(x) = 4x^3 + 7x + 9, \quad D(x) = 2x + 1$
5. $P(x) = x^4 - x^3 + 4x + 2, \quad D(x) = x^2 + 3$
6. $P(x) = 2x^5 + 4x^4 - 4x^3 - x - 3, \quad D(x) = x^2 - 2$

7–12 ■ Two polynomials P and D are given. Use either synthetic or long division to divide $P(x)$ by $D(x)$, and express the quotient $P(x)/D(x)$ in the form

$$\frac{P(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}$$

7. $P(x) = x^2 + 4x - 8, \quad D(x) = x + 3$
8. $P(x) = x^3 + 6x + 5, \quad D(x) = x - 4$
9. $P(x) = 4x^2 - 3x - 7, \quad D(x) = 2x - 1$
10. $P(x) = 6x^3 + x^2 - 12x + 5, \quad D(x) = 3x - 4$
11. $P(x) = 2x^4 - x^3 + 9x^2, \quad D(x) = x^2 + 4$
12. $P(x) = x^5 + x^4 - 2x^3 + x + 1, \quad D(x) = x^2 + x - 1$

13–22 ■ Find the quotient and remainder using long division.

13. $\dfrac{x^2 - 6x - 8}{x - 4}$
14. $\dfrac{x^3 - x^2 - 2x + 6}{x - 2}$
15. $\dfrac{4x^3 + 2x^2 - 2x - 3}{2x + 1}$
16. $\dfrac{x^3 + 3x^2 + 4x + 3}{3x + 6}$
17. $\dfrac{x^3 + 6x + 3}{x^2 - 2x + 2}$
18. $\dfrac{3x^4 - 5x^3 - 20x - 5}{x^2 + x + 3}$
19. $\dfrac{6x^3 + 2x^2 + 22x}{2x^2 + 5}$
20. $\dfrac{9x^2 - x + 5}{3x^2 - 7x}$
21. $\dfrac{x^6 + x^4 + x^2 + 1}{x^2 + 1}$
22. $\dfrac{2x^5 - 7x^4 - 13}{4x^2 - 6x + 8}$

23–36 ■ Find the quotient and remainder using synthetic division.

23. $\dfrac{x^2 - 5x + 4}{x - 3}$
24. $\dfrac{x^2 - 5x + 4}{x - 1}$
25. $\dfrac{3x^2 + 5x}{x - 6}$
26. $\dfrac{4x^2 - 3}{x + 5}$
27. $\dfrac{x^3 + 2x^2 + 2x + 1}{x + 2}$
28. $\dfrac{3x^3 - 12x^2 - 9x + 1}{x - 5}$
29. $\dfrac{x^3 - 8x + 2}{x + 3}$
30. $\dfrac{x^4 - x^3 + x^2 - x + 2}{x - 2}$

ALTERNATE EXAMPLE 6

(a) Find a polynomial of degree 3 that has zeros 1, 3, and 4.
(b) Find a polynomial of degree 4 that has zeros 1, 3, and 4.

ANSWERS

(a) $P(x) = (x - 1)(x - 3)(x - 4)$ works, as would $k(x - 1) \times (x - 3)(x - 4)$ for every nonzero constant k; $(x - 1) \times (x - 3)(x - 4) = x^3 - 8x^2 + 19x - 12$
(b) There are many choices here—we could multiply our previous answer by any factor $(x - k)$. If we want the ONLY zeros to be 1, 3, and 4 we would multiply by $(x - 1)$ or $(x - 3)$ or $(x - 4)$ to get a zero of multiplicity 2.

IN-CLASS MATERIALS

Students often miss the crucial idea that synthetic division is a technique that only works for divisors of the form $x - c$. They also tend to believe that synthetic division is a magic process that has nothing to do with the long division that they have just learned. They should be disabused of both notions. For example, divide the polynomial $x^3 - x^2 + x - 1$ by $x - 2$ using both methods, showing all work, and then have students point out the similarities between the two computations. They should see that both processes are essentially the same, the only difference being that synthetic division minimizes the amount of writing.

31. $\dfrac{x^5 + 3x^3 - 6}{x - 1}$

32. $\dfrac{x^3 - 9x^2 + 27x - 27}{x - 3}$

33. $\dfrac{2x^3 + 3x^2 - 2x + 1}{x - \frac{1}{2}}$

34. $\dfrac{6x^4 + 10x^3 + 5x^2 + x + 1}{x + \frac{2}{3}}$

35. $\dfrac{x^3 - 27}{x - 3}$

36. $\dfrac{x^4 - 16}{x + 2}$

37–49 ■ Use synthetic division and the Remainder Theorem to evaluate $P(c)$.

37. $P(x) = 4x^2 + 12x + 5, \quad c = -1$

38. $P(x) = 2x^2 + 9x + 1, \quad c = \frac{1}{2}$

39. $P(x) = x^3 + 3x^2 - 7x + 6, \quad c = 2$

40. $P(x) = x^3 - x^2 + x + 5, \quad c = -1$

41. $P(x) = x^3 + 2x^2 - 7, \quad c = -2$

42. $P(x) = 2x^3 - 21x^2 + 9x - 200, \quad c = 11$

43. $P(x) = 5x^4 + 30x^3 - 40x^2 + 36x + 14, \quad c = -7$

44. $P(x) = 6x^5 + 10x^3 + x + 1, \quad c = -2$

45. $P(x) = x^7 - 3x^2 - 1, \quad c = 3$

46. $P(x) = -2x^6 + 7x^5 + 40x^4 - 7x^2 + 10x + 112, \quad c = -3$

47. $P(x) = 3x^3 + 4x^2 - 2x + 1, \quad c = \frac{2}{3}$

48. $P(x) = x^3 - x + 1, \quad c = \frac{1}{4}$

49. $P(x) = x^3 + 2x^2 - 3x - 8, \quad c = 0.1$

50. Let

$$P(x) = 6x^7 - 40x^6 + 16x^5 - 200x^4 - 60x^3 - 69x^2 + 13x - 139$$

Calculate $P(7)$ by **(a)** using synthetic division and **(b)** substituting $x = 7$ into the polynomial and evaluating directly.

51–54 ■ Use the Factor Theorem to show that $x - c$ is a factor of $P(x)$ for the given value(s) of c.

51. $P(x) = x^3 - 3x^2 + 3x - 1, \quad c = 1$

52. $P(x) = x^3 + 2x^2 - 3x - 10, \quad c = 2$

53. $P(x) = 2x^3 + 7x^2 + 6x - 5, \quad c = \frac{1}{2}$

54. $P(x) = x^4 + 3x^3 - 16x^2 - 27x + 63, \quad c = 3, -3$

55–56 ■ Show that the given value(s) of c are zeros of $P(x)$, and find all other zeros of $P(x)$.

55. $P(x) = x^3 - x^2 - 11x + 15, \quad c = 3$

56. $P(x) = 3x^4 - x^3 - 21x^2 - 11x + 6, \quad c = \frac{1}{3}, -2$

57–60 ■ Find a polynomial of the specified degree that has the given zeros.

57. Degree 3; zeros $-1, 1, 3$

58. Degree 4; zeros $-2, 0, 2, 4$

59. Degree 4; zeros $-1, 1, 3, 5$

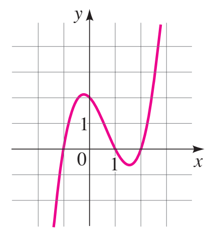
60. Degree 5; zeros $-2, -1, 0, 1, 2$

61. Find a polynomial of degree 3 that has zeros 1, -2, and 3, and in which the coefficient of x^2 is 3.

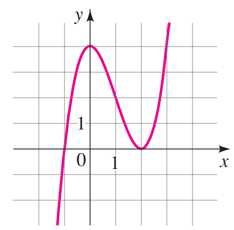
62. Find a polynomial of degree 4 that has integer coefficients and zeros 1, -1, 2, and $\frac{1}{2}$.

63–66 ■ Find the polynomial of the specified degree whose graph is shown.

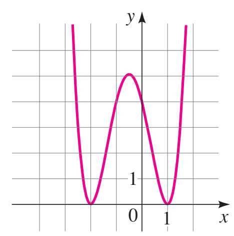
63. Degree 3



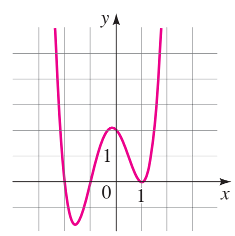
64. Degree 3



65. Degree 4



66. Degree 4



Discovery • Discussion

67. Impossible Division? Suppose you were asked to solve the following two problems on a test:

A. Find the remainder when $6x^{1000} - 17x^{562} + 12x + 26$ is divided by $x + 1$.

B. Is $x - 1$ a factor of $x^{567} - 3x^{400} + x^9 + 2$?

Obviously, it's impossible to solve these problems by dividing, because the polynomials are of such large degree. Use one or more of the theorems in this section to solve these problems *without* actually dividing.

IN-CLASS MATERIALS

One important application of polynomial division is finding asymptotes for rational functions. This is a good time to introduce the concept of a horizontal asymptote. A good example to discuss with students is $f(x) = \dfrac{2x^2 + 3x + 5}{x^2 - 4x + 2}$. Use long division to write this as $f(x) = 2 + \dfrac{11x + 1}{x^2 - 4x + 2}$. Now note what happens to the second term for large values of x. (If students have calculators, they can go ahead and try $x = 100$, $x = 100{,}000$, and $x = 1{,}000{,}000{,}000$. Show how, graphically, this corresponds to a horizontal asymptote. Now point out that the 2 came from only the highest-degree term in the numerator and the highest-degree term in the denominator. Now discuss the possibilities for horizontal asymptotes in the rational functions $\dfrac{x^2 + 2x + 2}{3x^2 + 2x + 2}$, $\dfrac{x^2 + 2x + 2}{x - 7}$, and $\dfrac{x^2 + 2x + 2}{x^5 - x + 4}$. In all cases, go ahead and do the long division, so the students see the possibilities. (The second one has no horizontal asymptote.)

68. Nested Form of a Polynomial Expand Q to prove that the polynomials P and Q are the same.

$$P(x) = 3x^4 - 5x^3 + x^2 - 3x + 5$$
$$Q(x) = (((3x - 5)x + 1)x - 3)x + 5$$

Try to evaluate $P(2)$ and $Q(2)$ in your head, using the forms given. Which is easier? Now write the polynomial $R(x) = x^5 - 2x^4 + 3x^3 - 2x^2 + 3x + 4$ in "nested" form, like the polynomial Q. Use the nested form to find $R(3)$ in your head.

Do you see how calculating with the nested form follows the same arithmetic steps as calculating the value of a polynomial using synthetic division?

SUGGESTED TIME AND EMPHASIS

$\frac{1}{2}$–1 class.
Essential material. Can be combined with Section 3.2.

3.3 Real Zeros of Polynomials

The Factor Theorem tells us that finding the zeros of a polynomial is really the same thing as factoring it into linear factors. In this section we study some algebraic methods that help us find the real zeros of a polynomial, and thereby factor the polynomial. We begin with the *rational* zeros of a polynomial.

Rational Zeros of Polynomials

To help us understand the next theorem, let's consider the polynomial

$$P(x) = (x - 2)(x - 3)(x + 4) \qquad \text{Factored form}$$
$$= x^3 - x^2 - 14x + 24 \qquad \text{Expanded form}$$

From the factored form we see that the zeros of P are 2, 3, and -4. When the polynomial is expanded, the constant 24 is obtained by multiplying $(-2) \times (-3) \times 4$. This means that the zeros of the polynomial are all factors of the constant term. The following generalizes this observation.

Rational Zeros Theorem

If the polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ has integer coefficients, then every rational zero of P is of the form

$$\frac{p}{q}$$

where p is a factor of the constant coefficient a_0
and q is a factor of the leading coefficient a_n.

■ **Proof** If p/q is a rational zero, in lowest terms, of the polynomial P, then we have

$$a_n\left(\frac{p}{q}\right)^n + a_{n-1}\left(\frac{p}{q}\right)^{n-1} + \cdots + a_1\left(\frac{p}{q}\right) + a_0 = 0$$
$$a_n p^n + a_{n-1}p^{n-1}q + \cdots + a_1 p q^{n-1} + a_0 q^n = 0 \qquad \text{Multiply by } q^n$$
$$p(a_n p^{n-1} + a_{n-1}p^{n-2}q + \cdots + a_1 q^{n-1}) = -a_0 q^n \qquad \text{Subtract } a_0q^n \text{ and factor LHS}$$

Now p is a factor of the left side, so it must be a factor of the right as well. Since p/q is in lowest terms, p and q have no factor in common, and so p must be a factor of a_0. A similar proof shows that q is a factor of a_n. ■

We see from the Rational Zeros Theorem that if the leading coefficient is 1 or -1, then the rational zeros must be factors of the constant term.

POINTS TO STRESS

1. The Rational Zeros Theorem: The rational zeros of a polynomial function are always quotients of factors of the constant and the leading terms.
2. Factoring large polynomials using the Rational Zeros Theorem and the quadratic formula.
3. Bounding the number and size of zeros of a polynomial function.

Evariste Galois (1811–1832) is one of the very few mathematicians to have an entire theory named in his honor. Not yet 21 when he died, he completely settled the central problem in the theory of equations by describing a criterion that reveals whether a polynomial equation can be solved by algebraic operations. Galois was one of the greatest mathematicians in the world at that time, although no one knew it but him. He repeatedly sent his work to the eminent mathematicians Cauchy and Poisson, who either lost his letters or did not understand his ideas. Galois wrote in a terse style and included few details, which probably played a role in his failure to pass the entrance exams at the Ecole Polytechnique in Paris. A political radical, Galois spent several months in prison for his revolutionary activities. His brief life came to a tragic end when he was killed in a duel over a love affair. The night before his duel, fearing he would die, Galois wrote down the essence of his ideas and entrusted them to his friend Auguste Chevalier. He concluded by writing "... there will, I hope, be people who will find it to their advantage to decipher all this mess." The mathematician Camille Jordan did just that, 14 years later.

Example 1 Finding Rational Zeros (Leading Coefficient 1)

Find the rational zeros of $P(x) = x^3 - 3x + 2$.

Solution Since the leading coefficient is 1, any rational zero must be a divisor of the constant term 2. So the possible rational zeros are ± 1 and ± 2. We test each of these possibilities.

$$\begin{aligned} P(1) &= (1)^3 - 3(1) + 2 = 0 \\ P(-1) &= (-1)^3 - 3(-1) + 2 = 4 \\ P(2) &= (2)^3 - 3(2) + 2 = 4 \\ P(-2) &= (-2)^3 - 3(-2) + 2 = 0 \end{aligned}$$

The rational zeros of P are 1 and -2. ■

ALTERNATE EXAMPLE 1
Find all rational zeros of the polynomial $P(x) = x^3 - 11x^2 + 23x + 35$.

ANSWER
$-1, 5, 7$

Example 2 Using the Rational Zeros Theorem to Factor a Polynomial

Factor the polynomial $P(x) = 2x^3 + x^2 - 13x + 6$.

Solution By the Rational Zeros Theorem the rational zeros of P are of the form

$$\text{possible rational zero of } P = \frac{\text{factor of constant term}}{\text{factor of leading coefficient}}$$

The constant term is 6 and the leading coefficient is 2, so

$$\text{possible rational zero of } P = \frac{\text{factor of 6}}{\text{factor of 2}}$$

The factors of 6 are $\pm 1, \pm 2, \pm 3, \pm 6$ and the factors of 2 are $\pm 1, \pm 2$. Thus, the possible rational zeros of P are

$$\pm\frac{1}{1}, \quad \pm\frac{2}{1}, \quad \pm\frac{3}{1}, \quad \pm\frac{6}{1}, \quad \pm\frac{1}{2}, \quad \pm\frac{2}{2}, \quad \pm\frac{3}{2}, \quad \pm\frac{6}{2}$$

Simplifying the fractions and eliminating duplicates, we get the following list of possible rational zeros:

$$\pm 1, \quad \pm 2, \quad \pm 3, \quad \pm 6, \quad \pm\frac{1}{2}, \quad \pm\frac{3}{2}$$

To check which of these *possible* zeros actually *are* zeros, we need to evaluate P at each of these numbers. An efficient way to do this is to use synthetic division.

Test if 1 is a zero

$$\begin{array}{r|rrrr} 1 & 2 & 1 & -13 & 6 \\ & & 2 & 3 & -10 \\ \hline & 2 & 3 & -10 & -4 \end{array}$$

Remainder is not 0, so 1 is not a zero.

Test if 2 is a zero

$$\begin{array}{r|rrrr} 2 & 2 & 1 & -13 & 6 \\ & & 4 & 10 & -6 \\ \hline & 2 & 5 & -3 & 0 \end{array}$$

Remainder is 0, so 2 is a zero.

ALTERNATE EXAMPLE 2
Factor the polynomial $P(x) = 3x^3 - 4x^2 - 13x - 6$.

ANSWER
$P(x) = (x + 1)(x - 3)(3x + 2)$

SAMPLE QUESTIONS
Text Questions
Consider $f(x) = x^6 - 2x^5 - x^4 + 4x^3 - x^2 - 2x + 1$.

(a) According to the Rational Zeros Theorem, how many possible rational zeros can this polynomial have?
(b) List all the rational zeros of $f(x)$. Ignore multiplicities and show your work.

Answers
(a) 2
(b) Both 1 and -1 are zeros of f. This can be shown by manually calculating $f(1)$ and $f(-1)$.

DRILL QUESTION

Factor $f(x) = x^3 - 6x + 4$.

Answer

$f(x) = (x - 2)(x + 1 + \sqrt{3}) \times (x + 1 - \sqrt{3})$

From the last synthetic division we see that 2 is a zero of P and that P factors as

$$\begin{aligned} P(x) &= 2x^3 + x^2 - 13x + 6 \\ &= (x - 2)(2x^2 + 5x - 3) \\ &= (x - 2)(2x - 1)(x + 3) \qquad \text{Factor } 2x^2 + 5x - 3 \end{aligned}$$

■

The following box explains how we use the Rational Zeros Theorem with synthetic division to factor a polynomial.

Finding the Rational Zeros of a Polynomial

1. **List Possible Zeros.** List all possible rational zeros using the Rational Zeros Theorem.
2. **Divide.** Use synthetic division to evaluate the polynomial at each of the candidates for rational zeros that you found in Step 1. When the remainder is 0, note the quotient you have obtained.
3. **Repeat.** Repeat Steps 1 and 2 for the quotient. Stop when you reach a quotient that is quadratic or factors easily, and use the quadratic formula or factor to find the remaining zeros.

ALTERNATE EXAMPLE 3a

Find all the real zeros of P.

$P(x) = -x^3 - 3x^2 + 13x + 15$

ANSWER

3, −1, −5

Example 3 Using the Rational Zeros Theorem and the Quadratic Formula

Let $P(x) = x^4 - 5x^3 - 5x^2 + 23x + 10$.

(a) Find the zeros of P.

(b) Sketch the graph of P.

1	1	−5	−5	23	10
		1	−4	−9	14
	1	−4	−9	14	24

2	1	−5	−5	23	10
		2	−6	−22	2
	1	−3	−11	1	12

5	1	−5	−5	23	10
		5	0	−25	−10
	1	0	−5	−2	0

Solution

(a) The leading coefficient of P is 1, so all the rational zeros are integers: They are divisors of the constant term 10. Thus, the possible candidates are

$$\pm 1, \quad \pm 2, \quad \pm 5, \quad \pm 10$$

Using synthetic division (see the margin) we find that 1 and 2 are not zeros, but that 5 is a zero and that P factors as

$$x^4 - 5x^3 - 5x^2 + 23x + 10 = (x - 5)(x^3 - 5x - 2)$$

We now try to factor the quotient $x^3 - 5x - 2$. Its possible zeros are the divisors of -2, namely,

$$\pm 1, \quad \pm 2$$

−2	1	0	−5	−2
		−2	4	2
	1	−2	−1	0

Since we already know that 1 and 2 are not zeros of the original polynomial P, we don't need to try them again. Checking the remaining candidates -1 and -2, we see that -2 is a zero (see the margin), and P factors as

$$\begin{aligned} x^4 - 5x^3 - 5x^2 + 23x + 10 &= (x - 5)(x^3 - 5x - 2) \\ &= (x - 5)(x + 2)(x^2 - 2x - 1) \end{aligned}$$

IN-CLASS MATERIALS

Students often misinterpret the Rational Zeros Theorem in two ways. Some believe that it classifies *all* the real zeros of a polynomial function, not just the rational ones. Others believe that it applies to *all* polynomial functions, not just the ones with integer coefficients.

Start with the simple quadratic $p(x) = x^2 - 2$, pointing out that the candidates for rational zeros are ± 1 and ± 2. None of these candidates are zeros of $p(x)$, but it is simple to find that there are two real zeros: $x = \pm\sqrt{2}$. Then move to a polynomial with one real zero and two irrational ones, such as $p(x) = x^3 - 6x + 4$. Ask students if they can come up with a polynomial with three real, irrational zeros. One example is $p(x) = x^3 - 3x - \sqrt{2}$ (the zeros are $-\sqrt{2}$ and $\frac{1}{2}\sqrt{2} \pm \frac{1}{2}\sqrt{6}$). This might be perceived as a bit of a cheat, so follow up by asking them if they can come up with a polynomial with integer coefficients and three real, irrational zeros. ($x^3 - 5x + 1$ works. The fact that there are three real roots can be determined from a graph, the fact that none are rational can be determined using the Rational Zeros Theorem.)

Now we use the quadratic formula to obtain the two remaining zeros of P:

$$x = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(-1)}}{2} = 1 \pm \sqrt{2}$$

The zeros of P are $5, -2, 1 + \sqrt{2}$, and $1 - \sqrt{2}$.

(b) Now that we know the zeros of P, we can use the methods of Section 3.1 to sketch the graph. If we want to use a graphing calculator instead, knowing the zeros allows us to choose an appropriate viewing rectangle—one that is wide enough to contain all the x-intercepts of P. Numerical approximations to the zeros of P are

$$5, \quad -2, \quad 2.4, \quad \text{and} \quad -0.4$$

So in this case we choose the rectangle $[-3, 6]$ by $[-50, 50]$ and draw the graph shown in Figure 1. ■

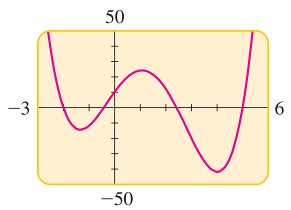


Figure 1
$P(x) = x^4 - 5x^3 - 5x^2 + 23x + 10$

Descartes' Rule of Signs and Upper and Lower Bounds for Roots

In some cases, the following rule—discovered by the French philosopher and mathematician René Descartes around 1637 (see page 112)—is helpful in eliminating candidates from lengthy lists of possible rational roots. To describe this rule, we need the concept of *variation in sign*. If $P(x)$ is a polynomial with real coefficients, written with descending powers of x (and omitting powers with coefficient 0), then a **variation in sign** occurs whenever adjacent coefficients have opposite signs. For example,

$$P(x) = 5x^7 - 3x^5 - x^4 + 2x^2 + x - 3$$

has three variations in sign.

Polynomial	Variations in sign
$x^2 + 4x + 1$	0
$2x^3 + x - 6$	1
$x^4 - 3x^2 - x + 4$	2

Descartes' Rule of Signs

Let P be a polynomial with real coefficients.

1. The number of positive real zeros of $P(x)$ is either equal to the number of variations in sign in $P(x)$ or is less than that by an even whole number.
2. The number of negative real zeros of $P(x)$ is either equal to the number of variations in sign in $P(-x)$ or is less than that by an even whole number.

Example 4 Using Descartes' Rule

Use Descartes' Rule of Signs to determine the possible number of positive and negative real zeros of the polynomial

$$P(x) = 3x^6 + 4x^5 + 3x^3 - x - 3$$

Solution The polynomial has one variation in sign and so it has one positive zero. Now

$$\begin{aligned} P(-x) &= 3(-x)^6 + 4(-x)^5 + 3(-x)^3 - (-x) - 3 \\ &= 3x^6 - 4x^5 - 3x^3 + x - 3 \end{aligned}$$

So, $P(-x)$ has three variations in sign. Thus, $P(x)$ has either three or one negative zero(s), making a total of either two or four real zeros. ■

ALTERNATE EXAMPLE 4
Find all the real zeros of the polynomial. Use the quadratic formula if necessary.
$P(x) = x^3 + 4x^2 + 3x - 2$

ANSWER
$-2, -1 \pm \sqrt{2}$

IN-CLASS MATERIALS

Ask students why we do not advocate using synthetic division to find the roots of a polynomial such as $p(x) = x^2 + 9x + 20$, $p(x) = x^2 + 9x - 7$, or even $p(x) = x^2 + \pi x - \sqrt[5]{2}$. Hopefully, you will arrive at the conclusion that the quadratic formula is easier to use and will find all the zeros, not just the rational ones. Point out that there actually is such a formula for third-degree polynomials, but that it is much harder to use. (For those interested, refer to Exercise 102.) There is one for fourth-degree polynomials as well. It has been proved that there is no such formula for arbitrary fifth-degree polynomials. In other words, we can find the exact roots for any polynomial up through a fourth-degree polynomial, but there are some polynomials, fifth-degree and higher, whose roots we can only approximate.

ALTERNATE EXAMPLE 5
Is it true or false that all the real zeros of the polynomial $P(x) = 4x^3 + 19x^2 + 11x - 7$ lie between -3 and 1?

ANSWER
False

ALTERNATE EXAMPLE 6
Factor completely the polynomial $2x^5 - 11x^4 - 10x^3 + 56x^2 + 88x + 35$.

ANSWER
$(x + 1)^3(x - 5)(2x - 7)$

We say that a is a **lower bound** and b is an **upper bound** for the zeros of a polynomial if every real zero c of the polynomial satisfies $a \le c \le b$. The next theorem helps us find such bounds for the zeros of a polynomial.

The Upper and Lower Bounds Theorem

Let P be a polynomial with real coefficients.

1. If we divide $P(x)$ by $x - b$ (with $b > 0$) using synthetic division, and if the row that contains the quotient and remainder has no negative entry, then b is an upper bound for the real zeros of P.
2. If we divide $P(x)$ by $x - a$ (with $a < 0$) using synthetic division, and if the row that contains the quotient and remainder has entries that are alternately nonpositive and nonnegative, then a is a lower bound for the real zeros of P.

A proof of this theorem is suggested in Exercise 91. The phrase "alternately nonpositive and nonnegative" simply means that the signs of the numbers alternate, with 0 considered to be positive or negative as required.

Example 5 Upper and Lower Bounds for Zeros of a Polynomial

Show that all the real zeros of the polynomial $P(x) = x^4 - 3x^2 + 2x - 5$ lie between -3 and 2.

Solution We divide $P(x)$ by $x - 2$ and $x + 3$ using synthetic division.

2	1	0	−3	2	−5
		2	4	2	8
	1	2	1	4	3

All entries positive

−3	1	0	−3	2	−5
		−3	9	−18	48
	1	−3	6	−16	43

Entries alternate in sign.

By the Upper and Lower Bounds Theorem, -3 is a lower bound and 2 is an upper bound for the zeros. Since neither -3 nor 2 is a zero (the remainders are not 0 in the division table), all the real zeros lie between these numbers. ■

Example 6 Factoring a Fifth-Degree Polynomial

Factor completely the polynomial

$$P(x) = 2x^5 + 5x^4 - 8x^3 - 14x^2 + 6x + 9$$

Solution The possible rational zeros of P are $\pm\frac{1}{2}, \pm1, \pm\frac{3}{2}, \pm3, \pm\frac{9}{2}$, and ±9. We check the positive candidates first, beginning with the smallest.

$\frac{1}{2}$	2	5	−8	−14	6	9
		1	3	$-\frac{5}{2}$	$-\frac{33}{4}$	$-\frac{9}{8}$
	2	6	−5	$-\frac{33}{2}$	$-\frac{9}{4}$	$\frac{63}{8}$

$\frac{1}{2}$ is not a zero

1	2	5	−8	−14	6	9
		2	7	−1	−15	−9
	2	7	−1	−15	−9	0

$P(1) = 0$

IN-CLASS MATERIALS

Point out that being able to find the zeros of a polynomial allows us to solve many types of problems. The text gives several examples of applied problems (and there are many more, of course). For example, we can now find the intersection points between two polynomial curves [if $f(x) = g(x)$, then $f(x) - g(x) = 0$]. If $p(x)$ is a polynomial with an inverse, we can find $p^{-1}(k)$ for a specific k by solving $p(x) - k = 0$. In addition, being able to factor polynomials is very important. For example, the graph of $f(x) = \dfrac{x^3 - 4x}{x^3 + 6x^2 + 11x + 6}$ has a hole at $x = -2$, vertical asymptotes at $x = -1$ and $x = -3$, and x-intercepts at (2, 0) and (0, 0). This information is easily obtained if we write $f(x)$ as $\dfrac{(x + 2)(x - 2)x}{(x + 1)(x + 2)(x + 3)}$.

So 1 is a zero, and $P(x) = (x - 1)(2x^4 + 7x^3 - x^2 - 15x - 9)$. We continue by factoring the quotient. We still have the same list of possible zeros except that $\frac{1}{2}$ has been eliminated.

1	2	7	−1	−15	−9
		2	9	8	−7
	2	9	8	−7	−16

1 is not a zero.

$\frac{3}{2}$	2	7	−1	−15	−9
		3	15	21	9
	2	10	14	6	0

$P(\frac{3}{2}) = 0$, all entries nonnegative

We see that $\frac{3}{2}$ is both a zero and an upper bound for the zeros of $P(x)$, so we don't need to check any further for positive zeros, because all the remaining candidates are greater than $\frac{3}{2}$.

$$\begin{aligned} P(x) &= (x - 1)(x - \tfrac{3}{2})(2x^3 + 10x^2 + 14x + 6) \\ &= (x - 1)(2x - 3)(x^3 + 5x^2 + 7x + 3) \end{aligned}$$

Factor 2 from last factor, multiply into second factor

By Descartes' Rule of Signs, $x^3 + 5x^2 + 7x + 3$ has no positive zero, so its only possible rational zeros are -1 and -3.

−1	1	5	7	3
		−1	−4	−3
	1	4	3	0

$P(-1) = 0$

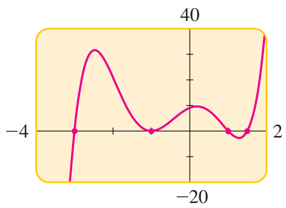


Figure 2
$P(x) = 2x^5 + 5x^4 - 8x^3 - 14x^2 + 6x + 9$
$= (x - 1)(2x - 3)(x + 1)^2(x + 3)$

Therefore

$$\begin{aligned} P(x) &= (x - 1)(2x - 3)(x + 1)(x^2 + 4x + 3) \\ &= (x - 1)(2x - 3)(x + 1)^2(x + 3) \end{aligned}$$

Factor quadratic

This means that the zeros of P are $1, \frac{3}{2}, -1$, and -3. The graph of the polynomial is shown in Figure 2. ■

Using Algebra and Graphing Devices to Solve Polynomial Equations

In Section 1.9 we used graphing devices to solve equations graphically. We can now use the algebraic techniques we've learned to select an appropriate viewing rectangle when solving a polynomial equation graphically.

Example 7 Solving a Fourth-Degree Equation Graphically

Find all real solutions of the following equation, correct to the nearest tenth.

$$3x^4 + 4x^3 - 7x^2 - 2x - 3 = 0$$

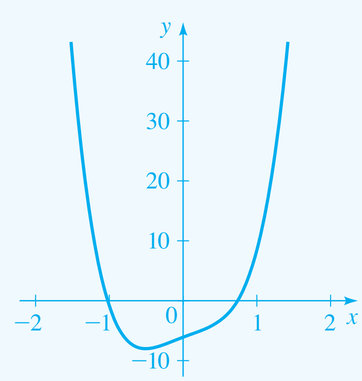
Solution To solve the equation graphically, we graph

$$P(x) = 3x^4 + 4x^3 - 7x^2 - 2x - 3$$

EXAMPLES
An example to use in demonstrating the remainder and factor theorems: $f(x) = x^3 - x^2 - 14x + 24$ has zeros $= -4, 2$, and 3; $f(0) = 24$, $f(1) = 10$, and $f(-1) = 36$.

ALTERNATE EXAMPLE 7
Find all real solutions of the following equation, correct to the nearest hundredth:
$10x^4 - x^2 + 4x - 6$

ANSWER
Roots to the nearest hundredth: $-1.03, 0.77$



We use the Upper and Lower Bounds Theorem to see where the roots can be found.

First we use the Upper and Lower Bounds Theorem to find two numbers between which all the solutions must lie. This allows us to choose a viewing rectangle that is certain to contain all the x-intercepts of P. We use synthetic division and proceed by trial and error.

To find an upper bound, we try the whole numbers, 1, 2, 3, . . . as potential candidates. We see that 2 is an upper bound for the roots.

2	3	4	−7	−2	−3
		6	20	26	48
	3	10	13	24	45

All positive

Now we look for a lower bound, trying the numbers −1, −2, and −3 as potential candidates. We see that −3 is a lower bound for the roots.

−3	3	4	−7	−2	−3
		−9	15	−24	78
	3	−5	8	−26	75

Entries alternate in sign.

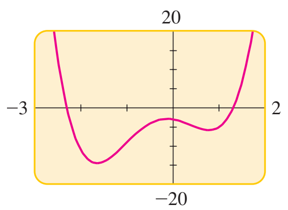


Figure 3
$y = 3x^4 + 4x^3 - 7x^2 - 2x - 3$

Thus, all the roots lie between −3 and 2. So the viewing rectangle $[-3, 2]$ by $[-20, 20]$ contains all the x-intercepts of P. The graph in Figure 3 has two x-intercepts, one between −3 and −2 and the other between 1 and 2. Zooming in, we find that the solutions of the equation, to the nearest tenth, are −2.3 and 1.3. ■

ALTERNATE EXAMPLE 8
A fuel tank consists of a cylindrical center section 10 ft long and two hemispherical end sections, as shown in Figure 4. If the tank has a volume of 400 ft^3, what is the radius r shown in the figure, correct to the nearest hundredth of a foot?

ANSWER
As in the text, we get the equation $\frac{4}{3}\pi r^3 + 10\pi r^2 - 400$. We graph to find its x-intercept to the nearest hundredth, $r = 3.01$ ft.

Example 8 Determining the Size of a Fuel Tank

A fuel tank consists of a cylindrical center section 4 ft long and two hemispherical end sections, as shown in Figure 4. If the tank has a volume of 100 ft^3, what is the radius r shown in the figure, correct to the nearest hundredth of a foot?

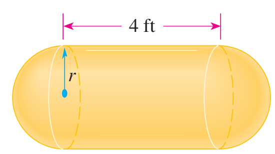


Figure 4

Solution Using the volume formula listed on the inside front cover of this book, we see that the volume of the cylindrical section of the tank is

Volume of a cylinder: $V = \pi r^2 h$

$$\pi \cdot r^2 \cdot 4$$

The two hemispherical parts together form a complete sphere whose volume is

Volume of a sphere: $V = \frac{4}{3}\pi r^3$

$$\tfrac{4}{3}\pi r^3$$

Because the total volume of the tank is 100 ft^3, we get the following equation:

$$\tfrac{4}{3}\pi r^3 + 4\pi r^2 = 100$$

A negative solution for r would be meaningless in this physical situation, and by substitution we can verify that $r = 3$ leads to a tank that is over 226 ft^3 in volume, much larger than the required 100 ft^3. Thus, we know the correct radius lies somewhere between 0 and 3 ft, and so we use a viewing rectangle of $[0, 3]$ by $[50, 150]$

EXAMPLE
A polynomial with many rational zeros: $f(x) = 6x^5 + 17x^4 - 40x^3 - 45x^2 + 14x + 8$

Factored form: $(2x - 1)(3x + 1)(x + 4)(x + 1)(x - 2)$

Zeros: $x = \frac{1}{2}, \frac{1}{3}, -4, -1$, and 2

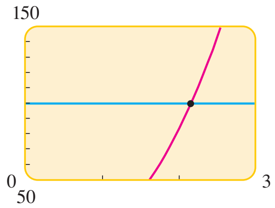


Figure 5
$y = \frac{4}{3}\pi x^3 + 4\pi x^2$ and $y = 100$

to graph the function $y = \frac{4}{3}\pi x^3 + 4\pi x^2$, as shown in Figure 5. Since we want the value of this function to be 100, we also graph the horizontal line $y = 100$ in the same viewing rectangle. The correct radius will be the x-coordinate of the point of intersection of the curve and the line. Using the cursor and zooming in, we see that at the point of intersection $x \approx 2.15$, correct to two decimal places. Thus, the tank has a radius of about 2.15 ft. ■

Note that we also could have solved the equation in Example 8 by first writing it as

$$\frac{4}{3}\pi r^3 + 4\pi r^2 - 100 = 0$$

and then finding the x-intercept of the function $y = \frac{4}{3}\pi x^3 + 4\pi x^2 - 100$.

3.3 Exercises

1–6 ■ List all possible rational zeros given by the Rational Zeros Theorem (but don't check to see which actually are zeros).

1. $P(x) = x^3 - 4x^2 + 3$

2. $Q(x) = x^4 - 3x^3 - 6x + 8$

3. $R(x) = 2x^5 + 3x^3 + 4x^2 - 8$

4. $S(x) = 6x^4 - x^2 + 2x + 12$

5. $T(x) = 4x^4 - 2x^2 - 7$

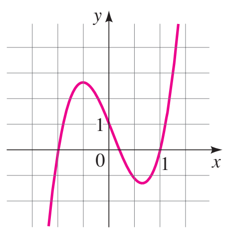
6. $U(x) = 12x^5 + 6x^3 - 2x - 8$

7–10 ■ A polynomial function P and its graph are given.

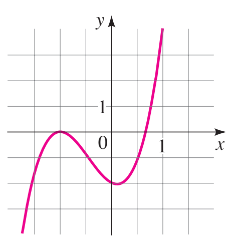
(a) List all possible rational zeros of P given by the Rational Zeros Theorem.

(b) From the graph, determine which of the possible rational zeros actually turn out to be zeros.

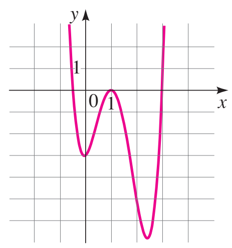
7. $P(x) = 5x^3 - x^2 - 5x + 1$



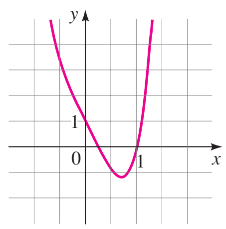
8. $P(x) = 3x^3 + 4x^2 - x - 2$



9. $P(x) = 2x^4 - 9x^3 + 9x^2 + x - 3$



10. $P(x) = 4x^4 - x^3 - 4x + 1$



11–40 ■ Find all rational zeros of the polynomial.

11. $P(x) = x^3 + 3x^2 - 4$

12. $P(x) = x^3 - 7x^2 + 14x - 8$

13. $P(x) = x^3 - 3x - 2$

14. $P(x) = x^3 + 4x^2 - 3x - 18$

15. $P(x) = x^3 - 6x^2 + 12x - 8$

16. $P(x) = x^3 - x^2 - 8x + 12$

17. $P(x) = x^3 - 4x^2 + x + 6$

18. $P(x) = x^3 - 4x^2 - 7x + 10$

19. $P(x) = x^3 + 3x^2 + 6x + 4$

EXAMPLE
A polynomial with one rational zero and four irrational zeros that can be found by elementary methods:

$$f(x) = 2x^5 - 10x^3 + 12x - x^4 + 5x^2 - 6 = (2x - 1)(x^2 - 2)(x^2 - 3)$$

Zeros: $x = \frac{1}{2}, \pm\sqrt{2}$, and $\pm\sqrt{3}$

20. $P(x) = x^3 - 2x^2 - 2x - 3$

21. $P(x) = x^4 - 5x^2 + 4$

22. $P(x) = x^4 - 2x^3 - 3x^2 + 8x - 4$

23. $P(x) = x^4 + 6x^3 + 7x^2 - 6x - 8$

24. $P(x) = x^4 - x^3 - 23x^2 - 3x + 90$

25. $P(x) = 4x^4 - 25x^2 + 36$

26. $P(x) = x^4 - x^3 - 5x^2 + 3x + 6$

27. $P(x) = x^4 + 8x^3 + 24x^2 + 32x + 16$

28. $P(x) = 2x^3 + 7x^2 + 4x - 4$

29. $P(x) = 4x^3 + 4x^2 - x - 1$

30. $P(x) = 2x^3 - 3x^2 - 2x + 3$

31. $P(x) = 4x^3 - 7x + 3$

32. $P(x) = 8x^3 + 10x^2 - x - 3$

33. $P(x) = 4x^3 + 8x^2 - 11x - 15$

34. $P(x) = 6x^3 + 11x^2 - 3x - 2$

35. $P(x) = 2x^4 - 7x^3 + 3x^2 + 8x - 4$

36. $P(x) = 6x^4 - 7x^3 - 12x^2 + 3x + 2$

37. $P(x) = x^5 + 3x^4 - 9x^3 - 31x^2 + 36$

38. $P(x) = x^5 - 4x^4 - 3x^3 + 22x^2 - 4x - 24$

39. $P(x) = 3x^5 - 14x^4 - 14x^3 + 36x^2 + 43x + 10$

40. $P(x) = 2x^6 - 3x^5 - 13x^4 + 29x^3 - 27x^2 + 32x - 12$

41–50 ■ Find all the real zeros of the polynomial. Use the quadratic formula if necessary, as in Example 3(a).

41. $P(x) = x^3 + 4x^2 + 3x - 2$

42. $P(x) = x^3 - 5x^2 + 2x + 12$

43. $P(x) = x^4 - 6x^3 + 4x^2 + 15x + 4$

44. $P(x) = x^4 + 2x^3 - 2x^2 - 3x + 2$

45. $P(x) = x^4 - 7x^3 + 14x^2 - 3x - 9$

46. $P(x) = x^5 - 4x^4 - x^3 + 10x^2 + 2x - 4$

47. $P(x) = 4x^3 - 6x^2 + 1$

48. $P(x) = 3x^3 - 5x^2 - 8x - 2$

49. $P(x) = 2x^4 + 15x^3 + 17x^2 + 3x - 1$

50. $P(x) = 4x^5 - 18x^4 - 6x^3 + 91x^2 - 60x + 9$

51–58 ■ A polynomial P is given.

(a) Find all the real zeros of P.

(b) Sketch the graph of P.

51. $P(x) = x^3 - 3x^2 - 4x + 12$

52. $P(x) = -x^3 - 2x^2 + 5x + 6$

53. $P(x) = 2x^3 - 7x^2 + 4x + 4$

54. $P(x) = 3x^3 + 17x^2 + 21x - 9$

55. $P(x) = x^4 - 5x^3 + 6x^2 + 4x - 8$

56. $P(x) = -x^4 + 10x^2 + 8x - 8$

57. $P(x) = x^5 - x^4 - 5x^3 + x^2 + 8x + 4$

58. $P(x) = x^5 - x^4 - 6x^3 + 14x^2 - 11x + 3$

59–64 ■ Use Descartes' Rule of Signs to determine how many positive and how many negative real zeros the polynomial can have. Then determine the possible total number of real zeros.

59. $P(x) = x^3 - x^2 - x - 3$

60. $P(x) = 2x^3 - x^2 + 4x - 7$

61. $P(x) = 2x^6 + 5x^4 - x^3 - 5x - 1$

62. $P(x) = x^4 + x^3 + x^2 + x + 12$

63. $P(x) = x^5 + 4x^3 - x^2 + 6x$

64. $P(x) = x^8 - x^5 + x^4 - x^3 + x^2 - x + 1$

65–68 ■ Show that the given values for a and b are lower and upper bounds for the real zeros of the polynomial.

65. $P(x) = 2x^3 + 5x^2 + x - 2$; $a = -3, b = 1$

66. $P(x) = x^4 - 2x^3 - 9x^2 + 2x + 8$; $a = -3, b = 5$

67. $P(x) = 8x^3 + 10x^2 - 39x + 9$; $a = -3, b = 2$

68. $P(x) = 3x^4 - 17x^3 + 24x^2 - 9x + 1$; $a = 0, b = 6$

69–72 ■ Find integers that are upper and lower bounds for the real zeros of the polynomial.

69. $P(x) = x^3 - 3x^2 + 4$

70. $P(x) = 2x^3 - 3x^2 - 8x + 12$

71. $P(x) = x^4 - 2x^3 + x^2 - 9x + 2$

72. $P(x) = x^5 - x^4 + 1$

73–78 ■ Find all rational zeros of the polynomial, and then find the irrational zeros, if any. Whenever appropriate, use the Rational Zeros Theorem, the Upper and Lower Bounds Theorem, Descartes' Rule of Signs, the quadratic formula, or other factoring techniques.

73. $P(x) = 2x^4 + 3x^3 - 4x^2 - 3x + 2$

74. $P(x) = 2x^4 + 15x^3 + 31x^2 + 20x + 4$

75. $P(x) = 4x^4 - 21x^2 + 5$

76. $P(x) = 6x^4 - 7x^3 - 8x^2 + 5x$

77. $P(x) = x^5 - 7x^4 + 9x^3 + 23x^2 - 50x + 24$

78. $P(x) = 8x^5 - 14x^4 - 22x^3 + 57x^2 - 35x + 6$

280 CHAPTER 3 Polynomial and Rational Functions

79–82 ■ Show that the polynomial does not have any rational zeros.

79. $P(x) = x^3 - x - 2$

80. $P(x) = 2x^4 - x^3 + x + 2$

81. $P(x) = 3x^3 - x^2 - 6x + 12$

82. $P(x) = x^{50} - 5x^{25} + x^2 - 1$

83–86 ■ The real solutions of the given equation are rational. List all possible rational roots using the Rational Zeros Theorem, and then graph the polynomial in the given viewing rectangle to determine which values are actually solutions. (All solutions can be seen in the given viewing rectangle.)

83. $x^3 - 3x^2 - 4x + 12 = 0$; $[-4, 4]$ by $[-15, 15]$

84. $x^4 - 5x^2 + 4 = 0$; $[-4, 4]$ by $[-30, 30]$

85. $2x^4 - 5x^3 - 14x^2 + 5x + 12 = 0$; $[-2, 5]$ by $[-40, 40]$

86. $3x^3 + 8x^2 + 5x + 2 = 0$; $[-3, 3]$ by $[-10, 10]$

87–90 ■ Use a graphing device to find all real solutions of the equation, correct to two decimal places.

87. $x^4 - x - 4 = 0$

88. $2x^3 - 8x^2 + 9x - 9 = 0$

89. $4.00x^4 + 4.00x^3 - 10.96x^2 - 5.88x + 9.09 = 0$

90. $x^5 + 2.00x^4 + 0.96x^3 + 5.00x^2 + 10.00x + 4.80 = 0$

91. Let $P(x)$ be a polynomial with real coefficients and let $b > 0$. Use the Division Algorithm to write

$$P(x) = (x - b) \cdot Q(x) + r$$

Suppose that $r \geq 0$ and that all the coefficients in $Q(x)$ are nonnegative. Let $z > b$.

(a) Show that $P(z) > 0$.

(b) Prove the first part of the Upper and Lower Bounds Theorem.

(c) Use the first part of the Upper and Lower Bounds Theorem to prove the second part. [*Hint:* Show that if $P(x)$ satisfies the second part of the theorem, then $P(-x)$ satisfies the first part.]

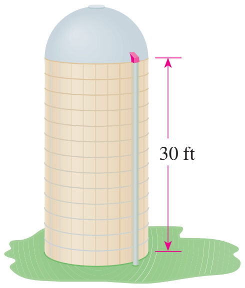
92. Show that the equation

$$x^5 - x^4 - x^3 - 5x^2 - 12x - 6 = 0$$

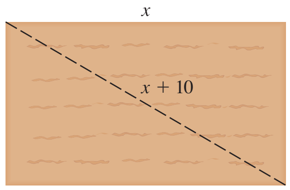
has exactly one rational root, and then prove that it must have either two or four irrational roots.

Applications

93. Volume of a Silo A grain silo consists of a cylindrical main section and a hemispherical roof. If the total volume of the silo (including the part inside the roof section) is 15,000 ft^3 and the cylindrical part is 30 ft tall, what is the radius of the silo, correct to the nearest tenth of a foot?



94. Dimensions of a Lot A rectangular parcel of land has an area of 5000 ft^2. A diagonal between opposite corners is measured to be 10 ft longer than one side of the parcel. What are the dimensions of the land, correct to the nearest foot?



95. Depth of Snowfall Snow began falling at noon on Sunday. The amount of snow on the ground at a certain location at time t was given by the function

$$h(t) = 11.60t - 12.41t^2 + 6.20t^3 - 1.58t^4 + 0.20t^5 - 0.01t^6$$

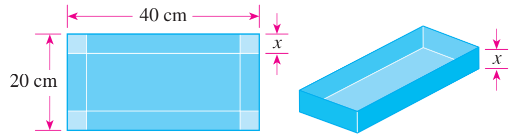
where t is measured in days from the start of the snowfall and $h(t)$ is the depth of snow in inches. Draw a graph of this function and use your graph to answer the following questions.

(a) What happened shortly after noon on Tuesday?

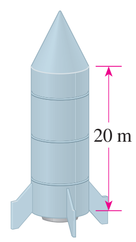
(b) Was there ever more than 5 in. of snow on the ground? If so, on what day(s)?

(c) On what day and at what time (to the nearest hour) did the snow disappear completely?

96. Volume of a Box An open box with a volume of 1500 cm^3 is to be constructed by taking a piece of cardboard 20 cm by 40 cm, cutting squares of side length x cm from each corner, and folding up the sides. Show that this can be done in two different ways, and find the exact dimensions of the box in each case.



97. Volume of a Rocket A rocket consists of a right circular cylinder of height 20 m surmounted by a cone whose height and diameter are equal and whose radius is the same as that of the cylindrical section. What should this radius be (correct to two decimal places) if the total volume is to be $500\pi/3\ \text{m}^3$?

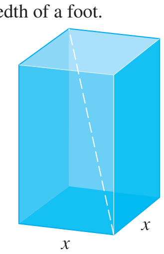


98. Volume of a Box A rectangular box with a volume of $2\sqrt{2}\ \text{ft}^3$ has a square base as shown below. The diagonal of the box (between a pair of opposite corners) is 1 ft longer than each side of the base.

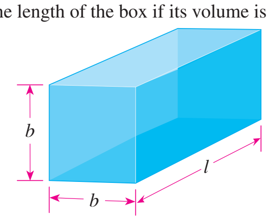
(a) If the base has sides of length x feet, show that

$$x^6 - 2x^5 - x^4 + 8 = 0$$

(b) Show that two different boxes satisfy the given conditions. Find the dimensions in each case, correct to the nearest hundredth of a foot.



99. Girth of a Box A box with a square base has length plus girth of 108 in. (Girth is the distance "around" the box.) What is the length of the box if its volume is $2200\ \text{in}^3$?



Discovery • Discussion

100. How Many Real Zeros Can a Polynomial Have? Give examples of polynomials that have the following properties, or explain why it is impossible to find such a polynomial.

(a) A polynomial of degree 3 that has no real zeros

(b) A polynomial of degree 4 that has no real zeros

(c) A polynomial of degree 3 that has three real zeros, only one of which is rational

(d) A polynomial of degree 4 that has four real zeros, none of which is rational

What must be true about the degree of a polynomial with integer coefficients if it has no real zeros?

101. The Depressed Cubic The most general cubic (third-degree) equation with rational coefficients can be written as

$$x^3 + ax^2 + bx + c = 0$$

(a) Show that if we replace x by $X - a/3$ and simplify, we end up with an equation that doesn't have an X^2 term, that is, an equation of the form

$$X^3 + pX + q = 0$$

This is called a *depressed cubic*, because we have "depressed" the quadratic term.

(b) Use the procedure described in part (a) to depress the equation $x^3 + 6x^2 + 9x + 4 = 0$.

102. The Cubic Formula The quadratic formula can be used to solve any quadratic (or second-degree) equation. You may have wondered if similar formulas exist for cubic (third-degree), quartic (fourth-degree), and higher-degree equations. For the depressed cubic $x^3 + px + q = 0$, Cardano (page 296) found the following formula for one solution:

$$x = \sqrt[3]{\frac{-q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{\frac{-q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

A formula for quartic equations was discovered by the Italian mathematician Ferrari in 1540. In 1824 the Norwegian mathematician Niels Henrik Abel proved that it is impossible to write a quintic formula, that is, a formula for fifth-degree equations. Finally, Galois (page 273) gave a criterion for determining which equations can be solved by a formula involving radicals.

Use the cubic formula to find a solution for the following equations. Then solve the equations using the methods you learned in this section. Which method is easier?

(a) $x^3 - 3x + 2 = 0$

(b) $x^3 - 27x - 54 = 0$

(c) $x^3 + 3x + 4 = 0$

282 CHAPTER 3 Polynomial and Rational Functions

DISCOVERY PROJECT

Zeroing in on a Zero

We have seen how to find the zeros of a polynomial algebraically and graphically. Let's work through a **numerical method** for finding the zeros. With this method we can find the value of any real zero to as many decimal places as we wish.

The Intermediate Value Theorem states: If P is a polynomial and if $P(a)$ and $P(b)$ are of opposite sign, then P has a zero between a and b. (See page 255.) The Intermediate Value Theorem is an example of an **existence theorem**—it tells us that a zero exists, but doesn't tell us exactly where it is. Nevertheless, we can use the theorem to zero in on the zero.

For example, consider the polynomial $P(x) = x^3 + 8x - 30$. Notice that $P(2) < 0$ and $P(3) > 0$. By the Intermediate Value Theorem P must have a zero between 2 and 3. To "trap" the zero in a smaller interval, we evaluate P at successive tenths between 2 and 3 until we find where P changes sign, as in Table 1. From the table we see that the zero we are looking for lies between 2.2 and 2.3, as shown in Figure 1.

Table 1

x	$P(x)$	
2.1	−3.94	
2.2	−1.75	} change of sign
2.3	0.57	

Table 2

x	$P(x)$	
2.26	−0.38	
2.27	−0.14	} change of sign
2.28	0.09	

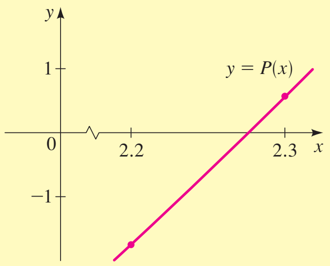


Figure 1

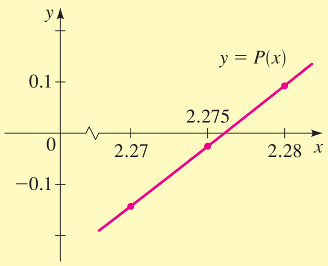


Figure 2

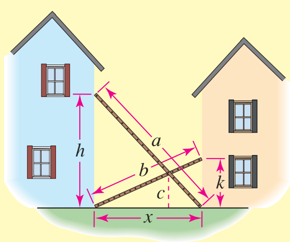
We can repeat this process by evaluating P at successive 100ths between 2.2 and 2.3, as in Table 2. By repeating this process over and over again, we can get a numerical value for the zero as accurately as we want. From Table 2 we see that the zero is between 2.27 and 2.28. To see whether it is closer to 2.27 or 2.28, we check the value of P halfway between these two numbers: $P(2.275) \approx -0.03$. Since this value is negative, the zero we are looking for lies between 2.275 and 2.28, as illustrated in Figure 2. Correct to the nearest 100th, the zero is 2.28.

1. (a) Show that $P(x) = x^2 - 2$ has a zero between 1 and 2.
(b) Find the zero of P to the nearest tenth.
(c) Find the zero of P to the nearest 100th.
(d) Explain why the zero you found is an approximation to $\sqrt{2}$. Repeat the process several times to obtain $\sqrt{2}$ correct to three decimal places. Compare your results to $\sqrt{2}$ obtained by a calculator.

2. Find a polynomial that has $\sqrt[3]{5}$ as a zero. Use the process described here to zero in on $\sqrt[3]{5}$ to four decimal places.

3. Show that the polynomial has a zero between the given integers, and then zero in on that zero, correct to two decimals.
(a) $P(x) = x^3 + x - 7$; between 1 and 2
(b) $P(x) = x^3 - x^2 - 5$; between 2 and 3
(c) $P(x) = 2x^4 - 4x^2 + 1$; between 1 and 2
(d) $P(x) = 2x^4 - 4x^2 + 1$; between -1 and 0

4. Find the indicated irrational zero, correct to two decimals.
(a) The positive zero of $P(x) = x^4 + 2x^3 + x^2 - 1$
(b) The negative zero of $P(x) = x^4 + 2x^3 + x^2 - 1$



5. In a passageway between two buildings, two ladders are propped up from the base of each building to the wall of the other so that they cross, as shown in the figure. If the ladders have lengths $a = 3$ m and $b = 2$ m and the crossing point is at height $c = 1$ m, then it can be shown that the distance x between the buildings is a solution of the equation

$$x^8 - 22x^6 + 163x^4 - 454x^2 + 385 = 0$$

(a) This equation has two positive solutions, which lie between 1 and 2. Use the technique of "zeroing in" to find both of these correct to the nearest tenth.
(b) Draw two scale diagrams, like the figure, one for each of the two values of x that you found in part (a). Measure the height of the crossing point on each. Which value of x seems to be the correct one?
(c) Here is how to get the above equation. First, use similar triangles to show that

$$\frac{1}{c} = \frac{1}{h} + \frac{1}{k}$$

Then use the Pythagorean Theorem to rewrite this as

$$\frac{1}{c} = \frac{1}{\sqrt{a^2 - x^2}} + \frac{1}{\sqrt{b^2 - x^2}}$$

Substitute $a = 3$, $b = 2$, and $c = 1$, then simplify to obtain the desired equation. [Note that you must square twice in this process to eliminate both square roots. This is why you obtain an extraneous solution in part (a). (See the *Warning* on page 53.)]

3.4 Complex Numbers

In Section 1.5 we saw that if the discriminant of a quadratic equation is negative, the equation has no real solution. For example, the equation

$$x^2 + 4 = 0$$

has no real solution. If we try to solve this equation, we get $x^2 = -4$, so

$$x = \pm\sqrt{-4}$$

But this is impossible, since the square of any real number is positive. [For example, $(-2)^2 = 4$, a positive number.] Thus, negative numbers don't have real square roots.

To make it possible to solve *all* quadratic equations, mathematicians invented an expanded number system, called the *complex number system*. First they defined the new number

$$i = \sqrt{-1}$$

This means $i^2 = -1$. A complex number is then a number of the form $a + bi$, where a and b are real numbers.

See the note on Cardano, page 296, for an example of how complex numbers are used to find real solutions of polynomial equations.

Definition of Complex Numbers

A **complex number** is an expression of the form

$$a + bi$$

where a and b are real numbers and $i^2 = -1$. The **real part** of this complex number is a and the **imaginary part** is b. Two complex numbers are **equal** if and only if their real parts are equal and their imaginary parts are equal.

Note that both the real and imaginary parts of a complex number are real numbers.

Example 1 Complex Numbers

The following are examples of complex numbers.

$3 + 4i$	Real part 3, imaginary part 4
$\frac{1}{2} - \frac{2}{3}i$	Real part $\frac{1}{2}$, imaginary part $-\frac{2}{3}$
$6i$	Real part 0, imaginary part 6
-7	Real part -7, imaginary part 0

■

A number such as $6i$, which has real part 0, is called a **pure imaginary number**. A real number like -7 can be thought of as a complex number with imaginary part 0.

In the complex number system every quadratic equation has solutions. The numbers $2i$ and $-2i$ are solutions of $x^2 = -4$ because

$$(2i)^2 = 2^2i^2 = 4(-1) = -4 \qquad \text{and} \qquad (-2i)^2 = (-2)^2i^2 = 4(-1) = -4$$

SUGGESTED TIME AND EMPHASIS

$\frac{1}{2}$–1 class.
Essential material.

ALTERNATE EXAMPLE 1
Write the real and imaginary parts of the complex number $4 + 9i$.

ANSWER
4, 9

POINTS TO STRESS

1. Arithmetic operations with complex numbers.
2. Complex numbers as roots of equations.

SAMPLE QUESTION
Text Question

Write out all solutions (real and complex) to the equation $z^2 = -9$.

Answer

$z = \pm 3i$

Although we use the term *imaginary* in this context, imaginary numbers should not be thought of as any less "real" (in the ordinary rather than the mathematical sense of that word) than negative numbers or irrational numbers. All numbers (except possibly the positive integers) are creations of the human mind—the numbers -1 and $\sqrt{2}$ as well as the number i. We study complex numbers because they complete, in a useful and elegant fashion, our study of the solutions of equations. In fact, imaginary numbers are useful not only in algebra and mathematics, but in the other sciences as well. To give just one example, in electrical theory the *reactance* of a circuit is a quantity whose measure is an imaginary number.

Arithmetic Operations on Complex Numbers

Complex numbers are added, subtracted, multiplied, and divided just as we would any number of the form $a + b\sqrt{c}$. The only difference we need to keep in mind is that $i^2 = -1$. Thus, the following calculations are valid.

$$(a + bi)(c + di) = ac + (ad + bc)i + bdi^2 \quad \text{Multiply and collect like terms}$$
$$= ac + (ad + bc)i + bd(-1) \quad i^2 = -1$$
$$= (ac - bd) + (ad + bc)i \quad \text{Combine real and imaginary parts}$$

We therefore define the sum, difference, and product of complex numbers as follows.

Adding, Subtracting, and Multiplying Complex Numbers

Definition	Description
Addition	
$(a + bi) + (c + di) = (a + c) + (b + d)i$	To add complex numbers, add the real parts and the imaginary parts.
Subtraction	
$(a + bi) - (c + di) = (a - c) + (b - d)i$	To subtract complex numbers, subtract the real parts and the imaginary parts.
Multiplication	
$(a + bi) \cdot (c + di) = (ac - bd) + (ad + bc)i$	Multiply complex numbers like binomials, using $i^2 = -1$.

ALTERNATE EXAMPLE 2

Express the following in the form $a + bi$.

(a) $(6 - 5i) + (2 + 3i)$
(b) $(6 - 5i) - (2 + 3i)$
(c) $(6 - 5i)(2 + 3i)$
(d) i^{18}

ANSWERS

(a) $8 - 2i$
(b) $4 - 8i$
(c) $27 + 8i$
(d) -1

Example 2 Adding, Subtracting, and Multiplying Complex Numbers

Graphing calculators can perform arithmetic operations on complex numbers.

```
(3+5i)+(4-2i)
                 7+3i
(3+5i)*(4-2i)
               22+14i
```

Express the following in the form $a + bi$.

(a) $(3 + 5i) + (4 - 2i)$
(b) $(3 + 5i) - (4 - 2i)$
(c) $(3 + 5i)(4 - 2i)$
(d) i^{23}

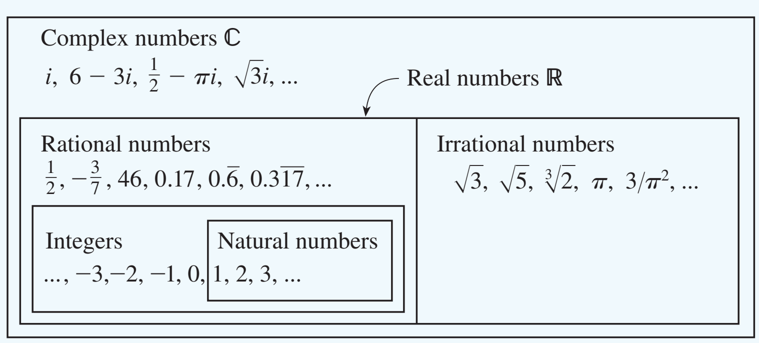
Solution

(a) According to the definition, we add the real parts and we add the imaginary parts.

$$(3 + 5i) + (4 - 2i) = (3 + 4) + (5 - 2)i = 7 + 3i$$

IN-CLASS MATERIALS

Students often believe that the relationship between real and complex numbers is similar to the relationship between rational and irrational numbers—they don't see that the number 5 can be thought of as complex $(5 + 0i)$ as well as real. Perhaps show them this figure:



(b) $(3 + 5i) - (4 - 2i) = (3 - 4) + [5 - (-2)]i = -1 + 7i$

(c) $(3 + 5i)(4 - 2i) = [3 \cdot 4 - 5(-2)] + [3(-2) + 5 \cdot 4]i = 22 + 14i$

(d) $i^{23} = i^{22+1} = (i^2)^{11}i = (-1)^{11}i = (-1)i = -i$ ■

Complex Conjugates

Number	Conjugate
$3 + 2i$	$3 - 2i$
$1 - i$	$1 + i$
$4i$	$-4i$
5	5

Division of complex numbers is much like rationalizing the denominator of a radical expression, which we considered in Section 1.2. For the complex number $z = a + bi$ we define its **complex conjugate** to be $\overline{z} = a - bi$. Note that

$$z \cdot \overline{z} = (a + bi)(a - bi) = a^2 + b^2$$

So the product of a complex number and its conjugate is always a nonnegative real number. We use this property to divide complex numbers.

Dividing Complex Numbers

To simplify the quotient $\dfrac{a + bi}{c + di}$, multiply the numerator and the denominator by the complex conjugate of the denominator:

$$\frac{a + bi}{c + di} = \left(\frac{a + bi}{c + di}\right)\left(\frac{c - di}{c - di}\right) = \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2}$$

Rather than memorize this entire formula, it's easier to just remember the first step and then multiply out the numerator and the denominator as usual.

Example 3 Dividing Complex Numbers

Express the following in the form $a + bi$.

(a) $\dfrac{3 + 5i}{1 - 2i}$ (b) $\dfrac{7 + 3i}{4i}$

Solution We multiply both the numerator and denominator by the complex conjugate of the denominator to make the new denominator a real number.

(a) The complex conjugate of $1 - 2i$ is $\overline{1 - 2i} = 1 + 2i$.

$$\frac{3 + 5i}{1 - 2i} = \left(\frac{3 + 5i}{1 - 2i}\right)\left(\frac{1 + 2i}{1 + 2i}\right) = \frac{-7 + 11i}{5} = -\frac{7}{5} + \frac{11}{5}i$$

(b) The complex conjugate of $4i$ is $-4i$. Therefore

$$\frac{7 + 3i}{4i} = \left(\frac{7 + 3i}{4i}\right)\left(\frac{-4i}{-4i}\right) = \frac{12 - 28i}{16} = \frac{3}{4} - \frac{7}{4}i$$ ■

ALTERNATE EXAMPLE 3a

Express the following quotient in the form $a + bi$.

$\dfrac{5 + 6i}{1 - 3i}$

ANSWER

$-\dfrac{13}{10} + \dfrac{21}{10} \cdot i$

ALTERNATE EXAMPLE 3b

Express the following quotient in the form $a + bi$.

$\dfrac{13 + 11i}{4i}$

ANSWER

$\dfrac{11}{4} - \dfrac{13}{4} \cdot i$

Square Roots of Negative Numbers

Just as every positive real number r has two square roots ($\sqrt{r}$ and $-\sqrt{r}$), every negative number has two square roots as well. If $-r$ is a negative number, then its square roots are $\pm i\sqrt{r}$, because $(i\sqrt{r})^2 = i^2 r = -r$ and $(-i\sqrt{r})^2 = i^2 r = -r$.

IN-CLASS MATERIALS

One doesn't have to think of complex numbers as a philosophical abstraction. Many applied fields use complex numbers, because the result of complex arithmetic leads to real-world understanding. One can think of complex numbers as points in the plane with the real and imaginary axes replacing the x- and y-axes. (In that sense, the complex numbers become a geometric extension of a number line.) Now we can model walking two feet north and one foot east as $1 + 2i$, and one foot north and three feet east as $3 + i$. Adding the numbers now has a physical significance: how far have you walked in total? Multiplication has a meaning too: when we multiply two complex numbers (thinking of them as points on the plane) we are multiplying their distances from the origin and adding their vector angles. So when we say $i^2 = -1$ we are really just saying that a 90° angle plus a 90° angle is a 180° angle. The statement $i = \sqrt{-1}$ is then a notational aid. Engineers represent waves of a fixed frequency as a magnitude and a phase angle. This interpretation of a complex number is well suited to that model.

Leonhard Euler (1707–1783) was born in Basel, Switzerland, the son of a pastor. At age 13 his father sent him to the University at Basel to study theology, but Euler soon decided to devote himself to the sciences. Besides theology he studied mathematics, medicine, astronomy, physics, and Asian languages. It is said that Euler could calculate as effortlessly as "men breathe or as eagles fly." One hundred years before Euler, Fermat (see page 652) had conjectured that $2^{2^n} + 1$ is a prime number for all n. The first five of these numbers are 5, 17, 257, 65537, and 4,294,967,297. It's easy to show that the first four are prime. The fifth was also thought to be prime until Euler, with his phenomenal calculating ability, showed that it is the product $641 \times 6{,}700{,}417$ and so is not prime. Euler published more than any other mathematician in history. His collected works comprise 75 large volumes. Although he was blind for the last 17 years of his life, he continued to work and publish. In his writings he popularized the use of the symbols π, e, and i, which you will find in this textbook. One of Euler's most lasting contributions is his development of complex numbers.

Square Roots of Negative Numbers

If $-r$ is negative, then the **principal square root** of $-r$ is

$$\sqrt{-r} = i\sqrt{r}$$

The two square roots of $-r$ are $i\sqrt{r}$ and $-i\sqrt{r}$.

We usually write $i\sqrt{b}$ instead of $\sqrt{b}\,i$ to avoid confusion with $\sqrt{bi}$.

ALTERNATE EXAMPLE 4b
Simplify $\sqrt{-36}$.

ANSWER
$6i$

Example 4 Square Roots of Negative Numbers

(a) $\sqrt{-1} = i\sqrt{1} = i$

(b) $\sqrt{-16} = i\sqrt{16} = 4i$

(c) $\sqrt{-3} = i\sqrt{3}$ ■

Special care must be taken when performing calculations involving square roots of negative numbers. Although $\sqrt{a}\cdot\sqrt{b} = \sqrt{ab}$ when a and b are positive, this is *not* true when both are negative. For example,

$$\sqrt{-2}\cdot\sqrt{-3} = i\sqrt{2}\cdot i\sqrt{3} = i^2\sqrt{6} = -\sqrt{6}$$

but $$\sqrt{(-2)(-3)} = \sqrt{6}$$

so $$\sqrt{-2}\cdot\sqrt{-3} \neq \sqrt{(-2)(-3)}$$

When multiplying radicals of negative numbers, express them first in the form $i\sqrt{r}$ (where $r > 0$) to avoid possible errors of this type.

ALTERNATE EXAMPLE 5
Evaluate $(\sqrt{63} - \sqrt{-7})(7 + \sqrt{-9})$ and express your answer in the form $a + bi$.

ANSWER
$24\sqrt{7} + i\cdot 2\sqrt{7}$

Example 5 Using Square Roots of Negative Numbers

Evaluate $(\sqrt{12} - \sqrt{-3})(3 + \sqrt{-4})$ and express in the form $a + bi$.

Solution

$$\begin{aligned}(\sqrt{12} - \sqrt{-3})(3 + \sqrt{-4}) &= (\sqrt{12} - i\sqrt{3})(3 + i\sqrt{4})\\ &= (2\sqrt{3} - i\sqrt{3})(3 + 2i)\\ &= (6\sqrt{3} + 2\sqrt{3}) + i(2\cdot 2\sqrt{3} - 3\sqrt{3})\\ &= 8\sqrt{3} + i\sqrt{3}\end{aligned}$$ ■

Complex Roots of Quadratic Equations

We have already seen that, if $a \neq 0$, then the solutions of the quadratic equation $ax^2 + bx + c = 0$ are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If $b^2 - 4ac < 0$, then the equation has no real solution. But in the complex number system, this equation will always have solutions, because negative numbers have square roots in this expanded setting.

IN-CLASS MATERIALS

The text states an important truth: every quadratic equation has two solutions (allowing for multiplicities) if complex numbers are considered. Equivalently, we can say that every quadratic expression $ax^2 + bx + c$ can be factored into two linear factors $(x - z_1)(x - z_2)$, where z_1 and z_2 are complex numbers (and possibly real). An important, easy-to-understand generalization is the Fundamental Theorem of Algebra: every nth degree polynomial can be factored into n linear factors, if we allow complex numbers. (If we do not, it can be shown that every nth degree polynomial can be factored into linear factors and irreducible quadratic factors.) The Fundamental Theorem will be covered explicitly in Section 3.5.

Example 6 Quadratic Equations with Complex Solutions

Solve each equation.

(a) $x^2 + 9 = 0$ (b) $x^2 + 4x + 5 = 0$

Solution

(a) The equation $x^2 + 9 = 0$ means $x^2 = -9$, so

$$x = \pm\sqrt{-9} = \pm i\sqrt{9} = \pm 3i$$

The solutions are therefore $3i$ and $-3i$.

(b) By the quadratic formula we have

$$x = \frac{-4 \pm \sqrt{4^2 - 4\cdot 5}}{2}$$

$$= \frac{-4 \pm \sqrt{-4}}{2}$$

$$= \frac{-4 \pm 2i}{2} = \frac{2(-2 \pm i)}{2} = -2 \pm i$$

So, the solutions are $-2 + i$ and $-2 - i$. ■

ALTERNATE EXAMPLE 6

Solve each equation.

(a) $x^2 + 16 = 0$

(b) $x^2 - 4x + 6.25 = 0$

ANSWERS

(a) $4i$ and $-4i$

(b) $2 + 1.5i$, $2 - 1.5i$

The two solutions of *any* quadratic equation that has real coefficients are complex conjugates of each other. To understand why this is true, think about the $\pm$ sign in the quadratic formula.

Example 7 Complex Conjugates as Solutions of a Quadratic

Show that the solutions of the equation

$$4x^2 - 24x + 37 = 0$$

are complex conjugates of each other.

Solution We use the quadratic formula to get

$$x = \frac{24 \pm \sqrt{(24)^2 - 4(4)(37)}}{2(4)}$$

$$= \frac{24 \pm \sqrt{-16}}{8} = \frac{24 \pm 4i}{8} = 3 \pm \frac{1}{2}i$$

So, the solutions are $3 + \frac{1}{2}i$ and $3 - \frac{1}{2}i$, and these are complex conjugates. ■

ALTERNATE EXAMPLE 7

Show that the solutions of the equation $2x^2 + 5x + 10 = 0$ are complex conjugates of each other.

ANSWER

The quadratic formula gives the solutions $\frac{1}{4}(-5 + \sqrt{55})$ and $\frac{1}{4}(-5 - \sqrt{55})$. These are conjugates of each other.

DRILL QUESTION

Simplify $\dfrac{3i + 2}{3 + 4i}$.

Answer

$\dfrac{18}{25} + \dfrac{1}{25}i$

3.4 Exercises

1–10 ■ Find the real and imaginary parts of the complex number.

1. $5 - 7i$ **2.** $-6 + 4i$

3. $\dfrac{-2 - 5i}{3}$ **4.** $\dfrac{4 + 7i}{2}$

5. 3 **6.** $-\frac{1}{2}$

7. $-\frac{2}{3}i$ **8.** $i\sqrt{3}$

9. $\sqrt{3} + \sqrt{-4}$ **10.** $2 - \sqrt{-5}$

11–22 ■ Perform the addition or subtraction and write the result in the form $a + bi$.

11. $(2 - 5i) + (3 + 4i)$

12. $(2 + 5i) + (4 - 6i)$

13. $(-6 + 6i) + (9 - i)$

14. $(3 - 2i) + \left(-5 - \frac{1}{3}i\right)$

15. $3i + (6 - 4i)$

IN-CLASS MATERIALS

There is a certain similarity to dividing complex numbers and rationalizing denominators. Make this similarity explicit by having the students do these two problems:

1. Rationalize the denominator of $\dfrac{8}{3 + \sqrt{2i}}$.

2. Simplify $\dfrac{8}{3 + 2i}$.

Answers

1. $\dfrac{24}{7} - \dfrac{8}{7}\sqrt{2}$

2. $\dfrac{24}{13} - \dfrac{16}{13}i$

16. $\left(\frac{1}{2} - \frac{1}{3}i\right) + \left(\frac{1}{2} + \frac{1}{3}i\right)$

17. $\left(7 - \frac{1}{2}i\right) - \left(5 + \frac{3}{2}i\right)$

18. $(-4 + i) - (2 - 5i)$

19. $(-12 + 8i) - (7 + 4i)$

20. $6i - (4 - i)$

21. $\frac{1}{3}i - \left(\frac{1}{4} - \frac{1}{6}i\right)$

22. $(0.1 - 1.1i) - (1.2 - 3.6i)$

23–56 ■ Evaluate the expression and write the result in the form $a + bi$.

23. $4(-1 + 2i)$

24. $2i\left(\frac{1}{2} - i\right)$

25. $(7 - i)(4 + 2i)$

26. $(5 - 3i)(1 + i)$

27. $(3 - 4i)(5 - 12i)$

28. $\left(\frac{2}{3} + 12i\right)\left(\frac{1}{6} + 24i\right)$

29. $(6 + 5i)(2 - 3i)$

30. $(-2 + i)(3 - 7i)$

31. $\dfrac{1}{i}$ **32.** $\dfrac{1}{1 + i}$

33. $\dfrac{2 - 3i}{1 - 2i}$ **34.** $\dfrac{5 - i}{3 + 4i}$

35. $\dfrac{26 + 39i}{2 - 3i}$ **36.** $\dfrac{25}{4 - 3i}$

37. $\dfrac{10i}{1 - 2i}$ **38.** $(2 - 3i)^{-1}$

39. $\dfrac{4 + 6i}{3i}$ **40.** $\dfrac{-3 + 5i}{15i}$

41. $\dfrac{1}{1 + i} - \dfrac{1}{1 - i}$ **42.** $\dfrac{(1 + 2i)(3 - i)}{2 + i}$

43. i^3 **44.** $(2i)^4$

45. i^{100} **46.** i^{1002}

47. $\sqrt{-25}$ **48.** $\sqrt{\dfrac{-9}{4}}$

49. $\sqrt{-3}\sqrt{-12}$ **50.** $\sqrt{\frac{1}{3}}\sqrt{-27}$

51. $(3 - \sqrt{-5})(1 + \sqrt{-1})$

52. $\dfrac{1 - \sqrt{-1}}{1 + \sqrt{-1}}$

53. $\dfrac{2 + \sqrt{-8}}{1 + \sqrt{-2}}$

54. $(\sqrt{3} - \sqrt{-4})(\sqrt{6} - \sqrt{-8})$

55. $\dfrac{\sqrt{-36}}{\sqrt{-2}\sqrt{-9}}$

56. $\dfrac{\sqrt{-7}\sqrt{-49}}{\sqrt{28}}$

57–70 ■ Find all solutions of the equation and express them in the form $a + bi$.

57. $x^2 + 9 = 0$ **58.** $9x^2 + 4 = 0$

59. $x^2 - 4x + 5 = 0$ **60.** $x^2 + 2x + 2 = 0$

61. $x^2 + x + 1 = 0$ **62.** $x^2 - 3x + 3 = 0$

63. $2x^2 - 2x + 1 = 0$ **64.** $2x^2 + 3 = 2x$

65. $t + 3 + \dfrac{3}{t} = 0$ **66.** $z + 4 + \dfrac{12}{z} = 0$

67. $6x^2 + 12x + 7 = 0$ **68.** $4x^2 - 16x + 19 = 0$

69. $\frac{1}{2}x^2 - x + 5 = 0$ **70.** $x^2 + \frac{1}{2}x + 1 = 0$

71–78 ■ Recall that the symbol $\bar{z}$ represents the complex conjugate of z. If $z = a + bi$ and $w = c + di$, prove each statement.

71. $\bar{z} + \bar{w} = \overline{z + w}$

72. $\overline{zw} = \bar{z} \cdot \bar{w}$

73. $(\bar{z})^2 = \overline{z^2}$

74. $\bar{\bar{z}} = z$

75. $z + \bar{z}$ is a real number

76. $z - \bar{z}$ is a pure imaginary number

77. $z \cdot \bar{z}$ is a real number

78. $z = \bar{z}$ if and only if z is real

Discovery • Discussion

79. Complex Conjugate Roots Suppose that the equation $ax^2 + bx + c = 0$ has real coefficients and complex roots. Why must the roots be complex conjugates of each other? (Think about how you would find the roots using the quadratic formula.)

80. Powers of i Calculate the first 12 powers of i, that is, $i, i^2, i^3, \ldots, i^{12}$. Do you notice a pattern? Explain how you would calculate any whole number power of i, using the pattern you have discovered. Use this procedure to calculate i^{4446}.

81. Complex Radicals The number 8 has one real cube root, $\sqrt[3]{8} = 2$. Calculate $(-1 + i\sqrt{3})^3$ and $(-1 - i\sqrt{3})^3$ to verify that 8 has at least two other complex cube roots. Can you find four fourth roots of 16?

EXAMPLES

Sample operations with complex numbers: Let $a = 3 + 2i$ and $b = 7 - 2i$. Then

$$a + b = 10$$
$$a - b = -4 + 4i$$
$$ab = 25 + 8i$$
$$\frac{a}{b} = \frac{17}{53} + \frac{20}{53}i$$

3.5 Complex Zeros and the Fundamental Theorem of Algebra

We have already seen that an nth-degree polynomial can have at most n real zeros. In the complex number system an nth-degree polynomial has exactly n zeros, and so can be factored into exactly n linear factors. This fact is a consequence of the Fundamental Theorem of Algebra, which was proved by the German mathematician C. F. Gauss in 1799 (see page 294).

The Fundamental Theorem of Algebra and Complete Factorization

The following theorem is the basis for much of our work in factoring polynomials and solving polynomial equations.

Fundamental Theorem of Algebra

Every polynomial

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \qquad (n \geq 1,\ a_n \neq 0)$$

with complex coefficients has at least one complex zero.

Because any real number is also a complex number, the theorem applies to polynomials with real coefficients as well.

The Fundamental Theorem of Algebra and the Factor Theorem together show that a polynomial can be factored completely into linear factors, as we now prove.

Complete Factorization Theorem

If $P(x)$ is a polynomial of degree $n \geq 1$, then there exist complex numbers a, $c_1, c_2, \ldots, c_n$ (with $a \neq 0$) such that

$$P(x) = a(x - c_1)(x - c_2)\cdots(x - c_n)$$

■ **Proof** By the Fundamental Theorem of Algebra, P has at least one zero. Let's call it c_1. By the Factor Theorem, $P(x)$ can be factored as

$$P(x) = (x - c_1) \cdot Q_1(x)$$

where $Q_1(x)$ is of degree $n - 1$. Applying the Fundamental Theorem to the quotient $Q_1(x)$ gives us the factorization

$$P(x) = (x - c_1) \cdot (x - c_2) \cdot Q_2(x)$$

where $Q_2(x)$ is of degree $n - 2$ and c_2 is a zero of $Q_1(x)$. Continuing this process for n steps, we get a final quotient $Q_n(x)$ of degree 0, a nonzero constant that we will call a. This means that P has been factored as

$$P(x) = a(x - c_1)(x - c_2)\cdots(x - c_n)$$ ■

SUGGESTED TIME AND EMPHASIS

$\frac{1}{2}$ class.
Essential material.

POINTS TO STRESS

1. The Complete Factorization Theorem.
2. The Conjugate Zeros Theorem.
3. The Linear and Quadratic Factors Theorem.

To actually find the complex zeros of an nth-degree polynomial, we usually first factor as much as possible, then use the quadratic formula on parts that we can't factor further.

Example 1 Factoring a Polynomial Completely

Let $P(x) = x^3 - 3x^2 + x - 3$.

(a) Find all the zeros of P.

(b) Find the complete factorization of P.

Solution

(a) We first factor P as follows.

$$\begin{aligned} P(x) &= x^3 - 3x^2 + x - 3 && \text{Given} \\ &= x^2(x-3) + (x-3) && \text{Group terms} \\ &= (x-3)(x^2+1) && \text{Factor } x - 3 \end{aligned}$$

We find the zeros of P by setting each factor equal to 0:

$$P(x) = (x-3)(x^2+1)$$

This factor is 0 when $x = 3$. This factor is 0 when $x = i$ or $-i$.

Setting $x - 3 = 0$, we see that $x = 3$ is a zero. Setting $x^2 + 1 = 0$, we get $x^2 = -1$, so $x = \pm i$. So the zeros of P are 3, i, and $-i$.

(b) Since the zeros are 3, i, and $-i$, by the Complete Factorization Theorem P factors as

$$\begin{aligned} P(x) &= (x-3)(x-i)[x-(-i)] \\ &= (x-3)(x-i)(x+i) \end{aligned}$$

■

Example 2 Factoring a Polynomial Completely

Let $P(x) = x^3 - 2x + 4$.

(a) Find all the zeros of P.

(b) Find the complete factorization of P.

Solution

(a) The possible rational zeros are the factors of 4, which are $\pm1, \pm2, \pm4$. Using synthetic division (see the margin) we find that -2 is a zero, and the polynomial factors as

-2	1	0	-2	4
		-2	4	-4
	1	-2	2	0

$$P(x) = (x+2)(x^2 - 2x + 2)$$

This factor is 0 when $x = -2$. Use the quadratic formula to find when this factor is 0.

ALTERNATE EXAMPLE 1

Find the complete factorization of the polynomial $P(x) = x^3 + x^2 + 81x + 81$ into linear factors with complex coefficients.

ANSWER

$P(x) = (x+1)\cdot(x+9i)\cdot(x-9i)$

SAMPLE QUESTIONS

Text Questions

Let $f(x) = x^3 + ax^2 + bx + c$.

(a) What is the maximum number of zeros this polynomial function can have?

(b) What is the minimum number of real zeros this polynomial function can have?

Answers

(a) 3

(b) 1

ALTERNATE EXAMPLE 2

Let $P(x) = x^4 - 3x^3 + 7x^2 + 21x - 26$.

(a) Find all the zeros of P.

(b) Find the complete factorization of P.

ANSWERS

(a) $1, -2, 2+3i, 2-3i$

(b) $P(x) = (x-1)(x+2)(x-2-3i)(x-2+3i)$

IN-CLASS MATERIALS

It is possible to use the techniques of the previous sections to solve polynomials with nonreal coefficients. If you want to demonstrate this fact, consider the polynomial $f(z) = z^3 + (2-3i)z^2 + (-3-6i) + 9i$, and use synthetic division to obtain $f(z) = (z-1)(z+3)(z-3i)$.

To find the zeros, we set each factor equal to 0. Of course, $x + 2 = 0$ means $x = -2$. We use the quadratic formula to find when the other factor is 0.

$$x^2 - 2x + 2 = 0 \qquad \text{Set factor equal to 0}$$

$$x = \frac{2 \pm \sqrt{4 - 8}}{2} \qquad \text{Quadratic formula}$$

$$x = \frac{2 \pm 2i}{2} \qquad \text{Take square root}$$

$$x = 1 \pm i \qquad \text{Simplify}$$

So the zeros of P are -2, $1 + i$, and $1 - i$.

(b) Since the zeros are -2, $1 + i$, and $1 - i$, by the Complete Factorization Theorem P factors as

$$\begin{aligned} P(x) &= [x - (-2)][x - (1 + i)][x - (1 - i)] \\ &= (x + 2)(x - 1 - i)(x - 1 + i) \end{aligned}$$

■

Zeros and Their Multiplicities

In the Complete Factorization Theorem the numbers $c_1, c_2, \ldots, c_n$ are the zeros of P. These zeros need not all be different. If the factor $x - c$ appears k times in the complete factorization of $P(x)$, then we say that c is a zero of **multiplicity k** (see page 259). For example, the polynomial

$$P(x) = (x - 1)^3(x + 2)^2(x + 3)^5$$

has the following zeros:

$$1 \text{ (multiplicity 3)}, \qquad -2 \text{ (multiplicity 2)}, \qquad -3 \text{ (multiplicity 5)}$$

The polynomial P has the same number of zeros as its degree—it has degree 10 and has 10 zeros, provided we count multiplicities. This is true for all polynomials, as we prove in the following theorem.

Zeros Theorem

Every polynomial of degree $n \geq 1$ has exactly n zeros, provided that a zero of multiplicity k is counted k times.

■ **Proof** Let P be a polynomial of degree n. By the Complete Factorization Theorem

$$P(x) = a(x - c_1)(x - c_2) \cdots (x - c_n)$$

Now suppose that c is a zero of P other than $c_1, c_2, \ldots, c_n$. Then

$$P(c) = a(c - c_1)(c - c_2) \cdots (c - c_n) = 0$$

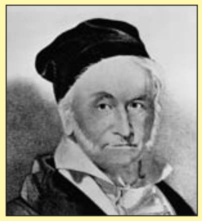
Thus, by the Zero-Product Property one of the factors $c - c_i$ must be 0, so $c = c_i$ for some i. It follows that P has exactly the n zeros $c_1, c_2, \ldots, c_n$. ■

IN-CLASS MATERIALS

Stress the power of the Complete Factorization Theorem, and how it dovetails with the Linear and Quadratic Factors Theorem. Once we allow complex numbers, we can view *all* polynomial functions as functions of the form $f(x) = a(x - c_1)(x - c_2) \cdots (x - c_n)$; simple products of linear factors. If we don't want to allow complex numbers (the preference of many students), we still can write all polynomials almost as simply, as the product of linear and (irreducible) quadratic factors.

ALTERNATE EXAMPLE 3
Find the complete factorization of the polynomial
$P(x) = 3x^5 + 54x^3 + 243x$.

ANSWER
$P(x) = 3x((x - 3i)(x + 3i))^2$


Corbis

Carl Friedrich Gauss (1777–1855) is considered the greatest mathematician of modern times. His contemporaries called him the "Prince of Mathematics." He was born into a poor family; his father made a living as a mason. As a very small child, Gauss found a calculation error in his father's accounts, the first of many incidents that gave evidence of his mathematical precocity. (See also page 834.) At 19 Gauss demonstrated that the regular 17-sided polygon can be constructed with straight-edge and compass alone. This was remarkable because, since the time of Euclid, it was thought that the only regular polygons constructible in this way were the triangle and pentagon. Because of this discovery Gauss decided to pursue a career in mathematics instead of languages, his other passion. In his doctoral dissertation, written at the age of 22, Gauss proved the Fundamental Theorem of Algebra: A polynomial of degree n with complex coefficients has n roots. His other accomplishments range over every branch of mathematics, as well as physics and astronomy.

Example 3 Factoring a Polynomial with Complex Zeros

Find the complete factorization and all five zeros of the polynomial

$$P(x) = 3x^5 + 24x^3 + 48x$$

Solution Since $3x$ is a common factor, we have

$$\begin{aligned} P(x) &= 3x(x^4 + 8x^2 + 16) \\ &= 3x(x^2 + 4)^2 \end{aligned}$$

This factor is 0 when $x = 0$. *This factor is 0 when $x = 2i$ or $x = -2i$.*

To factor $x^2 + 4$, note that $2i$ and $-2i$ are zeros of this polynomial. Thus $x^2 + 4 = (x - 2i)(x + 2i)$, and so

$$\begin{aligned} P(x) &= 3x[(x - 2i)(x + 2i)]^2 \\ &= 3x(x - 2i)^2(x + 2i)^2 \end{aligned}$$

0 is a zero of multiplicity 1. *$2i$ is a zero of multiplicity 2.* *$-2i$ is a zero of multiplicity 2.*

The zeros of P are 0, $2i$, and $-2i$. Since the factors $x - 2i$ and $x + 2i$ each occur twice in the complete factorization of P, the zeros $2i$ and $-2i$ are of multiplicity 2 (or *double* zeros). Thus, we have found all five zeros. ■

The following table gives further examples of polynomials with their complete factorizations and zeros.

Degree	Polynomial	Zero(s)	Number of zeros
1	$P(x) = x - 4$	4	1
2	$P(x) = x^2 - 10x + 25$ $= (x - 5)(x - 5)$	5 (multiplicity 2)	2
3	$P(x) = x^3 + x$ $= x(x - i)(x + i)$	$0, i, -i$	3
4	$P(x) = x^4 + 18x^2 + 81$ $= (x - 3i)^2(x + 3i)^2$	$3i$ (multiplicity 2), $-3i$ (multiplicity 2)	4
5	$P(x) = x^5 - 2x^4 + x^3$ $= x^3(x - 1)^2$	0 (multiplicity 3), 1 (multiplicity 2)	5

IN-CLASS MATERIALS

Point out that when graphing $y = f(x)$, the real zeros appear as x-intercepts, as expected. Remind students how the multiplicities of the real zeros can be seen. A multiplicity of 1 crosses the x-axis, an even multiplicity touches the x-axis, and an odd multiplicity greater than one crosses the x-axis and is flat there. (See Section 2.1.) Note that the complex zeros don't appear on the real plane.

Example 4 Finding Polynomials with Specified Zeros

(a) Find a polynomial $P(x)$ of degree 4, with zeros i, $-i$, 2, and -2 and with $P(3) = 25$.

(b) Find a polynomial $Q(x)$ of degree 4, with zeros -2 and 0, where -2 is a zero of multiplicity 3.

Solution

(a) The required polynomial has the form

$$\begin{aligned} P(x) &= a(x - i)(x - (-i))(x - 2)(x - (-2)) \\ &= a(x^2 + 1)(x^2 - 4) && \text{Difference of squares} \\ &= a(x^4 - 3x^2 - 4) && \text{Multiply} \end{aligned}$$

We know that $P(3) = a(3^4 - 3 \cdot 3^2 - 4) = 50a = 25$, so $a = \frac{1}{2}$. Thus

$$P(x) = \tfrac{1}{2}x^4 - \tfrac{3}{2}x^2 - 2$$

(b) We require

$$\begin{aligned} Q(x) &= a[x - (-2)]^3(x - 0) \\ &= a(x + 2)^3 x \\ &= a(x^3 + 6x^2 + 12x + 8)x && \text{Special Product Formula 4 (Section 1.3)} \\ &= a(x^4 + 6x^3 + 12x^2 + 8x) \end{aligned}$$

Since we are given no information about Q other than its zeros and their multiplicity, we can choose any number for a. If we use $a = 1$, we get

$$Q(x) = x^4 + 6x^3 + 12x^2 + 8x$$ ■

ALTERNATE EXAMPLE 4
Find the polynomial of degree 4, with zeros i, $-i$, 2, and -2 and with $P(5) = 273$.

ANSWER
$P(x) = \frac{1}{2}x^4 - \frac{3}{2}x^2 - 2$

Example 5 Finding All the Zeros of a Polynomial

Find all four zeros of $P(x) = 3x^4 - 2x^3 - x^2 - 12x - 4$.

Solution Using the Rational Zeros Theorem from Section 3.3, we obtain the following list of possible rational zeros: $\pm 1, \pm 2, \pm 4, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{4}{3}$. Checking these using synthetic division, we find that 2 and $-\frac{1}{3}$ are zeros, and we get the following factorization.

$$\begin{aligned} P(x) &= 3x^4 - 2x^3 - x^2 - 12x - 4 \\ &= (x - 2)(3x^3 + 4x^2 + 7x + 2) && \text{Factor } x - 2 \\ &= (x - 2)\left(x + \tfrac{1}{3}\right)(3x^2 + 3x + 6) && \text{Factor } x + \tfrac{1}{3} \\ &= 3(x - 2)\left(x + \tfrac{1}{3}\right)(x^2 + x + 2) && \text{Factor 3} \end{aligned}$$

The zeros of the quadratic factor are

$$x = \frac{-1 \pm \sqrt{1 - 8}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{7}}{2} \qquad \text{Quadratic formula}$$

so the zeros of $P(x)$ are

$$2, \quad -\frac{1}{3}, \quad -\frac{1}{2} + i\frac{\sqrt{7}}{2}, \quad \text{and} \quad -\frac{1}{2} - i\frac{\sqrt{7}}{2}$$ ■

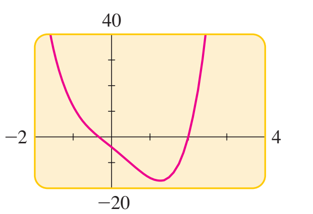


Figure 1
$P(x) = 3x^4 - 2x^3 - x^2 - 12x - 4$

Figure 1 shows the graph of the polynomial P in Example 5. The x-intercepts correspond to the real zeros of P. The imaginary zeros cannot be determined from the graph.

ALTERNATE EXAMPLE 5
Find all four zeros of $P(x) = 2x^4 - 7x^3 - 4x^2 - 50x - 25$.

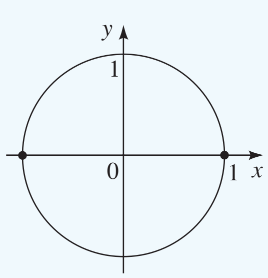
ANSWER
$5, -\frac{1}{2}, \frac{-1 + i\sqrt{19}}{2}, \frac{-1 - i\sqrt{19}}{2}$

IN-CLASS MATERIALS

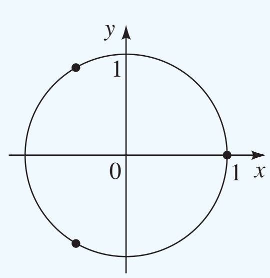
Exercise 70 discusses roots of unity—the zeros of polynomials of the form $f(x) = x^n - 1$ or, equivalently, the solutions of $x^n = 1$. It is easy to find the real solutions, 1 if n is odd and ± 1 if n is even. Exercise 70 prompts students to solve $x^n = 1$ for $n = 2, 3$, and 4. If your students have been exposed to trigonometric functions, then you can show them the general formula:

$$x^n = 1 \Leftrightarrow x = \cos\left(\frac{2\pi k}{n}\right) + i \sin\left(\frac{2\pi k}{n}\right) \text{ for } k = 0, \ldots, n.$$

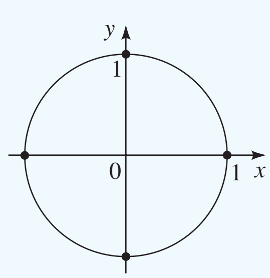
Interestingly enough, if you plot the solutions in the complex plane, there is a wonderful amount of symmetry. Even if you don't want to discuss the general formula, you can show your students where the roots of unity live. The complex roots of unity can be thought of as points that are evenly distributed around the unit circle.



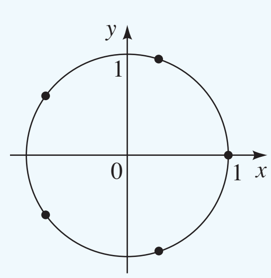
$n = 2$



$n = 3$



$n = 4$



$n = 5$

Gerolamo Cardano (1501–1576) is certainly one of the most colorful figures in the history of mathematics. He was the most well-known physician in Europe in his day, yet throughout his life he was plagued by numerous maladies, including ruptures, hemorrhoids, and an irrational fear of encountering rabid dogs. A doting father, his beloved sons broke his heart—his favorite was eventually beheaded for murdering his own wife. Cardano was also a compulsive gambler; indeed, this vice may have driven him to write the *Book on Games of Chance*, the first study of probability from a mathematical point of view.

In Cardano's major mathematical work, the *Ars Magna*, he detailed the solution of the general third- and fourth-degree polynomial equations. At the time of its publication, mathematicians were uncomfortable even with negative numbers, but Cardano's formulas paved the way for the acceptance not just of negative numbers, but also of imaginary numbers, because they occurred naturally in solving polynomial equations. For example, for the cubic equation

$$x^3 - 15x - 4 = 0$$

one of his formulas gives the solution

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$$

(See page 282, Exercise 102). This value for x actually turns out to be the *integer* 4, yet to find it Cardano had to use the imaginary number $\sqrt{-121} = 11i$.

Complex Zeros Come in Conjugate Pairs

As you may have noticed from the examples so far, the complex zeros of polynomials with real coefficients come in pairs. Whenever $a + bi$ is a zero, its complex conjugate $a - bi$ is also a zero.

Conjugate Zeros Theorem

If the polynomial P has real coefficients, and if the complex number z is a zero of P, then its complex conjugate $\bar{z}$ is also a zero of P.

■ **Proof** Let

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where each coefficient is real. Suppose that $P(z) = 0$. We must prove that $P(\bar{z}) = 0$. We use the facts that the complex conjugate of a sum of two complex numbers is the sum of the conjugates and that the conjugate of a product is the product of the conjugates (see Exercises 71 and 72 in Section 3.4).

$$\begin{aligned} P(\bar{z}) &= a_n(\bar{z})^n + a_{n-1}(\bar{z})^{n-1} + \cdots + a_1\bar{z} + a_0 \\ &= \overline{a_n}\,\overline{z^n} + \overline{a_{n-1}}\,\overline{z^{n-1}} + \cdots + \overline{a_1}\,\overline{z} + \overline{a_0} && \text{Because the coefficients are real} \\ &= \overline{a_n z^n} + \overline{a_{n-1}z^{n-1}} + \cdots + \overline{a_1 z} + \overline{a_0} \\ &= \overline{a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0} \\ &= \overline{P(z)} = \bar{0} = 0 \end{aligned}$$

This shows that $\bar{z}$ is also a zero of $P(x)$, which proves the theorem. ■

Example 6 A Polynomial with a Specified Complex Zero

Find a polynomial $P(x)$ of degree 3 that has integer coefficients and zeros $\frac{1}{2}$ and $3 - i$.

Solution Since $3 - i$ is a zero, then so is $3 + i$ by the Conjugate Zeros Theorem. This means that $P(x)$ has the form

$$\begin{aligned} P(x) &= a(x - \tfrac{1}{2})[x - (3 - i)][x - (3 + i)] \\ &= a(x - \tfrac{1}{2})[(x - 3) + i][(x - 3) - i] && \text{Regroup} \\ &= a(x - \tfrac{1}{2})[(x - 3)^2 - i^2] && \text{Difference of Squares Formula} \\ &= a(x - \tfrac{1}{2})(x^2 - 6x + 10) && \text{Expand} \\ &= a(x^3 - \tfrac{13}{2}x^2 + 13x - 5) && \text{Expand} \end{aligned}$$

To make all coefficients integers, we set $a = 2$ and get

$$P(x) = 2x^3 - 13x^2 + 26x - 10$$

Any other polynomial that satisfies the given requirements must be an integer multiple of this one. ■

ALTERNATE EXAMPLE 6

Find a polynomial $P(x)$ of degree 3 that has integer coefficients and zeros $\frac{1}{3}$ and $5 - 2i$.

ANSWER

$P(x) = 3(x - \frac{1}{3})(x - 5 - 2i) \times (x - 5 + 2i) = 3x^3 - 31x^2 + 97x - 29$

EXAMPLE

A polynomial with many rational zeros: $f(x) = 6x^5 + 17x^4 - 40x^3 - 45x^2 + 14x + 8$

Factored form: $(2x - 1)(3x + 1)(x + 4)(x + 1)(x - 2)$

Zeros: $x = \frac{1}{2}, \frac{1}{3}, -4, -1,$ and 2

Example 7 **Using Descartes' Rule to Count Real and Imaginary Zeros**

Without actually factoring, determine how many positive real zeros, negative real zeros, and imaginary zeros the following polynomial could have:

$$P(x) = x^4 + 6x^3 - 12x^2 - 14x - 24$$

Solution Since there is one change of sign, by Descartes' Rule of Signs, P has one positive real zero. Also, $P(-x) = x^4 - 6x^3 - 12x^2 + 14x - 24$ has three changes of sign, so there are either three or one negative real zero(s). So P has a total of either four or two real zeros. Since P is of degree 4, it has four zeros in all, which gives the following possibilities.

Positive real zeros	Negative real zeros	Imaginary zeros
1	3	0
1	1	2

■

Linear and Quadratic Factors

We have seen that a polynomial factors completely into linear factors if we use complex numbers. If we don't use complex numbers, then a polynomial with real coefficients can always be factored into linear and quadratic factors. We use this property in Section 9.8 when we study partial fractions. A quadratic polynomial with no real zeros is called **irreducible** over the real numbers. Such a polynomial cannot be factored without using complex numbers.

Linear and Quadratic Factors Theorem

Every polynomial with real coefficients can be factored into a product of linear and irreducible quadratic factors with real coefficients.

■ **Proof** We first observe that if $c = a + bi$ is a complex number, then

$$\begin{aligned}(x - c)(x - \bar{c}) &= [x - (a + bi)][x - (a - bi)] \\ &= [(x - a) - bi][(x - a) + bi] \\ &= (x - a)^2 - (bi)^2 \\ &= x^2 - 2ax + (a^2 + b^2)\end{aligned}$$

The last expression is a quadratic with *real* coefficients.

Now, if P is a polynomial with real coefficients, then by the Complete Factorization Theorem

$$P(x) = a(x - c_1)(x - c_2)\cdots(x - c_n)$$

Since the complex roots occur in conjugate pairs, we can multiply the factors corresponding to each such pair to get a quadratic factor with real coefficients. This results in P being factored into linear and irreducible quadratic factors. ■

ALTERNATE EXAMPLE 7

Without actually factoring, determine how many positive real zeros, negative real zeros, and imaginary zeros the following polynomial could have:
$P(x) = x^3 - 100x^2 + 32x + 50$

ANSWER

There are two sign changes. By Descartes' Rule of Signs, there are two or zero positive zeros.

$P(-x) = -x^3 - 100x^2 - 32x + 50$

There is one sign change. By Descartes' Rule of Signs, there is one negative zero.

There are three zeros in all. So here are the possibilities:

Positive Real Zeros	Negative Real Zeros	Complex Zeros
0	1	2
2	1	0

EXAMPLES

1. A polynomial that is the product of two irreducible quadratic terms:

$$f(x) = x^4 + 2x^3 + 9x^2 + 2x + 8 = (x^2 + 1)(x^2 + 2x + 8)$$

This can be factored by noting that $x = i$ is a zero, and therefore $x = -i$ is a zero, and then dividing by $(x^2 + 1)$.

2. A polynomial that is the product of two linear terms and an irreducible quadratic term:

$$f(x) = x^4 + 5x^3 + 10x^2 + 16x - 32 = (x^2 + 2x + 8)(x - 1)(x + 4)$$

ALTERNATE EXAMPLE 8
Factor the polynomial $P(x)$ completely into linear factors with complex coefficients:
$P(x) = x^4 + 9x^2 - 112$

ANSWER
$P(x) = (x - \sqrt{7})(x + \sqrt{7}) \times (x + 4i)(x - 4i)$

Example 8 Factoring a Polynomial into Linear and Quadratic Factors

Let $P(x) = x^4 + 2x^2 - 8$.

(a) Factor P into linear and irreducible quadratic factors with real coefficients.

(b) Factor P completely into linear factors with complex coefficients.

Solution

(a)
$$\begin{aligned} P(x) &= x^4 + 2x^2 - 8 \\ &= (x^2 - 2)(x^2 + 4) \\ &= (x - \sqrt{2})(x + \sqrt{2})(x^2 + 4) \end{aligned}$$

The factor $x^2 + 4$ is irreducible since it has only the imaginary zeros $\pm 2i$.

(b) To get the complete factorization, we factor the remaining quadratic factor.

$$\begin{aligned} P(x) &= (x - \sqrt{2})(x + \sqrt{2})(x^2 + 4) \\ &= (x - \sqrt{2})(x + \sqrt{2})(x - 2i)(x + 2i) \end{aligned}$$ ■

3.5 Exercises

1–12 ■ A polynomial P is given.
(a) Find all zeros of P, real and complex.
(b) Factor P completely.

1. $P(x) = x^4 + 4x^2$
2. $P(x) = x^5 + 9x^3$
3. $P(x) = x^3 - 2x^2 + 2x$
4. $P(x) = x^3 + x^2 + x$
5. $P(x) = x^4 + 2x^2 + 1$
6. $P(x) = x^4 - x^2 - 2$
7. $P(x) = x^4 - 16$
8. $P(x) = x^4 + 6x^2 + 9$
9. $P(x) = x^3 + 8$
10. $P(x) = x^3 - 8$
11. $P(x) = x^6 - 1$
12. $P(x) = x^6 - 7x^3 - 8$

13–30 ■ Factor the polynomial completely and find all its zeros. State the multiplicity of each zero.

13. $P(x) = x^2 + 25$
14. $P(x) = 4x^2 + 9$
15. $Q(x) = x^2 + 2x + 2$
16. $Q(x) = x^2 - 8x + 17$
17. $P(x) = x^3 + 4x$
18. $P(x) = x^3 - x^2 + x$
19. $Q(x) = x^4 - 1$
20. $Q(x) = x^4 - 625$
21. $P(x) = 16x^4 - 81$
22. $P(x) = x^3 - 64$
23. $P(x) = x^3 + x^2 + 9x + 9$
24. $P(x) = x^6 - 729$
25. $Q(x) = x^4 + 2x^2 + 1$
26. $Q(x) = x^4 + 10x^2 + 25$
27. $P(x) = x^4 + 3x^2 - 4$
28. $P(x) = x^5 + 7x^3$
29. $P(x) = x^5 + 6x^3 + 9x$
30. $P(x) = x^6 + 16x^3 + 64$

31–40 ■ Find a polynomial with integer coefficients that satisfies the given conditions.

31. P has degree 2, and zeros $1 + i$ and $1 - i$.
32. P has degree 2, and zeros $1 + i\sqrt{2}$ and $1 - i\sqrt{2}$.
33. Q has degree 3, and zeros 3, $2i$, and $-2i$.
34. Q has degree 3, and zeros 0 and i.
35. P has degree 3, and zeros 2 and i.
36. Q has degree 3, and zeros -3 and $1 + i$.
37. R has degree 4, and zeros $1 - 2i$ and 1, with 1 a zero of multiplicity 2.
38. S has degree 4, and zeros $2i$ and $3i$.
39. T has degree 4, zeros i and $1 + i$, and constant term 12.
40. U has degree 5, zeros $\frac{1}{2}$, -1, and $-i$, and leading coefficient 4; the zero -1 has multiplicity 2.

41–58 ■ Find all zeros of the polynomial.

41. $P(x) = x^3 + 2x^2 + 4x + 8$
42. $P(x) = x^3 - 7x^2 + 17x - 15$
43. $P(x) = x^3 - 2x^2 + 2x - 1$
44. $P(x) = x^3 + 7x^2 + 18x + 18$
45. $P(x) = x^3 - 3x^2 + 3x - 2$

DRILL QUESTION

Consider the polynomial $f(x) = x^5 - 2x^4 + 16x^3 + 8x^2 + 20x + 200$. It is a fact that -2 is a zero of this polynomial, and that $(x - 1 - 3i)^2$ is a factor of this polynomial. Using this information, factor the polynomial completely.

Answer

$f(x) = (x - 1 - 3i)^2(x - 1 + 3i)^2(x + 2)$

46. $P(x) = x^3 - x - 6$

47. $P(x) = 2x^3 + 7x^2 + 12x + 9$

48. $P(x) = 2x^3 - 8x^2 + 9x - 9$

49. $P(x) = x^4 + x^3 + 7x^2 + 9x - 18$

50. $P(x) = x^4 - 2x^3 - 2x^2 - 2x - 3$

51. $P(x) = x^5 - x^4 + 7x^3 - 7x^2 + 12x - 12$

52. $P(x) = x^5 + x^3 + 8x^2 + 8$ [*Hint:* Factor by grouping.]

53. $P(x) = x^4 - 6x^3 + 13x^2 - 24x + 36$

54. $P(x) = x^4 - x^2 + 2x + 2$

55. $P(x) = 4x^4 + 4x^3 + 5x^2 + 4x + 1$

56. $P(x) = 4x^4 + 2x^3 - 2x^2 - 3x - 1$

57. $P(x) = x^5 - 3x^4 + 12x^3 - 28x^2 + 27x - 9$

58. $P(x) = x^5 - 2x^4 + 2x^3 - 4x^2 + x - 2$

59–64 ■ A polynomial P is given.

(a) Factor P into linear and irreducible quadratic factors with real coefficients.

(b) Factor P completely into linear factors with complex coefficients.

59. $P(x) = x^3 - 5x^2 + 4x - 20$

60. $P(x) = x^3 - 2x - 4$

61. $P(x) = x^4 + 8x^2 - 9$

62. $P(x) = x^4 + 8x^2 + 16$

63. $P(x) = x^6 - 64$

64. $P(x) = x^5 - 16x$

65. By the Zeros Theorem, every nth-degree polynomial equation has exactly n solutions (including possibly some that are repeated). Some of these may be real and some may be imaginary. Use a graphing device to determine how many real and imaginary solutions each equation has.

(a) $x^4 - 2x^3 - 11x^2 + 12x = 0$

(b) $x^4 - 2x^3 - 11x^2 + 12x - 5 = 0$

(c) $x^4 - 2x^3 - 11x^2 + 12x + 40 = 0$

66–68 ■ So far we have worked only with polynomials that have real coefficients. These exercises involve polynomials with real and imaginary coefficients.

66. Find all solutions of the equation.

(a) $2x + 4i = 1$

(b) $x^2 - ix = 0$

(c) $x^2 + 2ix - 1 = 0$

(d) $ix^2 - 2x + i = 0$

67. (a) Show that $2i$ and $1 - i$ are both solutions of the equation

$$x^2 - (1 + i)x + (2 + 2i) = 0$$

but that their complex conjugates $-2i$ and $1 + i$ are not.

(b) Explain why the result of part (a) does not violate the Conjugate Zeros Theorem.

68. (a) Find the polynomial with *real* coefficients of the smallest possible degree for which i and $1 + i$ are zeros and in which the coefficient of the highest power is 1.

(b) Find the polynomial with *complex* coefficients of the smallest possible degree for which i and $1 + i$ are zeros and in which the coefficient of the highest power is 1.

Discovery • Discussion

69. Polynomials of Odd Degree The Conjugate Zeros Theorem says that the complex zeros of a polynomial with real coefficients occur in complex conjugate pairs. Explain how this fact proves that a polynomial with real coefficients and odd degree has at least one real zero.

70. Roots of Unity There are two square roots of 1, namely 1 and -1. These are the solutions of $x^2 = 1$. The fourth roots of 1 are the solutions of the equation $x^4 = 1$ or $x^4 - 1 = 0$. How many fourth roots of 1 are there? Find them. The cube roots of 1 are the solutions of the equation $x^3 = 1$ or $x^3 - 1 = 0$. How many cube roots of 1 are there? Find them. How would you find the sixth roots of 1? How many are there? Make a conjecture about the number of nth roots of 1.

3.6 Rational Functions

A rational function is a function of the form

$$r(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials. We assume that $P(x)$ and $Q(x)$ have no factor in common. Even though rational functions are constructed from polynomials, their graphs look quite different than the graphs of polynomial functions.

SUGGESTED TIME AND EMPHASIS

1 class.
Essential material.

POINTS TO STRESS

1. Various kinds of asymptotes and end behavior of functions, particularly rational functions.
2. Graphing rational functions.

Rational Functions and Asymptotes

Domains of rational expressions are discussed in Section 1.4.

The *domain* of a rational function consists of all real numbers x except those for which the denominator is zero. When graphing a rational function, we must pay special attention to the behavior of the graph near those x-values. We begin by graphing a very simple rational function.

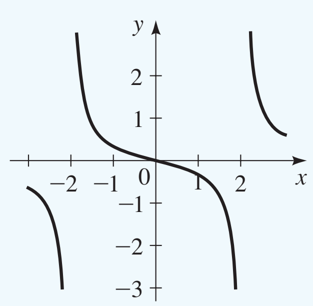
ALTERNATE EXAMPLE 1
For the graph of the rational function $r(x) = \dfrac{3}{x}$ evaluate the values of $r(x)$ as x approaches $-\infty, 0^-, 0^+, +\infty$.

ANSWER
$0, -\infty, \infty, 0$

IN-CLASS MATERIALS
This is a good time to remind students of parameters. For example, give each section of the class a different value of c and have them sketch

$$f(x) = \frac{x}{x^2 - c}$$

Note that when c is positive, there is a middle piece which disappears when $c = 0$ (the two vertical asymptotes become one). Note how the curve still gets large at $x = 0$ when c is small and negative.



$c = 4$

Example 1 A Simple Rational Function

Sketch a graph of the rational function $f(x) = \dfrac{1}{x}$.

Solution The function f is not defined for $x = 0$. The following tables show that when x is close to zero, the value of $|f(x)|$ is large, and the closer x gets to zero, the larger $|f(x)|$ gets.

For positive real numbers,

$$\frac{1}{\text{BIG NUMBER}} = \text{small number}$$
$$\frac{1}{\text{small number}} = \text{BIG NUMBER}$$

x	$f(x)$
-0.1	-10
-0.01	-100
-0.00001	$-100{,}000$
Approaching 0^-	Approaching $-\infty$

x	$f(x)$
0.1	10
0.01	100
0.00001	$100{,}000$
Approaching 0^+	Approaching ∞

We describe this behavior in words and in symbols as follows. The first table shows that as x approaches 0 from the left, the values of $y = f(x)$ decrease without bound. In symbols,

$$f(x) \to -\infty \quad \text{as} \quad x \to 0^- \qquad \text{"}y\text{ approaches negative infinity as }x\text{ approaches 0 from the left"}$$

The second table shows that as x approaches 0 from the right, the values of $f(x)$ increase without bound. In symbols,

$$f(x) \to \infty \quad \text{as} \quad x \to 0^+ \qquad \text{"}y\text{ approaches infinity as }x\text{ approaches 0 from the right"}$$

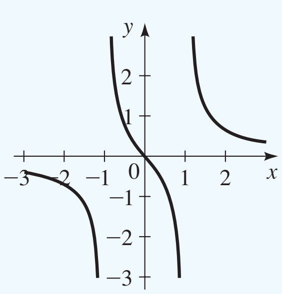
The next two tables show how $f(x)$ changes as $|x|$ becomes large.

x	$f(x)$
-10	-0.1
-100	-0.01
$-100{,}000$	-0.00001
Approaching $-\infty$	Approaching 0

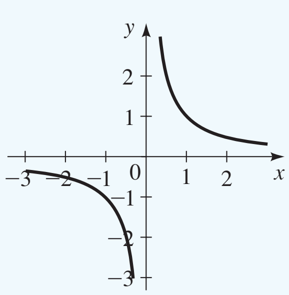
x	$f(x)$
10	0.1
100	0.01
$100{,}000$	0.00001
Approaching ∞	Approaching 0

These tables show that as $|x|$ becomes large, the value of $f(x)$ gets closer and closer to zero. We describe this situation in symbols by writing

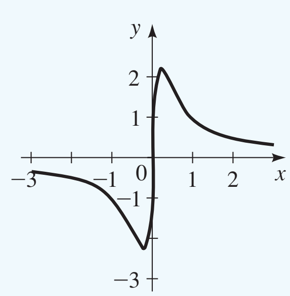
$$f(x) \to 0 \quad \text{as} \quad x \to -\infty \qquad \text{and} \qquad f(x) \to 0 \quad \text{as} \quad x \to \infty$$



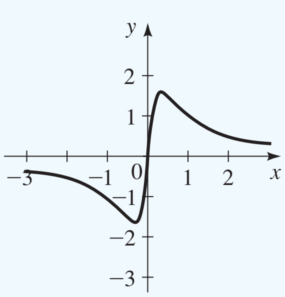
$c = 1$



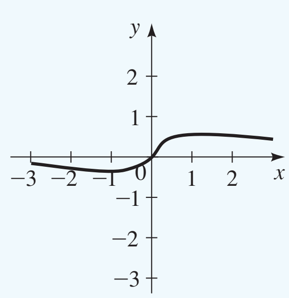
$c = 0$



$c = -0.05$



$c = -0.1$



$c = -1$

Using the information in these tables and plotting a few additional points, we obtain the graph shown in Figure 1.

x	$f(x) = \frac{1}{x}$
-2	$-\frac{1}{2}$
-1	-1
$-\frac{1}{2}$	-2
$\frac{1}{2}$	2
1	1
2	$\frac{1}{2}$

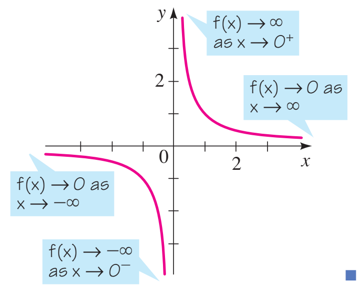


Figure 1
$f(x) = \frac{1}{x}$

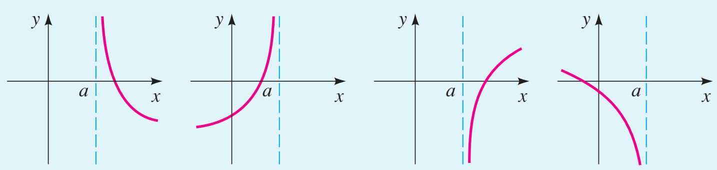
In Example 1 we used the following arrow notation.

Symbol	Meaning
$x \to a^-$	x approaches a from the left
$x \to a^+$	x approaches a from the right
$x \to -\infty$	x goes to negative infinity; that is, x decreases without bound
$x \to \infty$	x goes to infinity; that is, x increases without bound

The line $x = 0$ is called a *vertical asymptote* of the graph in Figure 1, and the line $y = 0$ is a *horizontal asymptote*. Informally speaking, an asymptote of a function is a line that the graph of the function gets closer and closer to as one travels along that line.

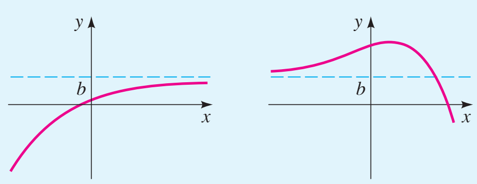
Definition of Vertical and Horizontal Asymptotes

1. The line $x = a$ is a **vertical asymptote** of the function $y = f(x)$ if y approaches $\pm\infty$ as x approaches a from the right or left.



$y \to \infty$ as $x \to a^+$ $y \to \infty$ as $x \to a^-$ $y \to -\infty$ as $x \to a^+$ $y \to -\infty$ as $x \to a^-$

2. The line $y = b$ is a **horizontal asymptote** of the function $y = f(x)$ if y approaches b as x approaches $\pm\infty$.



$y \to b$ as $x \to \infty$ $y \to b$ as $x \to -\infty$

A rational function has vertical asymptotes where the function is undefined, that is, where the denominator is zero.

Transformations of $y = \frac{1}{x}$

A rational function of the form

$$r(x) = \frac{ax + b}{cx + d}$$

can be graphed by shifting, stretching, and/or reflecting the graph of $f(x) = \frac{1}{x}$ shown in Figure 1, using the transformations studied in Section 2.4. (Such functions are called *linear fractional transformations*.)

Example 2 Using Transformations to Graph Rational Functions

Sketch a graph of each rational function.

(a) $r(x) = \dfrac{2}{x - 3}$

(b) $s(x) = \dfrac{3x + 5}{x + 2}$

Solution

(a) Let $f(x) = \frac{1}{x}$. Then we can express r in terms of f as follows:

$$r(x) = \frac{2}{x - 3}$$

$$= 2\left(\frac{1}{x - 3}\right) \qquad \text{Factor 2}$$

$$= 2(f(x - 3)) \qquad \text{Since } f(x) = \tfrac{1}{x}$$

From this form we see that the graph of r is obtained from the graph of f by shifting 3 units to the right and stretching vertically by a factor of 2. Thus, r has vertical asymptote $x = 3$ and horizontal asymptote $y = 0$. The graph of r is shown in Figure 2.

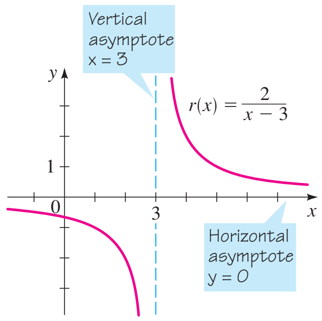


Figure 2

(b) Using long division (see the margin), we get $s(x) = 3 - \frac{1}{x + 2}$. Thus, we can express s in terms of f as follows:

$$\begin{array}{r} 3 \\ x + 2\,\overline{)\,3x + 5} \\ \underline{3x + 6} \\ -1 \end{array}$$

$$s(x) = 3 - \frac{1}{x + 2}$$

$$= -\frac{1}{x + 2} + 3 \qquad \text{Rearrange terms}$$

$$= -f(x + 2) + 3 \qquad \text{Since } f(x) = \tfrac{1}{x}$$

From this form we see that the graph of s is obtained from the graph of f by shifting 2 units to the left, reflecting in the x-axis, and shifting upward

ALTERNATE EXAMPLE 2

Sketch a graph of each rational function.

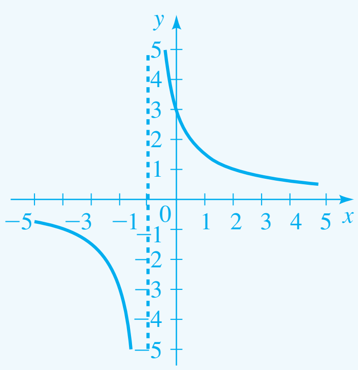
(a) $r(x) = 3/(x + 1)$

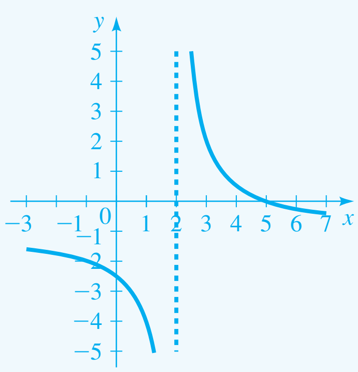
(b) $s(x) = -\dfrac{x - 5}{x - 2}$

ANSWERS

(a) We start with $f(x) = 1/x$ and shift the graph one unit to the left, and then stretch vertically by a factor of 3.

(b) $s(x) = \frac{3}{x - 2} - 1$. So we start with $f(x) = 1/x$, shift it two units to the right, stretch vertically by a factor of three, then shift vertically down by one.





IN-CLASS MATERIALS

A bit of care should be exercised when checking vertical asymptotes. For example, have students examine $f(x) = \dfrac{x^2 + 3x + 2}{x^2 - 1}$. If they are alert, they will notice an apparent x-intercept at $x = -1$, making it impossible to follow the text's dictum: "When choosing test values, we must make sure that there is no x-intercept between the test point and the vertical asymptote." The reason there is not a vertical asymptote at $x = -1$ is that there is a hole there, as seen when f is factored: $f(x) = \dfrac{(x + 1)(x + 2)}{(x + 1)(x - 1)}$. See Exercise 75 in the text.

3 units. Thus, s has vertical asymptote $x = -2$ and horizontal asymptote $y = 3$. The graph of s is shown in Figure 3.

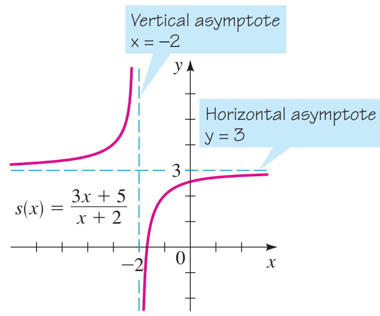


Figure 3

■

Asymptotes of Rational Functions

The methods of Example 2 work only for simple rational functions. To graph more complicated ones, we need to take a closer look at the behavior of a rational function near its vertical and horizontal asymptotes.

Example 3 Asymptotes of a Rational Function

Graph the rational function $r(x) = \dfrac{2x^2 - 4x + 5}{x^2 - 2x + 1}$.

Solution

VERTICAL ASYMPTOTE: We first factor the denominator

$$r(x) = \frac{2x^2 - 4x + 5}{(x - 1)^2}$$

The line $x = 1$ is a vertical asymptote because the denominator of r is zero when $x = 1$.

To see what the graph of r looks like near the vertical asymptote, we make tables of values for x-values to the left and to the right of 1. From the tables shown below we see that

$$y \to \infty \quad \text{as} \quad x \to 1^- \qquad \text{and} \qquad y \to \infty \quad \text{as} \quad x \to 1^+$$

$x \to 1^-$

x	y
0	5
0.5	14
0.9	302
0.99	30,002

Approaching 1^- — Approaching ∞

$x \to 1^+$

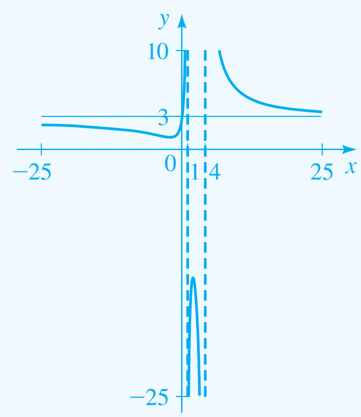
x	y
2	5
1.5	14
1.1	302
1.01	30,002

Approaching 1^+ — Approaching ∞

ALTERNATE EXAMPLE 3
Graph the rational function:

$$r(x) = \frac{3x^2 + x + 12}{x^2 - 5x + 4}$$

ANSWER
Factoring the denominator gives vertical asymptotes at $x = 1$ and $x = 4$. There is a horizontal asymptote at $y = 3$.



IN-CLASS MATERIALS

A good example to do with students is $f(x) = \dfrac{1}{x^2 + 1}$. A curve of this type is called a *Witch of Agnesi*. Its history may amuse your students. Italian mathematician Maria Agnesi (1718–1799) was a scholar whose first paper was published when she was nine years old. She called a particular curve *versiera* or "turning curve." John Colson from Cambridge confused the word with *avversiera* or "wife of the devil," and translated it "witch."

57050_03_ch03_p248-325.qxd 07/04/2008 05:18 PM Page 304

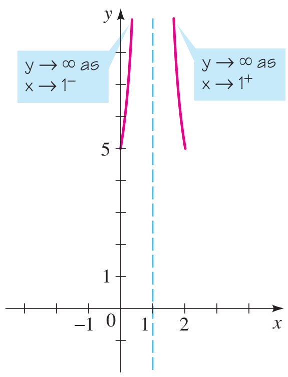


Figure 4

Thus, near the vertical asymptote $x = 1$, the graph of r has the shape shown in Figure 4.

HORIZONTAL ASYMPTOTE: The horizontal asymptote is the value y approaches as $x \to \pm\infty$. To help us find this value, we divide both numerator and denominator by x^2, the highest power of x that appears in the expression:

$$y = \frac{2x^2 - 4x + 5}{x^2 - 2x + 1} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \frac{2 - \frac{4}{x} + \frac{5}{x^2}}{1 - \frac{2}{x} + \frac{1}{x^2}}$$

The fractional expressions $\frac{4}{x}$, $\frac{5}{x^2}$, $\frac{2}{x}$, and $\frac{1}{x^2}$ all approach 0 as $x \to \pm\infty$ (see Exercise 79, Section 1.1). So as $x \to \pm\infty$, we have

These terms approach 0.

$$y = \frac{2 - \frac{4}{x} + \frac{5}{x^2}}{1 - \frac{2}{x} + \frac{1}{x^2}} \quad \to \quad \frac{2 - 0 + 0}{1 - 0 + 0} = 2$$

These terms approach 0.

Thus, the horizontal asymptote is the line $y = 2$.

Since the graph must approach the horizontal asymptote, we can complete it as in Figure 5.

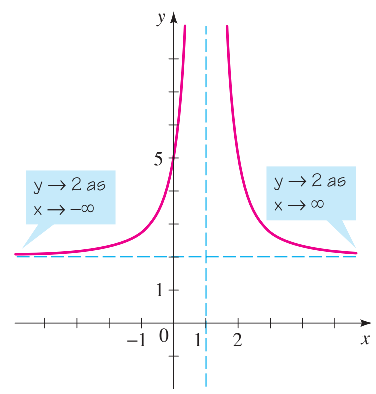


Figure 5

$$r(x) = \frac{2x^2 - 4x + 5}{x^2 - 2x + 1}$$

■

From Example 3 we see that the horizontal asymptote is determined by the leading coefficients of the numerator and denominator, since after dividing through by x^2 (the highest power of x) all other terms approach zero. In general, if

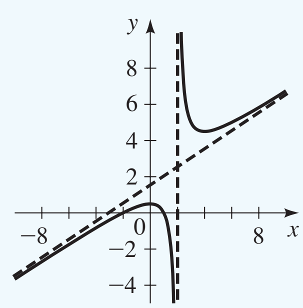
EXAMPLE

A rational function with a slant asymptote:

$$f(x) = \frac{x^2 + x - 2}{x - 2} = \frac{(x - 1)(x + 2)}{x - 2}$$

Intercepts: (1, 0), (−2, 0), (0, 1)

Asymptotes: $x = 2$, $y = x + \frac{3}{2}$



$r(x) = P(x)/Q(x)$ and the degrees of P and Q are the same (both n, say), then dividing both numerator and denominator by x^n shows that the horizontal asymptote is

$$y = \frac{\text{leading coefficient of } P}{\text{leading coefficient of } Q}$$

The following box summarizes the procedure for finding asymptotes.

Asymptotes of Rational Functions

Let r be the rational function

$$r(x) = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}$$

1. The vertical asymptotes of r are the lines $x = a$, where a is a zero of the denominator.

2. (a) If $n < m$, then r has horizontal asymptote $y = 0$.

(b) If $n = m$, then r has horizontal asymptote $y = \frac{a_n}{b_m}$.

(c) If $n > m$, then r has no horizontal asymptote.

Example 4 Asymptotes of a Rational Function

Find the vertical and horizontal asymptotes of $r(x) = \dfrac{3x^2 - 2x - 1}{2x^2 + 3x - 2}$.

Solution

VERTICAL ASYMPTOTES: We first factor

$$r(x) = \frac{3x^2 - 2x - 1}{(2x - 1)(x + 2)}$$

This factor is 0 when $x = \frac{1}{2}$.

This factor is 0 when $x = -2$.

The vertical asymptotes are the lines $x = \frac{1}{2}$ and $x = -2$.

HORIZONTAL ASYMPTOTE: The degrees of the numerator and denominator are the same and

$$\frac{\text{leading coefficient of numerator}}{\text{leading coefficient of denominator}} = \frac{3}{2}$$

Thus, the horizontal asymptote is the line $y = \frac{3}{2}$.

ALTERNATE EXAMPLE 4

Find the vertical and horizontal asymptotes of

$$r(x) = \frac{x^2 - 4x + 4}{9x^2 - 9x + 2}$$

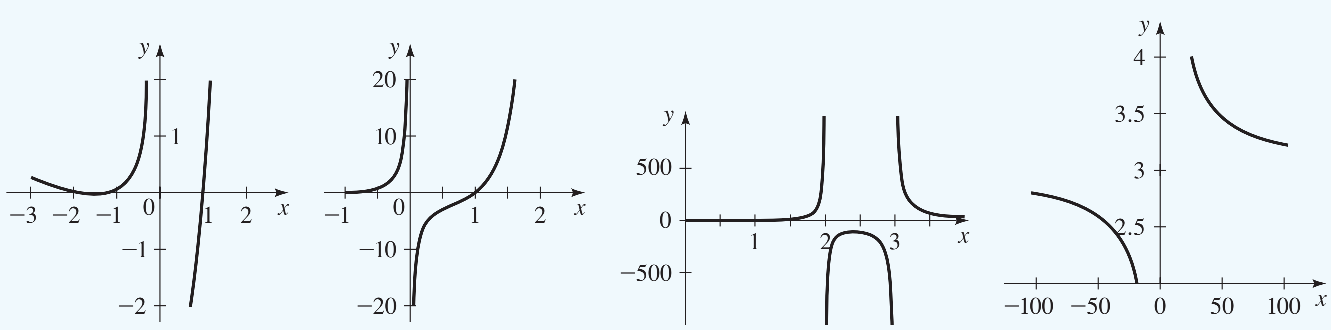
ANSWER

The vertical asymptotes are at $x = \frac{1}{3}$ and $x = \frac{2}{3}$. There is a horizontal asymptote at $y = \frac{1}{9}$. If students want to see these asymptotes, they should choose their graphing window carefully.

EXAMPLE

A rational function with many asymptotes and intercepts that are hard to find by inspecting a single viewing rectangle:

$$f(x) = \frac{3x^3 + 6x^2 - 3x - 6}{x^3 - 5x^2 + 6x} = \frac{3(x+1)(x-1)(x+2)}{x(x-2)(x-3)}$$



Intercepts: (1, 0), (−1, 0), (−2, 0)
Asymptotes: $x = 0$, $x = 2$, $x = 3$, $y = 3$

To confirm our results, we graph r using a graphing calculator (see Figure 6).

Graph is drawn using dot mode to avoid extraneous lines.

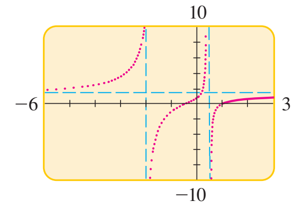


Figure 6

$$r(x) = \frac{3x^2 - 2x - 1}{2x^2 + 3x - 2}$$

Graphing Rational Functions

We have seen that asymptotes are important when graphing rational functions. In general, we use the following guidelines to graph rational functions.

Sketching Graphs of Rational Functions

1. **Factor.** Factor the numerator and denominator.
2. **Intercepts.** Find the x-intercepts by determining the zeros of the numerator, and the y-intercept from the value of the function at $x = 0$.
3. **Vertical Asymptotes.** Find the vertical asymptotes by determining the zeros of the denominator, and then see if $y \to \infty$ or $y \to -\infty$ on each side of each vertical asymptote by using test values.
4. **Horizontal Asymptote.** Find the horizontal asymptote (if any) by dividing both numerator and denominator by the highest power of x that appears in the denominator, and then letting $x \to \pm\infty$.
5. **Sketch the Graph.** Graph the information provided by the first four steps. Then plot as many additional points as needed to fill in the rest of the graph of the function.

A fraction is 0 if and only if its numerator is 0.

Example 5 Graphing a Rational Function

Graph the rational function $r(x) = \dfrac{2x^2 + 7x - 4}{x^2 + x - 2}$.

Solution We factor the numerator and denominator, find the intercepts and asymptotes, and sketch the graph.

FACTOR: $y = \dfrac{(2x - 1)(x + 4)}{(x - 1)(x + 2)}$

x-INTERCEPTS: The x-intercepts are the zeros of the numerator, $x = \frac{1}{2}$ and $x = -4$.

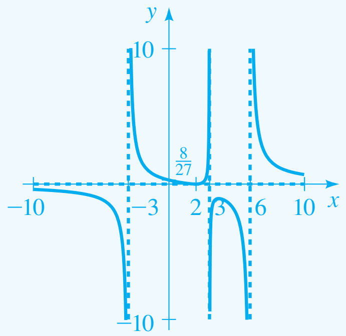
ALTERNATE EXAMPLE 5
Graph the rational function

$$r(x) = \frac{4x^2 - 16x + 16}{x^3 - 6x^2 - 9x + 54}$$

We factor to obtain

$$r(x) = \frac{4(x - 2)^2}{(x + 3)(x - 3)(x - 6)}$$

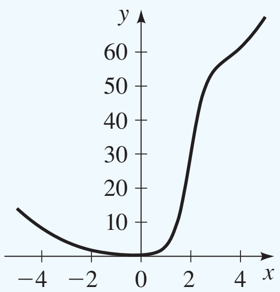
The x-intercept is at $x = 2$. The y-intercept is at $y = \frac{8}{27}$. The vertical asymptotes are at $x = -3$, $x = 3$, and $x = 6$. The horizontal asymptote is at $y = 0$. We plot a few additional points to obtain the graph:

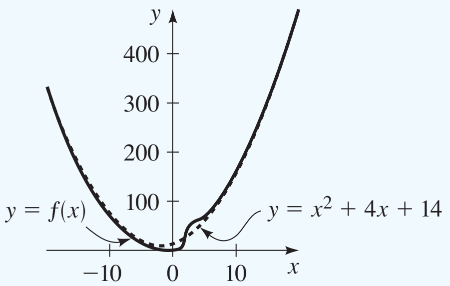


EXAMPLE
A rational function with no asymptote:

$$f(x) = \frac{x^4 + 3x^2 + 2}{x^2 - 4x + 5} = \frac{(x^2 + 1)(x^2 + 2)}{x^2 - 4x + 5}$$

Note that the end behavior of this function is similar to that of $y = x^2 + 4x + 14$.





y-INTERCEPT: To find the y-intercept, we substitute $x = 0$ into the original form of the function:

$$r(0) = \frac{2(0)^2 + 7(0) - 4}{(0)2 + (0) - 2} = \frac{-4}{-2} = 2$$

The y-intercept is 2.

VERTICAL ASYMPTOTES: The vertical asymptotes occur where the denominator is 0, that is, where the function is undefined. From the factored form we see that the vertical asymptotes are the lines $x = 1$ and $x = -2$.

BEHAVIOR NEAR VERTICAL ASYMPTOTES: We need to know whether $y \to \infty$ or $y \to -\infty$ on each side of each vertical asymptote. To determine the sign of y for x-values near the vertical asymptotes, we use test values. For instance, as $x \to 1^-$, we use a test value close to and to the left of 1 ($x = 0.9$, say) to check whether y is positive or negative to the left of $x = 1$:

When choosing test values, we must make sure that there is no x-intercept between the test point and the vertical asymptote.

$$y = \frac{(2(0.9) - 1)((0.9) + 4)}{((0.9) - 1)((0.9) + 2)} \quad \text{whose sign is} \quad \frac{(+)(+)}{(-)(+)} \quad \text{(negative)}$$

So $y \to -\infty$ as $x \to 1^-$. On the other hand, as $x \to 1^+$, we use a test value close to and to the right of 1 ($x = 1.1$, say), to get

$$y = \frac{(2(1.1) - 1)((1.1) + 4)}{((1.1) - 1)((1.1) + 2)} \quad \text{whose sign is} \quad \frac{(+)(+)}{(+)(+)} \quad \text{(positive)}$$

So $y \to \infty$ as $x \to 1^+$. The other entries in the following table are calculated similarly.

As $x \to$	-2^-	-2^+	1^-	1^+
the sign of $y = \frac{(2x-1)(x+4)}{(x-1)(x+2)}$ is	$\frac{(-)(+)}{(-)(-)}$	$\frac{(-)(+)}{(-)(+)}$	$\frac{(+)(+)}{(-)(+)}$	$\frac{(+)(+)}{(+)(+)}$
so $y \to$	$-\infty$	∞	$-\infty$	∞

HORIZONTAL ASYMPTOTE: The degrees of the numerator and denominator are the same and

$$\frac{\text{leading coefficient of numerator}}{\text{leading coefficient of denominator}} = \frac{2}{1} = 2$$

Thus, the horizontal asymptote is the line $y = 2$.

ADDITIONAL VALUES:

x	y
−6	0.93
−3	−1.75
−1	4.50
1.5	6.29
2	4.50
3	3.50

GRAPH:

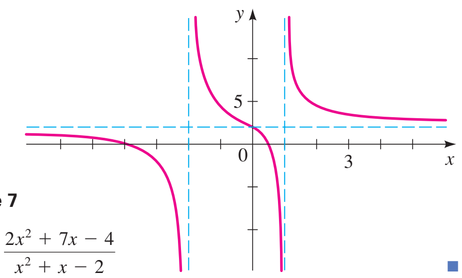


Figure 7

$r(x) = \frac{2x^2 + 7x - 4}{x^2 + x - 2}$

DRILL QUESTION

Find all the asymptotes of the rational function $f(x) = \dfrac{x^2 + 1}{2x^2 - 5x - 3}$.

Answer

Horizontal asymptote at $y = \frac{1}{2}$, vertical asymptotes at $x = -\frac{1}{2}$ and $x = 3$.

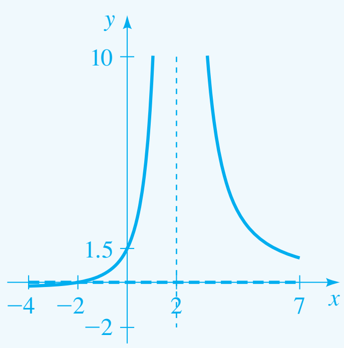
ALTERNATE EXAMPLE 6
Graph the rational function

$$r(x) = \frac{3x + 2}{x^2 - 4x + 4}$$

ANSWER

Factor: $r(x) = \dfrac{3x + 6}{(x - 2)^2}$

x-intercept: -2
y-intercept: $\frac{3}{2}$
Vertical asymptote: $x = 2$
Behavior near vertical asymptote:
$y \to +\infty$ as $x \to 2^-$,
$y \to +\infty$ as $x \to 2^+$
Horizontal asymptote: $y = 0$



Mathematics in the Modern World

Unbreakable Codes

If you read spy novels, you know about secret codes, and how the hero "breaks" the code. Today secret codes have a much more common use. Most of the information stored on computers is coded to prevent unauthorized use. For example, your banking records, medical records, and school records are coded. Many cellular and cordless phones code the signal carrying your voice so no one can listen in. Fortunately, because of recent advances in mathematics, today's codes are "unbreakable."

Modern codes are based on a simple principle: Factoring is much harder than multiplying. For example, try multiplying 78 and 93; now try factoring 9991. It takes a long time to factor 9991 because it is a product of two primes 97×103, so to factor it we had to find one of these primes. Now imagine trying to factor a number N that is the product of two primes p and q, each about 200 digits long. Even the fastest computers would take many millions of years to factor such a number! But the same computer would take less than a second to multiply two such numbers. This fact was used by Ron Rivest, Adi Shamir, and Leonard Adleman in the 1970s to devise the RSA code. Their code uses an extremely large number to encode a message but requires us to know its factors to decode it. As you can see, such a code is practically unbreakable.

(*continued*)

Example 6 Graphing a Rational Function

Graph the rational function $r(x) = \dfrac{5x + 21}{x^2 + 10x + 25}$.

Solution

FACTOR: $y = \dfrac{5x + 21}{(x + 5)^2}$

x-INTERCEPT: $-\frac{21}{5}$, from $5x + 21 = 0$

y-INTERCEPT: $\frac{21}{25}$, because $r(0) = \dfrac{5 \cdot 0 + 21}{0^2 + 10 \cdot 0 + 25} = \dfrac{21}{25}$

VERTICAL ASYMPTOTE: $x = -5$, from the zeros of the denominator

BEHAVIOR NEAR VERTICAL ASYMPTOTE:

As $x \to$	-5^-	-5^+
the sign of $y = \dfrac{5x + 21}{(x + 5)^2}$ is	$\dfrac{(-)}{(-)(-)}$	$\dfrac{(-)}{(+)(+)}$
so $y \to$	$-\infty$	$-\infty$

HORIZONTAL ASYMPTOTE: $y = 0$, because degree of numerator is less than degree of denominator

ADDITIONAL VALUES:

x	y
−15	−0.5
−10	−1.2
−3	1.5
−1	1.0
3	0.6
5	0.5
10	0.3

GRAPH:

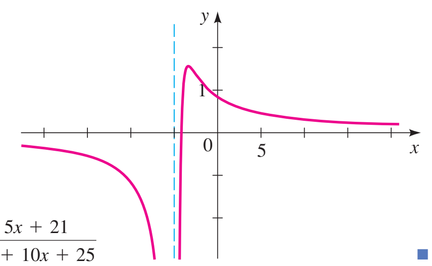


Figure 8
$r(x) = \dfrac{5x + 21}{x^2 + 10x + 25}$ ■

From the graph in Figure 8 we see that, contrary to the common misconception, a graph may cross a horizontal asymptote. The graph in Figure 8 crosses the x-axis (the horizontal asymptote) from below, reaches a maximum value near $x = -3$, and then approaches the x-axis from above as $x \to \infty$.

SAMPLE QUESTION
Text Question
What is a slant asymptote?

Answer
Answers will vary.

The RSA code is an example of a "public key encryption" code. In such codes, anyone can code a message using a publicly known procedure based on N, but to decode the message they must know p and q, the factors of N. When the RSA code was developed, it was thought that a carefully selected 80-digit number would provide an unbreakable code. But interestingly, recent advances in the study of factoring have made much larger numbers necessary.

Example 7 Graphing a Rational Function

Graph the rational function $r(x) = \dfrac{x^2 - 3x - 4}{2x^2 + 4x}$.

Solution

FACTOR: $y = \dfrac{(x + 1)(x - 4)}{2x(x + 2)}$

x-INTERCEPTS: -1 and 4, from $x + 1 = 0$ and $x - 4 = 0$

y-INTERCEPT: None, because $r(0)$ is undefined

VERTICAL ASYMPTOTES: $x = 0$ and $x = -2$, from the zeros of the denominator

BEHAVIOR NEAR VERTICAL ASYMPTOTES:

As $x \to$	-2^-	-2^+	0^-	0^+
the sign of $y = \dfrac{(x + 1)(x - 4)}{2x(x + 2)}$ is	$\dfrac{(-)(-)}{(-)(-)}$	$\dfrac{(-)(-)}{(-)(+)}$	$\dfrac{(+)(-)}{(-)(+)}$	$\dfrac{(+)(-)}{(+)(+)}$
so $y \to$	∞	$-\infty$	∞	$-\infty$

HORIZONTAL ASYMPTOTE: $y = \frac{1}{2}$, because degree of numerator and denominator are the same and

$$\frac{\text{leading coefficient of numerator}}{\text{leading coefficient of denominator}} = \frac{1}{2}$$

ADDITIONAL VALUES:

x	y
-3	2.33
-2.5	3.90
-0.5	1.50
1	-1.00
3	-0.13
5	0.09

GRAPH:

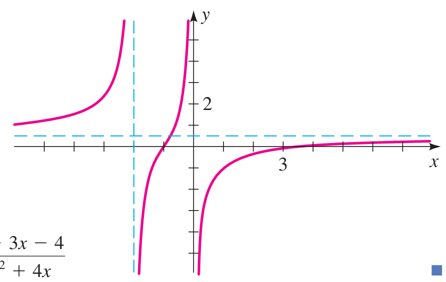


Figure 9

$r(x) = \dfrac{x^2 - 3x - 4}{2x^2 + 4x}$ ■

Slant Asymptotes and End Behavior

If $r(x) = P(x)/Q(x)$ is a rational function in which the degree of the numerator is one more than the degree of the denominator, we can use the Division Algorithm to express the function in the form

$$r(x) = ax + b + \frac{R(x)}{Q(x)}$$

where the degree of R is less than the degree of Q and $a \neq 0$. This means that as

ALTERNATE EXAMPLE 7

Graph the rational function

$r(x) = \dfrac{x^2 - 25}{2x^2 - 8}$

Factor:

$r(x) = \dfrac{(x + 5)(x - 5)}{2(x + 2)(x - 2)}$

x-intercepts: -5 and 5

y-intercept: $\frac{25}{8}$

Vertical asymptotes: $x = 2$ and $x = -2$

Behavior near vertical asymptotes:

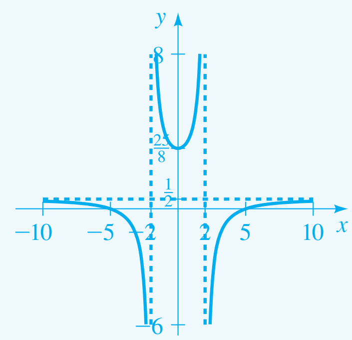
$y \to +\infty$ as $x \to 2^-$,

$y \to -\infty$ as $x \to 2^+$

$y \to -\infty$ as $x \to -2^-$,

$y \to +\infty$ as $x \to -2^+$

Horizontal asymptote: $y = \frac{1}{2}$



$x \to \pm\infty$, $R(x)/Q(x) \to 0$, so for large values of $|x|$, the graph of $y = r(x)$ approaches the graph of the line $y = ax + b$. In this situation we say that $y = ax + b$ is a **slant asymptote**, or an **oblique asymptote**.

Example 8 A Rational Function with a Slant Asymptote

Graph the rational function $r(x) = \dfrac{x^2 - 4x - 5}{x - 3}$.

Solution

FACTOR: $y = \dfrac{(x + 1)(x - 5)}{x - 3}$

x-INTERCEPTS: -1 and 5, from $x + 1 = 0$ and $x - 5 = 0$

y-INTERCEPTS: $\frac{5}{3}$, because $r(0) = \dfrac{0^2 - 4 \cdot 0 - 5}{0 - 3} = \dfrac{5}{3}$

HORIZONTAL ASYMPTOTE: None, because degree of numerator is greater than degree of denominator

VERTICAL ASYMPTOTE: $x = 3$, from the zero of the denominator

BEHAVIOR NEAR VERTICAL ASYMPTOTE: $y \to \infty$ as $x \to 3^-$ and $y \to -\infty$ as $x \to 3^+$

SLANT ASYMPTOTE: Since the degree of the numerator is one more than the degree of the denominator, the function has a slant asymptote. Dividing (see the margin), we obtain

$$r(x) = x - 1 - \frac{8}{x - 3}$$

$$\begin{array}{r} x - 1 \\ x - 3\overline{)x^2 - 4x - 5} \\ \underline{x^2 - 3x} \\ -x - 5 \\ \underline{-x + 3} \\ -8 \end{array}$$

Thus, $y = x - 1$ is the slant asymptote.

ADDITIONAL VALUES:

x	y
-2	-1.4
1	4
2	9
4	-5
6	2.33

GRAPH:

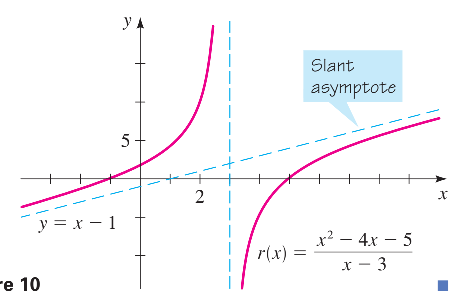


Figure 10

So far we have considered only horizontal and slant asymptotes as end behaviors for rational functions. In the next example we graph a function whose end behavior is like that of a parabola.

ALTERNATE EXAMPLE 8

Graph the rational function

$r(x) = \dfrac{x^2 - 3x - 1}{1 - x}$.

ANSWER

Factor: $r(x) = \dfrac{\left(x - \frac{3 + \sqrt{13}}{2}\right)\left(x - \frac{3 - \sqrt{13}}{2}\right)}{-(x - 1)}$

x-intercepts: $\dfrac{3 \pm \sqrt{13}}{2}$

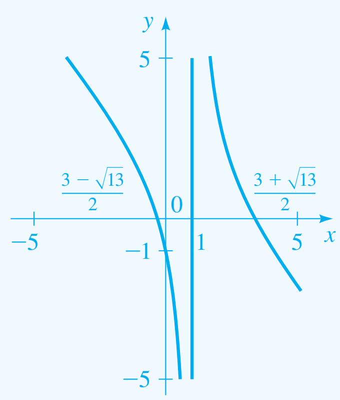
y-intercept: -1

Vertical asymptote: $x = 1$

Behavior near vertical asymptote: $y \to -\infty$ as $x \to 1^-$, $y \to +\infty$ as $x \to 1^+$

Slant asymptote: $r(x) = -x + 2 + 3/(x - 1)$

Slant asymptote at $y = -x + 2$



57050_03_ch03_p248-325.qxd 07/04/2008 05:19 PM Page 311

Example 9 End Behavior of a Rational Function

Graph the rational function

$$r(x) = \frac{x^3 - 2x^2 + 3}{x - 2}$$

and describe its end behavior.

Solution

FACTOR: $y = \dfrac{(x + 1)(x^2 - 3x + 3)}{x - 2}$

x-INTERCEPTS: -1, from $x + 1 = 0$ (The other factor in the numerator has no real zeros.)

y-INTERCEPTS: $-\frac{3}{2}$, because $r(0) = \dfrac{0^3 - 2 \cdot 0^2 + 3}{0 - 2} = -\dfrac{3}{2}$

VERTICAL ASYMPTOTE: $x = 2$, from the zero of the denominator

BEHAVIOR NEAR VERTICAL ASYMPTOTE: $y \to -\infty$ as $x \to 2^-$ and $y \to \infty$ as $x \to 2^+$

HORIZONTAL ASYMPTOTE: None, because degree of numerator is greater than degree of denominator

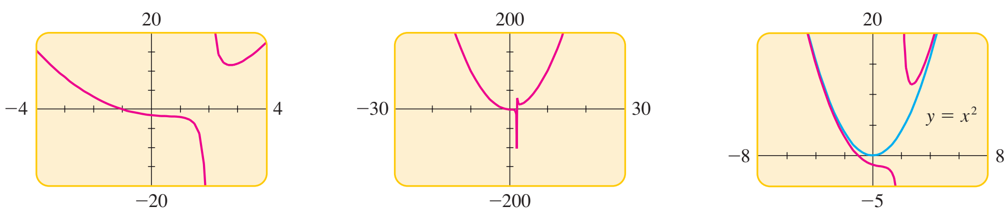
$$\begin{array}{r} x^2 \\ x - 2 \overline{)\, x^3 - 2x^2 + 0x + 3} \\ \underline{x^3 - 2x^2} \\ 3 \end{array}$$

END BEHAVIOR: Dividing (see the margin), we get

$$r(x) = x^2 + \frac{3}{x - 2}$$

This shows that the end behavior of r is like that of the parabola $y = x^2$ because $3/(x - 2)$ is small when $|x|$ is large. That is, $3/(x - 2) \to 0$ as $x \to \pm\infty$. This means that the graph of r will be close to the graph of $y = x^2$ for large $|x|$.

GRAPH: In Figure 11(a) we graph r in a small viewing rectangle; we can see the intercepts, the vertical asymptotes, and the local minimum. In Figure 11(b) we graph r in a larger viewing rectangle; here the graph looks almost like the graph of a parabola. In Figure 11(c) we graph both $y = r(x)$ and $y = x^2$; these graphs are very close to each other except near the vertical asymptote.



(a) (b) (c)

Figure 11

$r(x) = \dfrac{x^3 - 2x^2 + 3}{x - 2}$

Applications

Rational functions occur frequently in scientific applications of algebra. In the next example we analyze the graph of a function from the theory of electricity.

ALTERNATE EXAMPLE 9

Graph the rational function

$r(x) = \dfrac{x^4 - x^3 + 1}{x - 1}$.

ANSWER

Factor: Examining the graph of $x^4 - x^3 + 1$ shows us that the numerator has no real factors.

x-intercepts: none

y-intercept: -1

Vertical asymptote: $x = 1$

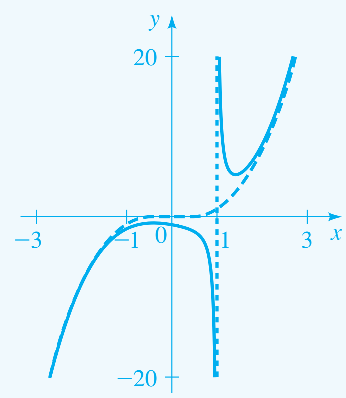
Behavior near vertical asymptote: $y \to -\infty$ as $x \to 1^-$, $y \to +\infty$ as $x \to 1^+$

Horizontal asymptote: none

End behavior:

$r(x) = x^3 + \dfrac{1}{x - 1}$.

The end behavior is like the power function $y = x^3$.



Example 10 Electrical Resistance

When two resistors with resistances R_1 and R_2 are connected in parallel, their combined resistance R is given by the formula

$$R = \frac{R_1 R_2}{R_1 + R_2}$$

Suppose that a fixed 8-ohm resistor is connected in parallel with a variable resistor, as shown in Figure 12. If the resistance of the variable resistor is denoted by x, then the combined resistance R is a function of x. Graph R and give a physical interpretation of the graph.

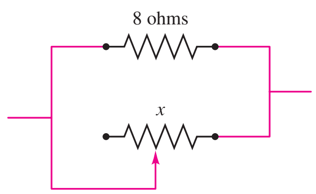


Figure 12

Solution Substituting $R_1 = 8$ and $R_2 = x$ into the formula gives the function

$$R(x) = \frac{8x}{8 + x}$$

Since resistance cannot be negative, this function has physical meaning only when $x > 0$. The function is graphed in Figure 13(a) using the viewing rectangle $[0, 20]$ by $[0, 10]$. The function has no vertical asymptote when x is restricted to positive values. The combined resistance R increases as the variable resistance x increases. If we widen the viewing rectangle to $[0, 100]$ by $[0, 10]$, we obtain the graph in Figure 13(b). For large x, the combined resistance R levels off, getting closer and closer to the horizontal asymptote $R = 8$. No matter how large the variable resistance x, the combined resistance is never greater than 8 ohms. ■

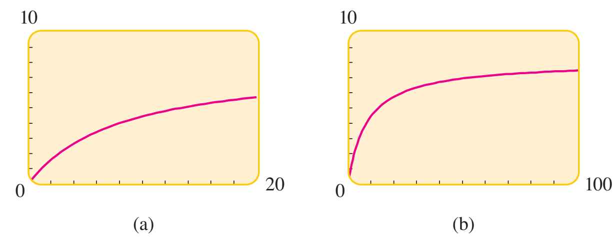


Figure 13
$R(x) = \frac{8x}{8 + x}$

3.6 Exercises

1–4 ■ A rational function is given. **(a)** Complete each table for the function. **(b)** Describe the behavior of the function near its vertical asymptote, based on Tables 1 and 2. **(c)** Determine the horizontal asymptote, based on Tables 3 and 4.

Table 1

x	$r(x)$
1.5	
1.9	
1.99	
1.999	

Table 2

x	$r(x)$
2.5	
2.1	
2.01	
2.001	

Table 3

x	$r(x)$
10	
50	
100	
1000	

Table 4

x	$r(x)$
−10	
−50	
−100	
−1000	

1. $r(x) = \frac{x}{x - 2}$

2. $r(x) = \frac{4x + 1}{x - 2}$

3. $r(x) = \frac{3x - 10}{(x - 2)^2}$

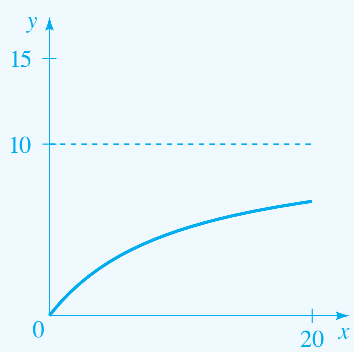
4. $r(x) = \frac{3x^2 + 1}{(x - 2)^2}$

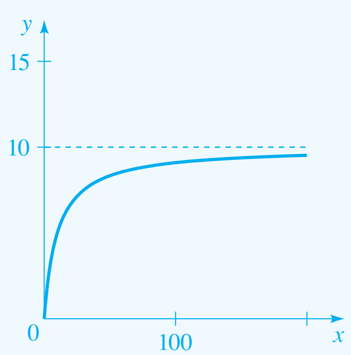
ALTERNATE EXAMPLE 10

When two resistors with resistances R_1 and R_2 are connected in parallel, their combined resistance R is given by the formula $R = \frac{R_1 R_2}{R_1 + R_2}$. Suppose that a fixed 10 ohm resistor is connected in parallel with a variable resistor. Graph R as a function of the resistance of the variable resistor, x.

ANSWER

Substituting $R_1 = 10$ and $R_2 = x$ into the formula gives: $R(x) = \frac{10x}{10 + x}$. The analysis proceeds as in the text. No matter how large the variable resistance x, the combined resistance is never greater than 10 ohms.





5–10 ■ Find the x- and y-intercepts of the rational function.

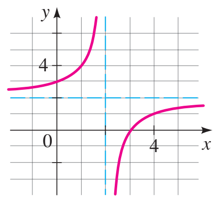
5. $r(x) = \dfrac{x - 1}{x + 4}$ **6.** $s(x) = \dfrac{3x}{x - 5}$

7. $t(x) = \dfrac{x^2 - x - 2}{x - 6}$ **8.** $r(x) = \dfrac{2}{x^2 + 3x - 4}$

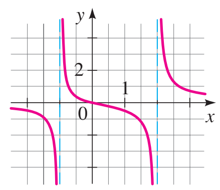
9. $r(x) = \dfrac{x^2 - 9}{x^2}$ **10.** $r(x) = \dfrac{x^3 + 8}{x^2 + 4}$

11–14 ■ From the graph, determine the x- and y-intercepts and the vertical and horizontal asymptotes.

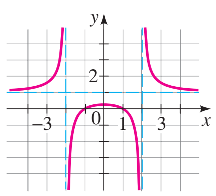
11.



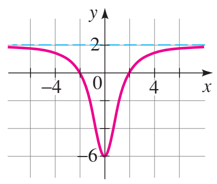
12.



13.



14.



15–24 ■ Find all horizontal and vertical asymptotes (if any).

15. $r(x) = \dfrac{3}{x + 2}$ **16.** $s(x) = \dfrac{2x + 3}{x - 1}$

17. $t(x) = \dfrac{x^2}{x^2 - x - 6}$ **18.** $r(x) = \dfrac{2x - 4}{x^2 + 2x + 1}$

19. $s(x) = \dfrac{6}{x^2 + 2}$ **20.** $t(x) = \dfrac{(x - 1)(x - 2)}{(x - 3)(x - 4)}$

21. $r(x) = \dfrac{6x - 2}{x^2 + 5x - 6}$ **22.** $s(x) = \dfrac{3x^2}{x^2 + 2x + 5}$

23. $t(x) = \dfrac{x^2 + 2}{x - 1}$ **24.** $r(x) = \dfrac{x^3 + 3x^2}{x^2 - 4}$

25–32 ■ Use transformations of the graph of $y = \frac{1}{x}$ to graph the rational function, as in Example 2.

25. $r(x) = \dfrac{1}{x - 1}$ **26.** $r(x) = \dfrac{1}{x + 4}$

27. $s(x) = \dfrac{3}{x + 1}$ **28.** $s(x) = \dfrac{-2}{x - 2}$

29. $t(x) = \dfrac{2x - 3}{x - 2}$ **30.** $t(x) = \dfrac{3x - 3}{x + 2}$

31. $r(x) = \dfrac{x + 2}{x + 3}$ **32.** $r(x) = \dfrac{2x - 9}{x - 4}$

33–56 ■ Find the intercepts and asymptotes, and then sketch a graph of the rational function. Use a graphing device to confirm your answer.

33. $r(x) = \dfrac{4x - 4}{x + 2}$ **34.** $r(x) = \dfrac{2x + 6}{-6x + 3}$

35. $s(x) = \dfrac{4 - 3x}{x + 7}$ **36.** $s(x) = \dfrac{1 - 2x}{2x + 3}$

37. $r(x) = \dfrac{18}{(x - 3)^2}$ **38.** $r(x) = \dfrac{x - 2}{(x + 1)^2}$

39. $s(x) = \dfrac{4x - 8}{(x - 4)(x + 1)}$ **40.** $s(x) = \dfrac{x + 2}{(x + 3)(x - 1)}$

41. $s(x) = \dfrac{6}{x^2 - 5x - 6}$ **42.** $s(x) = \dfrac{2x - 4}{x^2 + x - 2}$

43. $t(x) = \dfrac{3x + 6}{x^2 + 2x - 8}$ **44.** $t(x) = \dfrac{x - 2}{x^2 - 4x}$

45. $r(x) = \dfrac{(x - 1)(x + 2)}{(x + 1)(x - 3)}$ **46.** $r(x) = \dfrac{2x(x + 2)}{(x - 1)(x - 4)}$

47. $r(x) = \dfrac{x^2 - 2x + 1}{x^2 + 2x + 1}$ **48.** $r(x) = \dfrac{4x^2}{x^2 - 2x - 3}$

49. $r(x) = \dfrac{2x^2 + 10x - 12}{x^2 + x - 6}$ **50.** $r(x) = \dfrac{2x^2 + 2x - 4}{x^2 + x}$

51. $r(x) = \dfrac{x^2 - x - 6}{x^2 + 3x}$ **52.** $r(x) = \dfrac{x^2 + 3x}{x^2 - x - 6}$

53. $r(x) = \dfrac{3x^2 + 6}{x^2 - 2x - 3}$ **54.** $r(x) = \dfrac{5x^2 + 5}{x^2 + 4x + 4}$

55. $s(x) = \dfrac{x^2 - 2x + 1}{x^3 - 3x^2}$ **56.** $t(x) = \dfrac{x^3 - x^2}{x^3 - 3x - 2}$

57–64 ■ Find the slant asymptote, the vertical asymptotes, and sketch a graph of the function.

57. $r(x) = \dfrac{x^2}{x - 2}$ **58.** $r(x) = \dfrac{x^2 + 2x}{x - 1}$

59. $r(x) = \dfrac{x^2 - 2x - 8}{x}$ **60.** $r(x) = \dfrac{3x - x^2}{2x - 2}$

61. $r(x) = \dfrac{x^2 + 5x + 4}{x - 3}$ **62.** $r(x) = \dfrac{x^3 + 4}{2x^2 + x - 1}$

63. $r(x) = \dfrac{x^3 + x^2}{x^2 - 4}$ **64.** $r(x) = \dfrac{2x^3 + 2x}{x^2 - 1}$

65–68 ■ Graph the rational function f and determine all vertical asymptotes from your graph. Then graph f and g in a sufficiently large viewing rectangle to show that they have the same end behavior.

65. $f(x) = \dfrac{2x^2 + 6x + 6}{x + 3}, \quad g(x) = 2x$

66. $f(x) = \dfrac{-x^3 + 6x^2 - 5}{x^2 - 2x}, \quad g(x) = -x + 4$

67. $f(x) = \dfrac{x^3 - 2x^2 + 16}{x - 2}, \quad g(x) = x^2$

68. $f(x) = \dfrac{-x^4 + 2x^3 - 2x}{(x - 1)^2}, \quad g(x) = 1 - x^2$

69–74 ■ Graph the rational function and find all vertical asymptotes, x- and y-intercepts, and local extrema, correct to the nearest decimal. Then use long division to find a polynomial that has the same end behavior as the rational function, and graph both functions in a sufficiently large viewing rectangle to verify that the end behaviors of the polynomial and the rational function are the same.

69. $y = \dfrac{2x^2 - 5x}{2x + 3}$

70. $y = \dfrac{x^4 - 3x^3 + x^2 - 3x + 3}{x^2 - 3x}$

71. $y = \dfrac{x^5}{x^3 - 1}$

72. $y = \dfrac{x^4}{x^2 - 2}$

73. $r(x) = \dfrac{x^4 - 3x^3 + 6}{x - 3}$

74. $r(x) = \dfrac{4 + x^2 - x^4}{x^2 - 1}$

Applications

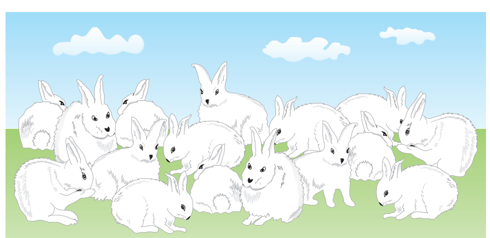
75. Population Growth Suppose that the rabbit population on Mr. Jenkins' farm follows the formula

$$p(t) = \frac{3000t}{t + 1}$$

where $t \geq 0$ is the time (in months) since the beginning of the year.

(a) Draw a graph of the rabbit population.

(b) What eventually happens to the rabbit population?



76. Drug Concentration After a certain drug is injected into a patient, the concentration c of the drug in the bloodstream is monitored. At time $t \geq 0$ (in minutes since the injection), the concentration (in mg/L) is given by

$$c(t) = \frac{30t}{t^2 + 2}$$

(a) Draw a graph of the drug concentration.

(b) What eventually happens to the concentration of drug in the bloodstream?

77. Drug Concentration A drug is administered to a patient and the concentration of the drug in the bloodstream is monitored. At time $t \geq 0$ (in hours since giving the drug), the concentration (in mg/L) is given by

$$c(t) = \frac{5t}{t^2 + 1}$$

Graph the function c with a graphing device.

(a) What is the highest concentration of drug that is reached in the patient's bloodstream?

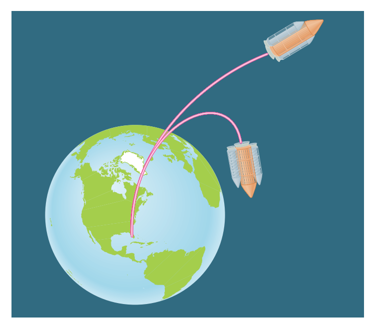
(b) What happens to the drug concentration after a long period of time?

(c) How long does it take for the concentration to drop below 0.3 mg/L?

78. Flight of a Rocket Suppose a rocket is fired upward from the surface of the earth with an initial velocity v (measured in m/s). Then the maximum height h (in meters) reached by the rocket is given by the function

$$h(v) = \frac{Rv^2}{2gR - v^2}$$

where $R = 6.4 \times 10^6$ m is the radius of the earth and $g = 9.8$ m/s^2 is the acceleration due to gravity. Use a graphing device to draw a graph of the function h. (Note that h and v must both be positive, so the viewing rectangle need not contain negative values.) What does the vertical asymptote represent physically?



79. The Doppler Effect As a train moves toward an observer (see the figure), the pitch of its whistle sounds higher to the observer than it would if the train were at rest, because the crests of the sound waves are compressed closer together. This phenomenon is called the *Doppler effect*. The observed pitch P is a function of the speed v of the train and is given by

$$P(v) = P_0\left(\frac{s_0}{s_0 - v}\right)$$

where P_0 is the actual pitch of the whistle at the source and $s_0 = 332$ m/s is the speed of sound in air. Suppose that a train has a whistle pitched at $P_0 = 440$ Hz. Graph the function $y = P(v)$ using a graphing device. How can the vertical asymptote of this function be interpreted physically?

80. Focusing Distance For a camera with a lens of fixed focal length F to focus on an object located a distance x from the lens, the film must be placed a distance y behind the lens, where F, x, and y are related by

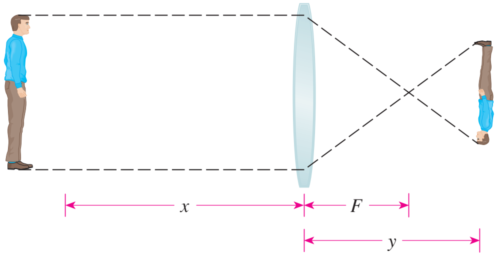
$$\frac{1}{x} + \frac{1}{y} = \frac{1}{F}$$

(See the figure.) Suppose the camera has a 55-mm lens ($F = 55$).

(a) Express y as a function of x and graph the function.

(b) What happens to the focusing distance y as the object moves far away from the lens?

(c) What happens to the focusing distance y as the object moves close to the lens?



Discovery • Discussion

81. Constructing a Rational Function from Its Asymptotes Give an example of a rational function that has vertical asymptote $x = 3$. Now give an example of one that has vertical asymptote $x = 3$ *and* horizontal asymptote $y = 2$. Now give an example of a rational function with vertical asymptotes $x = 1$ and $x = -1$, horizontal asymptote $y = 0$, and x-intercept 4.

82. A Rational Function with No Asymptote Explain how you can tell (without graphing it) that the function

$$r(x) = \frac{x^6 + 10}{x^4 + 8x^2 + 15}$$

has no x-intercept and no horizontal, vertical, or slant asymptote. What is its end behavior?'

83. Graphs with Holes In this chapter we adopted the convention that in rational functions, the numerator and denominator don't share a common factor. In this exercise we consider the graph of a rational function that doesn't satisfy this rule.

(a) Show that the graph of

$$r(x) = \frac{3x^2 - 3x - 6}{x - 2}$$

is the line $y = 3x + 3$ with the point $(2, 9)$ removed. [*Hint:* Factor. What is the domain of r?]

(b) Graph the rational functions:

$$s(x) = \frac{x^2 + x - 20}{x + 5}$$

$$t(x) = \frac{2x^2 - x - 1}{x - 1}$$

$$u(x) = \frac{x - 2}{x^2 - 2x}$$

84. Transformations of $y = 1/x^2$ In Example 2 we saw that some simple rational functions can be graphed by shifting, stretching, or reflecting the graph of $y = 1/x$. In this exercise we consider rational functions that can be graphed by transforming the graph of $y = 1/x^2$, shown on the following page.

(a) Graph the function

$$r(x) = \frac{1}{(x - 2)^2}$$

by transforming the graph of $y = 1/x^2$.

(b) Use long division and factoring to show that the function

$$s(x) = \frac{2x^2 + 4x + 5}{x^2 + 2x + 1}$$

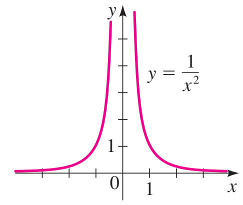
can be written as

$$s(x) = 2 + \frac{3}{(x + 1)^2}$$

Then graph s by transforming the graph of $y = 1/x^2$.

(c) One of the following functions can be graphed by transforming the graph of $y = 1/x^2$; the other cannot. Use transformations to graph the one that can be, and explain why this method doesn't work for the other one.

$$p(x) = \frac{2 - 3x^2}{x^2 - 4x + 4} \qquad q(x) = \frac{12x - 3x^2}{x^2 - 4x + 4}$$



3 Review

Concept Check

1. **(a)** Write the defining equation for a polynomial P of degree n.
(b) What does it mean to say that c is a zero of P?

2. Sketch graphs showing the possible end behaviors of polynomials of odd degree and of even degree.

3. What steps would you follow to graph a polynomial by hand?

4. **(a)** What is meant by a local maximum point or local minimum point of a polynomial?
(b) How many local extrema can a polynomial of degree n have?

5. State the Division Algorithm and identify the dividend, divisor, quotient, and remainder.

6. How does synthetic division work?

7. **(a)** State the Remainder Theorem.
(b) State the Factor Theorem.

8. **(a)** State the Rational Zeros Theorem.
(b) What steps would you take to find the rational zeros of a polynomial?

9. State Descartes' Rule of Signs.

10. **(a)** What does it mean to say that a is a lower bound and b is an upper bound for the zeros of a polynomial?
(b) State the Upper and Lower Bounds Theorem.

11. **(a)** What is a complex number?
(b) What are the real and imaginary parts of a complex number?
(c) What is the complex conjugate of a complex number?
(d) How do you add, subtract, multiply, and divide complex numbers?

12. **(a)** State the Fundamental Theorem of Algebra.
(b) State the Complete Factorization Theorem.
(c) What does it mean to say that c is a zero of multiplicity k of a polynomial P?
(d) State the Zeros Theorem.
(e) State the Conjugate Zeros Theorem.

13. **(a)** What is a rational function?
(b) What does it mean to say that $x = a$ is a vertical asymptote of $y = f(x)$?
(c) How do you locate a vertical asymptote?
(d) What does it mean to say that $y = b$ is a horizontal asymptote of $y = f(x)$?
(e) How do you locate a horizontal asymptote?
(f) What steps do you follow to sketch the graph of a rational function by hand?
(g) Under what circumstances does a rational function have a slant asymptote? If one exists, how do you find it?
(h) How do you determine the end behavior of a rational function?

Exercises

1–6 ■ Graph the polynomial by transforming an appropriate graph of the form $y = x^n$. Show clearly all x- and y-intercepts.

1. $P(x) = -x^3 + 64$ **2.** $P(x) = 2x^3 - 16$

3. $P(x) = 2(x + 1)^4 - 32$ **4.** $P(x) = 81 - (x - 3)^4$

5. $P(x) = 32 + (x - 1)^5$ **6.** $P(x) = -3(x + 2)^5 + 96$

7–10 ■ Use a graphing device to graph the polynomial. Find the x- and y-intercepts and the coordinates of all local extrema, correct to the nearest decimal. Describe the end behavior of the polynomial.

7. $P(x) = x^3 - 4x + 1$ **8.** $P(x) = -2x^3 + 6x^2 - 2$

9. $P(x) = 3x^4 - 4x^3 - 10x - 1$

10. $P(x) = x^5 + x^4 - 7x^3 - x^2 + 6x + 3$

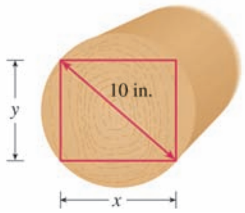
11. The strength S of a wooden beam of width x and depth y is given by the formula $S = 13.8xy^2$. A beam is to be cut from a log of diameter 10 in., as shown in the figure.

(a) Express the strength S of this beam as a function of x only.

(b) What is the domain of the function S?

(c) Draw a graph of S.

(d) What width will make the beam the strongest?

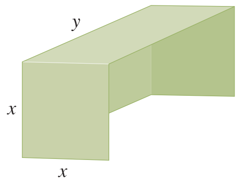


12. A small shelter for delicate plants is to be constructed of thin plastic material. It will have square ends and a rectangular top and back, with an open bottom and front, as shown in the figure. The total area of the four plastic sides is to be 1200 in^2.

(a) Express the volume V of the shelter as a function of the depth x.

(b) Draw a graph of V.

(c) What dimensions will maximize the volume of the shelter?



13–20 ■ Find the quotient and remainder.

13. $\dfrac{x^2 - 3x + 5}{x - 2}$ **14.** $\dfrac{x^2 + x - 12}{x - 3}$

15. $\dfrac{x^3 - x^2 + 11x + 2}{x - 4}$ **16.** $\dfrac{x^3 + 2x^2 - 10}{x + 3}$

17. $\dfrac{x^4 - 8x^2 + 2x + 7}{x + 5}$ **18.** $\dfrac{2x^4 + 3x^3 - 12}{x + 4}$

19. $\dfrac{2x^3 + x^2 - 8x + 15}{x^2 + 2x - 1}$ **20.** $\dfrac{x^4 - 2x^2 + 7x}{x^2 - x + 3}$

21–22 ■ Find the indicated value of the polynomial using the Remainder Theorem.

21. $P(x) = 2x^3 - 9x^2 - 7x + 13$; find $P(5)$

22. $Q(x) = x^4 + 4x^3 + 7x^2 + 10x + 15$; find $Q(-3)$

23. Show that $\frac{1}{2}$ is a zero of the polynomial

$$P(x) = 2x^4 + x^3 - 5x^2 + 10x - 4$$

24. Use the Factor Theorem to show that $x + 4$ is a factor of the polynomial

$$P(x) = x^5 + 4x^4 - 7x^3 - 23x^2 + 23x + 12$$

25. What is the remainder when the polynomial

$$P(x) = x^{500} + 6x^{201} - x^2 - 2x + 4$$

is divided by $x - 1$?

26. What is the remainder when $x^{101} - x^4 + 2$ is divided by $x + 1$?

27–28 ■ A polynomial P is given.

(a) List all possible rational zeros (without testing to see if they actually are zeros).

(b) Determine the possible number of positive and negative real zeros using Descartes' Rule of Signs.

27. $P(x) = x^5 - 6x^3 - x^2 + 2x + 18$

28. $P(x) = 6x^4 + 3x^3 + x^2 + 3x + 4$

29–36 ■ A polynomial P is given.

(a) Find all real zeros of P and state their multiplicities.

(b) Sketch the graph of P.

29. $P(x) = x^3 - 16x$ **30.** $P(x) = x^3 - 3x^2 - 4x$

31. $P(x) = x^4 + x^3 - 2x^2$ **32.** $P(x) = x^4 - 5x^2 + 4$

33. $P(x) = x^4 - 2x^3 - 7x^2 + 8x + 12$

34. $P(x) = x^4 - 2x^3 - 2x^2 + 8x - 8$

35. $P(x) = 2x^4 + x^3 + 2x^2 - 3x - 2$

36. $P(x) = 9x^5 - 21x^4 + 10x^3 + 6x^2 - 3x - 1$

37–46 ■ Evaluate the expression and write in the form $a + bi$.

37. $(2 - 3i) + (1 + 4i)$

38. $(3 - 6i) - (6 - 4i)$

39. $(2 + i)(3 - 2i)$

40. $4i(2 - \frac{1}{2}i)$

41. $\dfrac{4 + 2i}{2 - i}$

42. $\dfrac{8 + 3i}{4 + 3i}$

43. i^{25}

44. $(1 + i)^3$

45. $(1 - \sqrt{-1})(1 + \sqrt{-1})$

46. $\sqrt{-10} \cdot \sqrt{-40}$

47. Find a polynomial of degree 3 with constant coefficient 12 and zeros $-\frac{1}{2}$, 2, and 3.

48. Find a polynomial of degree 4 having integer coefficients and zeros $3i$ and 4, with 4 a double zero.

49. Does there exist a polynomial of degree 4 with integer coefficients that has zeros i, $2i$, $3i$, and $4i$? If so, find it. If not, explain why.

50. Prove that the equation $3x^4 + 5x^2 + 2 = 0$ has no real root.

51–60 ■ Find all rational, irrational, and complex zeros (and state their multiplicities). Use Descartes' Rule of Signs, the Upper and Lower Bounds Theorem, the quadratic formula, or other factoring techniques to help you whenever possible.

51. $P(x) = x^3 - 3x^2 - 13x + 15$

52. $P(x) = 2x^3 + 5x^2 - 6x - 9$

53. $P(x) = x^4 + 6x^3 + 17x^2 + 28x + 20$

54. $P(x) = x^4 + 7x^3 + 9x^2 - 17x - 20$

55. $P(x) = x^5 - 3x^4 - x^3 + 11x^2 - 12x + 4$

56. $P(x) = x^4 - 81$

57. $P(x) = x^6 - 64$

58. $P(x) = 18x^3 + 3x^2 - 4x - 1$

59. $P(x) = 6x^4 - 18x^3 + 6x^2 - 30x + 36$

60. $P(x) = x^4 + 15x^2 + 54$

61–64 ■ Use a graphing device to find all real solutions of the equation.

61. $2x^2 = 5x + 3$

62. $x^3 + x^2 - 14x - 24 = 0$

63. $x^4 - 3x^3 - 3x^2 - 9x - 2 = 0$

64. $x^5 = x + 3$

65–70 ■ Graph the rational function. Show clearly all x- and y-intercepts and asymptotes.

65. $r(x) = \dfrac{3x - 12}{x + 1}$

66. $r(x) = \dfrac{1}{(x + 2)^2}$

67. $r(x) = \dfrac{x - 2}{x^2 - 2x - 8}$

68. $r(x) = \dfrac{2x^2 - 6x - 7}{x - 4}$

69. $r(x) = \dfrac{x^2 - 9}{2x^2 + 1}$

70. $r(x) = \dfrac{x^3 + 27}{x + 4}$

71–74 ■ Use a graphing device to analyze the graph of the rational function. Find all x- and y-intercepts; and all vertical, horizontal, and slant asymptotes. If the function has no horizontal or slant asymptote, find a polynomial that has the same end behavior as the rational function.

71. $r(x) = \dfrac{x - 3}{2x + 6}$

72. $r(x) = \dfrac{2x - 7}{x^2 + 9}$

73. $r(x) = \dfrac{x^3 + 8}{x^2 - x - 2}$

74. $r(x) = \dfrac{2x^3 - x^2}{x + 1}$

75. Find the coordinates of all points of intersection of the graphs of

$$y = x^4 + x^2 + 24x \qquad \text{and} \qquad y = 6x^3 + 20$$

3 Test

1. Graph the polynomial $P(x) = -(x + 2)^3 + 27$, showing clearly all x- and y-intercepts.

2. (a) Use synthetic division to find the quotient and remainder when $x^4 - 4x^2 + 2x + 5$ is divided by $x - 2$.

(b) Use long division to find the quotient and remainder when $2x^5 + 4x^4 - x^3 - x^2 + 7$ is divided by $2x^2 - 1$.

3. Let $P(x) = 2x^3 - 5x^2 - 4x + 3$.

(a) List all possible rational zeros of P.

(b) Find the complete factorization of P.

(c) Find the zeros of P.

(d) Sketch the graph of P.

4. Perform the indicated operation and write the result in the form $a + bi$.

(a) $(3 - 2i) + (4 + 3i)$

(b) $(3 - 2i) - (4 + 3i)$

(c) $(3 - 2i)(4 + 3i)$

(d) $\dfrac{3 - 2i}{4 + 3i}$

(e) i^{48}

(f) $(\sqrt{2} - \sqrt{-2})(\sqrt{8} + \sqrt{-2})$

5. Find all real and complex zeros of $P(x) = x^3 - x^2 - 4x - 6$.

6. Find the complete factorization of $P(x) = x^4 - 2x^3 + 5x^2 - 8x + 4$.

7. Find a fourth-degree polynomial with integer coefficients that has zeros $3i$ and -1, with -1 a zero of multiplicity 2.

8. Let $P(x) = 2x^4 - 7x^3 + x^2 - 18x + 3$.

(a) Use Descartes' Rule of Signs to determine how many positive and how many negative real zeros P can have.

(b) Show that 4 is an upper bound and -1 is a lower bound for the real zeros of P.

(c) Draw a graph of P and use it to estimate the real zeros of P, correct to two decimal places.

(d) Find the coordinates of all local extrema of P, correct to two decimals.

9. Consider the following rational functions:

$$r(x) = \frac{2x - 1}{x^2 - x - 2} \qquad s(x) = \frac{x^3 + 27}{x^2 + 4} \qquad t(x) = \frac{x^3 - 9x}{x + 2} \qquad u(x) = \frac{x^2 + x - 6}{x^2 - 25}$$

(a) Which of these rational functions has a horizontal asymptote?

(b) Which of these functions has a slant asymptote?

(c) Which of these functions has no vertical asymptote?

(d) Graph $y = u(x)$, showing clearly any asymptotes and x- and y-intercepts the function may have.

(e) Use long division to find a polynomial P that has the same end behavior as t. Graph both P and t on the same screen to verify that they have the same end behavior.

Focus on Modeling
Fitting Polynomial Curves to Data

We have learned how to fit a line to data (see *Focus on Modeling*, page 239). The line models the increasing or decreasing trend in the data. If the data exhibits more variability, such as an increase followed by a decrease, then to model the data we need to use a curve rather than a line. Figure 1 shows a scatter plot with three possible models that appear to fit the data. Which model fits the data best?

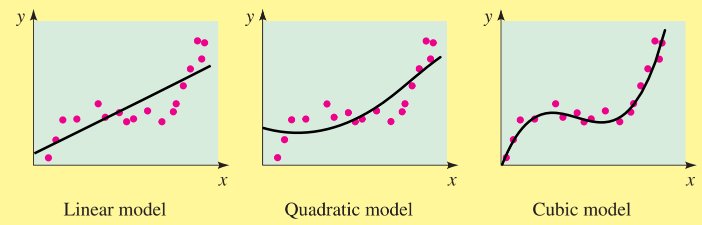


Figure 1

Polynomial Functions as Models

Polynomial functions are ideal for modeling data where the scatter plot has peaks or valleys (that is, local maxima or minima). For example, if the data have a single peak as in Figure 2(a), then it may be appropriate to use a quadratic polynomial to model the data. The more peaks or valleys the data exhibit, the higher the degree of the polynomial needed to model the data (see Figure 2).

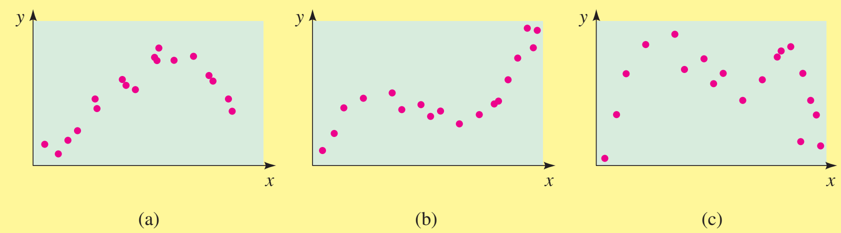


Figure 2

Graphing calculators are programmed to find the **polynomial of best fit** of a specified degree. As is the case for lines (see pages 239–240), a polynomial of a given degree fits the data *best* if the sum of the squares of the distances between the graph of the polynomial and the data points is minimized.

Ted Wood/The Image Bank/Getty Images

Example 1 **Rainfall and Crop Yield**

Rain is essential for crops to grow, but too much rain can diminish crop yields. The data give rainfall and cotton yield per acre for several seasons in a certain county.

(a) Make a scatter plot of the data. What degree polynomial seems appropriate for modeling the data?

(b) Use a graphing calculator to find the polynomial of best fit. Graph the polynomial on the scatter plot.

(c) Use the model you found to estimate the yield if there are 25 in. of rainfall.

Season	Rainfall (in.)	Yield (kg/acre)
1	23.3	5311
2	20.1	4382
3	18.1	3950
4	12.5	3137
5	30.9	5113
6	33.6	4814
7	35.8	3540
8	15.5	3850
9	27.6	5071
10	34.5	3881

Solution

(a) The scatter plot is shown in Figure 3. The data appear to have a peak, so it is appropriate to model the data by a quadratic polynomial (degree 2).

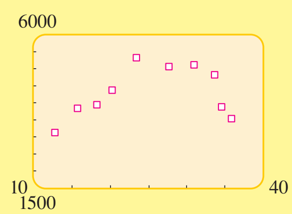
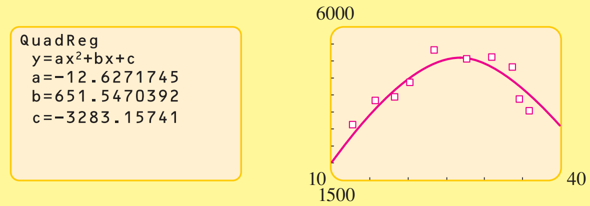


Figure 3
Scatter plot of yield vs. rainfall data

(b) Using a graphing calculator, we find that the quadratic polynomial of best fit is

$$y = -12.6x^2 + 651.5x - 3283.2$$

The calculator output and the scatter plot, together with the graph of the quadratic model, are shown in Figure 4.



(a) (b)

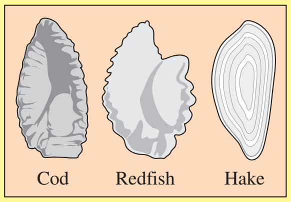
Figure 4

(c) Using the model with $x = 25$, we get

$$y = -12.6(25)^2 + 651.5(25) - 3283.2 \approx 5129.3$$

We estimate the yield to be about 5130 kg per acre. ■

Example 2 Length-at-Age Data for Fish



Otoliths for several fish species.

Otoliths (“earstones”) are tiny structures found in the heads of fish. Microscopic growth rings on the otoliths, not unlike growth rings on a tree, record the age of a fish. The table gives the lengths of rock bass of different ages, as determined by the otoliths. Scientists have proposed a cubic polynomial to model this data.

(a) Use a graphing calculator to find the cubic polynomial of best fit for the data.

(b) Make a scatter plot of the data and graph the polynomial from part (a).

(c) A fisherman catches a rock bass 20 in. long. Use the model to estimate its age.

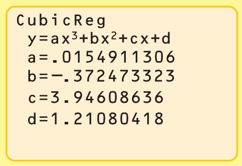
Age (yr)	Length (in.)	Age (yr)	Length (in.)
1	4.8	9	18.2
2	8.8	9	17.1
2	8.0	10	18.8
3	7.9	10	19.5
4	11.9	11	18.9
5	14.4	12	21.7
6	14.1	12	21.9
6	15.8	13	23.8
7	15.6	14	26.9
8	17.8	14	25.1

Solution

(a) Using a graphing calculator (see Figure 5(a)), we find the cubic polynomial of best fit

$$y = 0.0155x^3 - 0.372x^2 + 3.95x + 1.21$$

(b) The scatter plot of the data and the cubic polynomial are graphed in Figure 5(b).



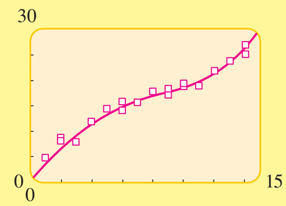


Figure 5 (a) (b)

(c) Moving the cursor along the graph of the polynomial, we find that $y = 20$ when $x \approx 10.8$. Thus, the fish is about 11 years old. ■

Problems

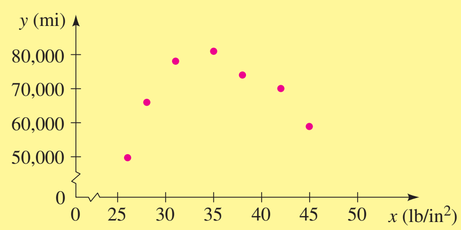
1. Tire Inflation and Treadware Car tires need to be inflated properly. Overinflation or underinflation can cause premature treadwear. The data and scatter plot show tire life for different inflation values for a certain type of tire.

(a) Find the quadratic polynomial that best fits the data.

(b) Draw a graph of the polynomial from part (a) together with a scatter plot of the data.

(c) Use your result from part (b) to estimate the pressure that gives the longest tire life.

Pressure (lb/in^2)	Tire life (mi)
26	50,000
28	66,000
31	78,000
35	81,000
38	74,000
42	70,000
45	59,000



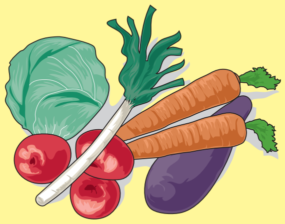
2. Too Many Corn Plants per Acre? The more corn a farmer plants per acre the greater the yield that he can expect, but only up to a point. Too many plants per acre can cause overcrowding and decrease yields. The data give crop yields per acre for various densities of corn plantings, as found by researchers at a university test farm.

(a) Find the quadratic polynomial that best fits the data.

(b) Draw a graph of the polynomial from part (a) together with a scatter plot of the data.

(c) Use your result from part (b) to estimate the yield for 37,000 plants per acre.

Density (plants/acre)	Crop yield (bushels/acre)
15,000	43
20,000	98
25,000	118
30,000	140
35,000	142
40,000	122
45,000	93
50,000	67



3. How Fast Can You List Your Favorite Things? If you are asked to make a list of objects in a certain category, how fast you can list them follows a predictable pattern. For example, if you try to name as many vegetables as you can, you'll probably think of several right away—for example, carrots, peas, beans, corn, and so on. Then after a pause you may think of ones you eat less frequently—perhaps zucchini, eggplant, and asparagus. Finally a few more exotic vegetables might come to mind—artichokes, jicama, bok choy, and the like. A psychologist performs this experiment on a number of subjects. The table below gives the average number of vegetables that the subjects named by a given number of seconds.

(a) Find the cubic polynomial that best fits the data.

(b) Draw a graph of the polynomial from part (a) together with a scatter plot of the data.

(c) Use your result from part (b) to estimate the number of vegetables that subjects would be able to name in 40 seconds.

(d) According to the model, how long (to the nearest 0.1 s) would it take a person to name five vegetables?

Seconds	Number of Vegetables
1	2
2	6
5	10
10	12
15	14
20	15
25	18
30	21

4. Clothing Sales Are Seasonal Clothing sales tend to vary by season with more clothes sold in spring and fall. The table gives sales figures for each month at a certain clothing store.

(a) Find the quartic (fourth-degree) polynomial that best fits the data.

(b) Draw a graph of the polynomial from part (a) together with a scatter plot of the data.

(c) Do you think that a quartic polynomial is a good model for these data? Explain.

Month	Sales ($)
January	8,000
February	18,000
March	22,000
April	31,000
May	29,000
June	21,000
July	22,000
August	26,000
September	38,000
October	40,000
November	27,000
December	15,000

5. Height of a Baseball A baseball is thrown upward and its height measured at 0.5-second intervals using a strobe light. The resulting data are given in the table.

(a) Draw a scatter plot of the data. What degree polynomial is appropriate for modeling the data?

(b) Find a polynomial model that best fits the data, and graph it on the scatter plot.

(c) Find the times when the ball is 20 ft above the ground.

(d) What is the maximum height attained by the ball?

Time (s)	Height (ft)
0	4.2
0.5	26.1
1.0	40.1
1.5	46.0
2.0	43.9
2.5	33.7
3.0	15.8

6. Torricelli's Law Water in a tank will flow out of a small hole in the bottom faster when the tank is nearly full than when it is nearly empty. According to Torricelli's Law, the height $h(t)$ of water remaining at time t is a quadratic function of t.

A certain tank is filled with water and allowed to drain. The height of the water is measured at different times as shown in the table.

(a) Find the quadratic polynomial that best fits the data.

(b) Draw a graph of the polynomial from part (a) together with a scatter plot of the data.

(c) Use your graph from part (b) to estimate how long it takes for the tank to drain completely.

Time (min)	Height (ft)
0	5.0
4	3.1
8	1.9
12	0.8
16	0.2