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1 Fundamentals

Chapter Overview

In this first chapter we review the real numbers, equations, and the coordinate plane. You are probably already familiar with these concepts, but it is helpful to get a fresh look at how these ideas work together to solve problems and model (or describe) real-world situations.

Let's see how all these ideas are used in the following real-life situation: Suppose you get paid \$8 an hour at your part-time job. We are interested in how much money you make.

To describe your pay we use *real numbers*. In fact, we use real numbers every day—to describe how tall we are, how much money we have, how cold (or warm) it is, and so on. In algebra, we express properties of the real numbers by using letters to stand for numbers. An important property is the distributive property:

$$A(B + C) = AB + AC$$

To see that this property makes sense, let's consider your pay if you work 6 hours one day and 5 hours the next. Your pay for those two days can be calculated in two different ways: $\$8(6 + 5)$ or $\$8 \cdot 6 + \$8 \cdot 5$, and both methods give the same answer. This and other properties of the real numbers constitute the rules for working with numbers, or the rules of algebra.

We can also model your pay for any number of hours by a formula. If you work x hours then your pay is y dollars, where y is given by the algebraic formula

$$y = 8x$$

So if you work 10 hours, your pay is $y = 8 \cdot 10 = 80$ dollars.

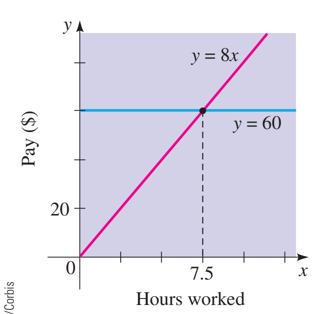
An *equation* is a sentence written in the language of algebra that expresses a fact about an unknown quantity x. For example, how many hours would you need to work to get paid 60 dollars? To answer this question we need to solve the equation

$$60 = 8x$$

We use the rules of algebra to find x. In this case we divide both sides of the equation by 8, so $x = \frac{60}{8} = 7.5$ hours.

The *coordinate plane* allows us to sketch a graph of an equation in two variables. For example, by graphing the equation $y = 8x$ we can "see" how pay increases with hours worked. We can also solve the equation $60 = 8x$ graphically by finding the value of x at which the graphs of $y = 8x$ and $y = 60$ intersect (see the figure).

In this chapter we will see many examples of how the real numbers, equations, and the coordinate plane all work together to help us solve real-life problems.



Bob Krist/Corbis

SUGGESTED TIME AND EMPHASIS

$\frac{1}{2}$–2 classes.
Essential material.

POINTS TO STRESS

1. The various subsets of the real number line.
2. The algebraic properties of real numbers.
3. Closed and open intervals of real numbers, and their unions and intersections.
4. Absolute value and distance.

1.1 Real Numbers

Let's review the types of numbers that make up the real number system. We start with the **natural numbers**:

$$1, 2, 3, 4, \ldots$$

The different types of real numbers were invented to meet specific needs. For example, natural numbers are needed for counting, negative numbers for describing debt or below-zero temperatures, rational numbers for concepts like "half a gallon of milk," and irrational numbers for measuring certain distances, like the diagonal of a square.

The **integers** consist of the natural numbers together with their negatives and 0:

$$\ldots, -3, -2, -1, 0, 1, 2, 3, 4, \ldots$$

We construct the **rational numbers** by taking ratios of integers. Thus, any rational number r can be expressed as

$$r = \frac{m}{n}$$

where m and n are integers and $n \neq 0$. Examples are:

$$\tfrac{1}{2} \qquad -\tfrac{3}{7} \qquad 46 = \tfrac{46}{1} \qquad 0.17 = \tfrac{17}{100}$$

(Recall that division by 0 is always ruled out, so expressions like $\frac{3}{0}$ and $\frac{0}{0}$ are undefined.) There are also real numbers, such as $\sqrt{2}$, that cannot be expressed as a ratio of integers and are therefore called **irrational numbers**. It can be shown, with varying degrees of difficulty, that these numbers are also irrational:

$$\sqrt{3} \qquad \sqrt{5} \qquad \sqrt[3]{2} \qquad \pi \qquad \frac{3}{\pi^2}$$

The set of all real numbers is usually denoted by the symbol $\mathbb{R}$. When we use the word *number* without qualification, we will mean "real number." Figure 1 is a diagram of the types of real numbers that we work with in this book.

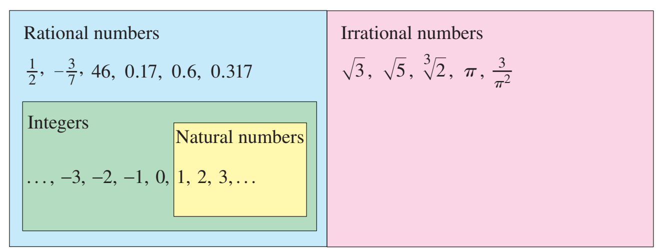


Figure 1
The real number system

A repeating decimal such as

$$x = 3.5474747\ldots$$

is a rational number. To convert it to a ratio of two integers, we write

$$\begin{aligned} 1000x &= 3547.47474747\ldots \\ 10x &= 35.47474747\ldots \\ \hline 990x &= 3512.0 \end{aligned}$$

Thus, $x = \frac{3512}{990}$. (The idea is to multiply x by appropriate powers of 10, and then subtract to eliminate the repeating part.)

Every real number has a decimal representation. If the number is rational, then its corresponding decimal is repeating. For example,

$$\tfrac{1}{2} = 0.5000\ldots = 0.5\overline{0} \qquad \tfrac{2}{3} = 0.66666\ldots = 0.\overline{6}$$

$$\tfrac{157}{495} = 0.3171717\ldots = 0.3\overline{17} \qquad \tfrac{9}{7} = 1.285714285714\ldots = 1.\overline{285714}$$

(The bar indicates that the sequence of digits repeats forever.) If the number is irrational, the decimal representation is nonrepeating:

$$\sqrt{2} = 1.414213562373095\ldots \qquad \pi = 3.141592653589793\ldots$$

SAMPLE QUESTIONS

Text Questions

Consider the numbers below from the text. Fill in each gray box with a label for the following list: integers, irrational numbers, natural numbers, and rational numbers.

1. $\frac{1}{2}, -\frac{3}{7}, 46, 0.17, 0.\overline{6}, 0.3\overline{17}, \ldots$
2. $\ldots, -3, -2, -1, 0,$
3. $1, 2, 3, \ldots$
4. $\sqrt{3}, \sqrt{5}, \sqrt[3]{2}, \pi, 3/\pi^2, \ldots$

Answers

1. Rational numbers
2. Integers
3. Natural numbers
4. Irrational numbers

If we stop the decimal expansion of any number at a certain place, we get an approximation to the number. For instance, we can write

$$\pi \approx 3.14159265$$

where the symbol $\approx$ is read "is approximately equal to." The more decimal places we retain, the better our approximation.

Properties of Real Numbers

We all know that $2 + 3 = 3 + 2$ and $5 + 7 = 7 + 5$ and $513 + 87 = 87 + 513$, and so on. In algebra, we express all these (infinitely many) facts by writing

$$a + b = b + a$$

where a and b stand for any two numbers. In other words, "$a + b = b + a$" is a concise way of saying that "when we add two numbers, the order of addition doesn't matter." This fact is called the *Commutative Property* for addition. From our experience with numbers we know that the properties in the following box are also valid.

Properties of Real Numbers

Property	Example	Description
Commutative Properties		
$a + b = b + a$	$7 + 3 = 3 + 7$	When we add two numbers, order doesn't matter.
$ab = ba$	$3 \cdot 5 = 5 \cdot 3$	When we multiply two numbers, order doesn't matter.
Associative Properties		
$(a + b) + c = a + (b + c)$	$(2 + 4) + 7 = 2 + (4 + 7)$	When we add three numbers, it doesn't matter which two we add first.
$(ab)c = a(bc)$	$(3 \cdot 7) \cdot 5 = 3 \cdot (7 \cdot 5)$	When we multiply three numbers, it doesn't matter which two we multiply first.
Distributive Property		
$a(b + c) = ab + ac$	$2 \cdot (3 + 5) = 2 \cdot 3 + 2 \cdot 5$	When we multiply a number by a sum of two numbers, we get the same result as multiplying the number by each of the terms and then adding the results.
$(b + c)a = ab + ac$	$(3 + 5) \cdot 2 = 2 \cdot 3 + 2 \cdot 5$	

The Distributive Property applies whenever we multiply a number by a sum. Figure 2 explains why this property works for the case in which all the numbers are positive integers, but the property is true for any real numbers a, b, and c.

The Distributive Property is crucial because it describes the way addition and multiplication interact with each other.

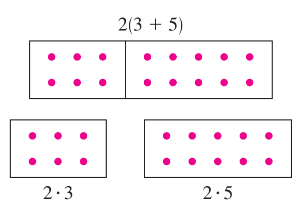


Figure 2
The Distributive Property

IN-CLASS MATERIALS

Point out that the set hierarchy in Figure 1 in the text isn't as simple as it may appear. For example, two irrational numbers can be added together to make a rational number ($a = 2 + \sqrt{3}$, $b = 3 - \sqrt{3}$) but two rational numbers, added together, are always rational (see Exercise 77). Similarly, mathematicians have long known that the numbers π and e ($\pi \approx 3.14159265\ldots$, $e \approx 2.718281828\ldots$) are irrational numbers, but it is unknown whether $\pi + e$ is rational or irrational. As people go further in mathematics, they break the real numbers down into other types of sets as transcendentals, algebraics, normals, and computables, to name a few.

ALTERNATE EXAMPLE 1b
Expand the expression: $(a + b)(c + d)$, using the properties of the real numbers.

ANSWER
$a \cdot c + b \cdot c + a \cdot d + b \cdot d$

Example 1 Using the Distributive Property

(a) $2(x + 3) = 2 \cdot x + 2 \cdot 3$ Distributive Property

$= 2x + 6$ Simplify

(b) $(a + b)(x + y) = (a + b)x + (a + b)y$ Distributive Property

$= (ax + bx) + (ay + by)$ Distributive Property

$= ax + bx + ay + by$ Associative Property of Addition

In the last step we removed the parentheses because, according to the Associative Property, the order of addition doesn't matter. ■

Don't assume that $-a$ is a negative number. Whether $-a$ is negative or positive depends on the value of a. For example, if $a = 5$, then $-a = -5$, a negative number, but if $a = -5$, then $-a = -(-5) = 5$ (Property 2), a positive number.

The number 0 is special for addition; it is called the **additive identity** because $a + 0 = a$ for any real number a. Every real number a has a **negative**, $-a$, that satisfies $a + (-a) = 0$. **Subtraction** is the operation that undoes addition; to subtract a number from another, we simply add the negative of that number. By definition

$$a - b = a + (-b)$$

To combine real numbers involving negatives, we use the following properties.

Properties of Negatives

Property	Example
1. $(-1)a = -a$	$(-1)5 = -5$
2. $-(-a) = a$	$-(-5) = 5$
3. $(-a)b = a(-b) = -(ab)$	$(-5)7 = 5(-7) = -(5 \cdot 7)$
4. $(-a)(-b) = ab$	$(-4)(-3) = 4 \cdot 3$
5. $-(a + b) = -a - b$	$-(3 + 5) = -3 - 5$
6. $-(a - b) = b - a$	$-(5 - 8) = 8 - 5$

Property 6 states the intuitive fact that $a - b$ and $b - a$ are negatives of each other. Property 5 is often used with more than two terms:

$$-(a + b + c) = -a - b - c$$

ALTERNATE EXAMPLE 2
Using the properties of negatives, simplify the expression $-(k + l - m)$.

ANSWER
$-k - l + m$

Example 2 Using Properties of Negatives

Let x, y, and z be real numbers.

(a) $-(x + 2) = -x - 2$ Property 5: $-(a + b) = -a - b$

(b) $-(x + y - z) = -x - y - (-z)$ Property 5: $-(a + b) = -a - b$

$= -x - y + z$ Property 2: $-(-a) = a$ ■

IN-CLASS MATERIALS

Students probably already know how to multiply binomial expressions together, some of them already using the acronym FOIL to avoid thinking about the process altogether. Use the properties in this section to demonstrate why FOIL works.

$(a + b)(c + d) = a(c + d) + b(c + d)$ Distributive Property

$= ac + ad + b(c + d)$ Distributive Property on the first term

$= ac + ad + bc + bd$ Distributive Property on the second term

The number 1 is special for multiplication; it is called the **multiplicative identity** because $a \cdot 1 = a$ for any real number a. Every nonzero real number a has an **inverse**, $1/a$, that satisfies $a \cdot (1/a) = 1$. **Division** is the operation that undoes multiplication; to divide by a number, we multiply by the inverse of that number. If $b \neq 0$, then, by definition,

$$a \div b = a \cdot \frac{1}{b}$$

We write $a \cdot (1/b)$ as simply a/b. We refer to a/b as the **quotient** of a and b or as the **fraction** a over b; a is the **numerator** and b is the **denominator** (or **divisor**). To combine real numbers using the operation of division, we use the following properties.

Properties of Fractions

Property	Example	Description
1. $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$	$\frac{2}{3} \cdot \frac{5}{7} = \frac{2 \cdot 5}{3 \cdot 7} = \frac{10}{21}$	When **multiplying fractions**, multiply numerators and denominators.
2. $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$	$\frac{2}{3} \div \frac{5}{7} = \frac{2}{3} \cdot \frac{7}{5} = \frac{14}{15}$	When **dividing fractions**, invert the divisor and multiply.
3. $\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$	$\frac{2}{5} + \frac{7}{5} = \frac{2+7}{5} = \frac{9}{5}$	When **adding fractions** with the **same denominator**, add the numerators.
4. $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$	$\frac{2}{5} + \frac{3}{7} = \frac{2 \cdot 7 + 3 \cdot 5}{35} = \frac{29}{35}$	When **adding fractions** with **different denominators**, find a common denominator. Then add the numerators.
5. $\frac{ac}{bc} = \frac{a}{b}$	$\frac{2 \cdot 5}{3 \cdot 5} = \frac{2}{3}$	**Cancel** numbers that are **common factors** in numerator and denominator.
6. If $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$	$\frac{2}{3} = \frac{6}{9}$, so $2 \cdot 9 = 3 \cdot 6$	**Cross multiply**.

When adding fractions with different denominators, we don't usually use Property 4. Instead we rewrite the fractions so that they have the smallest possible common denominator (often smaller than the product of the denominators), and then we use Property 3. This denominator is the **L**east **C**ommon **D**enominator (LCD) described in the next example.

Example 3 Using the LCD to Add Fractions

Evaluate: $\frac{5}{36} + \frac{7}{120}$

Solution Factoring each denominator into prime factors gives

$$36 = 2^2 \cdot 3^2 \quad \text{and} \quad 120 = 2^3 \cdot 3 \cdot 5$$

We find the least common denominator (LCD) by forming the product of all the factors that occur in these factorizations, using the highest power of each factor.

ALTERNATE EXAMPLE 3

Evaluate: $\frac{11}{441} + \frac{41}{2079}$

ANSWER

$\frac{650}{14553}$

IN-CLASS MATERIALS

Warn students that certain things cannot be rewritten. For example, an expression like $\frac{u}{u+1}$ cannot be simplified, while $\frac{u+1}{u}$ can be expressed as $1 + \frac{1}{u}$. Similarly, there is no way to simplify $a + bc$. Another common pitfall can be pointed out by asking the question "Is $-a$ a positive or negative number?"

Thus, the LCD is $2^3 \cdot 3^2 \cdot 5 = 360$. So

$$\frac{5}{36} + \frac{7}{120} = \frac{5 \cdot 10}{36 \cdot 10} + \frac{7 \cdot 3}{120 \cdot 3} \qquad \text{Use common denominator}$$

$$= \frac{50}{360} + \frac{21}{360} = \frac{71}{360} \qquad \text{Property 3: Adding fractions with the same denominator}$$

The Real Line

The real numbers can be represented by points on a line, as shown in Figure 3. The positive direction (toward the right) is indicated by an arrow. We choose an arbitrary reference point O, called the **origin**, which corresponds to the real number 0. Given any convenient unit of measurement, each positive number x is represented by the point on the line a distance of x units to the right of the origin, and each negative number $-x$ is represented by the point x units to the left of the origin. The number associated with the point P is called the coordinate of P, and the line is then called a **coordinate line**, or a **real number line**, or simply a **real line**. Often we identify the point with its coordinate and think of a number as being a point on the real line.

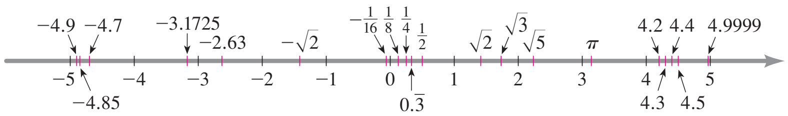


Figure 3 The real line

The real numbers are *ordered*. We say that ***a* is less than *b*** and write $a < b$ if $b - a$ is a positive number. Geometrically, this means that a lies to the left of b on the number line. Equivalently, we can say that ***b* is greater than *a*** and write $b > a$. The symbol $a \le b$ (or $b \ge a$) means that either $a < b$ or $a = b$ and is read "a is less than or equal to b." For instance, the following are true inequalities (see Figure 4):

$$7 < 7.4 < 7.5 \qquad -\pi < -3 \qquad \sqrt{2} < 2 \qquad 2 \le 2$$

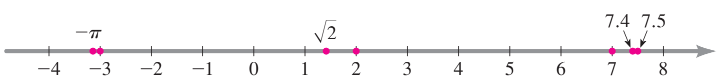


Figure 4

Sets and Intervals

A **set** is a collection of objects, and these objects are called the **elements** of the set. If S is a set, the notation $a \in S$ means that a is an element of S, and $b \notin S$ means that b is not an element of S. For example, if Z represents the set of integers, then $-3 \in Z$ but $\pi \notin Z$.

Some sets can be described by listing their elements within braces. For instance, the set A that consists of all positive integers less than 7 can be written as

$$A = \{1, 2, 3, 4, 5, 6\}$$

IN-CLASS MATERIALS

Ask the students why we never see intervals of the form $[3, \infty]$ or $(52, 3)$. Explain to them why these intervals are not well-formed.

We could also write A in **set-builder notation** as

$$A = \{x \mid x \text{ is an integer and } 0 < x < 7\}$$

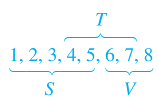
which is read "A is the set of all x such that x is an integer and $0 < x < 7$."

If S and T are sets, then their **union** $S \cup T$ is the set that consists of all elements that are in S *or* T (or in both). The **intersection** of S and T is the set $S \cap T$ consisting of all elements that are in both S *and* T. In other words, $S \cap T$ is the common part of S and T. The **empty set**, denoted by $\varnothing$, is the set that contains no element.

Example 4 Union and Intersection of Sets

If $S = \{1, 2, 3, 4, 5\}$, $T = \{4, 5, 6, 7\}$, and $V = \{6, 7, 8\}$, find the sets $S \cup T$, $S \cap T$, and $S \cap V$.

Solution



$S \cup T = \{1, 2, 3, 4, 5, 6, 7\}$ — All elements in S or T

$S \cap T = \{4, 5\}$ — Elements common to both S and T

$S \cap V = \varnothing$ — S and V have no element in common ■

ALTERNATE EXAMPLE 4
If $S = \{1, 3, 5, 6, 8\}$, $T = \{6, 8, 9, 10\}$, and $V = \{9, 10, 11\}$, find the sets $S \cup T$, $S \cap T$, and $S \cap V$.

ANSWER
$\{1, 3, 5, 6, 8, 9, 10\}$, $\{6, 8\}$, $\varnothing$

Certain sets of real numbers, called **intervals**, occur frequently in calculus and correspond geometrically to line segments. If $a < b$, then the **open interval** from a to b consists of all numbers between a and b and is denoted (a, b). The **closed interval** from a to b includes the endpoints and is denoted $[a, b]$. Using set-builder notation, we can write

$$(a, b) = \{x \mid a < x < b\} \qquad [a, b] = \{x \mid a \le x \le b\}$$

Figure 5
The open interval (a, b)

Figure 6
The closed interval $[a, b]$

Note that parentheses ( ) in the interval notation and open circles on the graph in Figure 5 indicate that endpoints are *excluded* from the interval, whereas square brackets [ ] and solid circles in Figure 6 indicate that the endpoints are *included*. Intervals may also include one endpoint but not the other, or they may extend infinitely far in one direction or both. The following table lists the possible types of intervals.

Notation	Set description	Graph
(a, b)	$\{x \mid a < x < b\}$	
$[a, b]$	$\{x \mid a \le x \le b\}$	
$[a, b)$	$\{x \mid a \le x < b\}$	
$(a, b]$	$\{x \mid a < x \le b\}$	
(a, ∞)	$\{x \mid a < x\}$	
$[a, \infty)$	$\{x \mid a \le x\}$	
$(-\infty, b)$	$\{x \mid x < b\}$	
$(-\infty, b]$	$\{x \mid x \le b\}$	
$(-\infty, \infty)$	$\mathbb{R}$ (set of all real numbers)	

The symbol ∞ ("infinity") does not stand for a number. The notation (a, ∞), for instance, simply indicates that the interval has no endpoint on the right but extends infinitely far in the positive direction.

EXAMPLE
A nontrivial union of intervals: Let $S = \{x \mid x > 0, x \ne 1/n$, where n is a positive integer$\}$ (the set of all positive numbers, except those of the form $1/n$ where n is a positive integer). This set is an infinite union of open intervals: $(1, \infty) \cup \left(\frac{1}{2}, 1\right) \cup \left(\frac{1}{3}, \frac{1}{2}\right) \cup \cdots$. Notice that each individual interval in the union is well-behaved and easy to understand. Also notice that the intersection of S with any positive open interval such as $(0.1, 5)$ becomes a finite union.

ALTERNATE EXAMPLE 5a
Is the number 0 included in the interval $[-3, 6)$?

ANSWER
Yes

ALTERNATE EXAMPLE 6a
Find $S \cap T$, if $S = (2, 5)$ and $T = [4, 7]$.

ANSWER
$[4, 5)$

DRILL QUESTIONS

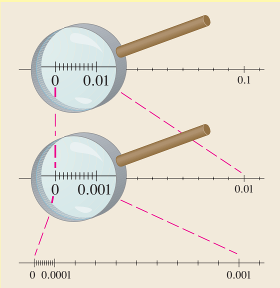
(a) Find $(2, 5] \cup [3, 8)$.
(b) Find $(2, 5] \cap [3, 8)$.

Answers

(a) $(2, 8)$
(b) $[3, 5]$

No Smallest or Largest Number in an Open Interval

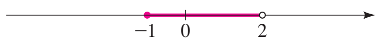
Any interval contains infinitely many numbers—every point on the graph of an interval corresponds to a real number. In the closed interval $[0, 1]$, the smallest number is 0 and the largest is 1, but the open interval $(0, 1)$ contains no smallest or largest number. To see this, note that 0.01 is close to zero, but 0.001 is closer, 0.0001 closer yet, and so on. So we can always find a number in the interval $(0, 1)$ closer to zero than any given number. Since 0 itself is not in the interval, the interval contains no smallest number. Similarly, 0.99 is close to 1, but 0.999 is closer, 0.9999 closer yet, and so on. Since 1 itself is not in the interval, the interval has no largest number.



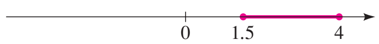
Example 5 Graphing Intervals

Express each interval in terms of inequalities, and then graph the interval.

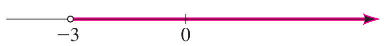
(a) $[-1, 2) = \{x \mid -1 \le x < 2\}$



(b) $[1.5, 4] = \{x \mid 1.5 \le x \le 4\}$



(c) $(-3, \infty) = \{x \mid -3 < x\}$



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Example 6 Finding Unions and Intersections of Intervals

Graph each set.

(a) $(1, 3) \cap [2, 7]$ (b) $(1, 3) \cup [2, 7]$

Solution

(a) The intersection of two intervals consists of the numbers that are in both intervals. Therefore

$$\begin{aligned}(1, 3) \cap [2, 7] &= \{x \mid 1 < x < 3 \text{ and } 2 \le x \le 7\} \\ &= \{x \mid 2 \le x < 3\} = [2, 3)\end{aligned}$$

This set is illustrated in Figure 7.

(b) The union of two intervals consists of the numbers that are in either one interval or the other (or both). Therefore

$$\begin{aligned}(1, 3) \cup [2, 7] &= \{x \mid 1 < x < 3 \text{ or } 2 \le x \le 7\} \\ &= \{x \mid 1 < x \le 7\} = (1, 7]\end{aligned}$$

This set is illustrated in Figure 8.

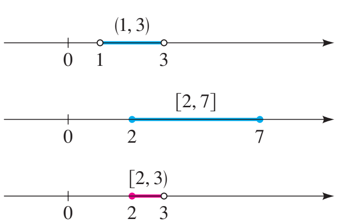


Figure 7
$(1, 3) \cap [2, 7] = [2, 3)$

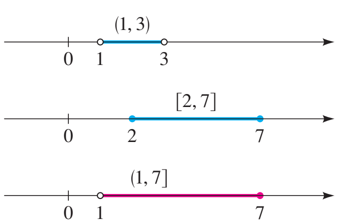


Figure 8
$(1, 3) \cup [2, 7] = (1, 7]$

■

Absolute Value and Distance

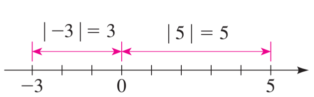


Figure 9

The **absolute value** of a number a, denoted by $|a|$, is the distance from a to 0 on the real number line (see Figure 9). Distance is always positive or zero, so we have $|a| \ge 0$ for every number a. Remembering that $-a$ is positive when a is negative, we have the following definition.

Definition of Absolute Value

If a is a real number, then the **absolute value** of a is

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

Example 7 Evaluating Absolute Values of Numbers

(a) $|3| = 3$

(b) $|-3| = -(-3) = 3$

(c) $|0| = 0$

(d) $|3 - \pi| = -(3 - \pi) = \pi - 3$ (since $3 < \pi \Rightarrow 3 - \pi < 0$) ■

When working with absolute values, we use the following properties.

Properties of Absolute Value

Property	Example	Description
1. $\lvert a \rvert \geq 0$	$\lvert -3 \rvert = 3 \geq 0$	The absolute value of a number is always positive or zero.
2. $\lvert a \rvert = \lvert -a \rvert$	$\lvert 5 \rvert = \lvert -5 \rvert$	A number and its negative have the same absolute value.
3. $\lvert ab \rvert = \lvert a \rvert \lvert b \rvert$	$\lvert -2 \cdot 5 \rvert = \lvert -2 \rvert \lvert 5 \rvert$	The absolute value of a product is the product of the absolute values.
4. $\left\lvert \frac{a}{b} \right\rvert = \frac{\lvert a \rvert}{\lvert b \rvert}$	$\left\lvert \frac{12}{-3} \right\rvert = \frac{\lvert 12 \rvert}{\lvert -3 \rvert}$	The absolute value of a quotient is the quotient of the absolute values.

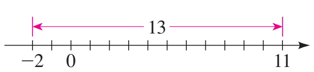


Figure 10

What is the distance on the real line between the numbers -2 and 11? From Figure 10 we see that the distance is 13. We arrive at this by finding either $|11 - (-2)| = 13$ or $|(-2) - 11| = 13$. From this observation we make the following definition (see Figure 11).

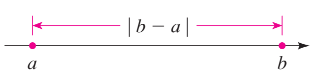


Figure 11
Length of a line segment $= |b - a|$

Distance between Points on the Real Line

If a and b are real numbers, then the **distance** between the points a and b on the real line is

$$d(a, b) = |b - a|$$

ALTERNATE EXAMPLE 7b
Evaluate the absolute value of the number $|-9|$.

ANSWER
9

ALTERNATE EXAMPLE 7d
Evaluate the absolute value of the number $|\sqrt{2} - 1|$.

ANSWER
$\sqrt{2} - 1$

IN-CLASS MATERIALS

Example 7(d) should be discussed: $|3 - \pi| = \pi - 3$. It touches on the ideas of distance, the absolute value of a negative number, and that we don't need to write out π to infinite precision to deduce that it is larger than 3.

ALTERNATE EXAMPLE 8
Find the distance between the numbers −8 and 5.

ANSWER
13

From Property 6 of negatives it follows that $|b - a| = |a - b|$. This confirms that, as we would expect, the distance from a to b is the same as the distance from b to a.

Example 8 Distance between Points on the Real Line

The distance between the numbers −8 and 2 is

$$d(a, b) = |-8 - 2| = |-10| = 10$$

We can check this calculation geometrically, as shown in Figure 12. ■

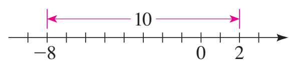


Figure 12

1.1 Exercises

1–2 ■ List the elements of the given set that are
(a) natural numbers
(b) integers
(c) rational numbers
(d) irrational numbers

1. $\{0, -10, 50, \frac{22}{7}, 0.538, \sqrt{7}, 1.2\overline{3}, -\frac{1}{3}, \sqrt[3]{2}\}$

2. $\{1.001, 0.333\ldots, -\pi, -11, 11, \frac{13}{15}, \sqrt{16}, 3.14, \frac{15}{3}\}$

3–10 ■ State the property of real numbers being used.

3. $7 + 10 = 10 + 7$

4. $2(3 + 5) = (3 + 5)2$

5. $(x + 2y) + 3z = x + (2y + 3z)$

6. $2(A + B) = 2A + 2B$

7. $(5x + 1)3 = 15x + 3$

8. $(x + a)(x + b) = (x + a)x + (x + a)b$

9. $2x(3 + y) = (3 + y)2x$

10. $7(a + b + c) = 7(a + b) + 7c$

11–14 ■ Rewrite the expression using the given property of real numbers.

11. Commutative Property of addition, $x + 3 = \square$

12. Associative Property of multiplication, $7(3x) = \square$

13. Distributive Property, $4(A + B) = \square$

14. Distributive Property, $5x + 5y = \square$

15–20 ■ Use properties of real numbers to write the expression without parentheses.

15. $3(x + y)$ **16.** $(a - b)8$

17. $4(2m)$ **18.** $\frac{4}{3}(-6y)$

19. $-\frac{5}{2}(2x - 4y)$ **20.** $(3a)(b + c - 2d)$

21–26 ■ Perform the indicated operations.

21. **(a)** $\frac{3}{10} + \frac{4}{15}$ **(b)** $\frac{1}{4} + \frac{1}{5}$

22. **(a)** $\frac{2}{3} - \frac{3}{5}$ **(b)** $1 + \frac{5}{8} - \frac{1}{6}$

23. **(a)** $\frac{2}{3}(6 - \frac{3}{2})$ **(b)** $0.25(\frac{8}{9} + \frac{1}{2})$

24. **(a)** $(3 + \frac{1}{4})(1 - \frac{4}{5})$ **(b)** $(\frac{1}{2} - \frac{1}{3})(\frac{1}{2} + \frac{1}{3})$

25. **(a)** $\dfrac{2}{\frac{2}{3}} - \dfrac{\frac{2}{3}}{2}$ **(b)** $\dfrac{\frac{1}{12}}{\frac{1}{8} - \frac{1}{9}}$

26. **(a)** $\dfrac{2 - \frac{3}{4}}{\frac{1}{2} - \frac{1}{3}}$ **(b)** $\dfrac{\frac{2}{5} + \frac{1}{2}}{\frac{1}{10} + \frac{3}{15}}$

27–28 ■ Place the correct symbol (<, >, or =) in the space.

27. **(a)** $3 \;\square\; \frac{7}{2}$ **(b)** $-3 \;\square\; -\frac{7}{2}$ **(c)** $3.5 \;\square\; \frac{7}{2}$

28. **(a)** $\frac{2}{3} \;\square\; 0.67$ **(b)** $\frac{2}{3} \;\square\; -0.67$ **(c)** $|0.67| \;\square\; |-0.67|$

29–32 ■ State whether each inequality is true or false.

29. **(a)** $-6 < -10$ **(b)** $\sqrt{2} > 1.41$

30. **(a)** $\dfrac{10}{11} < \dfrac{12}{13}$ **(b)** $-\dfrac{1}{2} < -1$

31. **(a)** $-\pi > -3$ **(b)** $8 \le 9$

32. **(a)** $1.1 > 1.\overline{1}$ **(b)** $8 \le 8$

33–34 ■ Write each statement in terms of inequalities.

33. **(a)** x is positive
(b) t is less than 4
(c) a is greater than or equal to π
(d) x is less than $\frac{1}{3}$ and is greater than −5
(e) The distance from p to 3 is at most 5

34. **(a)** y is negative
(b) z is greater than 1
(c) b is at most 8

(d) w is positive and is less than or equal to 17

(e) y is at least 2 units from π

35–38 ■ Find the indicated set if

$$A = \{1, 2, 3, 4, 5, 6, 7\} \qquad B = \{2, 4, 6, 8\}$$
$$C = \{7, 8, 9, 10\}$$

35. **(a)** $A \cup B$ **(b)** $A \cap B$

36. **(a)** $B \cup C$ **(b)** $B \cap C$

37. **(a)** $A \cup C$ **(b)** $A \cap C$

38. **(a)** $A \cup B \cup C$ **(b)** $A \cap B \cap C$

39–40 ■ Find the indicated set if

$$A = \{x \mid x \geq -2\} \qquad B = \{x \mid x < 4\}$$
$$C = \{x \mid -1 < x \leq 5\}$$

39. **(a)** $B \cup C$ **(b)** $B \cap C$

40. **(a)** $A \cap C$ **(b)** $A \cap B$

41–46 ■ Express the interval in terms of inequalities, and then graph the interval.

41. $(-3, 0)$ **42.** $(2, 8]$

43. $[2, 8)$ **44.** $[-6, -\frac{1}{2}]$

45. $[2, \infty)$ **46.** $(-\infty, 1)$

47–52 ■ Express the inequality in interval notation, and then graph the corresponding interval.

47. $x \leq 1$ **48.** $1 \leq x \leq 2$

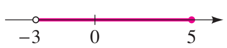
49. $-2 < x \leq 1$ **50.** $x \geq -5$

51. $x > -1$ **52.** $-5 < x < 2$

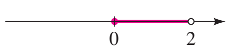
53–54 ■ Express each set in interval notation.

53. **(a)**

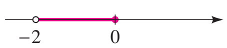
(b)



54. **(a)**



(b)



55–60 ■ Graph the set.

55. $(-2, 0) \cup (-1, 1)$ **56.** $(-2, 0) \cap (-1, 1)$

57. $[-4, 6] \cap [0, 8)$ **58.** $[-4, 6) \cup [0, 8)$

59. $(-\infty, -4) \cup (4, \infty)$ **60.** $(-\infty, 6] \cap (2, 10)$

61–66 ■ Evaluate each expression.

61. **(a)** $|100|$ **(b)** $|-73|$

62. **(a)** $|\sqrt{5} - 5|$ **(b)** $|10 - \pi|$

63. **(a)** $\big||-6| - |-4|\big|$ **(b)** $\dfrac{-1}{|-1|}$

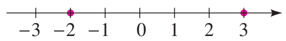
64. **(a)** $\big|2 - |-12|\big|$ **(b)** $-1 - \big|1 - |-1|\big|$

65. **(a)** $|(-2) \cdot 6|$ **(b)** $|(-\frac{1}{3})(-15)|$

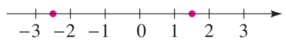
66. **(a)** $\left|\dfrac{-6}{24}\right|$ **(b)** $\left|\dfrac{7 - 12}{12 - 7}\right|$

67–70 ■ Find the distance between the given numbers.

67.



68.



69. **(a)** 2 and 17

(b) -3 and 21

(c) $\frac{11}{8}$ and $-\frac{3}{10}$

70. **(a)** $\frac{7}{15}$ and $-\frac{1}{21}$

(b) -38 and -57

(c) -2.6 and -1.8

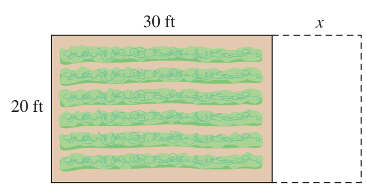
71–72 ■ Express each repeating decimal as a fraction. (See the margin note on page 2.)

71. **(a)** $0.\overline{7}$ **(b)** $0.2\overline{8}$ **(c)** $0.\overline{57}$

72. **(a)** $5.\overline{23}$ **(b)** $1.3\overline{7}$ **(c)** $2.1\overline{35}$

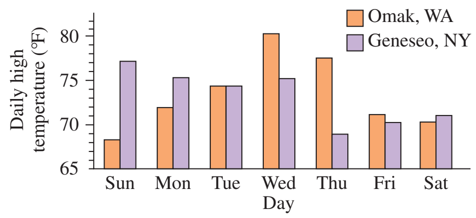
Applications

73. Area of a Garden Mary's backyard vegetable garden measures 20 ft by 30 ft, so its area is $20 \times 30 = 600$ ft^2. She decides to make it longer, as shown in the figure, so that the area increases to $A = 20(30 + x)$. Which property of real numbers tells us that the new area can also be written $A = 600 + 20x$?



74. Temperature Variation The bar graph shows the daily high temperatures for Omak, Washington, and Geneseo, New York, during a certain week in June. Let T_O represent the temperature in Omak and T_G the temperature in Geneseo. Calculate $T_O - T_G$ and $|T_O - T_G|$ for each day shown.

Which of these two values gives more information?

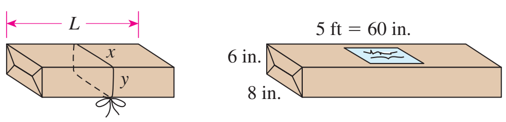


75. Mailing a Package The post office will only accept packages for which the length plus the "girth" (distance around) is no more than 108 inches. Thus, for the package in the figure, we must have

$$L + 2(x + y) \leq 108$$

(a) Will the post office accept a package that is 6 in. wide, 8 in. deep, and 5 ft long? What about a package that measures 2 ft by 2 ft by 4 ft?

(b) What is the greatest acceptable length for a package that has a square base measuring 9 in. by 9 in?



Discovery • Discussion

76. Signs of Numbers Let a, b, and c be real numbers such that $a > 0$, $b < 0$, and $c < 0$. Find the sign of each expression.

(a) $-a$ **(b)** $-b$ **(c)** bc
(d) $a - b$ **(e)** $c - a$ **(f)** $a + bc$
(g) $ab + ac$ **(h)** $-abc$ **(i)** ab^2

77. Sums and Products of Rational and Irrational Numbers Explain why the sum, the difference, and the product of two rational numbers are rational numbers. Is the product of two irrational numbers necessarily irrational? What about the sum?

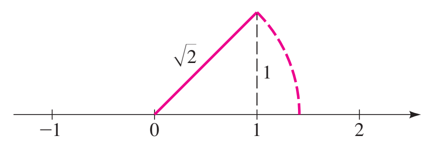
78. Combining Rational Numbers with Irrational Numbers Is $\frac{1}{2} + \sqrt{2}$ rational or irrational? Is $\frac{1}{2} \cdot \sqrt{2}$ rational or irrational? In general, what can you say about the sum of a rational and an irrational number? What about the product?

79. Limiting Behavior of Reciprocals Complete the tables. What happens to the size of the fraction $1/x$ as x gets large? As x gets small?

x	$1/x$
1	
2	
10	
100	
1000	

x	$1/x$
1.0	
0.5	
0.1	
0.01	
0.001	

80. Irrational Numbers and Geometry Using the following figure, explain how to locate the point $\sqrt{2}$ on a number line. Can you locate $\sqrt{5}$ by a similar method? What about $\sqrt{6}$? List some other irrational numbers that can be located this way.



81. Commutative and Noncommutative Operations We have seen that addition and multiplication are both commutative operations.

(a) Is subtraction commutative?

(b) Is division of nonzero real numbers commutative?

1.2 Exponents and Radicals

SUGGESTED TIME AND EMPHASIS

1–2 classes.
Essential material.

In this section we give meaning to expressions such as $a^{m/n}$ in which the exponent m/n is a rational number. To do this, we need to recall some facts about integer exponents, radicals, and nth roots.

Integer Exponents

A product of identical numbers is usually written in exponential notation. For example, $5 \cdot 5 \cdot 5$ is written as 5^3. In general, we have the following definition.

POINTS TO STRESS

1. Definitions of a^n in the cases where n is a positive integer, where n is zero, where n is rational, and where n is negative.
2. Algebraic properties of exponents.
3. Scientific notation and its relationship to significant digits.
4. Rationalizing denominators.

Exponential Notation

If a is any real number and n is a positive integer, then the ***n*th power** of a is

$$a^n = \underbrace{a \cdot a \cdot \cdots \cdot a}_{n \text{ factors}}$$

The number a is called the **base** and n is called the **exponent**.

Example 1 Exponential Notation

(a) $\left(\frac{1}{2}\right)^5 = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{32}$

(b) $(-3)^4 = (-3)\cdot(-3)\cdot(-3)\cdot(-3) = 81$

(c) $-3^4 = -(3\cdot3\cdot3\cdot3) = -81$ ■

Note the distinction between $(-3)^4$ and -3^4. In $(-3)^4$ the exponent applies to -3, but in -3^4 the exponent applies only to 3.

We can state several useful rules for working with exponential notation. To discover the rule for multiplication, we multiply 5^4 by 5^2:

$$5^4 \cdot 5^2 = \underbrace{(5\cdot5\cdot5\cdot5)}_{4 \text{ factors}}\underbrace{(5\cdot5)}_{2 \text{ factors}} = \underbrace{5\cdot5\cdot5\cdot5\cdot5\cdot5}_{6 \text{ factors}} = 5^6 = 5^{4+2}$$

It appears that *to multiply two powers of the same base, we add their exponents*. In general, for any real number a and any positive integers m and n, we have

$$a^m a^n = \underbrace{(a\cdot a\cdot\cdots\cdot a)}_{m \text{ factors}}\underbrace{(a\cdot a\cdot\cdots\cdot a)}_{n \text{ factors}} = \underbrace{a\cdot a\cdot a\cdot\cdots\cdot a}_{m+n \text{ factors}} = a^{m+n}$$

Thus $a^m a^n = a^{m+n}$.

We would like this rule to be true even when m and n are 0 or negative integers. For instance, we must have

$$2^0 \cdot 2^3 = 2^{0+3} = 2^3$$

But this can happen only if $2^0 = 1$. Likewise, we want to have

$$5^4 \cdot 5^{-4} = 5^{4+(-4)} = 5^{4-4} = 5^0 = 1$$

and this will be true if $5^{-4} = 1/5^4$. These observations lead to the following definition.

Zero and Negative Exponents

If $a \neq 0$ is any real number and n is a positive integer, then

$$a^0 = 1 \qquad \text{and} \qquad a^{-n} = \frac{1}{a^n}$$

Example 2 Zero and Negative Exponents

(a) $\left(\frac{4}{7}\right)^0 = 1$

(b) $x^{-1} = \dfrac{1}{x^1} = \dfrac{1}{x}$

(c) $(-2)^{-3} = \dfrac{1}{(-2)^3} = \dfrac{1}{-8} = -\dfrac{1}{8}$ ■

ALTERNATE EXAMPLE 1b

Evaluate $(-2)^6$ and -2^6.

ANSWER

64, −64

ALTERNATE EXAMPLE 2

Simplify:

(a) $\left(\dfrac{1}{3}\right)^{-1}$

(b) $(\sqrt{3} - \pi)^0$

ANSWERS

(a) 3

(b) 1

SAMPLE QUESTIONS

Text Questions

1. What is the definition of $a^{3/4}$?

2. Write 125,000,000 in scientific notation.

3. Why is scientific notation useful?

Answers

1. $a^{3/4} = \sqrt[4]{a^3}$

2. $125{,}000{,}000 = 1.25 \times 10^8$

3. It provides a compact way of writing very large numbers and very small numbers.

Familiarity with the following rules is essential for our work with exponents and bases. In the table the bases a and b are real numbers, and the exponents m and n are integers.

Laws of Exponents

Law	Example	Description
1. $a^m a^n = a^{m+n}$	$3^2 \cdot 3^5 = 3^{2+5} = 3^7$	To multiply two powers of the same number, add the exponents.
2. $\frac{a^m}{a^n} = a^{m-n}$	$\frac{3^5}{3^2} = 3^{5-2} = 3^3$	To divide two powers of the same number, subtract the exponents.
3. $(a^m)^n = a^{mn}$	$(3^2)^5 = 3^{2\cdot 5} = 3^{10}$	To raise a power to a new power, multiply the exponents.
4. $(ab)^n = a^n b^n$	$(3 \cdot 4)^2 = 3^2 \cdot 4^2$	To raise a product to a power, raise each factor to the power.
5. $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$	$\left(\frac{3}{4}\right)^2 = \frac{3^2}{4^2}$	To raise a quotient to a power, raise both numerator and denominator to the power.

■ **Proof of Law 3** If m and n are positive integers, we have

$$(a^m)^n = (\underbrace{a \cdot a \cdot \cdots \cdot a}_{m \text{ factors}})^n$$

$$= \underbrace{(\underbrace{a \cdot a \cdot \cdots \cdot a}_{m \text{ factors}})(\underbrace{a \cdot a \cdot \cdots \cdot a}_{m \text{ factors}}) \cdots (\underbrace{a \cdot a \cdot \cdots \cdot a}_{m \text{ factors}})}_{n \text{ groups of factors}}$$

$$= \underbrace{a \cdot a \cdot \cdots \cdot a}_{mn \text{ factors}} = a^{mn}$$

The cases for which $m \le 0$ or $n \le 0$ can be proved using the definition of negative exponents. ■

■ **Proof of Law 4** If n is a positive integer, we have

$$(ab)^n = \underbrace{(ab)(ab) \cdots (ab)}_{n \text{ factors}} = \underbrace{(a \cdot a \cdot \cdots \cdot a)}_{n \text{ factors}} \cdot \underbrace{(b \cdot b \cdot \cdots \cdot b)}_{n \text{ factors}} = a^n b^n$$

Here we have used the Commutative and Associative Properties repeatedly. If $n \le 0$, Law 4 can be proved using the definition of negative exponents. ■

You are asked to prove Laws 2 and 5 in Exercise 88.

ALTERNATE EXAMPLE 3b
Simplify the expression using the laws of exponents $x^3 x^5$.

ANSWER
x^8

Example 3 Using Laws of Exponents

(a) $x^4 x^7 = x^{4+7} = x^{11}$ — Law 1: $a^m a^n = a^{m+n}$

(b) $y^4 y^{-7} = y^{4-7} = y^{-3} = \frac{1}{y^3}$ — Law 1: $a^m a^n = a^{m+n}$

(c) $\frac{c^9}{c^5} = c^{9-5} = c^4$ — Law 2: $a^m/a^n = a^{m-n}$

(d) $(b^4)^5 = b^{4\cdot 5} = b^{20}$ Law 3: $(a^m)^n = a^{mn}$

(e) $(3x)^3 = 3^3x^3 = 27x^3$ Law 4: $(ab)^n = a^nb^n$

(f) $\left(\frac{x}{2}\right)^5 = \frac{x^5}{2^5} = \frac{x^5}{32}$ Law 5: $(a/b)^n = a^n/b^n$ ■

Example 4 Simplifying Expressions with Exponents

Simplify:

(a) $(2a^3b^2)(3ab^4)^3$ (b) $\left(\frac{x}{y}\right)^3\left(\frac{y^2x}{z}\right)^4$

Solution

(a)
$$\begin{aligned}(2a^3b^2)(3ab^4)^3 &= (2a^3b^2)[3^3a^3(b^4)^3] && \text{Law 4: } (ab)^n = a^nb^n\\ &= (2a^3b^2)(27a^3b^{12}) && \text{Law 3: } (a^m)^n = a^{mn}\\ &= (2)(27)a^3a^3b^2b^{12} && \text{Group factors with the same base}\\ &= 54a^6b^{14} && \text{Law 1: } a^ma^n = a^{m+n}\end{aligned}$$

(b)
$$\begin{aligned}\left(\frac{x}{y}\right)^3\left(\frac{y^2x}{z}\right)^4 &= \frac{x^3}{y^3}\,\frac{(y^2)^4x^4}{z^4} && \text{Laws 5 and 4}\\ &= \frac{x^3}{y^3}\,\frac{y^8x^4}{z^4} && \text{Law 3}\\ &= (x^3x^4)\left(\frac{y^8}{y^3}\right)\frac{1}{z^4} && \text{Group factors with the same base}\\ &= \frac{x^7y^5}{z^4} && \text{Laws 1 and 2}\end{aligned}$$
■

When simplifying an expression, you will find that many different methods will lead to the same result; you should feel free to use any of the rules of exponents to arrive at your own method. We now give two additional laws that are useful in simplifying expressions with negative exponents.

Laws of Exponents

Law	Example	Description
6. $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$	$\left(\frac{3}{4}\right)^{-2} = \left(\frac{4}{3}\right)^2$	To raise a fraction to a negative power, invert the fraction and change the sign of the exponent.
7. $\frac{a^{-n}}{b^{-m}} = \frac{b^m}{a^n}$	$\frac{3^{-2}}{4^{-5}} = \frac{4^5}{3^2}$	To move a number raised to a power from numerator to denominator or from denominator to numerator, change the sign of the exponent.

■ **Proof of Law 7** Using the definition of negative exponents and then Property 2 of fractions (page 5), we have

$$\frac{a^{-n}}{b^{-m}} = \frac{1/a^n}{1/b^m} = \frac{1}{a^n}\cdot\frac{b^m}{1} = \frac{b^m}{a^n}$$
■

You are asked to prove Law 6 in Exercise 88.

ALTERNATE EXAMPLE 4a

Simplify the expression $(5a^3b^2)(2ab^4)^3$.

ANSWER

$40a^6\cdot b^{14}$

DRILL QUESTIONS

(a) Simplify $\left(\frac{a}{3}\right)^{-3}a^5$.

(b) Simplify $\frac{\sqrt[3]{a^2}}{\sqrt[3]{a}}a^5$.

Answers

(a) $27a^2$

(b) $a^{16/3}$

IN-CLASS MATERIALS

One non-rigorous way to justify the way negative exponents work is to start a table such as this:

2^5	32
2^4	16
2^3	8
2^2	4
2^1	2

Note that every step we are dividing by two, and continuing the pattern we get $2^0 = 1$ and $2^{-1} = \frac{1}{2}$. After explaining this concept, take a minute to do 2^{-2} and 2^{-3}.

ALTERNATE EXAMPLE 5b

Eliminate negative exponents and simplify the expression

$$\left(\frac{y}{2z^2}\right)^{-2}.$$

ANSWER

$$\left(\frac{4z^4}{y^2}\right)$$

Mathematics in the Modern World

Although we are often unaware of its presence, mathematics permeates nearly every aspect of life in the modern world. With the advent of modern technology, mathematics plays an ever greater role in our lives. Today you were probably awakened by a digital alarm clock, made a phone call that used digital transmission, sent an e-mail message over the Internet, drove a car with digitally controlled fuel injection, listened to music on a CD player, then fell asleep in a room whose temperature is controlled by a digital thermostat. In each of these activities mathematics is crucially involved. In general, a property such as the intensity or frequency of sound, the oxygen level in the exhaust emission from a car, the colors in an image, or the temperature in your bedroom is transformed into sequences of numbers by sophisticated mathematical algorithms. These numerical data, which usually consist of many millions of bits (the digits 0 and 1), are then transmitted and reinterpreted. Dealing with such huge amounts of data was not feasible until the invention of computers, machines whose logical processes were invented by mathematicians.

The contributions of mathematics in the modern world are not limited to technological advances. The logical processes of mathematics are now used to analyze complex problems in the social, political, and life sciences in new and surprising ways. Advances in mathematics continue to be made, some of the most exciting of these just within the past decade.

In other *Mathematics in the Modern World*, we will describe in more detail how mathematics affects all of us in our everyday activities.

Example 5 Simplifying Expressions with Negative Exponents

Eliminate negative exponents and simplify each expression.

(a) $\dfrac{6st^{-4}}{2s^{-2}t^2}$ (b) $\left(\dfrac{y}{3z^3}\right)^{-2}$

Solution

(a) We use Law 7, which allows us to move a number raised to a power from the numerator to the denominator (or vice versa) by changing the sign of the exponent.

t^{-4} moves to denominator and becomes t^4.

$$\frac{6st^{-4}}{2s^{-2}t^2} = \frac{6ss^2}{2t^2t^4} \qquad \text{Law 7}$$

s^{-2} moves to numerator and becomes s^2.

$$= \frac{3s^3}{t^6} \qquad \text{Law 1}$$

(b) We use Law 6, which allows us to change the sign of the exponent of a fraction by inverting the fraction.

$$\left(\frac{y}{3z^3}\right)^{-2} = \left(\frac{3z^3}{y}\right)^2 \qquad \text{Law 6}$$

$$= \frac{9z^6}{y^2} \qquad \text{Laws 5 and 4}$$

■

Scientific Notation

Exponential notation is used by scientists as a compact way of writing very large numbers and very small numbers. For example, the nearest star beyond the sun, Proxima Centauri, is approximately 40,000,000,000,000 km away. The mass of a hydrogen atom is about 0.00000000000000000000000166 g. Such numbers are difficult to read and to write, so scientists usually express them in *scientific notation*.

Scientific Notation

A positive number x is said to be written in **scientific notation** if it is expressed as follows:

$$x = a \times 10^n \qquad \text{where } 1 \le a < 10 \text{ and } n \text{ is an integer}$$

For instance, when we state that the distance to the star Proxima Centauri is 4×10^{13} km, the positive exponent 13 indicates that the decimal point should be moved 13 places to the *right*:

$$4 \times 10^{13} = 40{,}000{,}000{,}000{,}000$$

Move decimal point 13 places to the right.

IN-CLASS MATERIALS

Write $\sqrt{ab} = \sqrt{a}\sqrt{b}$ on the board, and ask the students if this statement is always true. Point out (or have them discover) that this is a false statement when a and b are negative. Also, write $\sqrt{a^2 + b^2}$ on the board, and point out that it cannot be simplified.

When we state that the mass of a hydrogen atom is 1.66×10^{-24} g, the exponent -24 indicates that the decimal point should be moved 24 places to the *left*:

$$1.66 \times 10^{-24} = 0.00000000000000000000000166$$

Move decimal point 24 places to the left.

Example 6 Writing Numbers in Scientific Notation

(a) $\underbrace{327{,}900}_{\text{5 places}} = 3.279 \times 10^5$ (b) $\underbrace{0.000627}_{\text{4 places}} = 6.27 \times 10^{-4}$ ■

To use scientific notation on a calculator, press the key labeled EE or EXP or EEX to enter the exponent. For example, to enter the number 3.629×10^{15} on a TI-83 calculator, we enter

3.629 2ND EE 15

and the display reads

3.629E15

Scientific notation is often used on a calculator to display a very large or very small number. For instance, if we use a calculator to square the number 1,111,111, the display panel may show (depending on the calculator model) the approximation

1.234568 12 or 1.23468 E12

Here the final digits indicate the power of 10, and we interpret the result as

$$1.234568 \times 10^{12}$$

Example 7 Calculating with Scientific Notation

If $a \approx 0.00046$, $b \approx 1.697 \times 10^{22}$, and $c \approx 2.91 \times 10^{-18}$, use a calculator to approximate the quotient ab/c.

Solution We could enter the data using scientific notation, or we could use laws of exponents as follows:

$$\begin{aligned}\frac{ab}{c} &\approx \frac{(4.6 \times 10^{-4})(1.697 \times 10^{22})}{2.91 \times 10^{-18}} \\ &= \frac{(4.6)(1.697)}{2.91} \times 10^{-4+22+18} \\ &\approx 2.7 \times 10^{36}\end{aligned}$$

We state the answer correct to two significant figures because the least accurate of the given numbers is stated to two significant figures. ■

Radicals

We know what 2^n means whenever n is an integer. To give meaning to a power, such as $2^{4/5}$, whose exponent is a rational number, we need to discuss radicals.

The symbol $\sqrt{}$ means "the positive square root of." Thus

$$\sqrt{a} = b \quad \text{means} \quad b^2 = a \quad \text{and} \quad b \geq 0$$

It is true that the number 9 has two square roots, 3 and -3, but the notation $\sqrt{9}$ is reserved for the *positive* square root of 9 (sometimes called the *principal square root* of 9). If we want the negative root, we must write $-\sqrt{9}$, which is -3.

Since $a = b^2 \geq 0$, the symbol $\sqrt{a}$ makes sense only when $a \geq 0$. For instance,

$$\sqrt{9} = 3 \quad \text{because} \quad 3^2 = 9 \quad \text{and} \quad 3 \geq 0$$

ALTERNATE EXAMPLE 6
Put the following into scientific notation:
(a) 8625.126
(b) 0.000331

ANSWERS
(a) 8.625126×10^3
(b) 3.31×10^{-4}

ALTERNATE EXAMPLE 7
If $a \approx 0.00084$, $b \approx 2.195 \times 10^{21}$, and $c \approx 6.81 \times 10^{-19}$, use a calculator to approximate the quotient $\frac{ab}{c}$.

ANSWER
2.71×10^{36}

IN-CLASS MATERIALS

When George Everest measured the mountain that bears his name (he never actually climbed it) he used techniques that were state-of-the-art at the time. According to his measurements, Mount Everest was exactly 29,000 ft tall (to the nearest foot). He was afraid that people would think it was an estimate: that he was merely saying that the mountain was between 28,500 ft and 29,500 ft tall. He was proud of his work. Discuss how he could have reported his results in such a way that people would know that he was not estimating the height of the mountain. (Historically, he lied and said that it was 29,002 ft tall, so that people would know there were five significant figures. Since the 1950s the height has been listed at 29,028 ft, although as of 1999 some scientists want to add seven feet to that number.)

Answer

He could have said it was 2.9000×10^4 ft tall—the extra zeros imply that the zeros are significant.

Square roots are special cases of nth roots. The nth root of x is the number that, when raised to the nth power, gives x.

Definition of nth Root

If n is any positive integer, then the **principal nth root** of a is defined as follows:

$$\sqrt[n]{a} = b \quad \text{means} \quad b^n = a$$

If n is even, we must have $a \geq 0$ and $b \geq 0$.

Thus

$$\sqrt[4]{81} = 3 \quad \text{because} \quad 3^4 = 81 \quad \text{and} \quad 3 \geq 0$$

$$\sqrt[3]{-8} = -2 \quad \text{because} \quad (-2)^3 = -8$$

But $\sqrt{-8}$, $\sqrt[4]{-8}$, and $\sqrt[6]{-8}$ are not defined. (For instance, $\sqrt{-8}$ is not defined because the square of every real number is nonnegative.)

Notice that

$$\sqrt{4^2} = \sqrt{16} = 4 \quad \text{but} \quad \sqrt{(-4)^2} = \sqrt{16} = 4 = |-4|$$

So, the equation $\sqrt{a^2} = a$ is not always true; it is true only when $a \geq 0$. However, we can always write $\sqrt{a^2} = |a|$. This last equation is true not only for square roots, but for any even root. This and other rules used in working with nth roots are listed in the following box. In each property we assume that all the given roots exist.

Properties of nth Roots

Property	Example
1. $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$	$\sqrt[3]{-8 \cdot 27} = \sqrt[3]{-8}\sqrt[3]{27} = (-2)(3) = -6$
2. $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$	$\sqrt[4]{\dfrac{16}{81}} = \dfrac{\sqrt[4]{16}}{\sqrt[4]{81}} = \dfrac{2}{3}$
3. $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$	$\sqrt{\sqrt[3]{729}} = \sqrt[6]{729} = 3$
4. $\sqrt[n]{a^n} = a$ if n is odd	$\sqrt[3]{(-5)^3} = -5$, $\sqrt[5]{2^5} = 2$
5. $\sqrt[n]{a^n} = \|a\|$ if n is even	$\sqrt[4]{(-3)^4} = \|-3\| = 3$

Example 8 Simplifying Expressions Involving nth Roots

(a)
$$\begin{aligned}\sqrt[3]{x^4} &= \sqrt[3]{x^3 x} && \text{Factor out the largest cube}\\ &= \sqrt[3]{x^3}\sqrt[3]{x} && \text{Property 1: } \sqrt[3]{ab} = \sqrt[3]{a}\sqrt[3]{b}\\ &= x\sqrt[3]{x} && \text{Property 4: } \sqrt[3]{a^3} = a\end{aligned}$$

IN-CLASS MATERIALS

When people talk about groups of objects, they often use the word "dozen" for convenience. It is easier to think about 3 dozen eggs than it is to think about 36 eggs. Chemists use a similar word for atoms: a "mole." There is a mole of atoms in 22.4 L of gas (you can bring an empty 1 liter bottle to help them picture it) and a mole of atoms in 18 mL of water. (Bring in a graduated cylinder to demonstrate.) A mole of atoms is 6.02×10^{23} atoms. Ask the students which is bigger:

1. A mole or the number of inches along the Mississippi River
2. A mole or a trillion
3. A mole or the number of stars visible from Earth
4. A mole or the number of grains of sand on Earth
5. A mole or the number of sun-like stars in the universe

The answer for all five is "a mole."

(b) $\sqrt[4]{81x^8y^4} = \sqrt[4]{81}\sqrt[4]{x^8}\sqrt[4]{y^4}$ — Property 1: $\sqrt[n]{abc} = \sqrt[n]{a}\sqrt[n]{b}\sqrt[n]{c}$

$= 3\sqrt[4]{(x^2)^4}|y|$ — Property 5: $\sqrt[4]{a^4} = |a|$

$= 3x^2|y|$ — Property 5: $\sqrt[4]{a^4} = |a|$, $|x^2| = x^2$ ■

It is frequently useful to combine like radicals in an expression such as $2\sqrt{3} + 5\sqrt{3}$. This can be done by using the Distributive Property. Thus

$$2\sqrt{3} + 5\sqrt{3} = (2 + 5)\sqrt{3} = 7\sqrt{3}$$

The next example further illustrates this process.

Example 9 Combining Radicals

(a) $\sqrt{32} + \sqrt{200} = \sqrt{16\cdot 2} + \sqrt{100\cdot 2}$ — Factor out the largest squares

$= \sqrt{16}\sqrt{2} + \sqrt{100}\sqrt{2}$ — Property 1: $\sqrt{ab} = \sqrt{a}\sqrt{b}$

$= 4\sqrt{2} + 10\sqrt{2} = 14\sqrt{2}$ — Distributive Property

(b) If $b > 0$, then

$\sqrt{25b} - \sqrt{b^3} = \sqrt{25}\sqrt{b} - \sqrt{b^2}\sqrt{b}$ — Property 1: $\sqrt{ab} = \sqrt{a}\sqrt{b}$

$= 5\sqrt{b} - b\sqrt{b}$ — Property 5, $b > 0$

$= (5 - b)\sqrt{b}$ — Distributive Property ■

Avoid making the following error:

$$\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$$

For instance, if we let $a = 9$ and $b = 16$, then we see the error:

$$\sqrt{9+16} \stackrel{?}{=} \sqrt{9} + \sqrt{16}$$
$$\sqrt{25} \stackrel{?}{=} 3 + 4$$
$$5 \stackrel{?}{=} 7 \quad \text{Wrong!}$$

Rational Exponents

To define what is meant by a *rational exponent* or, equivalently, a *fractional exponent* such as $a^{1/3}$, we need to use radicals. In order to give meaning to the symbol $a^{1/n}$ in a way that is consistent with the Laws of Exponents, we would have to have

$$(a^{1/n})^n = a^{(1/n)n} = a^1 = a$$

So, by the definition of nth root,

$$a^{1/n} = \sqrt[n]{a}$$

In general, we define rational exponents as follows.

Definition of Rational Exponents

For any rational exponent m/n in lowest terms, where m and n are integers and $n > 0$, we define

$$a^{m/n} = (\sqrt[n]{a})^m \quad \text{or equivalently} \quad a^{m/n} = \sqrt[n]{a^m}$$

If n is even, then we require that $a \geq 0$.

With this definition it can be proved that *the Laws of Exponents also hold for rational exponents*.

ALTERNATE EXAMPLE 8b
Simplify the radical expression $\sqrt[4]{625p^8q^4}$.

ANSWER
$5p^2|q|$

ALTERNATE EXAMPLE 9b
Simplify the radical expression $\sqrt{125} + \sqrt{245}$.

ANSWER
$12\sqrt{5}$

IN-CLASS MATERIALS

There is a cube root magic trick that illustrates an interesting property of the cube function. A student takes a two-digit integer, cubes it, and gives you the answer (for example, 274,625). You knit your brow, think, and give the cube root (65) using the power of your mind. If the students are not impressed, stop there. If they are, do the trick a second time, getting the answer wrong by 1. Try it a third time, and again get the correct answer. Have them try to figure the trick out, and promise to reveal the secret if they do well on a future quiz or test.

The trick is as follows. Memorize the first ten cubes:

x	0	1	2	3	4	5	6	7	8	9
x^3	0	1	8	27	64	125	216	343	512	729

Notice that the last digit of each cube is unique, and that they aren't very well scrambled (2's cube ends in 8 and vice versa; 3's ends in 7 and vice versa). So the last digit of the cube root is easily obtained (274,625 ends in 5, so the number you were given is of the form ?5). Now the first digit can be obtained by looking at the first three digits of the cube, and seeing in which range they are. (274 is between 6^3 and 7^3, so the number you were given is 65.) Make sure to tell the students that if they do this trick for other people, they should not get the answer too fast, and they should get the last digit wrong occasionally. This will make it seem like they really can do cube roots in their heads.

ALTERNATE EXAMPLE 10b
Simplify $8^{1/3}$.

ANSWER
2

ALTERNATE EXAMPLE 11

Simplify $\dfrac{a^{1/3}b}{b^{1/5}a^{-2/3}b^{4/5}}$.

ANSWER
a

ALTERNATE EXAMPLE 12a
Simplify the expression
$(4\sqrt{x})(9\sqrt[3]{x})$.

ANSWER
$36x^{5/6}$

Diophantus lived in Alexandria about 250 A.D. His book *Arithmetica* is considered the first book on algebra. In it he gives methods for finding integer solutions of algebraic equations. *Arithmetica* was read and studied for more than a thousand years. Fermat (see page 652) made some of his most important discoveries while studying this book. Diophantus' major contribution is the use of symbols to stand for the unknowns in a problem. Although his symbolism is not as simple as what we use today, it was a major advance over writing everything in words. In Diophantus' notation the equation

$$x^5 - 7x^2 + 8x - 5 = 24$$

is written

ΔK$^{\upsilon}$α ς η ⫛ Δ$^{\upsilon}$ζ M̊ ε ι$^{\sigma}$ κ δ

Our modern algebraic notation did not come into common use until the 17th century.

Example 10 Using the Definition of Rational Exponents

(a) $4^{1/2} = \sqrt{4} = 2$

(b) $8^{2/3} = (\sqrt[3]{8})^2 = 2^2 = 4$ Alternative solution: $8^{2/3} = \sqrt[3]{8^2} = \sqrt[3]{64} = 4$

(c) $125^{-1/3} = \dfrac{1}{125^{1/3}} = \dfrac{1}{\sqrt[3]{125}} = \dfrac{1}{5}$

(d) $\dfrac{1}{\sqrt[3]{x^4}} = \dfrac{1}{x^{4/3}} = x^{-4/3}$ ■

Example 11 Using the Laws of Exponents with Rational Exponents

(a) $a^{1/3}a^{7/3} = a^{8/3}$ Law 1: $a^m a^n = a^{m+n}$

(b) $\dfrac{a^{2/5}a^{7/5}}{a^{3/5}} = a^{2/5+7/5-3/5} = a^{6/5}$ Law 1, Law 2: $\dfrac{a^m}{a^n} = a^{m-n}$

(c) $(2a^3b^4)^{3/2} = 2^{3/2}(a^3)^{3/2}(b^4)^{3/2}$ Law 4: $(abc)^n = a^n b^n c^n$

$= (\sqrt{2})^3 a^{3(3/2)} b^{4(3/2)}$ Law 3: $(a^m)^n = a^{mn}$

$= 2\sqrt{2}a^{9/2}b^6$

(d) $\left(\dfrac{2x^{3/4}}{y^{1/3}}\right)^3 \left(\dfrac{y^4}{x^{-1/2}}\right) = \dfrac{2^3(x^{3/4})^3}{(y^{1/3})^3}\cdot(y^4x^{1/2})$ Laws 5, 4, and 7

$= \dfrac{8x^{9/4}}{y}\cdot y^4x^{1/2}$ Law 3

$= 8x^{11/4}y^3$ Laws 1 and 2 ■

Example 12 Simplifying by Writing Radicals as Rational Exponents

(a) $(2\sqrt{x})(3\sqrt[3]{x}) = (2x^{1/2})(3x^{1/3})$ Definition of rational exponents

$= 6x^{1/2+1/3} = 6x^{5/6}$ Law 1

(b) $\sqrt{x\sqrt{x}} = (xx^{1/2})^{1/2}$ Definition of rational exponents

$= (x^{3/2})^{1/2}$ Law 1

$= x^{3/4}$ Law 3 ■

Rationalizing the Denominator

It is often useful to eliminate the radical in a denominator by multiplying both numerator and denominator by an appropriate expression. This procedure is called **rationalizing the denominator**. If the denominator is of the form $\sqrt{a}$, we multiply numerator and denominator by $\sqrt{a}$. In doing this we multiply the given quantity by 1, so we do not change its value. For instance,

$$\frac{1}{\sqrt{a}} = \frac{1}{\sqrt{a}}\cdot 1 = \frac{1}{\sqrt{a}}\cdot\frac{\sqrt{a}}{\sqrt{a}} = \frac{\sqrt{a}}{a}$$

EXAMPLES

1. A product whose answer will be given by a calculator in scientific notation:

$$(4{,}678{,}200{,}000)(6{,}006{,}200{,}000) = 28{,}098{,}204{,}840{,}000{,}000{,}000$$

2. A product whose answer contains too many significant figures for many calculators to compute accurately:

$$(415{,}210{,}709)(519{,}080{,}123) = 215{,}527{,}625{,}898{,}637{,}207$$

3. A factional exponent simplification that works out nicely:

$$\frac{\sqrt[3]{8^5}}{\sqrt[3]{8^4}} = \sqrt[3]{\frac{8^5}{8^4}} = \sqrt[3]{8} = 2$$

Note that the denominator in the last fraction contains no radical. In general, if the denominator is of the form $\sqrt[n]{a^m}$ with $m < n$, then multiplying the numerator and denominator by $\sqrt[n]{a^{n-m}}$ will rationalize the denominator, because (for $a > 0$)

$$\sqrt[n]{a^m}\sqrt[n]{a^{n-m}} = \sqrt[n]{a^{m+n-m}} = \sqrt[n]{a^n} = a$$

Example 13 Rationalizing Denominators

(a) $\dfrac{2}{\sqrt{3}} = \dfrac{2}{\sqrt{3}} \cdot \dfrac{\sqrt{3}}{\sqrt{3}} = \dfrac{2\sqrt{3}}{3}$

(b) $\dfrac{1}{\sqrt[3]{x^2}} = \dfrac{1}{\sqrt[3]{x^2}}\,\dfrac{\sqrt[3]{x}}{\sqrt[3]{x}} = \dfrac{\sqrt[3]{x}}{\sqrt[3]{x^3}} = \dfrac{\sqrt[3]{x}}{x}$

(c) $\sqrt[7]{\dfrac{1}{a^2}} = \dfrac{1}{\sqrt[7]{a^2}} = \dfrac{1}{\sqrt[7]{a^2}}\,\dfrac{\sqrt[7]{a^5}}{\sqrt[7]{a^5}} = \dfrac{\sqrt[7]{a^5}}{\sqrt[7]{a^7}} = \dfrac{\sqrt[7]{a^5}}{a}$ ■

ALTERNATE EXAMPLE 13b

Rationalize the denominator $\dfrac{1}{\sqrt[8]{x^7}}$.

ANSWER

$\dfrac{\sqrt[8]{x}}{x}$

1.2 Exercises

1–8 ■ Write each radical expression using exponents, and each exponential expression using radicals.

	Radical expression	Exponential expression
1.	$\dfrac{1}{\sqrt{5}}$	
2.	$\sqrt[3]{7^2}$	
3.		$4^{2/3}$
4.		$11^{-3/2}$
5.	$\sqrt[5]{5^3}$	
6.		$2^{-1.5}$
7.		$a^{2/5}$
8.	$\dfrac{1}{\sqrt{x^5}}$	

9–18 ■ Evaluate each expression.

9. **(a)** -3^2 **(b)** $(-3)^2$ **(c)** $(-3)^0$

10. **(a)** $5^2 \cdot \left(\frac{1}{5}\right)^3$ **(b)** $\dfrac{10^7}{10^4}$ **(c)** $\dfrac{3}{3^{-2}}$

11. **(a)** $\dfrac{4^{-3}}{2^{-8}}$ **(b)** $\dfrac{3^{-2}}{9}$ **(c)** $\left(\frac{1}{4}\right)^{-2}$

12. **(a)** $\left(\frac{2}{3}\right)^{-3}$ **(b)** $\left(\frac{3}{2}\right)^{-2} \cdot \frac{9}{16}$ **(c)** $\left(\frac{1}{2}\right)^4 \cdot \left(\frac{5}{2}\right)^{-2}$

13. **(a)** $\sqrt{16}$ **(b)** $\sqrt[4]{16}$ **(c)** $\sqrt[4]{1/16}$

14. **(a)** $\sqrt{64}$ **(b)** $\sqrt[3]{-64}$ **(c)** $\sqrt[5]{-32}$

15. **(a)** $\sqrt[3]{\dfrac{8}{27}}$ **(b)** $\sqrt[3]{\dfrac{-1}{64}}$ **(c)** $\dfrac{\sqrt[5]{-3}}{\sqrt[5]{96}}$

16. **(a)** $\sqrt{7}\sqrt{28}$ **(b)** $\dfrac{\sqrt{48}}{\sqrt{3}}$ **(c)** $\sqrt[4]{24}\sqrt[4]{54}$

17. **(a)** $\left(\frac{4}{9}\right)^{-1/2}$ **(b)** $(-32)^{2/5}$ **(c)** $-32^{2/5}$

18. **(a)** $1024^{-0.1}$ **(b)** $\left(-\frac{27}{8}\right)^{2/3}$ **(c)** $\left(\frac{25}{64}\right)^{-3/2}$

19–22 ■ Evaluate the expression using $x = 3$, $y = 4$, and $z = -1$.

19. $\sqrt{x^2 + y^2}$

20. $\sqrt[4]{x^3 + 14y + 2z}$

21. $(9x)^{2/3} + (2y)^{2/3} + z^{2/3}$

22. $(xy)^{2z}$

23–26 ■ Simplify the expression.

23. $\sqrt{32} + \sqrt{18}$

24. $\sqrt{75} + \sqrt{48}$

25. $\sqrt[5]{96} + \sqrt[5]{3}$

26. $\sqrt[4]{48} - \sqrt[4]{3}$

27–44 ■ Simplify the expression and eliminate any negative exponent(s).

27. a^9a^{-5}

28. $(3y^2)(4y^5)$

29. $(12x^2y^4)\left(\frac{1}{2}x^5y\right)$

30. $(6y)^3$

31. $\dfrac{x^9(2x)^4}{x^3}$

32. $\dfrac{a^{-3}b^4}{a^{-5}b^5}$

33. $b^4\left(\frac{1}{3}b^2\right)(12b^{-8})$

34. $(2s^3t^{-1})\left(\frac{1}{4}s^6\right)(16t^4)$

35. $(rs)^3(2s)^{-2}(4r)^4$

36. $(2u^2v^3)^3(3u^3v)^{-2}$

37. $\dfrac{(6y^3)^4}{2y^5}$

38. $\dfrac{(2x^3)^2(3x^4)}{(x^3)^4}$

39. $\dfrac{(x^2y^3)^4(xy^4)^{-3}}{x^2y}$

40. $\left(\dfrac{c^4d^3}{cd^2}\right)\left(\dfrac{d^2}{c^3}\right)^3$

41. $\frac{(xy^2z^3)^4}{(x^3y^2z)^3}$ **42.** $\left(\frac{xy^{-2}z^{-3}}{x^2y^3z^{-4}}\right)^{-3}$

43. $\left(\frac{q^{-1}rs^{-2}}{r^{-5}sq^{-8}}\right)^{-1}$ **44.** $(3ab^2c)\left(\frac{2a^2b}{c^3}\right)^{-2}$

45–52 ■ Simplify the expression. Assume the letters denote any real numbers.

45. $\sqrt[4]{x^4}$ **46.** $\sqrt[5]{x^{10}}$

47. $\sqrt[4]{16x^8}$ **48.** $\sqrt[3]{x^3y^6}$

49. $\sqrt{a^2b^6}$ **50.** $\sqrt[3]{a^2b}\sqrt[3]{a^4b}$

51. $\sqrt[3]{\sqrt{64x^6}}$ **52.** $\sqrt[4]{x^4y^2z^2}$

53–70 ■ Simplify the expression and eliminate any negative exponent(s). Assume that all letters denote positive numbers.

53. $x^{2/3}x^{1/5}$ **54.** $(2x^{3/2})(4x)^{-1/2}$

55. $(-3a^{1/4})(9a)^{-3/2}$ **56.** $(-2a^{3/4})(5a^{3/2})$

57. $(4b)^{1/2}(8b^{2/5})$ **58.** $(8x^6)^{-2/3}$

59. $(c^2d^3)^{-1/3}$ **60.** $(4x^6y^8)^{3/2}$

61. $(y^{3/4})^{2/3}$ **62.** $(a^{2/5})^{-3/4}$

63. $(2x^4y^{-4/5})^3(8y^2)^{2/3}$ **64.** $(x^{-5}y^3z^{10})^{-3/5}$

65. $\left(\frac{x^6y}{y^4}\right)^{5/2}$ **66.** $\left(\frac{-2x^{1/3}}{y^{1/2}z^{1/6}}\right)^4$

67. $\left(\frac{3a^{-2}}{4b^{-1/3}}\right)^{-1}$ **68.** $\frac{(y^{10}z^{-5})^{1/5}}{(y^{-2}z^3)^{1/3}}$

69. $\frac{(9st)^{3/2}}{(27s^3t^{-4})^{2/3}}$ **70.** $\left(\frac{a^2b^{-3}}{x^{-1}y^2}\right)^3\left(\frac{x^{-2}b^{-1}}{a^{3/2}y^{1/3}}\right)$

71–72 ■ Write each number in scientific notation.

71. **(a)** 69,300,000 **(b)** 7,200,000,000,000
(c) 0.000028536 **(d)** 0.0001213

72. **(a)** 129,540,000 **(b)** 7,259,000,000
(c) 0.0000000014 **(d)** 0.0007029

73–74 ■ Write each number in decimal notation.

73. **(a)** 3.19×10^5 **(b)** 2.721×10^8
(c) 2.670×10^{-8} **(d)** 9.999×10^{-9}

74. **(a)** 7.1×10^{14} **(b)** 6×10^{12}
(c) 8.55×10^{-3} **(d)** 6.257×10^{-10}

75–76 ■ Write the number indicated in each statement in scientific notation.

75. **(a)** A light-year, the distance that light travels in one year, is about 5,900,000,000,000 mi.
(b) The diameter of an electron is about 0.0000000000004 cm.
(c) A drop of water contains more than 33 billion billion molecules.

76. **(a)** The distance from the earth to the sun is about 93 million miles.
(b) The mass of an oxygen molecule is about 0.000000000000000000000053 g.
(c) The mass of the earth is about 5,970,000,000,000,000,000,000,000 kg.

77–82 ■ Use scientific notation, the Laws of Exponents, and a calculator to perform the indicated operations. State your answer correct to the number of significant digits indicated by the given data.

77. $(7.2 \times 10^{-9})(1.806 \times 10^{-12})$

78. $(1.062 \times 10^{24})(8.61 \times 10^{19})$

79. $\frac{1.295643 \times 10^9}{(3.610 \times 10^{-17})(2.511 \times 10^6)}$

80. $\frac{(73.1)(1.6341 \times 10^{28})}{0.0000000019}$

81. $\frac{(0.0000162)(0.01582)}{(594{,}621{,}000)(0.0058)}$ **82.** $\frac{(3.542 \times 10^{-6})^9}{(5.05 \times 10^4)^{12}}$

83–86 ■ Rationalize the denominator.

83. **(a)** $\frac{1}{\sqrt{10}}$ **(b)** $\sqrt{\frac{2}{x}}$ **(c)** $\sqrt{\frac{x}{3}}$

84. **(a)** $\sqrt{\frac{5}{12}}$ **(b)** $\sqrt{\frac{x}{6}}$ **(c)** $\sqrt{\frac{y}{2z}}$

85. **(a)** $\frac{2}{\sqrt[3]{x}}$ **(b)** $\frac{1}{\sqrt[4]{y^3}}$ **(c)** $\frac{x}{y^{2/5}}$

86. **(a)** $\frac{1}{\sqrt[4]{a}}$ **(b)** $\frac{a}{\sqrt[3]{b^2}}$ **(c)** $\frac{1}{c^{3/7}}$

87. Let a, b, and c be real numbers with $a > 0$, $b < 0$, and $c < 0$. Determine the sign of each expression.
(a) b^5 **(b)** b^{10} **(c)** ab^2c^3
(d) $(b - a)^3$ **(e)** $(b - a)^4$ **(f)** $\frac{a^3c^3}{b^6c^6}$

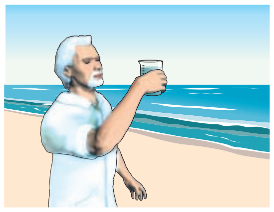
88. Prove the given Laws of Exponents for the case in which m and n are positive integers and $m > n$.
(a) Law 2 **(b)** Law 5 **(c)** Law 6

Applications

89. Distance to the Nearest Star Proxima Centauri, the star nearest to our solar system, is 4.3 light-years away. Use the information in Exercise 75(a) to express this distance in miles.

90. Speed of Light The speed of light is about 186,000 mi/s. Use the information in Exercise 76(a) to find how long it takes for a light ray from the sun to reach the earth.

91. Volume of the Oceans The average ocean depth is 3.7×10^3 m, and the area of the oceans is 3.6×10^{14} m^2. What is the total volume of the ocean in liters? (One cubic meter contains 1000 liters.)



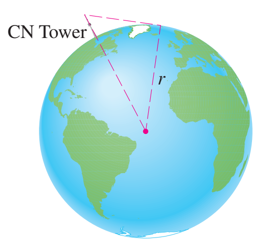
92. National Debt As of November 2004, the population of the United States was 2.949×10^8, and the national debt was 7.529×10^{12} dollars. How much was each person's share of the debt?

93. Number of Molecules A sealed room in a hospital, measuring 5 m wide, 10 m long, and 3 m high, is filled with pure oxygen. One cubic meter contains 1000 L, and 22.4 L of any gas contains 6.02×10^{23} molecules (Avogadro's number). How many molecules of oxygen are there in the room?

94. How Far Can You See? Due to the curvature of the earth, the maximum distance D that you can see from the top of a tall building of height h is estimated by the formula

$$D = \sqrt{2rh + h^2}$$

where $r = 3960$ mi is the radius of the earth and D and h are also measured in miles. How far can you see from the observation deck of the Toronto CN Tower, 1135 ft above the ground?

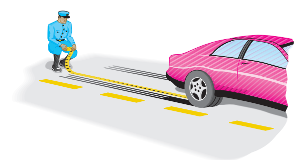


95. Speed of a Skidding Car Police use the formula $s = \sqrt{30fd}$ to estimate the speed s (in mi/h) at which a car is traveling if it skids d feet after the brakes are applied suddenly. The number f is the coefficient of friction of the road, which is a measure of the "slipperiness" of the road. The table gives some typical estimates for f.

	Tar	Concrete	Gravel
Dry	1.0	0.8	0.2
Wet	0.5	0.4	0.1

(a) If a car skids 65 ft on wet concrete, how fast was it moving when the brakes were applied?

(b) If a car is traveling at 50 mi/h, how far will it skid on wet tar?



96. Distance from the Earth to the Sun It follows from **Kepler's Third Law** of planetary motion that the average distance from a planet to the sun (in meters) is

$$d = \left(\frac{GM}{4\pi^2}\right)^{1/3} T^{2/3}$$

where $M = 1.99 \times 10^{30}$ kg is the mass of the sun, $G = 6.67 \times 10^{-11}$ N·m^2/kg^2 is the gravitational constant, and T is the period of the planet's orbit (in seconds). Use the fact that the period of the earth's orbit is about 365.25 days to find the distance from the earth to the sun.

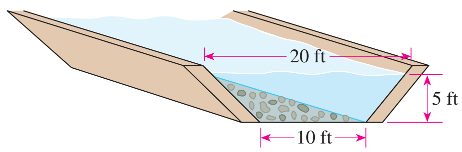
97. Flow Speed in a Channel The speed of water flowing in a channel, such as a canal or river bed, is governed by the **Manning Equation**

$$V = 1.486 \frac{A^{2/3} S^{1/2}}{p^{2/3} n}$$

Here V is the velocity of the flow in ft/s; A is the cross-sectional area of the channel in square feet; S is the downward slope of the channel; p is the wetted perimeter in feet (the distance from the top of one bank, down the side of the channel, across the bottom, and up to the top of the other bank); and n is the roughness coefficient (a measure of the roughness of the channel bottom). This equation is used to predict the capacity of flood channels to handle runoff from heavy rainfalls. For the canal shown in the figure, $A = 75\ \text{ft}^2$, $S = 0.050$, $p = 24.1$ ft, and $n = 0.040$.

(a) Find the speed with which water flows through this canal.

(b) How many cubic feet of water can the canal discharge per second? [*Hint:* Multiply V by A to get the volume of the flow per second.]



Discovery • Discussion

98. How Big Is a Billion? If you have a million (10^6) dollars in a suitcase, and you spend a thousand (10^3) dollars each day, how many years would it take you to use all the money? Spending at the same rate, how many years would it take you to empty a suitcase filled with a *billion* (10^9) dollars?

99. Easy Powers That Look Hard Calculate these expressions in your head. Use the Laws of Exponents to help you.

(a) $\dfrac{18^5}{9^5}$ **(b)** $20^6 \cdot (0.5)^6$

100. Limiting Behavior of Powers Complete the following tables. What happens to the nth root of 2 as n gets large? What about the nth root of $\frac{1}{2}$?

n	$2^{1/n}$
1	
2	
5	
10	
100	

n	$\left(\frac{1}{2}\right)^{1/n}$
1	
2	
5	
10	
100	

Construct a similar table for $n^{1/n}$. What happens to the nth root of n as n gets large?

101. Comparing Roots Without using a calculator, determine which number is larger in each pair.

(a) $2^{1/2}$ or $2^{1/3}$ **(b)** $\left(\frac{1}{2}\right)^{1/2}$ or $\left(\frac{1}{2}\right)^{1/3}$

(c) $7^{1/4}$ or $4^{1/3}$ **(d)** $\sqrt[3]{5}$ or $\sqrt{3}$

1.3 Algebraic Expressions

SUGGESTED TIME AND EMPHASIS

1–$1\frac{1}{2}$ classes.
Essential material.

A **variable** is a letter that can represent any number from a given set of numbers. If we start with variables such as x, y, and z and some real numbers, and combine them using addition, subtraction, multiplication, division, powers, and roots, we obtain an **algebraic expression**. Here are some examples:

$$2x^2 - 3x + 4 \qquad \sqrt{x} + 10 \qquad \frac{y - 2z}{y^2 + 4}$$

A **monomial** is an expression of the form ax^k, where a is a real number and k is a nonnegative integer. A **binomial** is a sum of two monomials and a **trinomial** is a sum of three monomials. In general, a sum of monomials is called a *polynomial*. For example, the first expression listed above is a polynomial, but the other two are not.

Polynomials

A **polynomial** in the variable x is an expression of the form

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where $a_0, a_1, \ldots, a_n$ are real numbers, and n is a nonnegative integer. If $a_n \neq 0$, then the polynomial has **degree** $\boldsymbol{n}$. The monomials $a_k x^k$ that make up the polynomial are called the **terms** of the polynomial.

POINTS TO STRESS

1. Definitions of polynomial, degree, terms, etc.
2. Algebraic operations with polynomials.
3. Special product formulas.
4. Factoring expressions by finding common factors and recognition of special cases.
5. Factoring quadratics by trial and error.

Note that the degree of a polynomial is the highest power of the variable that appears in the polynomial.

Polynomial	Type	Terms	Degree
$2x^2 - 3x + 4$	trinomial	$2x^2, -3x, 4$	2
$x^8 + 5x$	binomial	$x^8, 5x$	8
$3 - x + x^2 - \frac{1}{2}x^3$	four terms	$-\frac{1}{2}x^3, x^2, -x, 3$	3
$5x + 1$	binomial	$5x, 1$	1
$9x^5$	monomial	$9x^5$	5
6	monomial	6	0

Combining Algebraic Expressions

We **add** and **subtract** polynomials using the properties of real numbers that were discussed in Section 1.1. The idea is to combine **like terms** (that is, terms with the same variables raised to the same powers) using the Distributive Property. For instance,

Distributive Property

$ac + bc = (a + b)c$

$$5x^7 + 3x^7 = (5 + 3)x^7 = 8x^7$$

In subtracting polynomials we have to remember that if a minus sign precedes an expression in parentheses, then the sign of every term within the parentheses is changed when we remove the parentheses:

$$-(b + c) = -b - c$$

[This is simply a case of the Distributive Property, $a(b + c) = ab + ac$, with $a = -1$.]

Example 1 Adding and Subtracting Polynomials

(a) Find the sum $(x^3 - 6x^2 + 2x + 4) + (x^3 + 5x^2 - 7x)$.

(b) Find the difference $(x^3 - 6x^2 + 2x + 4) - (x^3 + 5x^2 - 7x)$.

Solution

(a) $(x^3 - 6x^2 + 2x + 4) + (x^3 + 5x^2 - 7x)$

$= (x^3 + x^3) + (-6x^2 + 5x^2) + (2x - 7x) + 4$ Group like terms

$= 2x^3 - x^2 - 5x + 4$ Combine like terms

(b) $(x^3 - 6x^2 + 2x + 4) - (x^3 + 5x^2 - 7x)$

$= x^3 - 6x^2 + 2x + 4 - x^3 - 5x^2 + 7x$ Distributive Property

$= (x^3 - x^3) + (-6x^2 - 5x^2) + (2x + 7x) + 4$ Group like terms

$= -11x^2 + 9x + 4$ Combine like terms ■

To find the **product** of polynomials or other algebraic expressions, we need to use the Distributive Property repeatedly. In particular, using it three times on the product of two binomials, we get

$$(a + b)(c + d) = a(c + d) + b(c + d) = ac + ad + bc + bd$$

ALTERNATE EXAMPLE 1b

Find the difference:
$(x^3 - 8x^2 + 4x + 7) - (x^3 + 7x^2 - 5x)$

ANSWER

$-15x^2 + 9x + 7$

SAMPLE QUESTIONS

Text Questions

(a) Give an example of an expression that is a polynomial.

(b) Give an example of an expression that is not a polynomial.

This says that we multiply the two factors by multiplying each term in one factor by each term in the other factor and adding these products. Schematically we have

$$(a+b)(c+d) = \underset{\text{F}}{ac} + \underset{\text{O}}{ad} + \underset{\text{I}}{bc} + \underset{\text{L}}{bd}$$

The acronym **FOIL** helps us remember that the product of two binomials is the sum of the products of the **F**irst terms, the **O**uter terms, the **I**nner terms, and the **L**ast terms.

In general, we can multiply two algebraic expressions by using the Distributive Property and the Laws of Exponents.

ALTERNATE EXAMPLE 2a
Find the product:
$(4x+1)(5x-4)$

ANSWER
$20x^2 - 11x - 4$

Example 2 Multiplying Algebraic Expressions

(a) $(2x+1)(3x-5) = \underset{\text{F}}{6x^2} - \underset{\text{O}}{10x} + \underset{\text{I}}{3x} - \underset{\text{L}}{5}$ Distributive Property

$= 6x^2 - 7x - 5$ Combine like terms

(b) $(x^2-3)(x^3+2x+1) = x^2(x^3+2x+1) - 3(x^3+2x+1)$ Distributive Property

$= x^5 + 2x^3 + x^2 - 3x^3 - 6x - 3$ Distributive Property

$= x^5 - x^3 + x^2 - 6x - 3$ Combine like terms

(c) $(1+\sqrt{x})(2-3\sqrt{x}) = 2 - 3\sqrt{x} + 2\sqrt{x} - 3(\sqrt{x})^2$ Distributive Property

$= 2 - \sqrt{x} - 3x$ Combine like terms ■

Certain types of products occur so frequently that you should memorize them. You can verify the following formulas by performing the multiplications.

See the Discovery Project on page 34 for a geometric interpretation of some of these formulas.

Special Product Formulas

If A and B are any real numbers or algebraic expressions, then

1. $(A+B)(A-B) = A^2 - B^2$ Sum and product of same terms
2. $(A+B)^2 = A^2 + 2AB + B^2$ Square of a sum
3. $(A-B)^2 = A^2 - 2AB + B^2$ Square of a difference
4. $(A+B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$ Cube of a sum
5. $(A-B)^3 = A^3 - 3A^2B + 3AB^2 - B^3$ Cube of a difference

The key idea in using these formulas (or any other formula in algebra) is the **Principle of Substitution**: We may substitute any algebraic expression for any letter in a formula. For example, to find $(x^2+y^3)^2$ we use Product Formula 2, substituting x^2 for A and y^3 for B, to get

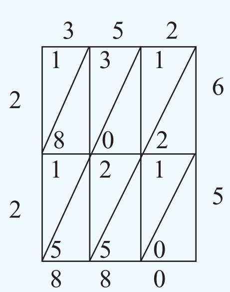
$$(x^2+y^3)^2 = (x^2)^2 + 2(x^2)(y^3) + (y^3)^2$$

$$(A+B)^2 = A^2 + 2AB + B^2$$

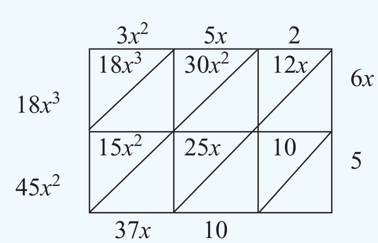
IN-CLASS MATERIALS

Remind students of the standard multiplication algorithm (some people don't learn it) by multiplying 352 by 65. Then multiply $3x^2+5x+2$ by $6x+5$ using the algorithm in the text. If your students are used to "lattice multiplication" this can also be done. One doesn't really need the "lattice" but the process can be put into the same form they are used to.

$$\begin{array}{r} 352 \\ \times\ 65 \\ \hline 1760 \\ 2112 \\ \hline 22880 \end{array}$$



$$\begin{array}{r} 3x^2 + 5x + 2 \\ \times \qquad 6x + 5 \\ \hline 15x^2 + 25x + 10 \\ 18x^3 + 30x^2 + 12x \\ \hline 18x^3 + 45x^2 + 37x + 10 \end{array}$$



Example 3 **Using the Special Product Formulas**

Use the Special Product Formulas to find each product.

(a) $(3x + 5)^2$ (b) $(x^2 - 2)^3$ (c) $(2x - \sqrt{y})(2x + \sqrt{y})$

Solution

(a) Substituting $A = 3x$ and $B = 5$ in Product Formula 2, we get

$$(3x + 5)^2 = (3x)^2 + 2(3x)(5) + 5^2 = 9x^2 + 30x + 25$$

(b) Substituting $A = x^2$ and $B = 2$ in Product Formula 5, we get

$$\begin{aligned}(x^2 - 2)^3 &= (x^2)^3 - 3(x^2)^2(2) + 3(x^2)(2)^2 - 2^3 \\ &= x^6 - 6x^4 + 12x^2 - 8\end{aligned}$$

(c) Substituting $A = 2x$ and $B = \sqrt{y}$ in Product Formula 1, we get

$$\begin{aligned}(2x - \sqrt{y})(2x + \sqrt{y}) &= (2x)^2 - (\sqrt{y})^2 \\ &= 4x^2 - y\end{aligned}$$

■

ALTERNATE EXAMPLE 3c
Find the product:
$(6x - \sqrt{y})(6x + \sqrt{y})$

ANSWER
$36x^2 - y$

Factoring

We use the Distributive Property to expand algebraic expressions. We sometimes need to reverse this process (again using the Distributive Property) by **factoring** an expression as a product of simpler ones. For example, we can write

FACTORING →

$$x^2 - 4 = (x - 2)(x + 2)$$

← EXPANDING

We say that $x - 2$ and $x + 2$ are **factors** of $x^2 - 4$.

The easiest type of factoring occurs when the terms have a common factor.

Example 4 **Factoring Out Common Factors**

Factor each expression.

(a) $3x^2 - 6x$ (b) $8x^4y^2 + 6x^3y^3 - 2xy^4$

(c) $(2x + 4)(x - 3) - 5(x - 3)$

Solution

(a) The greatest common factor of the terms $3x^2$ and $-6x$ is $3x$, so we have

$$3x^2 - 6x = 3x(x - 2)$$

(b) We note that

8, 6, and -2 have the greatest common factor 2

x^4, x^3, and x have the greatest common factor x

y^2, y^3, and y^4 have the greatest common factor y^2

So the greatest common factor of the three terms in the polynomial is $2xy^2$, and we have

$$\begin{aligned}8x^4y^2 + 6x^3y^3 - 2xy^4 &= (2xy^2)(4x^3) + (2xy^2)(3x^2y) + (2xy^2)(-y^2) \\ &= 2xy^2(4x^3 + 3x^2y - y^2)\end{aligned}$$

Check Your Answer

Multiplying gives

$3x(x - 2) = 3x^2 - 6x$ ✓

Check Your Answer

Multiplying gives

$2xy^2(4x^3 + 3x^2y - y^2) =$

$8x^4y^2 + 6x^3y^3 - 2xy^4$ ✓

ALTERNATE EXAMPLE 4a
Factor the expression:
$3x^2 - 21x.$

ANSWER
$3x(x - 7)$

DRILL QUESTION
Factor the polynomial:
$12x^3 + 18x^2y$

Answer
$6x^2(2x + 3y)$

IN-CLASS MATERIALS

This is an interesting product to look at with students:

$$(x - 1)(1 + x + x^2 + x^3 + x^4 + \cdots + x^n) = x^{n+1} - 1$$

Have them work it out for $n = 2, 3, 4$ and see the pattern. Once they see how it goes, you can derive

$$(1 + x + x^2 + x^3 + x^4 + \cdots + x^n) = \frac{x^{n+1} - 1}{x - 1}$$

and use it to estimate such things as $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ or the sum of any geometric series.

(c) The two terms have the common factor $x - 3$.

$$(2x + 4)(x - 3) - 5(x - 3) = [(2x + 4) - 5](x - 3) \quad \text{Distributive Property}$$
$$= (2x - 1)(x - 3) \quad \text{Simplify}$$
■

To factor a trinomial of the form $x^2 + bx + c$, we note that

$$(x + r)(x + s) = x^2 + (r + s)x + rs$$

so we need to choose numbers r and s so that $r + s = b$ and $rs = c$.

ALTERNATE EXAMPLE 5
Factor:
$x^2 + x - 6$

ANSWER
$(x - 2)(x + 3)$

Example 5 Factoring $x^2 + bx + c$ by Trial and Error

Factor: $x^2 + 7x + 12$

Solution We need to find two integers whose product is 12 and whose sum is 7. By trial and error we find that the two integers are 3 and 4. Thus, the factorization is

$$x^2 + 7x + 12 = \underbrace{(x + 3)(x + 4)}_{\text{factors of 12}}$$
■

Check Your Answer

Multiplying gives

$(x + 3)(x + 4) = x^2 + 7x + 12$ ✓

To factor a trinomial of the form $ax^2 + bx + c$ with $a \neq 1$, we look for factors of the form $px + r$ and $qx + s$:

$$ax^2 + bx + c = (px + r)(qx + s) = pqx^2 + (ps + qr)x + rs$$

Therefore, we try to find numbers p, q, r, and s such that $pq = a$, $rs = c$, $ps + qr = b$. If these numbers are all integers, then we will have a limited number of possibilities to try for p, q, r, and s.

$$ax^2 + bx + c = (\overset{\text{factors of } a}{p}x + \underset{\text{factors of } c}{r})(\overset{}{q}x + s)$$

(factors of a: p and q; factors of c: r and s)

ALTERNATE EXAMPLE 6
Factor:
$6x^2 - x - 2$

ANSWER
$(2x + 1)(3x - 2)$

Example 6 Factoring $ax^2 + bx + c$ by Trial and Error

Factor: $6x^2 + 7x - 5$

Solution We can factor 6 as $6 \cdot 1$ or $3 \cdot 2$, and -5 as $-5 \cdot 1$ or $5 \cdot (-1)$. By trying these possibilities, we arrive at the factorization

$$6x^2 + 7x - 5 = (3x + 5)(2x - 1)$$

(factors of 6: $3x$ and $2x$; factors of -5: 5 and -1)
■

Check Your Answer

Multiplying gives

$(3x + 5)(2x - 1) = 6x^2 + 7x - 5$ ✓

ALTERNATE EXAMPLE 7
Factor each expression.
(a) $x^2 + 5x + 6$
(b) $(2a - 1)^2 + 5(2a - 1) + 6$

ANSWERS
(a) $(x + 2)(x + 3)$
(b) $(2a + 1)(2a + 2)$

Example 7 Recognizing the Form of an Expression

Factor each expression.

(a) $x^2 - 2x - 3$ (b) $(5a + 1)^2 - 2(5a + 1) - 3$

Solution

(a) $x^2 - 2x - 3 = (x - 3)(x + 1)$ Trial and error

(b) This expression is of the form

$$\square^2 - 2\square - 3$$

IN-CLASS MATERIALS

After doing some more routine examples, show your students that grouping is not always obvious, as in examples like

$$2xa + 4y + ay + 8x = (2xa + ay) + (4y + 8x) = a(2x + y) + 4(y + 2x) = (2x + y)(a + 4)$$

Some of the standard formulas can also be awkward. For example, $x^2 - 2$ can be factored, even though we don't often refer to the number 2 as a "square."

where $\square$ represents $5a + 1$. This is the same form as the expression in part (a), so it will factor as $(\square - 3)(\square + 1)$.

$$(5a + 1)^2 - 2(5a + 1) - 3 = [(5a + 1) - 3][(5a + 1) + 1]$$
$$= (5a - 2)(5a + 2)$$ ■

Some special algebraic expressions can be factored using the following formulas. The first three are simply Special Product Formulas written backward.

Special Factoring Formulas

Formula	Name
1. $A^2 - B^2 = (A - B)(A + B)$	Difference of squares
2. $A^2 + 2AB + B^2 = (A + B)^2$	Perfect square
3. $A^2 - 2AB + B^2 = (A - B)^2$	Perfect square
4. $A^3 - B^3 = (A - B)(A^2 + AB + B^2)$	Difference of cubes
5. $A^3 + B^3 = (A + B)(A^2 - AB + B^2)$	Sum of cubes

Example 8 Factoring Differences of Squares

Factor each polynomial.

(a) $4x^2 - 25$ (b) $(x + y)^2 - z^2$

Solution

(a) Using the Difference of Squares Formula with $A = 2x$ and $B = 5$, we have

$$4x^2 - 25 = (2x)^2 - 5^2 = (2x - 5)(2x + 5)$$

$A^2 - B^2 = (A - B)(A + B)$

(b) We use the Difference of Squares Formula with $A = x + y$ and $B = z$.

$$(x + y)^2 - z^2 = (x + y - z)(x + y + z)$$ ■

Example 9 Factoring Differences and Sums of Cubes

Factor each polynomial.

(a) $27x^3 - 1$ (b) $x^6 + 8$

Solution

(a) Using the Difference of Cubes Formula with $A = 3x$ and $B = 1$, we get

$$27x^3 - 1 = (3x)^3 - 1^3 = (3x - 1)[(3x)^2 + (3x)(1) + 1^2]$$
$$= (3x - 1)(9x^2 + 3x + 1)$$

ALTERNATE EXAMPLE 8a
Factor the polynomial:
$4x^2 - 49$

ANSWER
$(2x - 7)(2x + 7)$

ALTERNATE EXAMPLE 8b
Factor the polynomial:
$x^6 + 125$

ANSWER
$(x^2 + 5)(x^4 - 5x^2 + 25)$

ALTERNATE EXAMPLE 9a
Factor the expression completely:
$2x^4 - 8x^2$

ANSWER
$2x^2(x - 2)(x + 2)$

IN-CLASS MATERIALS

Most people would look at $x^4 + 324$ and think it cannot be factored. (Point out that it could be if it was $x^4 - 324$.) It can be factored, although the factorization is not obvious:

$$\begin{aligned} x^4 + 324 &= x^4 + 36x^2 - 36x^2 + 324 \\ &= (x^2 + 18)^2 - 36x^2 \\ &= (x^2 + 18)^2 - (6x)^2 \\ &= (x^2 + 6x + 18)(x^2 - 6x + 18) \end{aligned}$$

ALTERNATE EXAMPLE 10b
Factor the trinomial:
$25x^2 - 10xy + y^2$

ANSWER
$(5x - y)^2$

ALTERNATE EXAMPLE 11
Factor:
$12x^3y^2 - 3xy^4$

ANSWER
$3xy^2(2x + y)(2x - y)$

Mathematics in the Modern World

Changing Words, Sound, and Pictures into Numbers

Pictures, sound, and text are routinely transmitted from one place to another via the Internet, fax machines, or modems. How can such things be transmitted through telephone wires? The key to doing this is to change them into numbers or bits (the digits 0 or 1). It's easy to see how to change text to numbers. For example, we could use the correspondence A = 00000001, B = 00000010, C = 00000011, D = 00000100, E = 00000101, and so on. The word "BED" then becomes 000000100000010100000100. By reading the digits in groups of eight, it is possible to translate this number back to the word "BED."

Changing sound to bits is more complicated. A sound wave can be graphed on an oscilloscope or a computer. The graph is then broken down mathematically into simpler components corresponding to the different frequencies of the original sound. (A branch of mathematics called Fourier analysis is used here.) The intensity of each component is a number, and the original sound can be reconstructed from these numbers. For example, music is stored on a CD as a sequence of bits; it may look like 101010001010010100101010 1000001011110101000101011.... (One second of music requires 1.5 million bits!) The CD player reconstructs the music from the numbers on the CD.

Changing pictures into numbers involves expressing the color and brightness of each dot (or pixel) into a number. This is done very efficiently using a branch of mathematics called wavelet theory. The FBI uses wavelets as a compact way to store the millions of fingerprints they need on file.

(b) Using the Sum of Cubes Formula with $A = x^2$ and $B = 2$, we have

$$x^6 + 8 = (x^2)^3 + 2^3 = (x^2 + 2)(x^4 - 2x^2 + 4)$$ ■

A trinomial is a perfect square if it is of the form

$$A^2 + 2AB + B^2 \qquad \text{or} \qquad A^2 - 2AB + B^2$$

So, we **recognize a perfect square** if the middle term ($2AB$ or $-2AB$) is plus or minus twice the product of the square roots of the outer two terms.

Example 10 Recognizing Perfect Squares

Factor each trinomial.

(a) $x^2 + 6x + 9$ (b) $4x^2 - 4xy + y^2$

Solution

(a) Here $A = x$ and $B = 3$, so $2AB = 2 \cdot x \cdot 3 = 6x$. Since the middle term is $6x$, the trinomial is a perfect square. By the Perfect Square Formula, we have

$$x^2 + 6x + 9 = (x + 3)^2$$

(b) Here $A = 2x$ and $B = y$, so $2AB = 2 \cdot 2x \cdot y = 4xy$. Since the middle term is $-4xy$, the trinomial is a perfect square. By the Perfect Square Formula, we have

$$4x^2 - 4xy + y^2 = (2x - y)^2$$ ■

When we factor an expression, the result can sometimes be factored further. In general, *we first factor out common factors*, then inspect the result to see if it can be factored by any of the other methods of this section. We repeat this process until we have factored the expression completely.

Example 11 Factoring an Expression Completely

Factor each expression completely.

(a) $2x^4 - 8x^2$ (b) $x^5y^2 - xy^6$

Solution

(a) We first factor out the power of x with the smallest exponent.

$$2x^4 - 8x^2 = 2x^2(x^2 - 4) \qquad \text{Common factor is } 2x^2$$

$$= 2x^2(x - 2)(x + 2) \qquad \text{Factor } x^2 - 4 \text{ as a difference of squares}$$

(b) We first factor out the powers of x and y with the smallest exponents.

$$x^5y^2 - xy^6 = xy^2(x^4 - y^4) \qquad \text{Common factor is } xy^2$$

$$= xy^2(x^2 + y^2)(x^2 - y^2) \qquad \text{Factor } x^4 - y^4 \text{ as a difference of squares}$$

$$= xy^2(x^2 + y^2)(x + y)(x - y) \qquad \text{Factor } x^2 - y^2 \text{ as a difference of squares}$$ ■

In the next example we factor out variables with fractional exponents. This type of factoring occurs in calculus.

IN-CLASS MATERIALS

This is a good opportunity to foreshadow polynomial division, which is covered in Section 3.2. Have students try to factor an expression such as $x^3 + 2x^2 - 21x + 18$. Now assume you have the hint that $x - 1$ is a factor. Show them how you can use the hint by dividing the polynomial by $x - 1$ to get a remaining quadratic which is easy to break down.

Answer

$(x - 3)(x + 6)(x - 1)$

Example 12 Factoring Expressions with Fractional Exponents

Factor each expression.

(a) $3x^{3/2} - 9x^{1/2} + 6x^{-1/2}$ (b) $(2 + x)^{-2/3}x + (2 + x)^{1/3}$

Solution

(a) Factor out the power of x with the *smallest exponent*, that is, $x^{-1/2}$.

$$3x^{3/2} - 9x^{1/2} + 6x^{-1/2} = 3x^{-1/2}(x^2 - 3x + 2) \quad \text{Factor out } 3x^{-1/2}$$

$$= 3x^{-1/2}(x - 1)(x - 2) \quad \text{Factor the quadratic } x^2 - 3x + 2$$

(b) Factor out the power of $2 + x$ with the *smallest exponent*, that is, $(2 + x)^{-2/3}$.

$$(2 + x)^{-2/3}x + (2 + x)^{1/3} = (2 + x)^{-2/3}[x + (2 + x)] \quad \text{Factor out } (2 + x)^{-2/3}$$

$$= (2 + x)^{-2/3}(2 + 2x) \quad \text{Simplify}$$

$$= 2(2 + x)^{-2/3}(1 + x) \quad \text{Factor out 2}$$

■

To factor out $x^{-1/2}$ from $x^{3/2}$, we *subtract* exponents:

$$x^{3/2} = x^{-1/2}(x^{3/2-(-1/2)})$$

$$= x^{-1/2}(x^{3/2+1/2})$$

$$= x^{-1/2}(x^2)$$

Check Your Answer

To see that you have factored correctly, multiply using the Laws of Exponents.

(a) $3x^{-1/2}(x^2 - 3x + 2)$

$= 3x^{3/2} - 9x^{1/2} + 6x^{-1/2}$ ✓

(b) $(2 + x)^{-2/3}[x + (2 + x)]$

$= (2 + x)^{-2/3}x + (2 + x)^{1/3}$ ✓

Polynomials with at least four terms can sometimes be factored by grouping terms. The following example illustrates the idea.

Example 13 Factoring by Grouping

Factor each polynomial.

(a) $x^3 + x^2 + 4x + 4$ (b) $x^3 - 2x^2 - 3x + 6$

Solution

(a)
$$x^3 + x^2 + 4x + 4 = (x^3 + x^2) + (4x + 4) \quad \text{Group terms}$$

$$= x^2(x + 1) + 4(x + 1) \quad \text{Factor out common factors}$$

$$= (x^2 + 4)(x + 1) \quad \text{Factor out } x + 1 \text{ from each term}$$

(b)
$$x^3 - 2x^2 - 3x + 6 = (x^3 - 2x^2) - (3x - 6) \quad \text{Group terms}$$

$$= x^2(x - 2) - 3(x - 2) \quad \text{Factor out common factors}$$

$$= (x^2 - 3)(x - 2) \quad \text{Factor out } x - 2 \text{ from each term}$$

■

ALTERNATE EXAMPLE 12a

Factor the expression:

$3x^{3/2} - 15x^{1/2} + 18x^{-1/2}$

ANSWER

$3x^{-1/2}(x - 2)(x - 3)$

ALTERNATE EXAMPLE 13b

Factor the polynomial:

$x^3 - 5x^2 - 2x + 10$

ANSWER

$(x^2 - 2)(x - 5)$

1.3 Exercises

1–6 ■ Complete the following table by stating whether the polynomial is a monomial, binomial, or trinomial; then list its terms and state its degree.

Polynomial	Type	Terms	Degree
1. $x^2 - 3x + 7$			
2. $2x^5 + 4x^2$			
3. -8			
4. $\frac{1}{2}x^7$			
5. $x - x^2 + x^3 - x^4$			
6. $\sqrt{2}x - \sqrt{3}$			

7–42 ■ Perform the indicated operations and simplify.

7. $(12x - 7) - (5x - 12)$ **8.** $(5 - 3x) + (2x - 8)$

9. $(3x^2 + x + 1) + (2x^2 - 3x - 5)$

10. $(3x^2 + x + 1) - (2x^2 - 3x - 5)$

11. $(x^3 + 6x^2 - 4x + 7) - (3x^2 + 2x - 4)$

12. $3(x - 1) + 4(x + 2)$

13. $8(2x + 5) - 7(x - 9)$

14. $4(x^2 - 3x + 5) - 3(x^2 - 2x + 1)$

15. $2(2 - 5t) + t^2(t - 1) - (t^4 - 1)$

16. $5(3t - 4) - (t^2 + 2) - 2t(t - 3)$

EXAMPLES

1. A cubic product: $(3x^3 - 2x^2 + 4x - 5)(x^3 - 2x^2 - 4x + 1) = 3x^6 - 8x^5 - 4x^4 - 2x^3 - 8x^2 + 24x - 5$

2. A fourth degree polynomial with integer factors: $(x - 1)(x - 1)(x + 2)(x - 3) = x^4 - 3x^3 - 3x^2 + 11x - 6$

3. A polynomial that can be factored nicely using the method of "In-Class Material" on page 29:
$x^4 + 64 = (x^2 - 4x + 8)(x^2 + 4x + 8)$

17. $\sqrt{x}(x - \sqrt{x})$ **18.** $x^{3/2}(\sqrt{x} - 1/\sqrt{x})$

19. $(3t - 2)(7t - 5)$ **20.** $(4x - 1)(3x + 7)$

21. $(x + 2y)(3x - y)$ **22.** $(4x - 3y)(2x + 5y)$

23. $(1 - 2y)^2$ **24.** $(3x + 4)^2$

25. $(2x^2 + 3y^2)^2$ **26.** $\left(c + \frac{1}{c}\right)^2$

27. $(2x - 5)(x^2 - x + 1)$ **28.** $(1 + 2x)(x^2 - 3x + 1)$

29. $(x^2 - a^2)(x^2 + a^2)$ **30.** $(x^{1/2} + y^{1/2})(x^{1/2} - y^{1/2})$

31. $\left(\sqrt{a} - \frac{1}{b}\right)\left(\sqrt{a} + \frac{1}{b}\right)$

32. $(\sqrt{h^2 + 1} + 1)(\sqrt{h^2 + 1} - 1)$

33. $(1 + a^3)^3$

34. $(1 - 2y)^3$

35. $(x^2 + x - 1)(2x^2 - x + 2)$

36. $(3x^3 + x^2 - 2)(x^2 + 2x - 1)$

37. $(1 + x^{4/3})(1 - x^{2/3})$ **38.** $(1 - b)^2(1 + b)^2$

39. $(3x^2y + 7xy^2)(x^2y^3 - 2y^2)$ **40.** $(x^4y - y^5)(x^2 + xy + y^2)$

41. $(x + y + z)(x - y - z)$ **42.** $(x^2 - y + z)(x^2 + y - z)$

43–48 ■ Factor out the common factor.

43. $-2x^3 + 16x$ **44.** $2x^4 + 4x^3 - 14x^2$

45. $y(y - 6) + 9(y - 6)$ **46.** $(z + 2)^2 - 5(z + 2)$

47. $2x^2y - 6xy^2 + 3xy$ **48.** $-7x^4y^2 + 14xy^3 + 21xy^4$

49–54 ■ Factor the trinomial.

49. $x^2 + 2x - 3$ **50.** $x^2 - 6x + 5$

51. $8x^2 - 14x - 15$ **52.** $6y^2 + 11y - 21$

53. $(3x + 2)^2 + 8(3x + 2) + 12$

54. $2(a + b)^2 + 5(a + b) - 3$

55–60 ■ Use a Special Factoring Formula to factor the expression.

55. $9a^2 - 16$ **56.** $(x + 3)^2 - 4$

57. $27x^3 + y^3$ **58.** $8s^3 - 125t^6$

59. $x^2 + 12x + 36$ **60.** $16z^2 - 24z + 9$

61–66 ■ Factor the expression by grouping terms.

61. $x^3 + 4x^2 + x + 4$ **62.** $3x^3 - x^2 + 6x - 2$

63. $2x^3 + x^2 - 6x - 3$ **64.** $-9x^3 - 3x^2 + 3x + 1$

65. $x^3 + x^2 + x + 1$ **66.** $x^5 + x^4 + x + 1$

67–70 ■ Factor the expression completely. Begin by factoring out the lowest power of each common factor.

67. $x^{5/2} - x^{1/2}$ **68.** $x^{-3/2} + 2x^{-1/2} + x^{1/2}$

69. $(x^2 + 1)^{1/2} + 2(x^2 + 1)^{-1/2}$

70. $2x^{1/3}(x - 2)^{2/3} - 5x^{4/3}(x - 2)^{-1/3}$

71–100 ■ Factor the expression completely.

71. $12x^3 + 18x$ **72.** $5ab - 8abc$

73. $x^2 - 2x - 8$ **74.** $y^2 - 8y + 15$

75. $2x^2 + 5x + 3$ **76.** $9x^2 - 36x - 45$

77. $6x^2 - 5x - 6$ **78.** $r^2 - 6rs + 9s^2$

79. $25s^2 - 10st + t^2$ **80.** $x^2 - 36$

81. $4x^2 - 25$ **82.** $49 - 4y^2$

83. $(a + b)^2 - (a - b)^2$

84. $\left(1 + \frac{1}{x}\right)^2 - \left(1 - \frac{1}{x}\right)^2$

85. $x^2(x^2 - 1) - 9(x^2 - 1)$ **86.** $(a^2 - 1)b^2 - 4(a^2 - 1)$

87. $8x^3 + 125$ **88.** $x^6 + 64$

89. $x^6 - 8y^3$ **90.** $27a^3 - b^6$

91. $x^3 + 2x^2 + x$ **92.** $3x^3 - 27x$

93. $y^3 - 3y^2 - 4y + 12$ **94.** $x^3 + 3x^2 - x - 3$

95. $2x^3 + 4x^2 + x + 2$ **96.** $3x^3 + 5x^2 - 6x - 10$

97. $(x - 1)(x + 2)^2 - (x - 1)^2(x + 2)$

98. $y^4(y + 2)^3 + y^5(y + 2)^4$

99. $(a^2 + 1)^2 - 7(a^2 + 1) + 10$

100. $(a^2 + 2a)^2 - 2(a^2 + 2a) - 3$

101–104 ■ Factor the expression completely. (This type of expression arises in calculus when using the "product rule.")

101. $5(x^2 + 4)^4(2x)(x - 2)^4 + (x^2 + 4)^5(4)(x - 2)^3$

102. $3(2x - 1)^2(2)(x + 3)^{1/2} + (2x - 1)^3(\frac{1}{2})(x + 3)^{-1/2}$

103. $(x^2 + 3)^{-1/3} - \frac{2}{3}x^2(x^2 + 3)^{-4/3}$

104. $\frac{1}{2}x^{-1/2}(3x + 4)^{1/2} - \frac{3}{2}x^{1/2}(3x + 4)^{-1/2}$

105. **(a)** Show that $ab = \frac{1}{2}[(a + b)^2 - (a^2 + b^2)]$.

(b) Show that $(a^2 + b^2)^2 - (a^2 - b^2)^2 = 4a^2b^2$.

(c) Show that

$$(a^2 + b^2)(c^2 + d^2) = (ac + bd)^2 + (ad - bc)^2$$

(d) Factor completely: $4a^2c^2 - (a^2 - b^2 + c^2)^2$.

106. Verify Special Factoring Formulas 4 and 5 by expanding their right-hand sides.

Applications

107. Volume of Concrete A culvert is constructed out of large cylindrical shells cast in concrete, as shown in the figure. Using the formula for the volume of a cylinder given on the inside back cover of this book, explain why the volume of the cylindrical shell is

$$V = \pi R^2 h - \pi r^2 h$$

Factor to show that

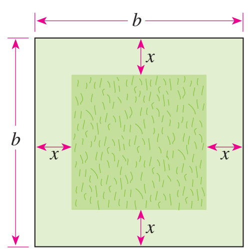
$$V = 2\pi \cdot \text{average radius} \cdot \text{height} \cdot \text{thickness}$$

Use the "unrolled" diagram to explain why this makes sense geometrically.

108. Mowing a Field A square field in a certain state park is mowed around the edges every week. The rest of the field is kept unmowed to serve as a habitat for birds and small animals (see the figure). The field measures b feet by b feet, and the mowed strip is x feet wide.

(a) Explain why the area of the mowed portion is $b^2 - (b - 2x)^2$.

(b) Factor the expression in (a) to show that the area of the mowed portion is also $4x(b - x)$.



Discovery • Discussion

109. Degrees of Sums and Products of Polynomials Make up several pairs of polynomials, then calculate the sum and product of each pair. Based on your experiments and observations, answer the following questions.

(a) How is the degree of the product related to the degrees of the original polynomials?

(b) How is the degree of the sum related to the degrees of the original polynomials?

110. The Power of Algebraic Formulas Use the Difference of Squares Formula to factor $17^2 - 16^2$. Notice that it is easy to calculate the factored form in your head, but not so easy to calculate the original form in this way. Evaluate each expression in your head:

(a) $528^2 - 527^2$ **(b)** $122^2 - 120^2$ **(c)** $1020^2 - 1010^2$

Now use the Special Product Formula

$$(A + B)(A - B) = A^2 - B^2$$

to evaluate these products in your head:

(d) $79 \cdot 51$ **(e)** $998 \cdot 1002$

111. Differences of Even Powers

(a) Factor the expressions completely: $A^4 - B^4$ and $A^6 - B^6$.

(b) Verify that $18{,}335 = 12^4 - 7^4$ and that $2{,}868{,}335 = 12^6 - 7^6$.

(c) Use the results of parts (a) and (b) to factor the integers 18,335 and 2,868,335. Then show that in both of these factorizations, all the factors are prime numbers.

112. Factoring $A^n - 1$ Verify these formulas by expanding and simplifying the right-hand side.

$$A^2 - 1 = (A - 1)(A + 1)$$
$$A^3 - 1 = (A - 1)(A^2 + A + 1)$$
$$A^4 - 1 = (A - 1)(A^3 + A^2 + A + 1)$$

Based on the pattern displayed in this list, how do you think $A^5 - 1$ would factor? Verify your conjecture. Now generalize the pattern you have observed to obtain a factoring formula for $A^n - 1$, where n is a positive integer.

113. Factoring $x^4 + ax^2 + b$ A trinomial of the form $x^4 + ax^2 + b$ can sometimes be factored easily. For example, $x^4 + 3x^2 - 4 = (x^2 + 4)(x^2 - 1)$. But $x^4 + 3x^2 + 4$ cannot be factored in this way. Instead, we can use the following method.

$x^4 + 3x^2 + 4 = (x^4 + 4x^2 + 4) - x^2$	Add and subtract x^2
$= (x^2 + 2)^2 - x^2$	Factor perfect square
$= [(x^2 + 2) - x][(x^2 + 2) + x]$	Difference of squares
$= (x^2 - x + 2)(x^2 + x + 2)$	

Factor the following using whichever method is appropriate.

(a) $x^4 + x^2 - 2$

(b) $x^4 + 2x^2 + 9$

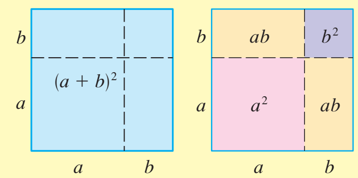
(c) $x^4 + 4x^2 + 16$

(d) $x^4 + 2x^2 + 1$

DISCOVERY PROJECT

Visualizing a Formula

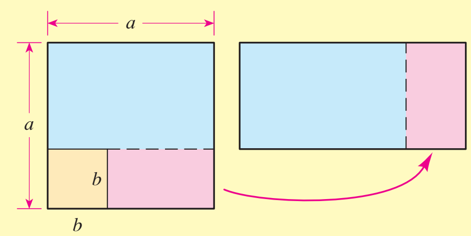
Many of the Special Product Formulas that we learned in this section can be "seen" as geometrical facts about length, area, and volume. For example, the figure shows how the formula for the square of a binomial can be interpreted as a fact about areas of squares and rectangles.



$$(a + b)^2 = a^2 + 2ab + b^2$$

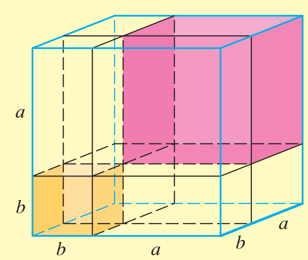
In the figure, a and b represent lengths, a^2, b^2, ab, and $(a + b)^2$ represent areas. The ancient Greeks always interpreted algebraic formulas in terms of geometric figures as we have done here.

1. Explain how the figure verifies the formula $a^2 - b^2 = (a + b)(a - b)$.



2. Find a figure that verifies the formula $(a - b)^2 = a^2 - 2ab + b^2$.

3. Explain how the figure verifies the formula
$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.



4. Is it possible to draw a geometric figure that verifies the formula for $(a + b)^4$? Explain.

5. **(a)** Expand $(a + b + c)^2$.

(b) Make a geometric figure that verifies the formula you found in part (a).

1.4 Rational Expressions

A quotient of two algebraic expressions is called a **fractional expression**. Here are some examples:

$$\frac{2x}{x-1} \qquad \frac{\sqrt{x}+3}{x+1} \qquad \frac{y-2}{y^2+4}$$

A **rational expression** is a fractional expression where both the numerator and denominator are polynomials. For example, the following are rational expressions:

$$\frac{2x}{x-1} \qquad \frac{x}{x^2+1} \qquad \frac{x^3-x}{x^2-5x+6}$$

In this section we learn how to perform algebraic operations on rational expressions.

The Domain of an Algebraic Expression

In general, an algebraic expression may not be defined for all values of the variable. The **domain** of an algebraic expression is the set of real numbers that the variable is permitted to have. The table in the margin gives some basic expressions and their domains.

Table 1

Expression	Domain
$\frac{1}{x}$	$\{x \mid x \neq 0\}$
$\sqrt{x}$	$\{x \mid x \geq 0\}$
$\frac{1}{\sqrt{x}}$	$\{x \mid x > 0\}$

Example 1 Finding the Domain of an Expression

Find the domains of the following expressions.

(a) $2x^2 + 3x - 1$ (b) $\dfrac{x}{x^2-5x+6}$ (c) $\dfrac{\sqrt{x}}{x-5}$

Solution

(a) This polynomial is defined for every x. Thus, the domain is the set $\mathbb{R}$ of real numbers.

(b) We first factor the denominator.

$$\frac{x}{x^2-5x+6} = \frac{x}{(x-2)(x-3)}$$

Denominator would be 0 if $x = 2$ or $x = 3$.

Since the denominator is zero when $x = 2$ or 3, the expression is not defined for these numbers. The domain is $\{x \mid x \neq 2 \text{ and } x \neq 3\}$.

(c) For the numerator to be defined, we must have $x \geq 0$. Also, we cannot divide by zero, so $x \neq 5$.

Must have $x \geq 0$ to take square root. $\qquad \dfrac{\sqrt{x}}{x-5} \qquad$ Denominator would be 0 if $x = 5$.

Thus, the domain is $\{x \mid x \geq 0 \text{ and } x \neq 5\}$. ■

SUGGESTED TIME AND EMPHASIS

$\frac{1}{2}$–1 class.

Review material.

ALTERNATE EXAMPLE 1

Find the domain of $\sqrt{x^2 + 3x + 2}$.

ANSWER

$\{x \mid x \leq -2 \text{ or } x \geq -1\}$

POINTS TO STRESS

1. Finding the domain of an algebraic expression.
2. Simplifying, adding, and subtracting rational expressions, including compound fractions.
3. Rationalizing numerators and denominators.

Simplifying Rational Expressions

To **simplify rational expressions**, we factor both numerator and denominator and use the following property of fractions:

$$\frac{AC}{BC} = \frac{A}{B}$$

This allows us to **cancel** common factors from the numerator and denominator.

ALTERNATE EXAMPLE 2
Simplify the fractional expression $\dfrac{x^2 - 64}{x^2 + x - 72}$.

ANSWER
$\dfrac{x + 8}{x + 9}$

Example 2 Simplifying Rational Expressions by Cancellation

Simplify: $\dfrac{x^2 - 1}{x^2 + x - 2}$

Solution

$$\frac{x^2 - 1}{x^2 + x - 2} = \frac{(x - 1)(x + 1)}{(x - 1)(x + 2)} \qquad \text{Factor}$$

$$= \frac{x + 1}{x + 2} \qquad \text{Cancel common factors}$$

■

We can't cancel the x^2's in $\dfrac{x^2 - 1}{x^2 + x - 2}$ because x^2 is not a factor.

Multiplying and Dividing Rational Expressions

To **multiply rational expressions**, we use the following property of fractions:

$$\frac{A}{B} \cdot \frac{C}{D} = \frac{AC}{BD}$$

This says that to multiply two fractions we multiply their numerators and multiply their denominators.

ALTERNATE EXAMPLE 3
Perform the indicated multiplication and simplify: $\dfrac{x^2 + 4x - 12}{x^2 + 8x + 16} \cdot \dfrac{2x + 8}{x - 2}$

ANSWER
$\dfrac{2(x + 6)}{x + 4}$

Example 3 Multiplying Rational Expressions

Perform the indicated multiplication and simplify: $\dfrac{x^2 + 2x - 3}{x^2 + 8x + 16} \cdot \dfrac{3x + 12}{x - 1}$

Solution We first factor.

$$\frac{x^2 + 2x - 3}{x^2 + 8x + 16} \cdot \frac{3x + 12}{x - 1} = \frac{(x - 1)(x + 3)}{(x + 4)^2} \cdot \frac{3(x + 4)}{x - 1} \qquad \text{Factor}$$

$$= \frac{3(x - 1)(x + 3)(x + 4)}{(x - 1)(x + 4)^2} \qquad \text{Property of fractions}$$

$$= \frac{3(x + 3)}{x + 4} \qquad \text{Cancel common factors}$$

■

To **divide rational expressions**, we use the following property of fractions:

$$\frac{A}{B} \div \frac{C}{D} = \frac{A}{B} \cdot \frac{D}{C}$$

SAMPLE QUESTION
Text Question
What is a rational expression?

Answer
A rational expression is a fractional expression where both the numerator and denominator are polynomials.

This says that to divide a fraction by another fraction we invert the divisor and multiply.

Example 4 Dividing Rational Expressions

Perform the indicated division and simplify: $\dfrac{x-4}{x^2-4} \div \dfrac{x^2-3x-4}{x^2+5x+6}$

Solution

$$\frac{x-4}{x^2-4} \div \frac{x^2-3x-4}{x^2+5x+6} = \frac{x-4}{x^2-4} \cdot \frac{x^2+5x+6}{x^2-3x-4} \quad \text{Invert and multiply}$$

$$= \frac{(x-4)(x+2)(x+3)}{(x-2)(x+2)(x-4)(x+1)} \quad \text{Factor}$$

$$= \frac{x+3}{(x-2)(x+1)} \quad \text{Cancel common factors}$$

■

ALTERNATE EXAMPLE 4

Perform the indicated division and simplify:

$\dfrac{x-5}{x^2-16} \div \dfrac{x^2-2x-15}{x^2+11x+28}$

ANSWER

$\dfrac{x+7}{(x-4)(x+3)}$

Adding and Subtracting Rational Expressions

Avoid making the following error:

$$\frac{A}{B+C} \cancel{=} \frac{A}{B} + \frac{A}{C}$$

For instance, if we let $A = 2$, $B = 1$, and $C = 1$, then we see the error:

$$\frac{2}{1+1} \stackrel{?}{=} \frac{2}{1} + \frac{2}{1}$$

$$\frac{2}{2} \stackrel{?}{=} 2 + 2$$

$$1 \stackrel{?}{=} 4 \quad \text{Wrong!}$$

To **add or subtract rational expressions**, we first find a common denominator and then use the following property of fractions:

$$\frac{A}{C} + \frac{B}{C} = \frac{A+B}{C}$$

Although any common denominator will work, it is best to use the **least common denominator** (LCD) as explained in Section 1.1. The LCD is found by factoring each denominator and taking the product of the distinct factors, using the highest power that appears in any of the factors.

Example 5 Adding and Subtracting Rational Expressions

Perform the indicated operations and simplify:

(a) $\dfrac{3}{x-1} + \dfrac{x}{x+2}$ (b) $\dfrac{1}{x^2-1} - \dfrac{2}{(x+1)^2}$

Solution

(a) Here the LCD is simply the product $(x-1)(x+2)$.

$$\frac{3}{x-1} + \frac{x}{x+2} = \frac{3(x+2)}{(x-1)(x+2)} + \frac{x(x-1)}{(x-1)(x+2)} \quad \text{Write fractions using LCD}$$

$$= \frac{3x+6+x^2-x}{(x-1)(x+2)} \quad \text{Add fractions}$$

$$= \frac{x^2+2x+6}{(x-1)(x+2)} \quad \text{Combine terms in numerator}$$

ALTERNATE EXAMPLE 5a

Perform the addition and simplify:

$\dfrac{5}{x-1} + \dfrac{x}{x+3}$

ANSWER

$\dfrac{x^2+4x+15}{(x-1)(x+3)}$

IN-CLASS MATERIALS

One of the most persistent mistakes students make is confusing the following two expressions:

$$\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c} \qquad \frac{a}{b+c} = \frac{a}{b} + \frac{a}{c} \leftarrow \text{Wrong!}$$

Make sure students understand the difference between these two cases. Perhaps give them the following expressions to simplify:

$$\frac{2+3}{5} = \frac{2}{5} + \frac{3}{5} \qquad \frac{u+1}{u} = 1 + \frac{1}{u} \qquad \frac{x^3+x+\sqrt[3]{x}}{x} = x^2 + 1 + x^{-2/3}$$

$$\frac{5}{2+3} = \frac{5}{2+3} \qquad \frac{u}{u+1} = \frac{u}{u+1} \qquad \frac{x}{x^3+x+\sqrt[3]{x}} = \frac{x}{x^3+x+\sqrt[3]{x}}$$

(b) The LCD of $x^2 - 1 = (x-1)(x+1)$ and $(x+1)^2$ is $(x-1)(x+1)^2$.

$$\begin{aligned}
\frac{1}{x^2-1} - \frac{2}{(x+1)^2} &= \frac{1}{(x-1)(x+1)} - \frac{2}{(x+1)^2} && \text{Factor}\\
&= \frac{(x+1) - 2(x-1)}{(x-1)(x+1)^2} && \text{Combine fractions using LCD}\\
&= \frac{x+1-2x+2}{(x-1)(x+1)^2} && \text{Distributive Property}\\
&= \frac{3-x}{(x-1)(x+1)^2} && \text{Combine terms in numerator}
\end{aligned}$$

■

Compound Fractions

A **compound fraction** is a fraction in which the numerator, the denominator, or both, are themselves fractional expressions.

Example 6 Simplifying a Compound Fraction

Simplify: $\dfrac{\dfrac{x}{y} + 1}{1 - \dfrac{y}{x}}$

Solution 1 We combine the terms in the numerator into a single fraction. We do the same in the denominator. Then we invert and multiply.

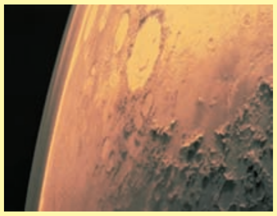
$$\begin{aligned}
\frac{\dfrac{x}{y} + 1}{1 - \dfrac{y}{x}} &= \frac{\dfrac{x+y}{y}}{\dfrac{x-y}{x}} = \frac{x+y}{y} \cdot \frac{x}{x-y}\\
&= \frac{x(x+y)}{y(x-y)}
\end{aligned}$$

Solution 2 We find the LCD of all the fractions in the expression, then multiply numerator and denominator by it. In this example the LCD of all the fractions is xy. Thus

$$\begin{aligned}
\frac{\dfrac{x}{y} + 1}{1 - \dfrac{y}{x}} &= \frac{\dfrac{x}{y} + 1}{1 - \dfrac{y}{x}} \cdot \frac{xy}{xy} && \text{Multiply numerator and denominator by } xy\\
&= \frac{x^2 + xy}{xy - y^2} && \text{Simplify}\\
&= \frac{x(x+y)}{y(x-y)} && \text{Factor}
\end{aligned}$$

■

Mathematics in the Modern World



NASA

Error-Correcting Codes

The pictures sent back by the *Pathfinder* spacecraft from the surface of Mars on July 4, 1997, were astoundingly clear. But few watching these pictures were aware of the complex mathematics used to accomplish that feat. The distance to Mars is enormous, and the background noise (or static) is many times stronger than the original signal emitted by the spacecraft. So, when scientists receive the signal, it is full of errors. To get a clear picture, the errors must be found and corrected. This same problem of errors is routinely encountered in transmitting bank records when you use an ATM machine, or voice when you are talking on the telephone.

To understand how errors are found and corrected, we must first understand that to transmit pictures, sound, or text we transform them into bits (the digits 0 or 1; see page 30). To help the receiver recognize errors, the message is "coded" by inserting additional bits. For example, suppose you want to transmit the message "10100." A very simple-minded code is as follows: Send each digit a million times. The person receiving the message reads it in blocks of a million digits. If the first block is mostly 1's, he concludes that you are probably trying to transmit a 1, and so on. To say that this code is

(continued)

ALTERNATE EXAMPLE 6

Simplify the fraction

$$\frac{\dfrac{u}{v} + 1}{\dfrac{v}{u} - 1}.$$

ANSWER

$$\frac{u(u+v)}{v(v-u)}$$

IN-CLASS MATERIALS

Ask your students why we like to rationalize the denominator. Note that it is often a matter of context, or even a matter of taste! There is nothing inherently "simpler" about $\dfrac{\sqrt{2}}{2}$ as opposed to $\dfrac{1}{\sqrt{2}}$ if they are just sitting there as numbers. However, we prefer to add $\dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{3}}{3} + \dfrac{1}{6}$ than to add $\dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}} + \dfrac{1}{6}$. See if they can come up with other reasons or instances of when it is convenient to rationalize a denominator.

not efficient is a bit of an understatement; it requires sending a million times more data than the original message. Another method inserts "check digits." For example, for each block of eight digits insert a ninth digit; the inserted digit is 0 if there is an even number of 1's in the block and 1 if there is an odd number. So, if a single digit is wrong (a 0 changed to a 1, or vice versa), the check digits allow us to recognize that an error has occurred. This method does not tell us where the error is, so we can't correct it. Modern error correcting codes use interesting mathematical algorithms that require inserting relatively few digits but which allow the receiver to not only recognize, but also correct, errors. The first error correcting code was developed in the 1940s by Richard Hamming at MIT. It is interesting to note that the English language has a built-in error correcting mechanism; to test it, try reading this error-laden sentence: Gve mo libty ox giv ne deth.

The next two examples show situations in calculus that require the ability to work with fractional expressions.

Example 7 Simplifying a Compound Fraction

Simplify: $\dfrac{\dfrac{1}{a+h}-\dfrac{1}{a}}{h}$

Solution We begin by combining the fractions in the numerator using a common denominator.

$$\frac{\dfrac{1}{a+h}-\dfrac{1}{a}}{h}=\frac{\dfrac{a-(a+h)}{a(a+h)}}{h}$$ Combine fractions in the numerator

$$=\frac{a-(a+h)}{a(a+h)}\cdot\frac{1}{h}$$ Property 2 of fractions (invert divisor and multiply)

$$=\frac{a-a-h}{a(a+h)}\cdot\frac{1}{h}$$ Distributive Property

$$=\frac{-h}{a(a+h)}\cdot\frac{1}{h}$$ Simplify

$$=\frac{-1}{a(a+h)}$$ Property 5 of fractions (cancel common factors) ■

ALTERNATE EXAMPLE 7
Simplify the compound fraction: $\dfrac{\dfrac{1}{(b+h)^2}-\dfrac{1}{b^2}}{h}$

ANSWER
$-\dfrac{2b+h}{(b+h)^2b^2}$

Example 8 Simplifying a Compound Fraction

Simplify: $\dfrac{(1+x^2)^{1/2}-x^2(1+x^2)^{-1/2}}{1+x^2}$

Solution 1 Factor $(1+x^2)^{-1/2}$ from the numerator.

Factor out the power of $1+x^2$ with the *smallest* exponent, in this case $(1+x^2)^{-1/2}$.

$$\frac{(1+x^2)^{1/2}-x^2(1+x^2)^{-1/2}}{1+x^2}=\frac{(1+x^2)^{-1/2}[(1+x^2)-x^2]}{1+x^2}$$

$$=\frac{(1+x^2)^{-1/2}}{1+x^2}=\frac{1}{(1+x^2)^{3/2}}$$

Solution 2 Since $(1+x^2)^{-1/2}=1/(1+x^2)^{1/2}$ is a fraction, we can clear all fractions by multiplying numerator and denominator by $(1+x^2)^{1/2}$.

$$\frac{(1+x^2)^{1/2}-x^2(1+x^2)^{-1/2}}{1+x^2}=\frac{(1+x^2)^{1/2}-x^2(1+x^2)^{-1/2}}{1+x^2}\cdot\frac{(1+x^2)^{1/2}}{(1+x^2)^{1/2}}$$

$$=\frac{(1+x^2)-x^2}{(1+x^2)^{3/2}}=\frac{1}{(1+x^2)^{3/2}}$$ ■

ALTERNATE EXAMPLE 8
Simplify the compound fraction: $\dfrac{(2+x^2)^{1/2}-x^2(2+x^2)^{-1/2}}{2+x^2}$

ANSWER
$\dfrac{2}{(2+x^2)^{3/2}}$

DRILL QUESTION
Simplify: $\dfrac{(x+2)/(x-3)}{x/(x-2)}$

Answer
$\dfrac{(x-2)(x+2)}{x(x-3)}$ or $\dfrac{x^2-4}{x^2-3x}$

IN-CLASS MATERIALS

This is a good time to start talking about magnitudes. For example, look at $\dfrac{x+6}{x^2+4}$ and ask the questions, "What is happening to this fraction when x gets large? What happens when x gets close to zero? What happens when x is large and negative, such as $-1{,}000{,}000$?" The idea is not yet to be rigorous, but to give students a feel for the idea that a large denominator yields a small fraction, and vice versa. You can pursue this idea with fractions like $\dfrac{x^2+4}{x}$ and $\dfrac{6}{x+(3/x)}$.

Rationalizing the Denominator or the Numerator

If a fraction has a denominator of the form $A + B\sqrt{C}$, we may rationalize the denominator by multiplying numerator and denominator by the **conjugate radical** $A - B\sqrt{C}$. This is effective because, by Special Product Formula 1 in Section 1.3, the product of the denominator and its conjugate radical does not contain a radical:

$$(A + B\sqrt{C})(A - B\sqrt{C}) = A^2 - B^2C$$

ALTERNATE EXAMPLE 9
Rationalize the denominator:
$\dfrac{12}{1 - \sqrt{7}}$

ANSWER
$-2 - 2\sqrt{7}$

Example 9 Rationalizing the Denominator

Rationalize the denominator: $\dfrac{1}{1 + \sqrt{2}}$

Solution We multiply both the numerator and the denominator by the conjugate radical of $1 + \sqrt{2}$, which is $1 - \sqrt{2}$.

$$\frac{1}{1+\sqrt{2}} = \frac{1}{1+\sqrt{2}} \cdot \frac{1-\sqrt{2}}{1-\sqrt{2}}$$ Multiply numerator and denominator by the conjugate radical

$$= \frac{1-\sqrt{2}}{1^2 - (\sqrt{2})^2}$$ Special Product Formula 1

$$= \frac{1-\sqrt{2}}{1-2} = \frac{1-\sqrt{2}}{-1} = \sqrt{2} - 1$$ ■

Special Product Formula 1
$(a + b)(a - b) = a^2 - b^2$

ALTERNATE EXAMPLE 10
Rationalize the numerator:
$\dfrac{\sqrt{1+h} - 1}{h}$

ANSWER
$\dfrac{1}{\sqrt{1+h} + 1}$

Example 10 Rationalizing the Numerator

Rationalize the numerator: $\dfrac{\sqrt{4+h} - 2}{h}$

Solution We multiply numerator and denominator by the conjugate radical $\sqrt{4+h} + 2$.

$$\frac{\sqrt{4+h}-2}{h} = \frac{\sqrt{4+h}-2}{h} \cdot \frac{\sqrt{4+h}+2}{\sqrt{4+h}+2}$$ Multiply numerator and denominator by the conjugate radical

$$= \frac{(\sqrt{4+h})^2 - 2^2}{h(\sqrt{4+h}+2)}$$ Special Product Formula 1

$$= \frac{4+h-4}{h(\sqrt{4+h}+2)}$$

$$= \frac{h}{h(\sqrt{4+h}+2)} = \frac{1}{\sqrt{4+h}+2}$$ Property 5 of fractions (cancel common factors) ■

Special Product Formula 1
$(a + b)(a - b) = a^2 - b^2$

Avoiding Common Errors

Don't make the mistake of applying properties of multiplication to the operation of addition. Many of the common errors in algebra involve doing just that. The following table states several properties of multiplication and illustrates the error in applying them to addition.

EXAMPLES

1. A compound fraction to simplify: $\dfrac{x + (3/b)}{b + (2x/6)} = 3\dfrac{xb + 3}{b(3b + x)}$

2. A denominator to rationalize: $\dfrac{x}{\sqrt{x} + \sqrt{3}} = \dfrac{x(\sqrt{x} - \sqrt{3})}{x - 3}$

Correct multiplication property	Common error with addition
$(a \cdot b)^2 = a^2 \cdot b^2$	$(a+b)^2 \neq a^2 + b^2$
$\sqrt{a \cdot b} = \sqrt{a}\sqrt{b} \quad (a, b \geq 0)$	$\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$
$\sqrt{a^2 \cdot b^2} = a \cdot b \quad (a, b \geq 0)$	$\sqrt{a^2 + b^2} \neq a + b$
$\dfrac{1}{a} \cdot \dfrac{1}{b} = \dfrac{1}{a \cdot b}$	$\dfrac{1}{a} + \dfrac{1}{b} \neq \dfrac{1}{a+b}$
$\dfrac{ab}{a} = b$	$\dfrac{a+b}{a} \neq b$
$a^{-1} \cdot b^{-1} = (a \cdot b)^{-1}$	$a^{-1} + b^{-1} \neq (a+b)^{-1}$

To verify that the equations in the right-hand column are wrong, simply substitute numbers for a and b and calculate each side. For example, if we take $a = 2$ and $b = 2$ in the fourth error, we find that the left-hand side is

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{2} + \frac{1}{2} = 1$$

whereas the right-hand side is

$$\frac{1}{a+b} = \frac{1}{2+2} = \frac{1}{4}$$

Since $1 \neq \frac{1}{4}$, the stated equation is wrong. You should similarly convince yourself of the error in each of the other equations. (See Exercise 97.)

1.4 Exercises

1–6 ■ Find the domain of the expression.

1. $4x^2 - 10x + 3$

2. $-x^4 + x^3 + 9x$

3. $\dfrac{2x+1}{x-4}$

4. $\dfrac{2t^2 - 5}{3t + 6}$

5. $\sqrt{x+3}$

6. $\dfrac{1}{\sqrt{x-1}}$

7–16 ■ Simplify the rational expression.

7. $\dfrac{3(x+2)(x-1)}{6(x-1)^2}$

8. $\dfrac{4(x^2-1)}{12(x+2)(x-1)}$

9. $\dfrac{x-2}{x^2-4}$

10. $\dfrac{x^2 - x - 2}{x^2 - 1}$

11. $\dfrac{x^2 + 6x + 8}{x^2 + 5x + 4}$

12. $\dfrac{x^2 - x - 12}{x^2 + 5x + 6}$

13. $\dfrac{y^2 + y}{y^2 - 1}$

14. $\dfrac{y^2 - 3y - 18}{2y^2 + 5y + 3}$

15. $\dfrac{2x^3 - x^2 - 6x}{2x^2 - 7x + 6}$

16. $\dfrac{1 - x^2}{x^3 - 1}$

17–30 ■ Perform the multiplication or division and simplify.

17. $\dfrac{4x}{x^2 - 4} \cdot \dfrac{x+2}{16x}$

18. $\dfrac{x^2 - 25}{x^2 - 16} \cdot \dfrac{x+4}{x+5}$

19. $\dfrac{x^2 - x - 12}{x^2 - 9} \cdot \dfrac{3+x}{4-x}$

20. $\dfrac{x^2 + 2x - 3}{x^2 - 2x - 3} \cdot \dfrac{3-x}{3+x}$

21. $\dfrac{t-3}{t^2+9} \cdot \dfrac{t+3}{t^2-9}$

22. $\dfrac{x^2 - x - 6}{x^2 + 2x} \cdot \dfrac{x^3 + x^2}{x^2 - 2x - 3}$

23. $\dfrac{x^2 + 7x + 12}{x^2 + 3x + 2} \cdot \dfrac{x^2 + 5x + 6}{x^2 + 6x + 9}$

24. $\dfrac{x^2 + 2xy + y^2}{x^2 - y^2} \cdot \dfrac{2x^2 - xy - y^2}{x^2 - xy - 2y^2}$

25. $\dfrac{2x^2 + 3x + 1}{x^2 + 2x - 15} \div \dfrac{x^2 + 6x + 5}{2x^2 - 7x + 3}$

26. $\dfrac{4y^2 - 9}{2y^2 + 9y - 18} \div \dfrac{2y^2 + y - 3}{y^2 + 5y - 6}$

27. $\dfrac{\dfrac{x^3}{x + 1}}{\dfrac{x}{x^2 + 2x + 1}}$ **28.** $\dfrac{\dfrac{2x^2 - 3x - 2}{x^2 - 1}}{\dfrac{2x^2 + 5x + 2}{x^2 + x - 2}}$

29. $\dfrac{x/y}{z}$ **30.** $\dfrac{x}{y/z}$

31–50 ▪ Perform the addition or subtraction and simplify.

31. $2 + \dfrac{x}{x + 3}$ **32.** $\dfrac{2x - 1}{x + 4} - 1$

33. $\dfrac{1}{x + 5} + \dfrac{2}{x - 3}$ **34.** $\dfrac{1}{x + 1} + \dfrac{1}{x - 1}$

35. $\dfrac{1}{x + 1} - \dfrac{1}{x + 2}$ **36.** $\dfrac{x}{x - 4} - \dfrac{3}{x + 6}$

37. $\dfrac{x}{(x + 1)^2} + \dfrac{2}{x + 1}$ **38.** $\dfrac{5}{2x - 3} - \dfrac{3}{(2x - 3)^2}$

39. $u + 1 + \dfrac{u}{u + 1}$ **40.** $\dfrac{2}{a^2} - \dfrac{3}{ab} + \dfrac{4}{b^2}$

41. $\dfrac{1}{x^2} + \dfrac{1}{x^2 + x}$ **42.** $\dfrac{1}{x} + \dfrac{1}{x^2} + \dfrac{1}{x^3}$

43. $\dfrac{2}{x + 3} - \dfrac{1}{x^2 + 7x + 12}$ **44.** $\dfrac{x}{x^2 - 4} + \dfrac{1}{x - 2}$

45. $\dfrac{1}{x + 3} + \dfrac{1}{x^2 - 9}$

46. $\dfrac{x}{x^2 + x - 2} - \dfrac{2}{x^2 - 5x + 4}$

47. $\dfrac{2}{x} + \dfrac{3}{x - 1} - \dfrac{4}{x^2 - x}$

48. $\dfrac{x}{x^2 - x - 6} - \dfrac{1}{x + 2} - \dfrac{2}{x - 3}$

49. $\dfrac{1}{x^2 + 3x + 2} - \dfrac{1}{x^2 - 2x - 3}$

50. $\dfrac{1}{x + 1} - \dfrac{2}{(x + 1)^2} + \dfrac{3}{x^2 - 1}$

51–60 ▪ Simplify the compound fractional expression.

51. $\dfrac{\dfrac{x}{y} - \dfrac{y}{x}}{\dfrac{1}{x^2} - \dfrac{1}{y^2}}$ **52.** $x - \dfrac{y}{\dfrac{x}{y} + \dfrac{y}{x}}$

53. $\dfrac{1 + \dfrac{1}{c - 1}}{1 - \dfrac{1}{c - 1}}$ **54.** $1 + \dfrac{1}{1 + \dfrac{1}{1 + x}}$

55. $\dfrac{\dfrac{5}{x - 1} - \dfrac{2}{x + 1}}{\dfrac{x}{x - 1} + \dfrac{1}{x + 1}}$ **56.** $\dfrac{\dfrac{a - b}{a} - \dfrac{a + b}{b}}{\dfrac{a - b}{b} + \dfrac{a + b}{a}}$

57. $\dfrac{x^{-2} - y^{-2}}{x^{-1} + y^{-1}}$ **58.** $\dfrac{x^{-1} + y^{-1}}{(x + y)^{-1}}$

59. $\dfrac{1}{1 + a^n} + \dfrac{1}{1 + a^{-n}}$ **60.** $\dfrac{\left(a + \dfrac{1}{b}\right)^m \left(a - \dfrac{1}{b}\right)^n}{\left(b + \dfrac{1}{a}\right)^m \left(b - \dfrac{1}{a}\right)^n}$

61–66 ▪ Simplify the fractional expression. (Expressions like these arise in calculus.)

61. $\dfrac{\dfrac{1}{a + h} - \dfrac{1}{a}}{h}$

62. $\dfrac{(x + h)^{-3} - x^{-3}}{h}$

63. $\dfrac{\dfrac{1 - (x + h)}{2 + (x + h)} - \dfrac{1 - x}{2 + x}}{h}$

64. $\dfrac{(x + h)^3 - 7(x + h) - (x^3 - 7x)}{h}$

65. $\sqrt{1 + \left(\dfrac{x}{\sqrt{1 - x^2}}\right)^2}$ **66.** $\sqrt{1 + \left(x^3 - \dfrac{1}{4x^3}\right)^2}$

67–72 ▪ Simplify the expression. (This type of expression arises in calculus when using the "quotient rule.")

67. $\dfrac{3(x + 2)^2(x - 3)^2 - (x + 2)^3(2)(x - 3)}{(x - 3)^4}$

68. $\dfrac{2x(x + 6)^4 - x^2(4)(x + 6)^3}{(x + 6)^8}$

69. $\dfrac{2(1 + x)^{1/2} - x(1 + x)^{-1/2}}{x + 1}$

70. $\dfrac{(1 - x^2)^{1/2} + x^2(1 - x^2)^{-1/2}}{1 - x^2}$

71. $\dfrac{3(1 + x)^{1/3} - x(1 + x)^{-2/3}}{(1 + x)^{2/3}}$

72. $\dfrac{(7 - 3x)^{1/2} + \frac{3}{2}x(7 - 3x)^{-1/2}}{7 - 3x}$

73–78 ■ Rationalize the denominator.

73. $\dfrac{1}{2 - \sqrt{3}}$ **74.** $\dfrac{2}{3 - \sqrt{5}}$

75. $\dfrac{2}{\sqrt{2} + \sqrt{7}}$ **76.** $\dfrac{1}{\sqrt{x} + 1}$

77. $\dfrac{y}{\sqrt{3} + \sqrt{y}}$ **78.** $\dfrac{2(x - y)}{\sqrt{x} - \sqrt{y}}$

79–84 ■ Rationalize the numerator.

79. $\dfrac{1 - \sqrt{5}}{3}$ **80.** $\dfrac{\sqrt{3} + \sqrt{5}}{2}$

81. $\dfrac{\sqrt{r} + \sqrt{2}}{5}$ **82.** $\dfrac{\sqrt{x} - \sqrt{x + h}}{h\sqrt{x}\sqrt{x + h}}$

83. $\sqrt{x^2 + 1} - x$ **84.** $\sqrt{x + 1} - \sqrt{x}$

85–92 ■ State whether the given equation is true for all values of the variables. (Disregard any value that makes a denominator zero.)

85. $\dfrac{16 + a}{16} = 1 + \dfrac{a}{16}$ **86.** $\dfrac{b}{b - c} = 1 - \dfrac{b}{c}$

87. $\dfrac{2}{4 + x} = \dfrac{1}{2} + \dfrac{2}{x}$ **88.** $\dfrac{x + 1}{y + 1} = \dfrac{x}{y}$

89. $\dfrac{x}{x + y} = \dfrac{1}{1 + y}$ **90.** $2\left(\dfrac{a}{b}\right) = \dfrac{2a}{2b}$

91. $\dfrac{-a}{b} = -\dfrac{a}{b}$ **92.** $\dfrac{1 + x + x^2}{x} = \dfrac{1}{x} + 1 + x$

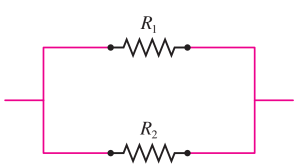
Applications

93. Electrical Resistance If two electrical resistors with resistances R_1 and R_2 are connected in parallel (see the figure), then the total resistance R is given by

$$R = \frac{1}{\dfrac{1}{R_1} + \dfrac{1}{R_2}}$$

(a) Simplify the expression for R.

(b) If $R_1 = 10$ ohms and $R_2 = 20$ ohms, what is the total resistance R?



94. Average Cost A clothing manufacturer finds that the cost of producing x shirts is $500 + 6x + 0.01x^2$ dollars.

(a) Explain why the average cost per shirt is given by the rational expression

$$A = \frac{500 + 6x + 0.01x^2}{x}$$

(b) Complete the table by calculating the average cost per shirt for the given values of x.

x	Average cost
10	
20	
50	
100	
200	
500	
1000	

Discovery • Discussion

95. Limiting Behavior of a Rational Expression The rational expression

$$\frac{x^2 - 9}{x - 3}$$

is not defined for $x = 3$. Complete the tables and determine what value the expression approaches as x gets closer and closer to 3. Why is this reasonable? Factor the numerator of the expression and simplify to see why.

x	$\dfrac{x^2 - 9}{x - 3}$
2.80	
2.90	
2.95	
2.99	
2.999	

x	$\dfrac{x^2 - 9}{x - 3}$
3.20	
3.10	
3.05	
3.01	
3.001	

96. Is This Rationalization? In the expression $2/\sqrt{x}$ we would eliminate the radical if we were to square both numerator and denominator. Is this the same thing as rationalizing the denominator?

97. Algebraic Errors The left-hand column in the table lists some common algebraic errors. In each case, give an example using numbers that show that the formula is not valid. An example of this type, which shows that a statement is false, is called a *counterexample*.

Algebraic error	Counterexample
$\frac{1}{a}+\frac{1}{b} \not= \frac{1}{a+b}$	$\frac{1}{2}+\frac{1}{2} \neq \frac{1}{2+2}$
$(a+b)^2 \not= a^2+b^2$	
$\sqrt{a^2+b^2} \not= a+b$	
$\frac{a+b}{a} \not= b$	
$(a^3+b^3)^{1/3} \not= a+b$	
$a^m/a^n \not= a^{m/n}$	
$a^{-1/n} \not= \frac{1}{a^n}$	

98. The Form of an Algebraic Expression An algebraic expression may look complicated, but its "form" is always simple; it must be a sum, a product, a quotient, or a power. For example, consider the following expressions:

$$(1+x^2)^2+\left(\frac{x+2}{x+1}\right)^3 \qquad (1+x)\left(1+\frac{x+5}{1+x^4}\right)$$

$$\frac{5-x^3}{1+\sqrt{1+x^2}} \qquad \sqrt{\frac{1+x}{1-x}}$$

With appropriate choices for A and B, the first has the form $A+B$, the second AB, the third A/B, and the fourth $A^{1/2}$. Recognizing the form of an expression helps us expand, simplify, or factor it correctly. Find the form of the following algebraic expressions.

(a) $x+\sqrt{1+\frac{1}{x}}$ **(b)** $(1+x^2)(1+x)^3$

(c) $\sqrt[3]{x^4(4x^2+1)}$ **(d)** $\frac{1-2\sqrt{1+x}}{1+\sqrt{1+x^2}}$

SUGGESTED TIME AND EMPHASIS

$1\frac{1}{2}$–2 classes.
Essential material.

1.5 Equations

An equation is a statement that two mathematical expressions are equal. For example,

$$3+5=8$$

is an equation. Most equations that we study in algebra contain variables, which are symbols (usually letters) that stand for numbers. In the equation

$$4x+7=19$$

the letter x is the variable. We think of x as the "unknown" in the equation, and our goal is to find the value of x that makes the equation true. The values of the unknown that make the equation true are called the **solutions** or **roots** of the equation, and the process of finding the solutions is called **solving the equation**.

$x=3$ is a solution of the equation $4x+7=19$, because substituting $x=3$ makes the equation true:

$x = 3$

$4(3)+7=19$ ✓

Two equations with exactly the same solutions are called **equivalent equations**. To solve an equation, we try to find a simpler, equivalent equation in which the variable stands alone on one side of the "equal" sign. Here are the properties that we use to solve an equation. (In these properties, A, B, and C stand for any algebraic expressions, and the symbol $\Leftrightarrow$ means "is equivalent to.")

Properties of Equality

Property	Description
1. $A=B \Leftrightarrow A+C=B+C$	Adding the same quantity to both sides of an equation gives an equivalent equation.
2. $A=B \Leftrightarrow CA=CB \quad (C \neq 0)$	Multiplying both sides of an equation by the same nonzero quantity gives an equivalent equation.

POINTS TO STRESS

1. Solving equations using the techniques of adding constants to both sides of the equation, multiplying both sides of the equation by a constant, and raising both sides of the equation to the same nonzero power.
2. Avoiding the pitfalls of accidentally multiplying or dividing by zero, or introducing extraneous solutions.
3. Solving quadratic equations using the techniques of factoring, completing the square, and the quadratic formula, and the use of the discriminant to determine the nature of the solutions to a quadratic equation.
4. Solving generalized quadratic equations.
5. Solving equations involving fractional expressions.

These properties require that you *perform the same operation on both sides of an equation* when solving it. Thus, if we say "*add* −7" when solving an equation, that is just a short way of saying "*add* −7 to each side of the equation."

Linear Equations

The simplest type of equation is a *linear equation*, or first-degree equation, which is an equation in which each term is either a constant or a nonzero multiple of the variable.

Linear Equations

A **linear equation** in one variable is an equation equivalent to one of the form

$$ax + b = 0$$

where a and b are real numbers and x is the variable.

Here are some examples that illustrate the difference between linear and nonlinear equations.

Linear equations	Nonlinear equations	
$4x - 5 = 3$	$x^2 + 2x = 8$	Not linear; contains the square of the variable
$2x = \frac{1}{2}x - 7$	$\sqrt{x} - 6x = 0$	Not linear; contains the square root of the variable
$x - 6 = \frac{x}{3}$	$\frac{3}{x} - 2x = 1$	Not linear; contains the reciprocal of the variable

Example 1 Solving a Linear Equation

Solve the equation $7x - 4 = 3x + 8$.

Solution We solve this by changing it to an equivalent equation with all terms that have the variable x on one side and all constant terms on the other.

$7x - 4 = 3x + 8$	Given equation
$(7x - 4) + 4 = (3x + 8) + 4$	Add 4
$7x = 3x + 12$	Simplify
$7x - 3x = (3x + 12) - 3x$	Subtract 3x
$4x = 12$	Simplify
$\frac{1}{4} \cdot 4x = \frac{1}{4} \cdot 12$	Multiply by $\frac{1}{4}$
$x = 3$	Simplify ■

Because it is important to CHECK YOUR ANSWER, we do this in many of our examples. In these checks, LHS stands for "left-hand side" and RHS stands for "right-hand side" of the original equation.

Check Your Answer

$x = 3$:

LHS $= 7(3) - 4$ $= 17$ (x = 3)

RHS $= 3(3) + 8$ $= 17$ (x = 3)

LHS = RHS ✓

ALTERNATE EXAMPLE 1
Solve the equation
$6x - 5 = 2x + 11$.

ANSWER
$x = 4$

SAMPLE QUESTIONS
Text Questions

1. Give two different reasons why it is important to check your answer after you have solved an equation such as: $2 + \frac{5}{x - 4} = \frac{x + 1}{x - 4}$.
2. Solve: $\frac{1}{x + 4} = \frac{1}{2x}$.

Answers

1. First, we must make sure that our answer is not extraneous. Second, we should check that we have not made an error in calculation.
2. $x = 4$

Many formulas in the sciences involve several variables, and it is often necessary to express one of the variables in terms of the others. In the next example we solve for a variable in Newton's Law of Gravity.

Example 2 Solving for One Variable in Terms of Others

Solve for the variable M in the equation

$$F = G\frac{mM}{r^2}$$

This is Newton's Law of Gravity. It gives the gravitational force F between two masses m and M that are a distance r apart. The constant G is the universal gravitational constant.

Solution Although this equation involves more than one variable, we solve it as usual by isolating M on one side and treating the other variables as we would numbers.

$$F = \left(\frac{Gm}{r^2}\right)M \qquad \text{Factor M from RHS}$$

$$\left(\frac{r^2}{Gm}\right)F = \left(\frac{r^2}{Gm}\right)\left(\frac{Gm}{r^2}\right)M \qquad \text{Multiply by reciprocal of } \frac{Gm}{r^2}$$

$$\frac{r^2F}{Gm} = M \qquad \text{Simplify}$$

The solution is $M = \dfrac{r^2F}{Gm}$. ■

Example 3 Solving for One Variable in Terms of Others

The surface area A of the closed rectangular box shown in Figure 1 can be calculated from the length l, the width w, and the height h according to the formula

$$A = 2lw + 2wh + 2lh$$

Solve for w in terms of the other variables in this equation.

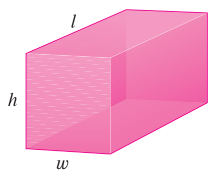


Figure 1
A closed rectangular box

Solution Although this equation involves more than one variable, we solve it as usual by isolating w on one side, treating the other variables as we would numbers.

$$A = (2lw + 2wh) + 2lh \qquad \text{Collect terms involving } w$$

$$A - 2lh = 2lw + 2wh \qquad \text{Subtract } 2lh$$

$$A - 2lh = (2l + 2h)w \qquad \text{Factor } w \text{ from RHS}$$

$$\frac{A - 2lh}{2l + 2h} = w \qquad \text{Divide by } 2l + 2h$$

The solution is $w = \dfrac{A - 2lh}{2l + 2h}$. ■

Quadratic Equations

Linear equations are first-degree equations like $2x + 1 = 5$ or $4 - 3x = 2$. Quadratic equations are second-degree equations like $x^2 + 2x - 3 = 0$ or $2x^2 + 3 = 5x$.

ALTERNATE EXAMPLE 3

The surface area A of the closed rectangular box can be calculated from the length l, the width w, and the height h according to the formula

$$A = 2lw + 2wh + 2lh$$

Solve for l in terms of the other variables in this equation.

ANSWER

$$\frac{(A - 2wh)}{(2w + 2h)} = l$$

DRILL QUESTIONS

1. Solve $x^2 + 3x = -2$ for x.
2. Solve $(x + 3)^2 + 2(x + 3) = -1$ for x.

Answers

1. $x = -2$ or $x = -1$
2. $x = -4$

Quadratic Equations

$x^2 - 2x - 8 = 0$

$3x + 10 = 4x^2$

$\frac{1}{2}x^2 + \frac{1}{3}x - \frac{1}{6} = 0$

Quadratic Equations

A **quadratic equation** is an equation of the form

$$ax^2 + bx + c = 0$$

where a, b, and c are real numbers with $a \neq 0$.

Some quadratic equations can be solved by factoring and using the following basic property of real numbers.

Zero-Product Property

$$AB = 0 \quad \text{if and only if} \quad A = 0 \quad \text{or} \quad B = 0$$

This means that if we can factor the left-hand side of a quadratic (or other) equation, then we can solve it by setting each factor equal to 0 in turn. This method works only when the right-hand side of the equation is 0.

Example 4 Solving a Quadratic Equation by Factoring

Solve the equation $x^2 + 5x = 24$.

Solution We must first rewrite the equation so that the right-hand side is 0.

$$x^2 + 5x = 24$$

$$x^2 + 5x - 24 = 0 \qquad \text{Subtract 24}$$

$$(x - 3)(x + 8) = 0 \qquad \text{Factor}$$

$$x - 3 = 0 \quad \text{or} \quad x + 8 = 0 \qquad \text{Zero-Product Property}$$

$$x = 3 \qquad x = -8 \qquad \text{Solve}$$

The solutions are $x = 3$ and $x = -8$. ■

Check Your Answers

$x = 3$:

$(3)^2 + 5(3) = 9 + 15 = 24$ ✓

$x = -8$:

$(-8)^2 + 5(-8) = 64 - 40 = 24$ ✓

ALTERNATE EXAMPLE 4

Solve the equation $x^2 + 4x = 32$.

ANSWER

4, −8

Do you see why one side of the equation must be 0 in Example 4? Factoring the equation as $x(x + 5) = 24$ does not help us find the solutions, since 24 can be factored in infinitely many ways, such as $6 \cdot 4$, $\frac{1}{2} \cdot 48$, $\left(-\frac{2}{5}\right) \cdot (-60)$, and so on.

A quadratic equation of the form $x^2 - c = 0$, where c is a positive constant, factors as $(x - \sqrt{c})(x + \sqrt{c}) = 0$, and so the solutions are $x = \sqrt{c}$ and $x = -\sqrt{c}$. We often abbreviate this as $x = \pm\sqrt{c}$.

Solving a Simple Quadratic Equation

The solutions of the equation $x^2 = c$ are $x = \sqrt{c}$ and $x = -\sqrt{c}$.

IN-CLASS MATERIALS

There are two main anomalies that students may encounter when solving linear equations: $0 = 0$ and $1 = 0$. While it is not a good idea to dwell on the anomalies, as opposed to spending time on the cases where the students will spend a majority of their time, it is good to address them, because they do come up, in math class and in real life. Have them attempt to solve the following equations, and check their work:

$$3x + 4 = x + 6$$
$$3x + 4 = x + 6 + 2x - 2$$
$$3x + 4 = x + 6 + 2x + 2$$

By the time this example is presented, most students should be able to obtain $x = 1$ for the first equation, and to check this answer. The second one gives $0 = 0$. Point out that $0 = 0$ is always a true statement. (If you like, introduce the term "tautology.") It is a true statement if $x = 1$, it is a true statement if $x = 2$, it is a true statement if the author Lemony Snicket writes another book, it is a true statement if he does not write another book. So students can test $x = 1$, $x = 2$, $x = -\sqrt{2}$ in the second equation, and all of them will yield truth. The third statement gives $1 = 0$ (actually, it gives $0 = 4$, but we can multiply both sides of the equation by $\frac{1}{4}$). This is a false statement, no matter what value we assign to x.

ALTERNATE EXAMPLE 5b
Solve the equation
$(x - 5)^2 = 3$.

ANSWER
$x = 5 \pm \sqrt{3}$

Example 5 Solving Simple Quadratics

Solve each equation.

(a) $x^2 = 5$ (b) $(x - 4)^2 = 5$

Solution

(a) From the principle in the preceding box, we get $x = \pm\sqrt{5}$.

(b) We can take the square root of each side of this equation as well.

$$(x - 4)^2 = 5$$
$$x - 4 = \pm\sqrt{5} \qquad \text{Take the square root}$$
$$x = 4 \pm \sqrt{5} \qquad \text{Add 4}$$

The solutions are $x = 4 + \sqrt{5}$ and $x = 4 - \sqrt{5}$. ■

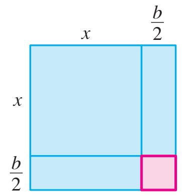
See page 30 for how to recognize when a quadratic expression is a perfect square.

Completing the Square

Area of blue region is

$$x^2 + 2\left(\frac{b}{2}\right)x = x^2 + bx$$

Add a small square of area $(b/2)^2$ to "complete" the square.



As we saw in Example 5, if a quadratic equation is of the form $(x \pm a)^2 = c$, then we can solve it by taking the square root of each side. In an equation of this form the left-hand side is a *perfect square*: the square of a linear expression in x. So, if a quadratic equation does not factor readily, then we can solve it using the technique of **completing the square**. This means that we add a constant to an expression to make it a perfect square. For example, to make $x^2 - 6x$ a perfect square we must add 9, since $x^2 - 6x + 9 = (x - 3)^2$.

Completing the Square

To make $x^2 + bx$ a perfect square, add $\left(\frac{b}{2}\right)^2$, the square of half the coefficient of x. This gives the perfect square

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$$

ALTERNATE EXAMPLE 6
Solve the equation
$5x^2 - 20x + 10 = 0$.

ANSWER
$x = 2 + \sqrt{2}, x = 2 - \sqrt{2}$

Example 6 Solving Quadratic Equations by Completing the Square

Solve each equation.

(a) $x^2 - 8x + 13 = 0$ (b) $3x^2 - 12x + 6 = 0$

When completing the square, make sure the coefficient of x^2 is 1. If it isn't, you must factor this coefficient from both terms that contain x:

$$ax^2 + bx = a\left(x^2 + \frac{b}{a}x\right)$$

Then complete the square inside the parentheses. Remember that the term added inside the parentheses is multiplied by a.

Solution

(a)
$$x^2 - 8x + 13 = 0 \qquad \text{Given equation}$$
$$x^2 - 8x = -13 \qquad \text{Subtract 13}$$
$$x^2 - 8x + 16 = -13 + 16 \qquad \text{Complete the square: add } \left(\frac{-8}{2}\right)^2 = 16$$
$$(x - 4)^2 = 3 \qquad \text{Perfect square}$$
$$x - 4 = \pm\sqrt{3} \qquad \text{Take square root}$$
$$x = 4 \pm \sqrt{3} \qquad \text{Add 4}$$

IN-CLASS MATERIALS

The zero-product property is the basis of various methods of solving quadratic equations. Many errors students make in solving these equations stem from misunderstanding the zero-product property. Students should see the difference between these two equations:

$$(x - 2)(x - 3)(x - 4) = 0$$
$$(x - 2)(x - 3)(x - 4) = -24$$

Note how the solution to the first is immediate, while the solution to the second requires some work. ($x = 0$ is the only real solution to the second.)

François Viète (1540–1603) had a successful political career before taking up mathematics late in life. He became one of the most famous French mathematicians of the 16th century. Viète introduced a new level of abstraction in algebra by using letters to stand for *known* quantities in an equation. Before Viète's time, each equation had to be solved on its own. For instance, the quadratic equations

$$3x^2 + 2x + 8 = 0$$

$$5x^2 - 6x + 4 = 0$$

had to be solved separately by completing the square. Viète's idea was to consider all quadratic equations at once by writing

$$ax^2 + bx + c = 0$$

where a, b, and c are known quantities. Thus, he made it possible to write a *formula* (in this case, the quadratic formula) involving a, b, and c that can be used to solve all such equations in one fell swoop.

Viète's mathematical genius proved quite valuable during a war between France and Spain. To communicate with their troops, the Spaniards used a complicated code that Viète managed to decipher. Unaware of Viète's accomplishment, the Spanish king, Philip II, protested to the Pope, claiming that the French were using witchcraft to read his messages.

(b) After subtracting 6 from each side of the equation, we must factor the coefficient of x^2 (the 3) from the left side to put the equation in the correct form for completing the square.

$$\begin{aligned} 3x^2 - 12x + 6 &= 0 && \text{Given equation} \\ 3x^2 - 12x &= -6 && \text{Subtract 6} \\ 3(x^2 - 4x) &= -6 && \text{Factor 3 from LHS} \end{aligned}$$

Now we complete the square by adding $(-2)^2 = 4$ *inside* the parentheses. Since everything inside the parentheses is multiplied by 3, this means that we are actually adding $3 \cdot 4 = 12$ to the left side of the equation. Thus, we must add 12 to the right side as well.

$$\begin{aligned} 3(x^2 - 4x + 4) &= -6 + 3 \cdot 4 && \text{Complete the square: add 4} \\ 3(x - 2)^2 &= 6 && \text{Perfect square} \\ (x - 2)^2 &= 2 && \text{Divide by 3} \\ x - 2 &= \pm\sqrt{2} && \text{Take square root} \\ x &= 2 \pm \sqrt{2} && \text{Add 2} \end{aligned}$$

■

We can use the technique of completing the square to derive a formula for the roots of the general quadratic equation $ax^2 + bx + c = 0$.

The Quadratic Formula

The roots of the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$, are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

■ **Proof** First, we divide each side of the equation by a and move the constant to the right side, giving

$$x^2 + \frac{b}{a}x = -\frac{c}{a} \qquad \text{Divide by } a$$

We now complete the square by adding $(b/2a)^2$ to each side of the equation:

$$\begin{aligned} x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 &= -\frac{c}{a} + \left(\frac{b}{2a}\right)^2 && \text{Complete the square: Add } \left(\frac{b}{2a}\right)^2 \\ \left(x + \frac{b}{2a}\right)^2 &= \frac{-4ac + b^2}{4a^2} && \text{Perfect square} \\ x + \frac{b}{2a} &= \pm\frac{\sqrt{b^2 - 4ac}}{2a} && \text{Take square root} \\ x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} && \text{Subtract } \frac{b}{2a} \end{aligned}$$

■

The quadratic formula could be used to solve the equations in Examples 4 and 6. You should carry out the details of these calculations.

IN-CLASS MATERIALS

Go through an example where the appearance of a quadratic equation may be unexpected. For example, assume that if we charge \$200 for a pair of hand-made shoes, we can sell 100 pairs. For every \$5 we raise the price, we can sell one fewer pair. (Conversely, for every \$5 we lower the price, we can sell one more pair.) At what prices will our revenue be \$23,520? (Answer: \$280, \$420.) Point out that this is somewhat artificial—in practice we would be more interested in finding a maximum revenue, or (ideally) maximum profit. Later (in Chapter 8) students will be able to solve such problems.

ALTERNATE EXAMPLE 7
Find all solutions of the equation $2x^2 - 3x - 1 = 0$.

ANSWER
$x = \frac{3 \pm \sqrt{17}}{4}$

Example 7 Using the Quadratic Formula

Find all solutions of each equation.

(a) $3x^2 - 5x - 1 = 0$ (b) $4x^2 + 12x + 9 = 0$ (c) $x^2 + 2x + 2 = 0$

Solution

(a) In this quadratic equation $a = 3$, $b = -5$, and $c = -1$.

$b = -5$

$$3x^2 - 5x - 1 = 0$$

$a = 3$ $c = -1$

By the quadratic formula,

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(3)(-1)}}{2(3)} = \frac{5 \pm \sqrt{37}}{6}$$

If approximations are desired, we can use a calculator to obtain

$$x = \frac{5 + \sqrt{37}}{6} \approx 1.8471 \quad \text{and} \quad x = \frac{5 - \sqrt{37}}{6} \approx -0.1805$$

Another Method
$4x^2 + 12x + 9 = 0$
$(2x + 3)^2 = 0$
$2x + 3 = 0$
$x = -\frac{3}{2}$

(b) Using the quadratic formula with $a = 4$, $b = 12$, and $c = 9$ gives

$$x = \frac{-12 \pm \sqrt{(12)^2 - 4 \cdot 4 \cdot 9}}{2 \cdot 4} = \frac{-12 \pm 0}{8} = -\frac{3}{2}$$

This equation has only one solution, $x = -\frac{3}{2}$.

(c) Using the quadratic formula with $a = 1$, $b = 2$, and $c = 2$ gives

$$x = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 2}}{2} = \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2\sqrt{-1}}{2} = -1 \pm \sqrt{-1}$$

Since the square of any real number is nonnegative, $\sqrt{-1}$ is undefined in the real number system. The equation has no real solution. ■

In Section 3.4 we study the complex number system, in which the square roots of negative numbers do exist. The equation in Example 7(c) does have solutions in the complex number system.

The quantity $b^2 - 4ac$ that appears under the square root sign in the quadratic formula is called the *discriminant* of the equation $ax^2 + bx + c = 0$ and is given the symbol D. If $D < 0$, then $\sqrt{b^2 - 4ac}$ is undefined, and the quadratic equation has no real solution, as in Example 7(c). If $D = 0$, then the equation has only one real solution, as in Example 7(b). Finally, if $D > 0$, then the equation has two distinct real solutions, as in Example 7(a). The following box summarizes these observations.

The Discriminant

The **discriminant** of the general quadratic $ax^2 + bx + c = 0$ $(a \neq 0)$ is $D = b^2 - 4ac$.

1. If $D > 0$, then the equation has two distinct real solutions.
2. If $D = 0$, then the equation has exactly one real solution.
3. If $D < 0$, then the equation has no real solution.

IN-CLASS MATERIALS

A resistor is an electrical device that resists the flow of electrical current in an electric circuit. It allows the circuit designer to control the current for a given voltage, among other applications. For two resistors in series, the total resistance is easy to find: $R = R_1 + R_2$. Have students solve the following straightforward problem: “Two resistors are in series. The total desired resistance is 100 Ω. The first resistor needs to have a resistance 45 Ω greater than the second. What should their resistances be?”

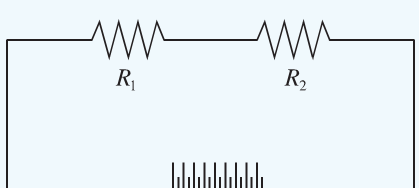
Answer

The equation is $R_1 + (R_1 - 45) = 100$, so $R_1 = 72.5$ and $R_2 = 27.5$.

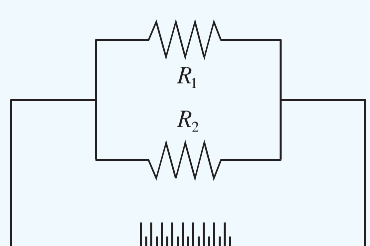
Things get more complicated if the resistors are in parallel. The parallel resistor equation is $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$. Have the students try to set up and solve the same problem, only with the resistors in parallel.

Answer

$R_1 = 225$ and $R_2 = 180$. There is also an extraneous solution.



Resistors in Series



Resistors in Parallel

Example 8 **Using the Discriminant**

Use the discriminant to determine how many real solutions each equation has.

(a) $x^2 + 4x - 1 = 0$ (b) $4x^2 - 12x + 9 = 0$ (c) $\frac{1}{3}x^2 - 2x + 4 = 0$

Solution

(a) The discriminant is $D = 4^2 - 4(1)(-1) = 20 > 0$, so the equation has two distinct real solutions.

(b) The discriminant is $D = (-12)^2 - 4 \cdot 4 \cdot 9 = 0$, so the equation has exactly one real solution.

(c) The discriminant is $D = (-2)^2 - 4(\frac{1}{3})4 = -\frac{4}{3} < 0$, so the equation has no real solution. ■

Now let's consider a real-life situation that can be modeled by a quadratic equation.

ALTERNATE EXAMPLE 8
Use the discriminant to determine how many real solutions the equation has:
$2x^2 - 4x + 1 = 0$

ANSWER
2

Example 9 **The Path of a Projectile**

An object thrown or fired straight upward at an initial speed of v_0 ft/s will reach a height of h feet after t seconds, where h and t are related by the formula

$$h = -16t^2 + v_0 t$$

Suppose that a bullet is shot straight upward with an initial speed of 800 ft/s. Its path is shown in Figure 2.

(a) When does the bullet fall back to ground level?
(b) When does it reach a height of 6400 ft?
(c) When does it reach a height of 2 mi?
(d) How high is the highest point the bullet reaches?

This formula depends on the fact that acceleration due to gravity is constant near the earth's surface. Here we neglect the effect of air resistance.

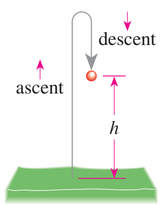
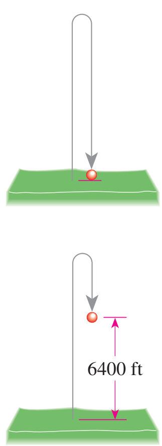


Figure 2



Solution Since the initial speed in this case is $v_0 = 800$ ft/s, the formula is

$$h = -16t^2 + 800t$$

(a) Ground level corresponds to $h = 0$, so we must solve the equation

$$0 = -16t^2 + 800t \qquad \text{Set } h = 0$$
$$0 = -16t(t - 50) \qquad \text{Factor}$$

Thus, $t = 0$ or $t = 50$. This means the bullet starts ($t = 0$) at ground level and returns to ground level after 50 s.

(b) Setting $h = 6400$ gives the equation

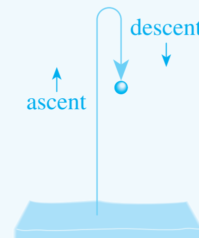
$$6400 = -16t^2 + 800t \qquad \text{Set } h = 6400$$
$$16t^2 - 800t + 6400 = 0 \qquad \text{All terms to LHS}$$
$$t^2 - 50t + 400 = 0 \qquad \text{Divide by 16}$$
$$(t - 10)(t - 40) = 0 \qquad \text{Factor}$$
$$t = 10 \quad \text{or} \quad t = 40 \qquad \text{Solve}$$

The bullet reaches 6400 ft after 10 s (on its ascent) and again after 40 s (on its descent to earth).

ALTERNATE EXAMPLE 9
An object thrown or fired straight upward at an initial speed of v_0 ft/s will reach a height of h feet after t seconds, where h and t are related by the formula:

$h = -16t^2 + v_0 t$

Suppose that a bullet is shot straight upward with an initial speed of 240 ft/s. Its path is shown in the picture below. When does it reach a height of 800 ft?



ANSWER
$t = 5$, $t = 10$

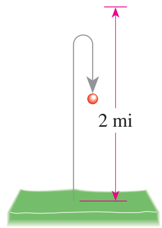
IN-CLASS MATERIALS

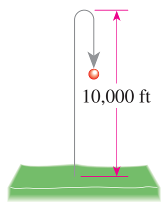
One can demonstrate an equation involving two square roots. For example: The square root of a number, plus one more than that number, is exactly 5. Find the number.

Answer

$$\sqrt{x} + (x + 1) = 5 \quad \Leftrightarrow \quad \sqrt{x} = 4 - x \quad \Leftrightarrow \quad x = (4 - x)^2 \quad \Leftrightarrow \quad x^2 - 9x + 16 = 0.$$

The solutions to the quadratic are approximately 2.438 and 6.562. The latter proves to be extraneous, leaving 2.438 as the answer.





(c) Two miles is $2 \times 5280 = 10{,}560$ ft.

$$10{,}560 = -16t^2 + 800t \qquad \text{Set } h = 10{,}560$$
$$16t^2 - 800t + 10{,}560 = 0 \qquad \text{All terms to LHS}$$
$$t^2 - 50t + 660 = 0 \qquad \text{Divide by 16}$$

The discriminant of this equation is $D = (-50)^2 - 4(660) = -140$, which is negative. Thus, the equation has no real solution. The bullet never reaches a height of 2 mi.

(d) Each height the bullet reaches is attained twice, once on its ascent and once on its descent. The only exception is the highest point of its path, which is reached only once. This means that for the highest value of h, the following equation has only one solution for t:

$$h = -16t^2 + 800t$$
$$16t^2 - 800t + h = 0 \qquad \text{All terms to LHS}$$

This in turn means that the discriminant D of the equation is 0, and so

$$D = (-800)^2 - 4(16)h = 0$$
$$640{,}000 - 64h = 0$$
$$h = 10{,}000$$

The maximum height reached is 10,000 ft. ■

Other Types of Equations

So far we have learned how to solve linear and quadratic equations. Now we study other types of equations, including those that involve higher powers, fractional expressions, and radicals.

ALTERNATE EXAMPLE 10

Solve the equation $\dfrac{10}{x} + \dfrac{98}{x+9} = 9$.

ANSWER

5, −2

Example 10 An Equation Involving Fractional Expressions

Solve the equation $\dfrac{3}{x} + \dfrac{5}{x+2} = 2$.

Solution We eliminate the denominators by multiplying each side by the lowest common denominator.

$$\left(\frac{3}{x} + \frac{5}{x+2}\right)x(x+2) = 2x(x+2) \qquad \text{Multiply by LCD } x(x+2)$$
$$3(x+2) + 5x = 2x^2 + 4x \qquad \text{Expand}$$
$$8x + 6 = 2x^2 + 4x \qquad \text{Expand LHS}$$
$$0 = 2x^2 - 4x - 6 \qquad \text{Subtract } 8x + 6$$
$$0 = x^2 - 2x - 3 \qquad \text{Divide both sides by 2}$$
$$0 = (x-3)(x+1) \qquad \text{Factor}$$
$$x - 3 = 0 \quad \text{or} \quad x + 1 = 0 \qquad \text{Zero-Product Property}$$
$$x = 3 \qquad x = -1 \qquad \text{Solve}$$

Check Your Answers

$x = 3$:

$\text{LHS} = \dfrac{3}{3} + \dfrac{5}{3+2}$

$= 1 + 1 = 2$

$\text{RHS} = 2$

LHS = RHS ✓

$x = -1$:

$\text{LHS} = \dfrac{3}{-1} + \dfrac{5}{-1+2}$

$= -3 + 5 = 2$

$\text{RHS} = 2$

LHS = RHS ✓

IN-CLASS MATERIALS

It has been proven mathematically that any polynomial can be factored into the products of linear factors and irreducible quadratic factors. It has also been proven that if the degree of the polynomial is greater than four there is no formula (analogous to the quadratic formula) that will allow us to find those factors in general.

We must check our answers because multiplying by an expression that contains the variable can introduce extraneous solutions. From *Check Your Answers* we see that the solutions are $x = 3$ and -1. ■

When you solve an equation that involves radicals, you must be especially careful to check your final answers. The next example demonstrates why.

Check Your Answers

$x = -\frac{1}{4}$:

$\text{LHS} = 2\left(-\frac{1}{4}\right) = -\frac{1}{2}$

$\text{RHS} = 1 - \sqrt{2 - \left(-\frac{1}{4}\right)}$

$= 1 - \sqrt{\frac{9}{4}}$

$= 1 - \frac{3}{2} = -\frac{1}{2}$

LHS = RHS ✓

$x = 1$:

$\text{LHS} = 2(1) = 2$

$\text{RHS} = 1 - \sqrt{2 - 1}$

$= 1 - 1 = 0$

LHS ≠ RHS ✗

Example 11 An Equation Involving a Radical

Solve the equation $2x = 1 - \sqrt{2 - x}$.

Solution To eliminate the square root, we first isolate it on one side of the equal sign, then square.

$2x - 1 = -\sqrt{2 - x}$	Subtract 1
$(2x - 1)^2 = 2 - x$	Square each side
$4x^2 - 4x + 1 = 2 - x$	Expand LHS
$4x^2 - 3x - 1 = 0$	Add $-2 + x$
$(4x + 1)(x - 1) = 0$	Factor
$4x + 1 = 0$ or $x - 1 = 0$	Zero-Product Property
$x = -\frac{1}{4}$ $\quad$ $x = 1$	Solve

The values $x = -\frac{1}{4}$ and $x = 1$ are only potential solutions. We must check them to see if they satisfy the original equation. From *Check Your Answers* we see that $x = -\frac{1}{4}$ is a solution but $x = 1$ is not. The only solution is $x = -\frac{1}{4}$. ■

ALTERNATE EXAMPLE 11
Solve the equation
$2x = 2 - \sqrt{6 - x}$.

ANSWER
$x = -\frac{1}{4}$

When we solve an equation, we may end up with one or more **extraneous solutions**, that is, potential solutions that do not satisfy the original equation. In Example 11, the value $x = 1$ is an extraneous solution. Extraneous solutions may be introduced when we square each side of an equation because the operation of squaring can turn a false equation into a true one. For example, $-1 \neq 1$, but $(-1)^2 = 1^2$. Thus, the squared equation may be true for more values of the variable than the original equation. That is why you must always check your answers to make sure that each satisfies the original equation.

An equation of the form $aW^2 + bW + c = 0$, where W is an algebraic expression, is an equation of **quadratic type**. We solve equations of quadratic type by substituting for the algebraic expression, as we see in the next two examples.

Example 12 A Fourth-Degree Equation of Quadratic Type

Find all solutions of the equation $x^4 - 8x^2 + 8 = 0$.

Solution If we set $W = x^2$, then we get a quadratic equation in the new variable W:

$(x^2)^2 - 8x^2 + 8 = 0$	Write x^4 as $(x^2)^2$
$W^2 - 8W + 8 = 0$	Let $W = x^2$
$W = \dfrac{-(-8) \pm \sqrt{(-8)^2 - 4 \cdot 8}}{2} = 4 \pm 2\sqrt{2}$	Quadratic formula
$x^2 = 4 \pm 2\sqrt{2}$	$W = x^2$
$x = \pm\sqrt{4 \pm 2\sqrt{2}}$	Take square roots

ALTERNATE EXAMPLE 12
Find all solutions of the equation
$x^4 - 8x^2 + 13 = 0$.

ANSWER
$x = \pm\sqrt{4 \pm \sqrt{3}}$

EXAMPLES

1. An equation with an extraneous solution: $\dfrac{x^2}{x + 1} = \dfrac{x + 2}{x + 1}$

This simplifies to $x^2 - x - 2 = 0$, either by multiplying by $x + 1$ directly or by first subtracting the RHS from the LHS and simplifying. The students can verify that $x = -2$ and $x = 2$ are solutions, and that $x = -1$ is an extraneous solution.

2. Quadratic equations with zero, one, and two solutions:

$x^2 - 6x + 11 = 3$ has two solutions, $x = 2$ and $x = 4$.
$x^2 - 6x + 12 = 3$ has one solution, $x = 3$.
$x^2 - 5x + 13 = 3$ has no real solution.

3. An equation in quadratic form:

$$(x^2 + 4x + 5)^2 - 3(x^2 + 4x + 5) + 2 = 0 \Rightarrow x = -3, x = -2, \text{ or } x = -1.$$

ALTERNATE EXAMPLE 13
Find all the solutions of the equation $x^{1/3} + x^{1/6} - 6 = 0$.

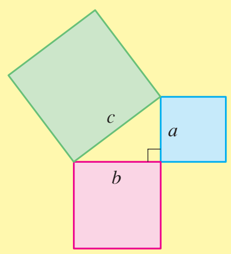
ANSWER
64

ALTERNATE EXAMPLE 14
Solve $|4x - 11| = 25$.

ANSWER
$x = 9, x = -7/2$

Pythagoras (circa 580–500 B.C.) founded a school in Croton in southern Italy, which was devoted to the study of arithmetic, geometry, music, and astronomy. The Pythagoreans, as they were called, were a secret society with peculiar rules and initiation rites. They wrote nothing down, and were not to reveal to anyone what they had learned from the Master. Although women were barred by law from attending public meetings, Pythagoras allowed women in his school, and his most famous student was Theano (whom he later married).

According to Aristotle, the Pythagoreans were convinced that "the principles of mathematics are the principles of all things." Their motto was "Everything is Number," by which they meant *whole* numbers. The outstanding contribution of Pythagoras is the theorem that bears his name: In a right triangle the area of the square on the hypotenuse is equal to the sum of the areas of the square on the other two sides.



$c^2 = a^2 + b^2$

The converse of Pythagoras's Theorem is also true: A triangle whose sides a, b, and c satisfy $a^2 + b^2 = c^2$ is a right triangle.

So, there are four solutions:

$$\sqrt{4 + 2\sqrt{2}}, \qquad \sqrt{4 - 2\sqrt{2}}, \qquad -\sqrt{4 + 2\sqrt{2}}, \qquad -\sqrt{4 - 2\sqrt{2}}$$

Using a calculator, we obtain the approximations $x \approx 2.61, 1.08, -2.61, -1.08$. ■

Example 13 An Equation Involving Fractional Powers

Find all solutions of the equation $x^{1/3} + x^{1/6} - 2 = 0$.

Solution This equation is of quadratic type because if we let $W = x^{1/6}$, then $W^2 = (x^{1/6})^2 = x^{1/3}$.

$$x^{1/3} + x^{1/6} - 2 = 0$$

$$W^2 + W - 2 = 0 \qquad \text{Let } W = x^{1/6}$$

$$(W - 1)(W + 2) = 0 \qquad \text{Factor}$$

$$W - 1 = 0 \quad \text{or} \quad W + 2 = 0 \qquad \text{Zero-Product Property}$$

$$W = 1 \qquad W = -2 \qquad \text{Solve}$$

$$x^{1/6} = 1 \qquad x^{1/6} = -2 \qquad W = x^{1/6}$$

$$x = 1^6 = 1 \qquad x = (-2)^6 = 64 \qquad \text{Take the 6th power}$$

From *Check Your Answers* we see that $x = 1$ is a solution but $x = 64$ is not. The only solution is $x = 1$. ■

Check Your Answers

$x = 1$:

LHS $= 1^{1/3} + 1^{1/6} - 2 = 0$

RHS $= 0$

LHS = RHS ✓

$x = 64$:

LHS $= 64^{1/3} + 64^{1/6} - 2$
$= 4 + 2 - 2 = 4$

RHS $= 0$

LHS $\neq$ RHS ✗

When solving equations that involve absolute values, we usually take cases.

Example 14 An Absolute Value Equation

Solve the equation $|2x - 5| = 3$.

Solution By the definition of absolute value, $|2x - 5| = 3$ is equivalent to

$$2x - 5 = 3 \quad \text{or} \quad 2x - 5 = -3$$

$$2x = 8 \qquad 2x = 2$$

$$x = 4 \qquad x = 1$$

The solutions are $x = 1$, $x = 4$. ■

1.5 Exercises

1–4 ■ Determine whether the given value is a solution of the equation.

1. $4x + 7 = 9x - 3$
(a) $x = -2$ **(b)** $x = 2$

2. $1 - [2 - (3 - x)] = 4x - (6 + x)$
(a) $x = 2$ **(b)** $x = 4$

3. $\dfrac{1}{x} - \dfrac{1}{x - 4} = 1$
(a) $x = 2$ **(b)** $x = 4$

4. $\dfrac{x^{3/2}}{x - 6} = x - 8$
(a) $x = 4$ **(b)** $x = 8$

5–22 ■ The given equation is either linear or equivalent to a linear equation. Solve the equation.

5. $2x + 7 = 31$
6. $5x - 3 = 4$
7. $\frac{1}{2}x - 8 = 1$
8. $3 + \frac{1}{3}x = 5$
9. $-7w = 15 - 2w$
10. $5t - 13 = 12 - 5t$
11. $\frac{1}{2}y - 2 = \frac{1}{3}y$
12. $\dfrac{z}{5} = \dfrac{3}{10}z + 7$
13. $2(1 - x) = 3(1 + 2x) + 5$
14. $\dfrac{2}{3}y + \dfrac{1}{2}(y - 3) = \dfrac{y + 1}{4}$
15. $x - \frac{1}{3}x - \frac{1}{2}x - 5 = 0$
16. $2x - \dfrac{x}{2} + \dfrac{x + 1}{4} = 6x$
17. $\dfrac{1}{x} = \dfrac{4}{3x} + 1$
18. $\dfrac{2x - 1}{x + 2} = \dfrac{4}{5}$
19. $\dfrac{3}{x + 1} - \dfrac{1}{2} = \dfrac{1}{3x + 3}$
20. $\dfrac{4}{x - 1} + \dfrac{2}{x + 1} = \dfrac{35}{x^2 - 1}$
21. $(t - 4)^2 = (t + 4)^2 + 32$
22. $\sqrt{3}x + \sqrt{12} = \dfrac{x + 5}{\sqrt{3}}$

23–36 ■ Solve the equation for the indicated variable.

23. $PV = nRT$; for R
24. $F = G\dfrac{mM}{r^2}$; for m
25. $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2}$; for R_1
26. $P = 2l + 2w$; for w
27. $\dfrac{ax + b}{cx + d} = 2$; for x
28. $a - 2[b - 3(c - x)] = 6$; for x
29. $a^2x + (a - 1) = (a + 1)x$; for x
30. $\dfrac{a + 1}{b} = \dfrac{a - 1}{b} + \dfrac{b + 1}{a}$; for a
31. $V = \frac{1}{3}\pi r^2 h$; for r
32. $F = G\dfrac{mM}{r^2}$; for r
33. $a^2 + b^2 = c^2$; for b
34. $A = P\left(1 + \dfrac{i}{100}\right)^2$; for i
35. $h = \frac{1}{2}gt^2 + v_0 t$; for t
36. $S = \dfrac{n(n + 1)}{2}$; for n

37–44 ■ Solve the equation by factoring.

37. $x^2 + x - 12 = 0$
38. $x^2 + 3x - 4 = 0$
39. $x^2 - 7x + 12 = 0$
40. $x^2 + 8x + 12 = 0$
41. $4x^2 - 4x - 15 = 0$
42. $2y^2 + 7y + 3 = 0$
43. $3x^2 + 5x = 2$
44. $6x(x - 1) = 21 - x$

45–52 ■ Solve the equation by completing the square.

45. $x^2 + 2x - 5 = 0$
46. $x^2 - 4x + 2 = 0$
47. $x^2 + 3x - \frac{7}{4} = 0$
48. $x^2 = \frac{3}{4}x - \frac{1}{8}$
49. $2x^2 + 8x + 1 = 0$
50. $3x^2 - 6x - 1 = 0$
51. $4x^2 - x = 0$
52. $-2x^2 + 6x + 3 = 0$

53–68 ■ Find all real solutions of the quadratic equation.

53. $x^2 - 2x - 15 = 0$
54. $x^2 + 30x + 200 = 0$
55. $x^2 + 3x + 1 = 0$
56. $x^2 - 6x + 1 = 0$
57. $2x^2 + x - 3 = 0$
58. $3x^2 + 7x + 4 = 0$
59. $2y^2 - y - \frac{1}{2} = 0$
60. $\theta^2 - \frac{3}{2}\theta + \frac{9}{16} = 0$
61. $4x^2 + 16x - 9 = 0$
62. $w^2 = 3(w - 1)$
63. $3 + 5z + z^2 = 0$
64. $x^2 - \sqrt{5}x + 1 = 0$
65. $\sqrt{6}x^2 + 2x - \sqrt{3/2} = 0$
66. $3x^2 + 2x + 2 = 0$
67. $25x^2 + 70x + 49 = 0$
68. $5x^2 - 7x + 5 = 0$

69–74 ■ Use the discriminant to determine the number of real solutions of the equation. Do not solve the equation.

69. $x^2 - 6x + 1 = 0$
70. $3x^2 = 6x - 9$
71. $x^2 + 2.20x + 1.21 = 0$
72. $x^2 + 2.21x + 1.21 = 0$
73. $4x^2 + 5x + \frac{13}{8} = 0$
74. $x^2 + rx - s = 0 \quad (s > 0)$

75–98 ■ Find all real solutions of the equation.

75. $\dfrac{1}{x - 1} + \dfrac{1}{x + 2} = \dfrac{5}{4}$
76. $\dfrac{10}{x} - \dfrac{12}{x - 3} + 4 = 0$
77. $\dfrac{x^2}{x + 100} = 50$
78. $\dfrac{1}{x - 1} - \dfrac{2}{x^2} = 0$

79. $\dfrac{x+5}{x-2} = \dfrac{5}{x+2} + \dfrac{28}{x^2-4}$ **80.** $\dfrac{x}{2x+7} - \dfrac{x+1}{x+3} = 1$

81. $\sqrt{2x+1} + 1 = x$ **82.** $\sqrt{5-x} + 1 = x - 2$

83. $2x + \sqrt{x+1} = 8$ **84.** $\sqrt{\sqrt{x-5} + x} = 5$

85. $x^4 - 13x^2 + 40 = 0$ **86.** $x^4 - 5x^2 + 4 = 0$

87. $2x^4 + 4x^2 + 1 = 0$ **88.** $x^6 - 2x^3 - 3 = 0$

89. $x^{4/3} - 5x^{2/3} + 6 = 0$ **90.** $\sqrt{x} - 3\sqrt[4]{x} - 4 = 0$

91. $4(x+1)^{1/2} - 5(x+1)^{3/2} + (x+1)^{5/2} = 0$

92. $x^{1/2} + 3x^{-1/2} = 10x^{-3/2}$

93. $x^{1/2} - 3x^{1/3} = 3x^{1/6} - 9$ **94.** $x - 5\sqrt{x} + 6 = 0$

95. $|2x| = 3$ **96.** $|3x + 5| = 1$

97. $|x - 4| = 0.01$ **98.** $|x - 6| = -1$

Applications

99–100 ■ Falling-Body Problems Suppose an object is dropped from a height h_0 above the ground. Then its height after t seconds is given by $h = -16t^2 + h_0$, where h is measured in feet. Use this information to solve the problem.

99. If a ball is dropped from 288 ft above the ground, how long does it take to reach ground level?

100. A ball is dropped from the top of a building 96 ft tall.

(a) How long will it take to fall half the distance to ground level?

(b) How long will it take to fall to ground level?

101–102 ■ Falling-Body Problems Use the formula $h = -16t^2 + v_0 t$ discussed in Example 9.

101. A ball is thrown straight upward at an initial speed of $v_0 = 40$ ft/s.

(a) When does the ball reach a height of 24 ft?

(b) When does it reach a height of 48 ft?

(c) What is the greatest height reached by the ball?

(d) When does the ball reach the highest point of its path?

(e) When does the ball hit the ground?

102. How fast would a ball have to be thrown upward to reach a maximum height of 100 ft? [*Hint:* Use the discriminant of the equation $16t^2 - v_0 t + h = 0$.]

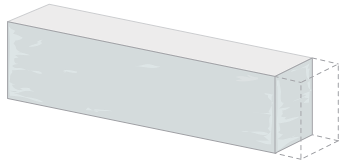
103. Shrinkage in Concrete Beams As concrete dries, it shrinks—the higher the water content, the greater the shrinkage. If a concrete beam has a water content of w kg/m^3, then it will shrink by a factor

$$S = \frac{0.032w - 2.5}{10{,}000}$$

where S is the fraction of the original beam length that disappears due to shrinkage.

(a) A beam 12.025 m long is cast in concrete that contains 250 kg/m^3 water. What is the shrinkage factor S? How long will the beam be when it has dried?

(b) A beam is 10.014 m long when wet. We want it to shrink to 10.009 m, so the shrinkage factor should be $S = 0.00050$. What water content will provide this amount of shrinkage?



104. The Lens Equation If F is the focal length of a convex lens and an object is placed at a distance x from the lens, then its image will be at a distance y from the lens, where F, x, and y are related by the *lens equation*

$$\frac{1}{F} = \frac{1}{x} + \frac{1}{y}$$

Suppose that a lens has a focal length of 4.8 cm, and that the image of an object is 4 cm closer to the lens than the object itself. How far from the lens is the object?

105. Fish Population The fish population in a certain lake rises and falls according to the formula

$$F = 1000(30 + 17t - t^2)$$

Here F is the number of fish at time t, where t is measured in years since January 1, 2002, when the fish population was first estimated.

(a) On what date will the fish population again be the same as on January 1, 2002?

(b) By what date will all the fish in the lake have died?

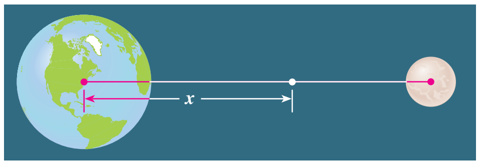
106. Fish Population A large pond is stocked with fish. The fish population P is modeled by the formula $P = 3t + 10\sqrt{t} + 140$, where t is the number of days since the fish were first introduced into the pond. How many days will it take for the fish population to reach 500?

107. Profit A small-appliance manufacturer finds that the profit P (in dollars) generated by producing x microwave ovens per week is given by the formula $P = \frac{1}{10}x(300 - x)$ provided that $0 \le x \le 200$. How many ovens must be manufactured in a given week to generate a profit of $1250?

108. Gravity If an imaginary line segment is drawn between the centers of the earth and the moon, then the net gravitational force F acting on an object situated on this line segment is

$$F = \frac{-K}{x^2} + \frac{0.012K}{(239 - x)^2}$$

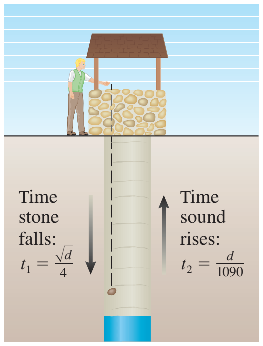
where $K > 0$ is a constant and x is the distance of the object from the center of the earth, measured in thousands of miles. How far from the center of the earth is the "dead spot" where no net gravitational force acts upon the object? (Express your answer to the nearest thousand miles.)



109. Depth of a Well One method for determining the depth of a well is to drop a stone into it and then measure the time it takes until the splash is heard. If d is the depth of the well (in feet) and t_1 the time (in seconds) it takes for the stone to fall, then $d = 16t_1^2$, so $t_1 = \sqrt{d}/4$. Now if t_2 is the time it takes for the sound to travel back up, then $d = 1090t_2$ because the speed of sound is 1090 ft/s. So $t_2 = d/1090$. Thus, the total time elapsed between dropping the stone and hearing the splash is

$$t_1 + t_2 = \frac{\sqrt{d}}{4} + \frac{d}{1090}$$

How deep is the well if this total time is 3 s?



Discovery • Discussion

110. A Family of Equations The equation

$$3x + k - 5 = kx - k + 1$$

is really a **family of equations**, because for each value of k, we get a different equation with the unknown x. The letter k is called a **parameter** for this family. What value should we pick for k to make the given value of x a solution of the resulting equation?

(a) $x = 0$ **(b)** $x = 1$ **(c)** $x = 2$

111. Proof That 0 = 1? The following steps appear to give equivalent equations, which seem to prove that $1 = 0$. Find the error.

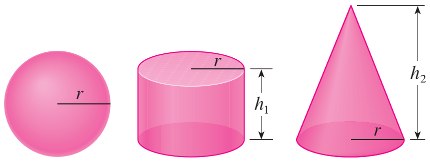
$x = 1$	Given
$x^2 = x$	Multiply by x
$x^2 - x = 0$	Subtract x
$x(x - 1) = 0$	Factor
$\frac{x(x-1)}{x-1} = \frac{0}{x-1}$	Divide by x − 1
$x = 0$	Simplify
$1 = 0$	Given x = 1

112. Volumes of Solids The sphere, cylinder, and cone shown here all have the same radius r and the same volume V.

(a) Use the volume formulas given on the inside front cover of this book, to show that

$$\tfrac{4}{3}\pi r^3 = \pi r^2 h_1 \quad \text{and} \quad \tfrac{4}{3}\pi r^3 = \tfrac{1}{3}\pi r^2 h_2$$

(b) Solve these equations for h_1 and h_2.



113. Relationship between Roots and Coefficients The quadratic formula gives us the roots of a quadratic equation from its coefficients. We can also obtain the coefficients from the roots. For example, find the roots of the equation $x^2 - 9x + 20 = 0$ and show that the product of the roots is the constant term 20 and the sum of the roots is 9, the negative of the coefficient of x. Show that the same relationship between roots and coefficients holds for the following equations:

$$x^2 - 2x - 8 = 0$$

$$x^2 + 4x + 2 = 0$$

Use the quadratic formula to prove that in general, if the equation $x^2 + bx + c = 0$ has roots r_1 and r_2, then $c = r_1 r_2$ and $b = -(r_1 + r_2)$.

114. Solving an Equation in Different Ways We have learned several different ways to solve an equation in this section. Some equations can be tackled by more than one method. For example, the equation $x - \sqrt{x} - 2 = 0$ is of quadratic type: We can solve it by letting $\sqrt{x} = u$ and $x = u^2$, and factoring. Or we could solve for $\sqrt{x}$, square each side, and then solve the resulting quadratic equation.

Solve the following equations using both methods indicated, and show that you get the same final answers.

(a) $x - \sqrt{x} - 2 = 0$ quadratic type; solve for the radical, and square

(b) $\dfrac{12}{(x-3)^2} + \dfrac{10}{x-3} + 1 = 0$ quadratic type; multiply by LCD

SUGGESTED TIME AND EMPHASIS

1 class.
Essential material.

1.6 Modeling with Equations

Many problems in the sciences, economics, finance, medicine, and numerous other fields can be translated into algebra problems; this is one reason that algebra is so useful. In this section we use equations as mathematical models to solve real-life problems.

Guidelines for Modeling with Equations

We will use the following guidelines to help us set up equations that model situations described in words. To show how the guidelines can help you set up equations, we note them in the margin as we work each example in this section.

Guidelines for Modeling with Equations

1. **Identify the Variable.** Identify the quantity that the problem asks you to find. This quantity can usually be determined by a careful reading of the question posed at the end of the problem. Then **introduce notation** for the variable (call it x or some other letter).
2. **Express All Unknown Quantities in Terms of the Variable.** Read each sentence in the problem again, and express all the quantities mentioned in the problem in terms of the variable you defined in Step 1. To organize this information, it is sometimes helpful to **draw a diagram** or **make a table**.
3. **Set Up the Model.** Find the crucial fact in the problem that gives a relationship between the expressions you listed in Step 2. **Set up an equation** (or **model**) that expresses this relationship.
4. **Solve the Equation and Check Your Answer.** Solve the equation, check your answer, and express it as a sentence that answers the question posed in the problem.

The following example illustrates how these guidelines are used to translate a "word problem" into the language of algebra.

POINTS TO STRESS

1. Solving applied problems described verbally.
2. Presenting the solution process in a clear, organized way.

Example 1 **Renting a Car**

A car rental company charges \$30 a day and 15¢ a mile for renting a car. Helen rents a car for two days and her bill comes to \$108. How many miles did she drive?

Solution We are asked to find the number of miles Helen has driven. So we let

Identify the variable

$$x = \text{number of miles driven}$$

Then we translate all the information given in the problem into the language of algebra.

Express all unknown quantities in terms of the variable

In Words	In Algebra
Number of miles driven	x
Mileage cost (at \$0.15 per mile)	$0.15x$
Daily cost (at \$30 per day)	$2(30)$

Now we set up the model.

Set up the model

$$\text{mileage cost} + \text{daily cost} = \text{total cost}$$

Solve

$$0.15x + 2(30) = 108$$

$$0.15x = 48 \qquad \text{Substract 60}$$

$$x = \frac{48}{0.15} \qquad \text{Divide by 0.15}$$

$$x = 320 \qquad \text{Calculator}$$

Helen drove her rental car 320 miles. ■

Check Your Answer

total cost = mileage cost + daily cost

= 0.15(320) + 2(30)

= 108 ✓

Constructing Models

In the examples and exercises that follow, we construct equations that model problems in many different real-life situations.

Example 2 **Interest on an Investment**

Mary inherits \$100,000 and invests it in two certificates of deposit. One certificate pays 6% and the other pays $4\frac{1}{2}$% simple interest annually. If Mary's total interest is \$5025 per year, how much money is invested at each rate?

Solution The problem asks for the amount she has invested at each rate. So we let

Identify the variable

$$x = \text{the amount invested at 6\%}$$

Since Mary's total inheritance is \$100,000, it follows that she invested $100,000 - x$ at $4\frac{1}{2}$%. We translate all the information given into the language of algebra.

ALTERNATE EXAMPLE 1

Express the given quantity in terms of the indicated variable.

The average of three test scores if the first two scores are 66 and 84; s = third test score

ANSWER

$$\frac{150 + s}{3}$$

ALTERNATE EXAMPLE 2

Mary inherits \$100,000 and invests it in two certificates of deposit. One certificate pays 6% and the other pays $4\frac{1}{2}$% simple interest annually.

If Mary's total interest is \$5,775 per year, how much money is invested at each rate?

ANSWER

\$85,000; \$15,000

SAMPLE QUESTION
Text Question

Discuss one of the problems presented in this section. You don't have to know all the specific numbers, but describe the problem and the method of solution.

Express all unknown quantities in terms of the variable

In Words	In Algebra
Amount invested at 6%	x
Amount invested at $4\frac{1}{2}\%$	$100{,}000 - x$
Interest earned at 6%	$0.06x$
Interest earned at $4\frac{1}{2}\%$	$0.045(100{,}000 - x)$

We use the fact that Mary's total interest is $5025 to set up the model.

Set up the model

interest at 6% + interest at $4\frac{1}{2}\%$ = total interest

Solve

$$0.06x + 0.045(100{,}000 - x) = 5025$$

$$0.06x + 4500 - 0.045x = 5025 \qquad \text{Multiply}$$

$$0.015x + 4500 = 5025 \qquad \text{Combine the } x\text{-terms}$$

$$0.015x = 525 \qquad \text{Subtract 4500}$$

$$x = \frac{525}{0.015} = 35{,}000 \qquad \text{Divide by 0.015}$$

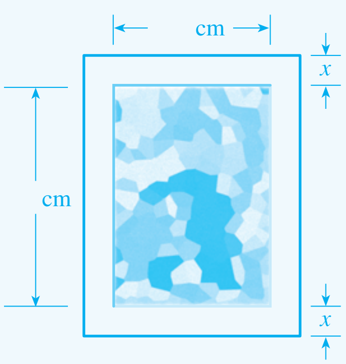
So Mary has invested $35,000 at 6% and the remaining $65,000 at $4\frac{1}{2}\%$. ■

Check Your Answer

$$\text{total interest} = 6\% \text{ of } \$35{,}000 + 4\tfrac{1}{2}\% \text{ of } \$65{,}000$$
$$= \$2100 + \$2925 = \$5025 \quad ✓$$

ALTERNATE EXAMPLE 3

A poster has a rectangular printed area 140 cm by 180 cm, and a blank strip of uniform width around the four edges. The perimeter of the poster is $1\frac{1}{2}$ times the perimeter of the printed area. Find the width of the blank strip and the dimensions of the poster.



ANSWER

40 cm, 260 cm × 220 cm

Example 3 Dimensions of a Poster

In a problem such as this, which involves geometry, it is essential to draw a diagram like the one shown in Figure 1.

A poster has a rectangular printed area 100 cm by 140 cm, and a blank strip of uniform width around the four edges. The perimeter of the poster is $1\frac{1}{2}$ times the perimeter of the printed area. What is the width of the blank strip, and what are the dimensions of the poster?

Solution We are asked to find the width of the blank strip. So we let

Identify the variable

$$x = \text{the width of the blank strip}$$

Then we translate the information in Figure 1 into the language of algebra:

Express all unknown quantities in terms of the variable

In Words	In Algebra
Width of blank strip	x
Perimeter of printed area	$2(100) + 2(140) = 480$
Width of poster	$100 + 2x$
Length of poster	$140 + 2x$
Perimeter, of poster	$2(100 + 2x) + 2(140 + 2x)$

Now we use the fact that the perimeter of the poster is $1\frac{1}{2}$ times the perimeter of the printed area to set up the model.

Set up the model

$$\text{perimeter of poster} = \tfrac{3}{2} \cdot \text{perimeter of printed area}$$

$$2(100 + 2x) + 2(140 + 2x) = \tfrac{3}{2} \cdot 480$$

Solve

$$480 + 8x = 720 \qquad \text{Expand and combine like terms on LHS}$$

$$8x = 240 \qquad \text{Subtract 480}$$

$$x = 30 \qquad \text{Divide by 8}$$

The blank strip is 30 cm wide, so the dimensions of the poster are

$$100 + 30 + 30 = 160 \text{ cm wide}$$

by

$$140 + 30 + 30 = 200 \text{ cm long}$$

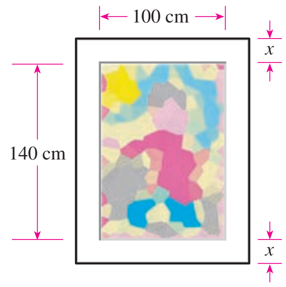


Figure 1

Example 4 Dimensions of a Building Lot

A rectangular building lot is 8 ft longer than it is wide and has an area of 2900 ft^2. Find the dimensions of the lot.

Solution We are asked to find the width and length of the lot. So let

Identify the variable

$$w = \text{width of lot}$$

Then we translate the information given in the problem into the language of algebra (see Figure 2 on page 62).

Express all unknown quantities in terms of the variable

In Words	In Algebra
Width of lot	w
Length of lot	$w + 8$

Now we set up the model.

ALTERNATE EXAMPLE 4
A rectangular building lot is 6 ft longer than it is wide and has an area of 3960 ft^2. Find the dimensions of the lot.

ANSWER
(60, 66)

IN-CLASS MATERIALS

Here is a simple-sounding problem: A person is currently making $25,000 a year (after taxes) working at a large company, and has an opportunity to quit the job and make $35,000 independently. Is this a good deal financially? Elicit other considerations such as: Insurance tends to cost $400 per month for an individual—the company is no longer paying for that. Taxes will be roughly 30%. The large company probably pays some percentage (say 2%) towards a retirement account. Based on the discussion that your particular class has, come up with a linear equation that converts an annual gross income for an independent contractor to a net income, and then figure out how much the independent contractor would have to charge in order to net $25,000. Other hard-to-quantify considerations may come up such as paid vacations, sick leave, "being one's own boss," the tax benefits of having a home office, quality of life, and so forth. Acknowledge these considerations, and perhaps (for purposes of comparison) try to quantify them. Perhaps assume that the tax benefits for the home office add 5% to the gross income. Ask the class if the pleasure of not having a boss is worth, say, $2000 a year. (On the other hand, would they get any work done at all without a boss standing over them?) This is a real-life modeling situation that, every year, more and more people find themselves thinking about.

Set up the model

$$\text{width of lot} \cdot \text{length of lot} = \text{area of lot}$$

$$w(w+8) = 2900$$

Solve

$$w^2 + 8w = 2900 \qquad \text{Expand}$$

$$w^2 + 8w - 2900 = 0 \qquad \text{Subtract 2900}$$

$$(w-50)(w+58) = 0 \qquad \text{Factor}$$

$$w = 50 \quad \text{or} \quad w = -58 \qquad \text{Zero-Product Property}$$

Since the width of the lot must be a positive number, we conclude that $w = 50$ ft. The length of the lot is $w + 8 = 50 + 8 = 58$ ft.

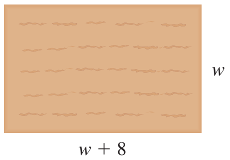


Figure 2

■

Example 5 Determining the Height of a Building Using Similar Triangles

A man 6 ft tall wishes to find the height of a certain four-story building. He measures its shadow and finds it to be 28 ft long, while his own shadow is $3\frac{1}{2}$ ft long. How tall is the building?

Solution The problem asks for the height of the building. So let

Identify the variable

$$h = \text{the height of the building}$$

We use the fact that the triangles in Figure 3 are similar. Recall that for any pair of similar triangles the ratios of corresponding sides are equal. Now we translate these observations into the language of algebra.

Express all unknown quantities in terms of the variable

In Words	In Algebra
Height of building	h
Ratio of height to base in large triangle	$\frac{h}{28}$
Ratio of height to base in small triangle	$\frac{6}{3.5}$

Since the large and small triangles are similar, we get the equation

Set up the model

$$\text{ratio of height to base in large triangle} = \text{ratio of height to base in small triangle}$$

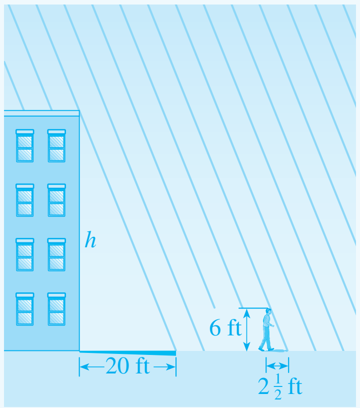
$$\frac{h}{28} = \frac{6}{3.5}$$

Solve

$$h = \frac{6 \cdot 28}{3.5} = 48$$

ALTERNATE EXAMPLE 5

A man 6 ft tall wishes to find the height of a certain four-story building. He measures its shadow and finds it to be 20 ft long, while his own shadow is $2\frac{1}{2}$ ft long. How tall is the building?



ANSWER

$h = 48$

IN-CLASS MATERIALS

This section is particularly suited to having students come up with their own problems to solve, or have others solve. Once they understand the concept, with a little (or a lot of) thought, they should be able to come up with practical, relevant problems. If students need a little prompting (try to get away without such prompting) you can wonder how many recordable CDs one would have to buy to store a hard drive full of music, how many people one could invite to a party that has a firm budget of \$200 (have them list fixed costs and per-person costs), how many miles one could drive a car (paying all expenses) for \$2000/year, etc.

The building is 48 ft tall.

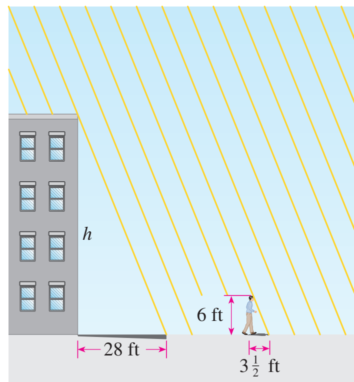


Figure 3

■

Example 6 Mixtures and Concentration

A manufacturer of soft drinks advertises their orange soda as "naturally flavored," although it contains only 5% orange juice. A new federal regulation stipulates that to be called "natural" a drink must contain at least 10% fruit juice. How much pure orange juice must this manufacturer add to 900 gal of orange soda to conform to the new regulation?

Solution The problem asks for the amount of pure orange juice to be added. So let

Identify the variable

$$x = \text{the amount (in gallons) of pure orange juice to be added}$$

In any problem of this type—in which two different substances are to be mixed—drawing a diagram helps us organize the given information (see Figure 4).

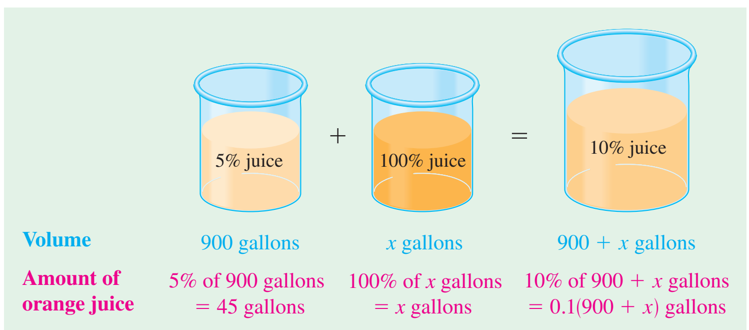


Figure 4

ALTERNATE EXAMPLE 6
A manufacturer of soft drinks advertises their orange soda as *naturally flavored*, although it contains only 4% orange juice. A new federal regulation stipulates that to be called *natural* a drink must contain at least 20% fruit juice.

Identify the amount of pure orange juice which the manufacturer must add to 80 gallons of orange soda to conform to the new regulation.

ANSWER
$x = 16$

EXAMPLES

1. A straightforward mixture problem: A child makes 2 quarts of chocolate milk consisting of 30% syrup and 70% milk. It is far too sweet. How much milk would you have to add to get a mixture that is 5% syrup?

2. A straightforward job-sharing problem: If it takes Mike 2 hours to mow the lawn, and Al 3 hours to mow the lawn, how long does it take the two of them, working together?

ANSWERS

1. 10 quarts

2. 1.2 hours

We now translate the information in the figure into the language of algebra.

Express all unknown quantities in terms of the variable

In Words	In Algebra
Amount of orange juice to be added	x
Amount of the mixture	$900 + x$
Amount of orange juice in the first vat	$0.05(900) = 45$
Amount of orange juice in the second vat	$1 \cdot x = x$
Amount of orange juice in the mix ture	$0.10(900 + x)$

To set up the model, we use the fact that the total amount of orange juice in the mixture is equal to the orange juice in the first two vats.

Set up the model

$$\boxed{\text{amount of orange juice in first vat}} + \boxed{\text{amount of orange juice in second vat}} = \boxed{\text{amount of orange juice in mixture}}$$

$$45 + x = 0.1(900 + x) \qquad \text{From Figure 4}$$

Solve

$$45 + x = 90 + 0.1x \qquad \text{Multiply}$$

$$0.9x = 45 \qquad \text{Subtract 0.1x and 45}$$

$$x = \frac{45}{0.9} = 50 \qquad \text{Divide by 0.9}$$

The manufacturer should add 50 gal of pure orange juice to the soda. ■

Check Your Answer

$$\text{amount of juice before mixing} = 5\% \text{ of } 900 \text{ gal} + 50 \text{ gal pure juice}$$

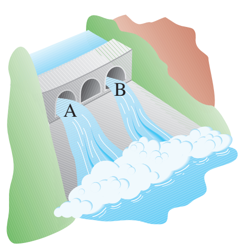
$$= 45 \text{ gal} + 50 \text{ gal} = 95 \text{ gal}$$

$$\text{amount of juice after mixing} = 10\% \text{ of } 950 \text{ gal} = 95 \text{ gal}$$

Amounts are equal. ✓

ALTERNATE EXAMPLE 7
It takes 8 hours for a cheap pump to clear out a flooded basement. It only takes 5 hours for an expensive pump to do so. How long would it take if we owned one of each?

ANSWER
40/13 or approximately 3 hours and 5 minutes.



Example 7 Time Needed to Do a Job

Because of an anticipated heavy rainstorm, the water level in a reservoir must be lowered by 1 ft. Opening spillway A lowers the level by this amount in 4 hours, whereas opening the smaller spillway B does the job in 6 hours. How long will it take to lower the water level by 1 ft if both spillways are opened?

Solution We are asked to find the time needed to lower the level by 1 ft if both spillways are open. So let

Identify the variable

x = the time (in hours) it takes to lower the water level by 1 ft if both spillways are open

Finding an equation relating x to the other quantities in this problem is not easy.

Certainly x is not simply $4 + 6$, because that would mean that together the two spillways require longer to lower the water level than either spillway alone. Instead, *we look at the fraction of the job that can be done in one hour by each spillway.*

Express all unknown quantities in terms of the variable

In Words	In Algebra
Time it takes to lower level 1 ft with A and B together	x h
Distance A lowers level in 1 h	$\frac{1}{4}$ ft
Distance B lowers level in 1 h	$\frac{1}{6}$ ft
Distance A and B together lower levels in 1 h	$\frac{1}{x}$ ft

Now we set up the model.

Set up the model

$$\text{fraction done by A} + \text{fraction done by B} = \text{fraction done by both}$$

$$\frac{1}{4} + \frac{1}{6} = \frac{1}{x}$$

Solve

$$3x + 2x = 12 \qquad \text{Multiply by the LCD, } 12x$$

$$5x = 12 \qquad \text{Add}$$

$$x = \frac{12}{5} \qquad \text{Divide by 5}$$

It will take $2\frac{2}{5}$ hours, or 2 h 24 min to lower the water level by 1 ft if both spillways are open. ■

The next example deals with distance, rate (speed), and time. The formula to keep in mind here is

$$\text{distance} = \text{rate} \times \text{time}$$

where the rate is either the constant speed or average speed of a moving object. For example, driving at 60 mi/h for 4 hours takes you a distance of $60 \cdot 4 = 240$ mi.

Example 8 A Distance-Speed-Time Problem

A jet flew from New York to Los Angeles, a distance of 4200 km. The speed for the return trip was 100 km/h faster than the outbound speed. If the total trip took 13 hours, what was the jet's speed from New York to Los Angeles?

Solution We are asked for the speed of the jet from New York to Los Angeles. So let

Identify the variable

$$s = \text{speed from New York to Los Angeles}$$

Then $$s + 100 = \text{speed from Los Angeles to New York}$$

Now we organize the information in a table. We fill in the "Distance" column first, since we know that the cities are 4200 km apart. Then we fill in the "Speed" column, since we have expressed both speeds (rates) in terms of the variable s. Finally, we calculate the entries for the "Time" column, using

$$\text{time} = \frac{\text{distance}}{\text{rate}}$$

ALTERNATE EXAMPLE 8

A jet flew from New York to Los Angeles, a distance of 4200 km. The speed for the return trip was 140 km/h faster than the outbound speed. If the total trip took 11 hours, what was the speed from New York to Los Angeles?

ANSWER

700 km/h

Express all unknown quantities in terms of the variable

	Distance (km)	Speed (km/h)	Time (h)
N.Y. to L.A.	4200	s	$\frac{4200}{s}$
L.A. to N.Y.	4200	$s + 100$	$\frac{4200}{s+100}$

The total trip took 13 hours, so we have the model

Set up the model

$$\text{time from N.Y. to L.A.} + \text{time from L.A. to N.Y.} = \text{total time}$$

$$\frac{4200}{s} + \frac{4200}{s+100} = 13$$

Multiplying by the common denominator, $s(s + 100)$, we get

$$4200(s + 100) + 4200s = 13s(s + 100)$$

$$8400s + 420{,}000 = 13s^2 + 1300s$$

$$0 = 13s^2 - 7100s - 420{,}000$$

Although this equation does factor, with numbers this large it is probably quicker to use the quadratic formula and a calculator.

Solve

$$s = \frac{7100 \pm \sqrt{(-7100)^2 - 4(13)(-420{,}000)}}{2(13)}$$

$$= \frac{7100 \pm 8500}{26}$$

$$s = 600 \qquad \text{or} \qquad s = \frac{-1400}{26} \approx -53.8$$

Since s represents speed, we reject the negative answer and conclude that the jet's speed from New York to Los Angeles was 600 km/h. ■

Example 9 Energy Expended in Bird Flight

Ornithologists have determined that some species of birds tend to avoid flights over large bodies of water during daylight hours, because air generally rises over land and falls over water in the daytime, so flying over water requires more energy. A bird is released from point A on an island, 5 mi from B, the nearest point on a straight shoreline. The bird flies to a point C on the shoreline and then flies along the shoreline to its nesting area D, as shown in Figure 5. Suppose the bird has 170 kcal of energy reserves. It uses 10 kcal/mi flying over land and 14 kcal/mi flying over water.

(a) Where should the point C be located so that the bird uses exactly 170 kcal of energy during its flight?

(b) Does the bird have enough energy reserves to fly directly from A to D?

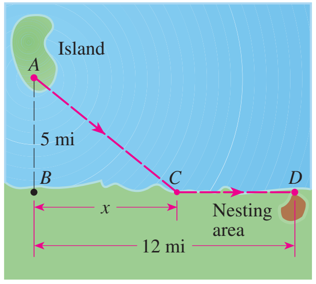
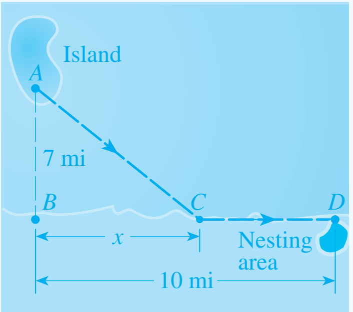


Figure 5

ALTERNATE EXAMPLE 9

Ornithologists have determined that some species of birds tend to avoid flights over large bodies of water during daylight hours, because air generally rises over land and falls over water in the daytime, so flying over water requires more energy. A bird is released from point A on an island, 7 mi from B, the nearest point on a straight shoreline. The bird flies to a point C on the shoreline and then flies along the shoreline to its nesting area D. Suppose the bird has 170 kcal of energy reserves. It uses 10 kcal/mi flying over land and 14 kcal/mi flying over water. Where should the point C be located so that the bird uses exactly 170 kcal of energy during its flight, if the distance between B and D is 10 miles?



ANSWER

$x = 5\frac{1}{4}, x = 9\frac{1}{3}$

Solution

(a) We are asked to find the location of C. So let

Identify the variable

$$x = \text{distance from } B \text{ to } C$$

From the figure, and from the fact that

$$\text{energy used} = \text{energy per mile} \times \text{miles flown}$$

we determine the following:

Express all unknown quantities in terms of the variable

In Words	In Algebra	
Distance from B to C	x	
Distance flown over water (from A to C)	$\sqrt{x^2 + 25}$	Pythagorean Theorem
Distance flown over land (from C to D)	$12 - x$	
Energy used over water	$14\sqrt{x^2 + 25}$	
Energy used over land	$10(12 - x)$	

Now we set up the model.

Set up the model

$$\boxed{\text{total energy used}} = \boxed{\text{energy used over water}} + \boxed{\text{energy used over land}}$$

$$170 = 14\sqrt{x^2 + 25} + 10(12 - x)$$

To solve this equation, we eliminate the square root by first bringing all other terms to the left of the equal sign and then squaring each side.

Solve

$$170 - 10(12 - x) = 14\sqrt{x^2 + 25} \qquad \text{Isolate square-root term on RHS}$$

$$50 + 10x = 14\sqrt{x^2 + 25} \qquad \text{Simplify LHS}$$

$$(50 + 10x)^2 = (14)^2(x^2 + 25) \qquad \text{Square each side}$$

$$2500 + 1000x + 100x^2 = 196x^2 + 4900 \qquad \text{Expand}$$

$$0 = 96x^2 - 1000x + 2400 \qquad \text{All terms to RHS}$$

This equation could be factored, but because the numbers are so large it is easier to use the quadratic formula and a calculator:

$$x = \frac{1000 \pm \sqrt{(-1000)^2 - 4(96)(2400)}}{2(96)}$$

$$= \frac{1000 \pm 280}{192} = 6\tfrac{2}{3} \quad \text{or} \quad 3\tfrac{3}{4}$$

Point C should be either $6\frac{2}{3}$ mi or $3\frac{3}{4}$ mi from B so that the bird uses exactly 170 kcal of energy during its flight.

(b) By the Pythagorean Theorem (see page 54), the length of the route directly from A to D is $\sqrt{5^2 + 12^2} = 13$ mi, so the energy the bird requires for that route is $14 \times 13 = 182$ kcal. This is more energy than the bird has available, so it can't use this route. ■

1.6 Exercises

1–12 ■ Express the given quantity in terms of the indicated variable.

1. The sum of three consecutive integers; n = first integer of the three

2. The sum of three consecutive integers; n = middle integer of the three

3. The average of three test scores if the first two scores are 78 and 82; s = third test score

4. The average of four quiz scores if each of the first three scores is 8; q = fourth quiz score

5. The interest obtained after one year on an investment at $2\frac{1}{2}\%$ simple interest per year; x = number of dollars invested

6. The total rent paid for an apartment if the rent is \$795 a month; n = number of months

7. The area (in ft^2) of a rectangle that is three times as long as it is wide; w = width of the rectangle (in ft)

8. The perimeter (in cm) of a rectangle that is 5 cm longer than it is wide; w = width of the rectangle (in cm)

9. The distance (in mi) that a car travels in 45 min; s = speed of the car (in mi/h)

10. The time (in hours) it takes to travel a given distance at 55 mi/h; d = given distance (in mi)

11. The concentration (in oz/gal) of salt in a mixture of 3 gal of brine containing 25 oz of salt, to which some pure water has been added; x = volume of pure water added (in gal)

12. The value (in cents) of the change in a purse that contains twice as many nickels as pennies, four more dimes than nickels, and as many quarters as dimes and nickels combined; p = number of pennies

Applications

13. Number Problem Find three consecutive integers whose sum is 156.

14. Number Problem Find four consecutive odd integers whose sum is 416.

15. Number Problem Find two numbers whose sum is 55 and whose product is 684.

16. Number Problem The sum of the squares of two consecutive even integers is 1252. Find the integers.

17. Investments Phyllis invested \$12,000, a portion earning a simple interest rate of $4\frac{1}{2}\%$ per year and the rest earning a rate of 4% per year. After one year the total interest earned on these investments was \$525. How much money did she invest at each rate?

18. Investments If Ben invests \$4000 at 4% interest per year, how much additional money must he invest at $5\frac{1}{2}\%$ annual interest to ensure that the interest he receives each year is $4\frac{1}{2}\%$ of the total amount invested?

19. Investments What annual rate of interest would you have to earn on an investment of \$3500 to ensure receiving \$262.50 interest after one year?

20. Investments Jack invests \$1000 at a certain annual interest rate, and he invests another \$2000 at an annual rate that is one-half percent higher. If he receives a total of \$190 interest in one year, at what rate is the \$1000 invested?

21. Salaries An executive in an engineering firm earns a monthly salary plus a Christmas bonus of \$8500. If she earns a total of \$97,300 per year, what is her monthly salary?

22. Salaries A woman earns 15% more than her husband. Together they make \$69,875 per year. What is the husband's annual salary?

23. Inheritance Craig is saving to buy a vacation home. He inherits some money from a wealthy uncle, then combines this with the \$22,000 he has already saved and doubles the total in a lucky investment. He ends up with \$134,000, just enough to buy a cabin on the lake. How much did he inherit?

24. Overtime Pay Helen earns \$7.50 an hour at her job, but if she works more than 35 hours in a week she is paid $1\frac{1}{2}$ times her regular salary for the overtime hours worked. One week her gross pay was \$352.50. How many overtime hours did she work that week?

25. Labor Costs A plumber and his assistant work together to replace the pipes in an old house. The plumber charges \$45 an hour for his own labor and \$25 an hour for his assistant's labor. The plumber works twice as long as his assistant on this job, and the labor charge on the final bill is \$4025. How long did the plumber and his assistant work on this job?

26. Career Home Runs During his major league career, Hank Aaron hit 41 more home runs than Babe Ruth hit during his career. Together they hit 1469 home runs. How many home runs did Babe Ruth hit?

27. A Riddle A movie star, unwilling to give his age, posed the following riddle to a gossip columnist. "Seven years ago, I was eleven times as old as my daughter. Now I am four times as old as she is." How old is the star?

28. A Riddle A father is four times as old as his daughter. In 6 years, he will be three times as old as she is. How old is the daughter now?

29. Value of Coins A change purse contains an equal number of pennies, nickels, and dimes. The total value of the coins is \$1.44. How many coins of each type does the purse contain?

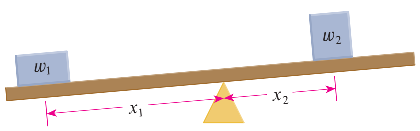
30. Value of Coins Mary has \$3.00 in nickels, dimes, and quarters. If she has twice as many dimes as quarters and five more nickels than dimes, how many coins of each type does she have?

31. Law of the Lever The figure shows a lever system, similar to a seesaw that you might find in a children's playground. For the system to balance, the product of the weight and its distance from the fulcrum must be the same on each side; that is

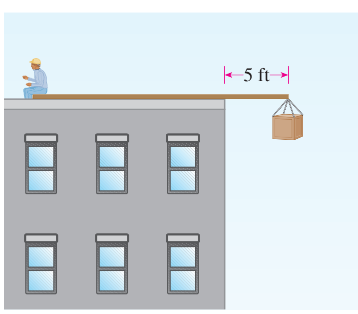
$$w_1x_1 = w_2x_2$$

This equation is called the **law of the lever**, and was first discovered by Archimedes (see page 748).

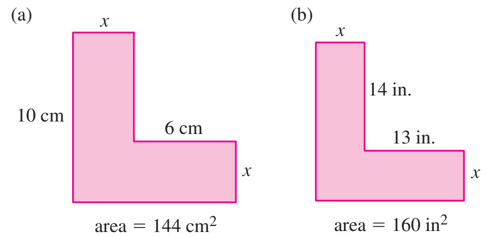
A woman and her son are playing on a seesaw. The boy is at one end, 8 ft from the fulcrum. If the son weighs 100 lb and the mother weighs 125 lb, where should the woman sit so that the seesaw is balanced?



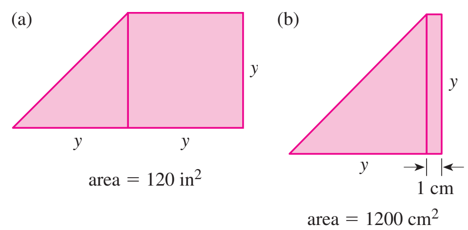
32. Law of the Lever A plank 30 ft long rests on top of a flat-roofed building, with 5 ft of the plank projecting over the edge, as shown in the figure. A worker weighing 240 lb sits on one end of the plank. What is the largest weight that can be hung on the projecting end of the plank if it is to remain in balance? (Use the law of the lever stated in Exercise 31.)



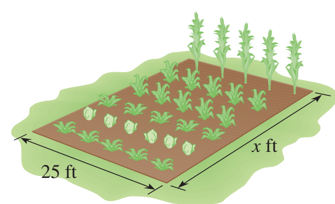
33. Length and Area Find the length x in the figure. The area of the shaded region is given.



34. Length and Area Find the length y in the figure. The area of the shaded region is given.



35. Length of a Garden A rectangular garden is 25 ft wide. If its area is 1125 ft^2, what is the length of the garden?



36. Width of a Pasture A pasture is twice as long as it is wide. Its area is 115,200 ft^2. How wide is the pasture?

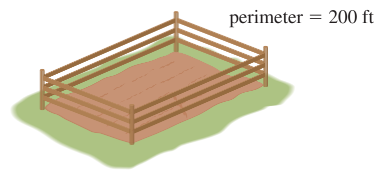
37. Dimensions of a Lot A square plot of land has a building 60 ft long and 40 ft wide at one corner. The rest of the land outside the building forms a parking lot. If the parking lot has area 12,000 ft^2, what are the dimensions of the entire plot of land?

38. Dimensions of a Lot A half-acre building lot is five times as long as it is wide. What are its dimensions? [*Note:* 1 acre = 43,560 ft^2.]

39. Dimensions of a Garden A rectangular garden is 10 ft longer than it is wide. Its area is 875 ft^2. What are its dimensions?

40. Dimensions of a Room A rectangular bedroom is 7 ft longer than it is wide. Its area is 228 ft^2. What is the width of the room?

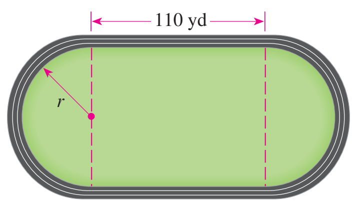
41. Dimensions of a Garden A farmer has a rectangular garden plot surrounded by 200 ft of fence. Find the length and width of the garden if its area is 2400 ft^2.



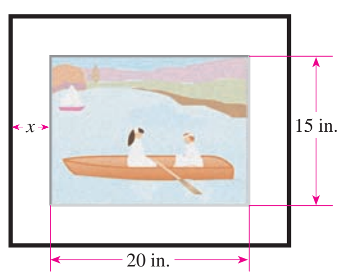
42. Dimensions of a Lot A parcel of land is 6 ft longer than it is wide. Each diagonal from one corner to the opposite corner is 174 ft long. What are the dimensions of the parcel?

43. Dimensions of a Lot A rectangular parcel of land is 50 ft wide. The length of a diagonal between opposite corners is 10 ft more than the length of the parcel. What is the length of the parcel?

44. Dimensions of a Track A running track has the shape shown in the figure, with straight sides and semicircular ends. If the length of the track is 440 yd and the two straight parts are each 110 yd long, what is the radius of the semicircular parts (to the nearest yard)?

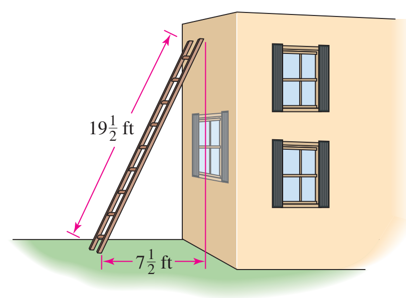


45. Framing a Painting Al paints with watercolors on a sheet of paper 20 in. wide by 15 in. high. He then places this sheet on a mat so that a uniformly wide strip of the mat shows all around the picture. The perimeter of the mat is 102 in. How wide is the strip of the mat showing around the picture?

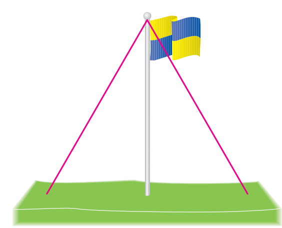


46. Width of a Lawn A factory is to be built on a lot measuring 180 ft by 240 ft. A local building code specifies that a lawn of uniform width and equal in area to the factory must surround the factory. What must the width of this lawn be, and what are the dimensions of the factory?

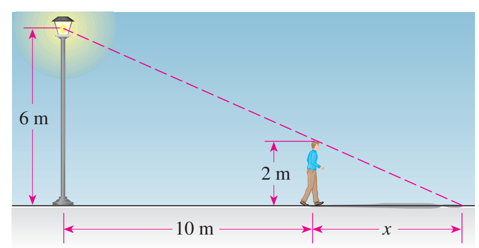
47. Reach of a Ladder A $19\frac{1}{2}$-foot ladder leans against a building. The base of the ladder is $7\frac{1}{2}$ ft from the building. How high up the building does the ladder reach?



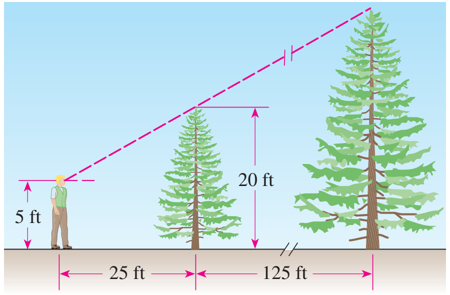
48. Height of a Flagpole A flagpole is secured on opposite sides by two guy wires, each of which is 5 ft longer than the pole. The distance between the points where the wires are fixed to the ground is equal to the length of one guy wire. How tall is the flagpole (to the nearest inch)?



49. Length of a Shadow A man is walking away from a lamppost with a light source 6 m above the ground. The man is 2 m tall. How long is the man's shadow when he is 10 m from the lamppost? [*Hint:* Use similar triangles.]



50. Height of a Tree A woodcutter determines the height of a tall tree by first measuring a smaller one 125 ft away, then moving so that his eyes are in the line of sight along the tops of the trees, and measuring how far he is standing from the small tree (see the figure). Suppose the small tree is 20 ft tall, the man is 25 ft from the small tree, and his eye level is 5 ft above the ground. How tall is the taller tree?



51. Buying a Cottage A group of friends decides to buy a vacation home for \$120,000, sharing the cost equally. If they can find one more person to join them, each person's contribution will drop by \$6000. How many people are in the group?

52. Mixture Problem What quantity of a 60% acid solution must be mixed with a 30% solution to produce 300 mL of a 50% solution?

53. Mixture Problem A jeweler has five rings, each weighing 18 g, made of an alloy of 10% silver and 90% gold. He decides to melt down the rings and add enough silver to reduce the gold content to 75%. How much silver should he add?

54. Mixture Problem A pot contains 6 L of brine at a concentration of 120 g/L. How much of the water should be boiled off to increase the concentration to 200 g/L?

55. Mixture Problem The radiator in a car is filled with a solution of 60% antifreeze and 40% water. The manufacturer of the antifreeze suggests that, for summer driving, optimal cooling of the engine is obtained with only 50% antifreeze. If the capacity of the radiator is 3.6 L, how much coolant should be drained and replaced with water to reduce the antifreeze concentration to the recommended level?

56. Mixture Problem A health clinic uses a solution of bleach to sterilize petri dishes in which cultures are grown. The sterilization tank contains 100 gal of a solution of 2% ordinary household bleach mixed with pure distilled water. New research indicates that the concentration of bleach should be 5% for complete sterilization. How much of the solution should be drained and replaced with bleach to increase the bleach content to the recommended level?

57. Mixture Problem A bottle contains 750 mL of fruit punch with a concentration of 50% pure fruit juice. Jill drinks 100 mL of the punch and then refills the bottle with an equal amount of a cheaper brand of punch. If the concentration of juice in the bottle is now reduced to 48%, what was the concentration in the punch that Jill added?

58. Mixture Problem A merchant blends tea that sells for \$3.00 a pound with tea that sells for \$2.75 a pound to produce 80 lb of a mixture that sells for \$2.90 a pound. How many pounds of each type of tea does the merchant use in the blend?

59. Sharing a Job Candy and Tim share a paper route. It takes Candy 70 min to deliver all the papers, and it takes Tim 80 min. How long does it take the two when they work together?

60. Sharing a Job Stan and Hilda can mow the lawn in 40 min if they work together. If Hilda works twice as fast as Stan, how long does it take Stan to mow the lawn alone?

61. Sharing a Job Betty and Karen have been hired to paint the houses in a new development. Working together the women can paint a house in two-thirds the time that it takes Karen working alone. Betty takes 6 h to paint a house alone. How long does it take Karen to paint a house working alone?

62. Sharing a Job Next-door neighbors Bob and Jim use hoses from both houses to fill Bob's swimming pool. They know it takes 18 h using both hoses. They also know that Bob's hose, used alone, takes 20% less time than Jim's hose alone. How much time is required to fill the pool by each hose alone?

63. Sharing a Job Henry and Irene working together can wash all the windows of their house in 1 h 48 min. Working alone, it takes Henry $1\frac{1}{2}$ h more than Irene to do the job. How long does it take each person working alone to wash all the windows?

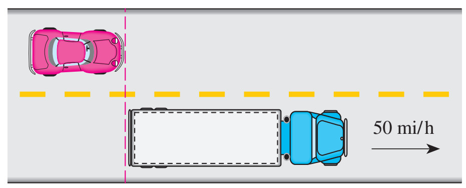
64. Sharing a Job Jack, Kay, and Lynn deliver advertising flyers in a small town. If each person works alone, it takes Jack 4 h to deliver all the flyers, and it takes Lynn 1 h longer than it takes Kay. Working together, they can deliver all the flyers in 40% of the time it takes Kay working alone. How long does it take Kay to deliver all the flyers alone?

65. Distance, Speed, and Time Wendy took a trip from Davenport to Omaha, a distance of 300 mi. She traveled part of the way by bus, which arrived at the train station just in time for Wendy to complete her journey by train. The bus averaged 40 mi/h and the train 60 mi/h. The entire trip took $5\frac{1}{2}$ h. How long did Wendy spend on the train?

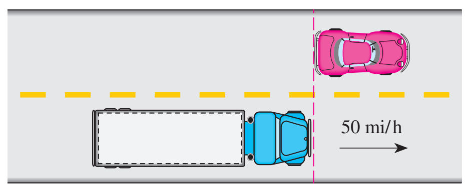
66. Distance, Speed, and Time Two cyclists, 90 mi apart, start riding toward each other at the same time. One cycles twice as fast as the other. If they meet 2 h later, at what average speed is each cyclist traveling?

67. Distance, Speed, and Time A pilot flew a jet from Montreal to Los Angeles, a distance of 2500 mi. On the return trip the average speed was 20% faster than the outbound speed. The round-trip took 9 h 10 min. What was the speed from Montreal to Los Angeles?

68. Distance, Speed, and Time A woman driving a car 14 ft long is passing a truck 30 ft long. The truck is traveling at 50 mi/h. How fast must the woman drive her car so that she can pass the truck completely in 6 s, from the position shown in figure (a) to the position shown in figure (b)? [*Hint:* Use feet and seconds instead of miles and hours.]



(a)



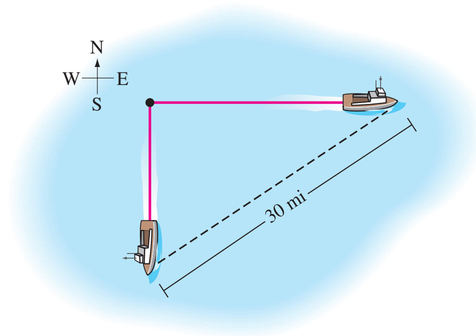
(b)

69. Distance, Speed, and Time A salesman drives from Ajax to Barrington, a distance of 120 mi, at a steady speed. He then increases his speed by 10 mi/h to drive the 150 mi from Barrington to Collins. If the second leg of his trip took 6 min more time than the first leg, how fast was he driving between Ajax and Barrington?

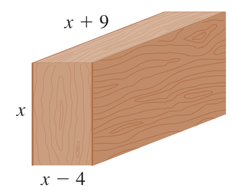
70. Distance, Speed, and Time Kiran drove from Tortula to Cactus, a distance of 250 mi. She increased her speed by 10 mi/h for the 360-mi trip from Cactus to Dry Junction. If the total trip took 11 h, what was her speed from Tortula to Cactus?

71. Distance, Speed, and Time It took a crew 2 h 40 min to row 6 km upstream and back again. If the rate of flow of the stream was 3 km/h, what was the rowing speed of the crew in still water?

72. Speed of a Boat Two fishing boats depart a harbor at the same time, one traveling east, the other south. The eastbound boat travels at a speed 3 mi/h faster than the southbound boat. After two hours the boats are 30 mi apart. Find the speed of the southbound boat.

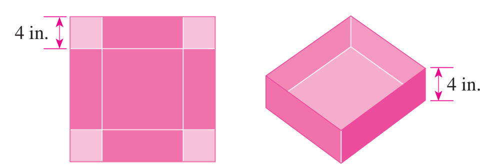


73. Dimensions of a Box A large plywood box has a volume of 180 ft^3. Its length is 9 ft greater than its height, and its width is 4 ft less than its height. What are the dimensions of the box?



74. Radius of a Sphere A jeweler has three small solid spheres made of gold, of radius 2 mm, 3 mm, and 4 mm. He decides to melt these down and make just one sphere out of them. What will the radius of this larger sphere be?

75. Dimensions of a Box A box with a square base and no top is to be made from a square piece of cardboard by cutting 4-in. squares from each corner and folding up the sides, as shown in the figure. The box is to hold 100 in^3. How big a piece of cardboard is needed?



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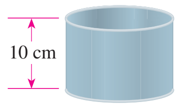
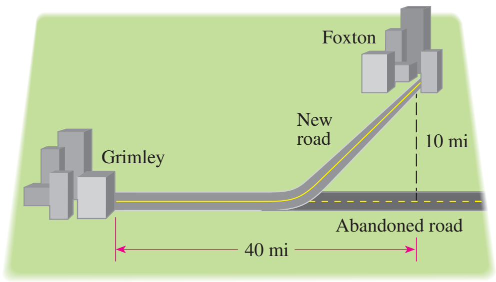
76. Dimensions of a Can A cylindrical can has a volume of 40π cm^3 and is 10 cm tall. What is its diameter? [*Hint:* Use the volume formula listed on the inside back cover of this book.]



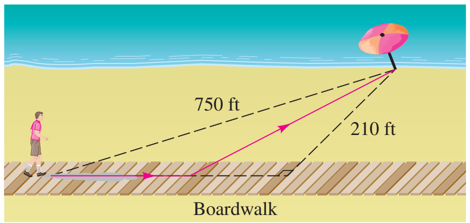
77. Radius of a Tank A spherical tank has a capacity of 750 gallons. Using the fact that one gallon is about 0.1337 ft^3, find the radius of the tank (to the nearest hundredth of a foot).

78. Dimensions of a Lot A city lot has the shape of a right triangle whose hypotenuse is 7 ft longer than one of the other sides. The perimeter of the lot is 392 ft. How long is each side of the lot?

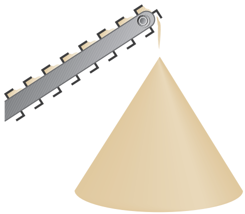
79. Construction Costs The town of Foxton lies 10 mi north of an abandoned east-west road that runs through Grimley, as shown in the figure. The point on the abandoned road closest to Foxton is 40 mi from Grimley. County officials are about to build a new road connecting the two towns. They have determined that restoring the old road would cost \$100,000 per mile, whereas building a new road would cost \$200,000 per mile. How much of the abandoned road should be used (as indicated in the figure) if the officials intend to spend exactly \$6.8 million? Would it cost less than this amount to build a new road connecting the towns directly?



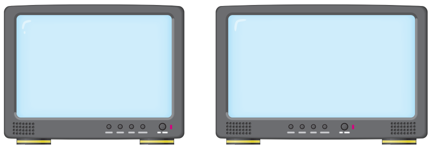
80. Distance, Speed, and Time A boardwalk is parallel to and 210 ft inland from a straight shoreline. A sandy beach lies between the boardwalk and the shoreline. A man is standing on the boardwalk, exactly 750 ft across the sand from his beach umbrella, which is right at the shoreline. The man walks 4 ft/s on the boardwalk and 2 ft/s on the sand. How far should he walk on the boardwalk before veering off onto the sand if he wishes to reach his umbrella in exactly 4 min 45 s?



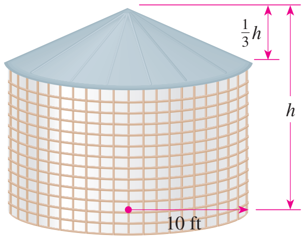
81. Volume of Grain Grain is falling from a chute onto the ground, forming a conical pile whose diameter is always three times its height. How high is the pile (to the nearest hundredth of a foot) when it contains 1000 ft^3 of grain?



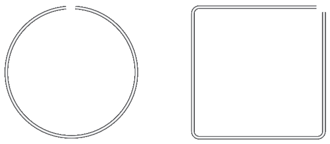
82. TV Monitors Two television monitors sitting beside each other on a shelf in an appliance store have the same screen height. One has a conventional screen, which is 5 in. wider than it is high. The other has a wider, high-definition screen, which is 1.8 times as wide as it is high. The diagonal measure of the wider screen is 14 in. more than the diagonal measure of the smaller. What is the height of the screens, correct to the nearest 0.1 in.?



83. Dimensions of a Structure A storage bin for corn consists of a cylindrical section made of wire mesh, surmounted by a conical tin roof, as shown in the figure. The height of the roof is one-third the height of the entire structure. If the total volume of the structure is 1400π ft^3 and its radius is 10 ft, what is its height? [*Hint:* Use the volume formulas listed on the inside front cover of this book.]



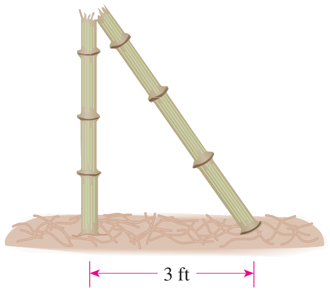
84. Comparing Areas A wire 360 in. long is cut into two pieces. One piece is formed into a square and the other into a circle. If the two figures have the same area, what are the lengths of the two pieces of wire (to the nearest tenth of an inch)?



85. An Ancient Chinese Problem This problem is taken from a Chinese mathematics textbook called *Chui-chang suan-shu*, or *Nine Chapters on the Mathematical Art*, which was written about 250 B.C.

A 10-ft-long stem of bamboo is broken in such a way that its tip touches the ground 3 ft from the base of the stem, as shown in the figure. What is the height of the break?

[*Hint:* Use the Pythagorean Theorem.]



Discovery • Discussion

86. Historical Research Read the biographical notes on Pythagoras (page 54), Euclid (page 532), and Archimedes (page 748). Choose one of these mathematicians and find out more about him from the library or on the Internet. Write a short essay on your findings. Include both biographical information and a description of the mathematics for which he is famous.

87. A Babylonian Quadratic Equation The ancient Babylonians knew how to solve quadratic equations. Here is a problem from a cuneiform tablet found in a Babylonian school dating back to about 2000 B.C.

> I have a reed, I know not its length. I broke from it one cubit, and it fit 60 times along the length of my field. I restored to the reed what I had broken off, and it fit 30 times along the width of my field. The area of my field is 375 square nindas. What was the original length of the reed?

Solve this problem. Use the fact that 1 ninda = 12 cubits.

DISCOVERY PROJECT

Equations through the Ages

Equations have been used to solve problems throughout recorded history, in every civilization. (See, for example, Exercise 85 on page 74.) Here is a problem from ancient Babylon (ca. 2000 B.C.).

> I found a stone but did not weigh it. After I added a seventh, and then added an eleventh of the result, I weighed it and found it weighed 1 mina. What was the original weight of the stone?

The British Museum

The answer given on the cuneiform tablet is $\frac{2}{3}$ mina, 8 sheqel, and $22\frac{1}{2}$ se, where 1 mina $=$ 60 sheqel, and 1 sheqel $=$ 180 se.

In ancient Egypt, knowing how to solve word problems was a highly prized secret. The Rhind Papyrus (ca. 1850 B.C.) contains many such problems (see page 716). Problem 32 in the Papyrus states

> A quantity, its third, its quarter, added together become 2. What is the quantity?

The answer in Egyptian notation is $1 + \overline{4} + \overline{76}$, where the bar indicates "reciprocal," much like our notation 4^{-1}.

The Greek mathematician Diophantus (ca. 250 A.D., see page 20) wrote the book *Arithmetica*, which contains many word problems and equations. The Indian mathematician Bhaskara (12th century A.D., see page 144) and the Chinese mathematician Chang Ch'iu-Chien (6th century A.D.) also studied and wrote about equations. Of course, equations continue to be important today.

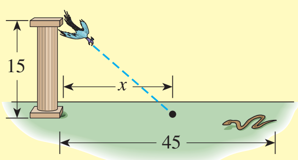
1. Solve the Babylonian problem and show that their answer is correct.

2. Solve the Egyptian problem and show that their answer is correct.

3. The ancient Egyptians and Babylonians used equations to solve practical problems. From the examples given here, do you think that they may have enjoyed posing and solving word problems just for fun?

4. Solve this problem from 12th-century India.

> A peacock is perched at the top of a 15-cubit pillar, and a snake's hole is at the foot of the pillar. Seeing the snake at a distance of 45 cubits from its hole, the peacock pounces obliquely upon the snake as it slithers home. At how many cubits from the snake's hole do they meet, assuming that each has traveled an equal distance?



5. Consider this problem from 6th-century China.

> If a rooster is worth 5 coins, a hen 3 coins, and three chicks together one coin, how many roosters, hens, and chicks, totaling 100, can be bought for 100 coins?

This problem has several answers. Use trial and error to find at least one answer. Is this a practical problem or more of a riddle? Write a short essay to support your opinion.

6. Write a short essay explaining how equations affect your own life in today's world.

SUGGESTED TIME AND EMPHASIS

1 class.
Essential material.

1.7 Inequalities

x	$4x + 7 \le 19$
1	$11 \le 19$ ✓
2	$15 \le 19$ ✓
3	$19 \le 19$ ✓
4	$23 \le 19$ ✗
5	$27 \le 19$ ✗

Some problems in algebra lead to **inequalities** instead of equations. An inequality looks just like an equation, except that in the place of the equal sign is one of the symbols, $<$, $>$, $\le$, or $\ge$. Here is an example of an inequality:

$$4x + 7 \le 19$$

The table in the margin shows that some numbers satisfy the inequality and some numbers don't.

To **solve** an inequality that contains a variable means to find all values of the variable that make the inequality true. Unlike an equation, an inequality generally has infinitely many solutions, which form an interval or a union of intervals on the real line. The following illustration shows how an inequality differs from its corresponding equation:

		Solution	**Graph**
Equation:	$4x + 7 = 19$	$x = 3$	
Inequality:	$4x + 7 \le 19$	$x \le 3$	

To solve inequalities, we use the following rules to isolate the variable on one side of the inequality sign. These rules tell us when two inequalities are *equivalent* (the symbol $\Leftrightarrow$ means "is equivalent to"). In these rules the symbols A, B, and C stand for real numbers or algebraic expressions. Here we state the rules for inequalities involving the symbol $\le$, but they apply to all four inequality symbols.

Rules for Inequalities

Rule	Description
1. $A \le B \quad \Leftrightarrow \quad A + C \le B + C$	**Adding** the same quantity to each side of an inequality gives an equivalent inequality.
2. $A \le B \quad \Leftrightarrow \quad A - C \le B - C$	**Subtracting** the same quantity from each side of an inequality gives an equivalent inequality.
3. If $C > 0$, then $A \le B \quad \Leftrightarrow \quad CA \le CB$	**Multiplying** each side of an inequality by the same *positive* quantity gives an equivalent inequality.
4. If $C < 0$, then $A \le B \quad \Leftrightarrow \quad CA \ge CB$	**Multiplying** each side of an inequality by the same *negative* quantity *reverses the direction* of the inequality.
5. If $A > 0$ and $B > 0$, then $A \le B \quad \Leftrightarrow \quad \frac{1}{A} \ge \frac{1}{B}$	**Taking reciprocals** of each side of an inequality involving *positive* quantities *reverses the direction* of the inequality.
6. If $A \le B$ and $C \le D$, then $A + C \le B + D$	Inequalities can be added.

POINTS TO STRESS

1. Definition of inequalities.
2. Manipulation of linear and nonlinear inequalities.
3. Solving inequalities of the form $|A| < B$ and $|A| > B$.

Pay special attention to Rules 3 and 4. Rule 3 says that we can multiply (or divide) each side of an inequality by a *positive* number, but Rule 4 says that if we multiply each side of an inequality by a *negative* number, then we reverse the direction of the inequality. For example, if we start with the inequality

$$3 < 5$$

and multiply by 2, we get

$$6 < 10$$

but if we multiply by -2, we get

$$-6 > -10$$

Linear Inequalities

An inequality is **linear** if each term is constant or a multiple of the variable.

Example 1 Solving a Linear Inequality

Solve the inequality $3x < 9x + 4$ and sketch the solution set.

Solution

$$3x < 9x + 4$$

$$3x - 9x < 9x + 4 - 9x \qquad \text{Subtract } 9x$$

$$-6x < 4 \qquad \text{Simplify}$$

$$\left(-\tfrac{1}{6}\right)(-6x) > \left(-\tfrac{1}{6}\right)(4) \qquad \text{Multiply by } -\tfrac{1}{6} \text{ (or divide by } -6)$$

$$x > -\tfrac{2}{3} \qquad \text{Simplify}$$

Multiplying by the negative number $-\frac{1}{6}$ *reverses* the direction of the inequality.

The solution set consists of all numbers greater than $-\frac{2}{3}$. In other words the solution of the inequality is the interval $\left(-\frac{2}{3}, \infty\right)$. It is graphed in Figure 1. ■

Figure 1

ALTERNATE EXAMPLE 1
Solve the inequality $3x \le 9x + 4$.

ANSWER
$\left[-\frac{2}{3}, \infty\right)$

Example 2 Solving a Pair of Simultaneous Inequalities

Solve the inequalities $4 \le 3x - 2 < 13$.

Solution The solution set consists of all values of x that satisfy both of the inequalities $4 \le 3x - 2$ and $3x - 2 < 13$. Using Rules 1 and 3, we see that the following inequalities are equivalent:

$$4 \le 3x - 2 < 13$$

$$6 \le 3x < 15 \qquad \text{Add 2}$$

$$2 \le x < 5 \qquad \text{Divide by 3}$$

Therefore, the solution set is $[2, 5)$, as shown in Figure 2. ■

Figure 2

ALTERNATE EXAMPLE 2
Solve the inequalities $5 \le 2x - 5 < 9$.

ANSWER
$[5, 7)$

Nonlinear Inequalities

To solve inequalities involving squares and other powers of the variable, we use factoring, together with the following principle.

SAMPLE QUESTION
Text Question
Solve the inequality $(x - 1)(x - 3) < 0$.

Answer
$1 < x < 3$

The Sign of a Product or Quotient

If a product or a quotient has an *even* number of *negative* factors, then its value is *positive*.

If a product or a quotient has an *odd* number of *negative* factors, then its value is *negative*.

Example 3 A Quadratic Inequality

Solve the inequality $x^2 - 5x + 6 \le 0$.

Solution First we factor the left side.

$$(x - 2)(x - 3) \le 0$$

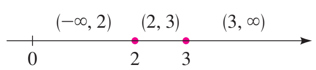


Figure 3

We know that the corresponding equation $(x - 2)(x - 3) = 0$ has the solutions 2 and 3. As shown in Figure 3, the numbers 2 and 3 divide the real line into three intervals: $(-\infty, 2)$, $(2, 3)$, and $(3, \infty)$. On each of these intervals we determine the signs of the factors using **test values**. We choose a number inside each interval and check the sign of the factors $x - 2$ and $x - 3$ at the value selected. For instance, if we use the test value $x = 1$ for the interval $(-\infty, 2)$ shown in Figure 4, then substitution in the factors $x - 2$ and $x - 3$ gives

$$x - 2 = 1 - 2 = -1 < 0$$

and $$x - 3 = 1 - 3 = -2 < 0$$

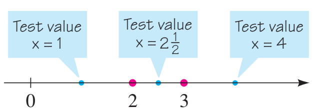


Figure 4

So both factors are negative on this interval. (The factors $x - 2$ and $x - 3$ change sign only at 2 and 3, respectively, so they maintain their signs over the length of each interval. That is why using a single test value on each interval is sufficient.)

Using the test values $x = 2\frac{1}{2}$ and $x = 4$ for the intervals $(2, 3)$ and $(3, \infty)$ (see Figure 4), respectively, we construct the following sign table. The final row of the table is obtained from the fact that the expression in the last row is the product of the two factors.

Interval	$(-\infty, 2)$	$(2, 3)$	$(3, \infty)$
Sign of $x - 2$	−	+	+
Sign of $x - 3$	−	−	+
Sign of $(x - 2)(x - 3)$	+	−	+

If you prefer, you can represent this information on a real number line, as in the following sign diagram. The vertical lines indicate the points at which the real line is divided into intervals:

DRILL QUESTION

Solve $|3x + 5| = 14$.

Answer

$x = -\frac{19}{3}$ or $x = 3$

	$(-\infty, 2)$	2	$(2, 3)$	3	$(3, \infty)$
Sign of $x - 2$	$-$		$+$		$+$
Sign of $x - 3$	$-$		$-$		$+$
Sign of $(x - 2)(x - 3)$	$+$		$-$		$+$

We read from the table or the diagram that $(x - 2)(x - 3)$ is negative on the interval $(2, 3)$. Thus, the solution of the inequality $(x - 2)(x - 3) \leq 0$ is

$$\{x \mid 2 \leq x \leq 3\} = [2, 3]$$

We have included the endpoints 2 and 3 because we seek values of x such that the product is either less than *or equal to* zero. The solution is illustrated in Figure 5. ■

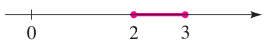


Figure 5

Example 3 illustrates the following guidelines for solving an inequality that can be factored.

Guidelines for Solving Nonlinear Inequalities

1. **Move All Terms to One Side.** If necessary, rewrite the inequality so that all nonzero terms appear on one side of the inequality sign. If the nonzero side of the inequality involves quotients, bring them to a common denominator.
2. **Factor.** Factor the nonzero side of the inequality.
3. **Find the Intervals.** Determine the values for which each factor is zero. These numbers will divide the real line into intervals. List the intervals determined by these numbers.
4. **Make a Table or Diagram.** Use test values to make a table or diagram of the signs of each factor on each interval. In the last row of the table determine the sign of the product (or quotient) of these factors.
5. **Solve.** Determine the solution of the inequality from the last row of the sign table. Be sure to check whether the inequality is satisfied by some or all of the endpoints of the intervals (this may happen if the inequality involves $\leq$ or $\geq$).

The factoring technique described in these guidelines works only if all nonzero terms appear on one side of the inequality symbol. If the inequality is not written in this form, first rewrite it, as indicated in Step 1. This technique is illustrated in the examples that follow.

IN-CLASS MATERIALS

Just as a quadratic equation can have zero, one, or two solutions, quadratic inequalities can have zero, one, or infinitely many solutions. A good way to clarify this point is to solve the following with your students:

$$x^2 + 2x \leq 0$$
$$x^2 + 2x + 1 \leq 0$$
$$x^2 + 2x + 1 < 0$$
$$x^2 + 2x + 2 \leq 0$$

ALTERNATE EXAMPLE 4

Solve: $\dfrac{6+x}{6-x} \geq 1$

ANSWER

$[0, 6)$

It is tempting to multiply both sides of the inequality by $1 - x$ (as you would if this were an *equation*). But this doesn't work because we don't know if $1 - x$ is positive or negative, so we can't tell if the inequality needs to be reversed. (See Exercise 110.)

Example 4 An Inequality Involving a Quotient

Solve: $\dfrac{1+x}{1-x} \geq 1$

Solution First we move all nonzero terms to the left side, and then we simplify using a common denominator.

Terms to one side

$$\frac{1+x}{1-x} \geq 1$$

$$\frac{1+x}{1-x} - 1 \geq 0 \qquad \text{Subtract 1 (to move all terms to LHS)}$$

$$\frac{1+x}{1-x} - \frac{1-x}{1-x} \geq 0 \qquad \text{Common denominator } 1-x$$

$$\frac{1+x-1+x}{1-x} \geq 0 \qquad \text{Combine the fractions}$$

$$\frac{2x}{1-x} \geq 0 \qquad \text{Simplify}$$

The numerator is zero when $x = 0$ and the denominator is zero when $x = 1$, so we construct the following sign diagram using these values to define intervals on the real line.

Make a diagram

	$(-\infty, 0)$	$(0, 1)$	$(1, \infty)$
Sign of $2x$	$-$	$+$	$+$
Sign of $1 - x$	$+$	$+$	$-$
Sign of $\dfrac{2x}{1-x}$	$-$	$+$	$-$

Solve

From the diagram we see that the solution set is $\{x \mid 0 \leq x < 1\} = [0, 1)$. We include the endpoint 0 because the original inequality requires the quotient to be greater than *or equal to* 1. However, we do not include the other endpoint 1, since the quotient in the inequality is not defined at 1. Always check the endpoints of solution intervals to determine whether they satisfy the original inequality.

The solution set $[0, 1)$ is illustrated in Figure 6. ■

Figure 6

ALTERNATE EXAMPLE 5

Solve the inequality $x < \dfrac{8}{x-2}$.

ANSWER

$(-\infty, -2) \cup (2, 4)$

Example 5 Solving an Inequality with Three Factors

Solve the inequality $x < \dfrac{2}{x-1}$.

Solution After moving all nonzero terms to one side of the inequality, we use a common denominator to combine the terms.

Terms to one side

$$x - \frac{2}{x-1} < 0 \qquad \text{Subtract } \frac{2}{x-1}$$

$$\frac{x(x-1)}{x-1} - \frac{2}{x-1} < 0 \qquad \text{Common denominator } x-1$$

$$\frac{x^2-x-2}{x-1} < 0 \qquad \text{Combine fractions}$$

Factor

$$\frac{(x+1)(x-2)}{x-1} < 0 \qquad \text{Factor numerator}$$

IN-CLASS MATERIALS

A nice comparison is the difference between these two inequalities:

$$\frac{x}{x^2+4} < 0 \qquad \frac{x}{x^2-4} < 0$$

Find the intervals

The factors in this quotient change sign at -1, 1, and 2, so we must examine the intervals $(-\infty, -1)$, $(-1, 1)$, $(1, 2)$, and $(2, \infty)$. Using test values, we get the following sign diagram.

Make a diagram

	-1		1		2	
Sign of $x + 1$	$-$		$+$		$+$	$+$
Sign of $x - 2$	$-$		$-$		$-$	$+$
Sign of $x - 1$	$-$		$-$		$+$	$+$
Sign of $\dfrac{(x+1)(x-2)}{x-1}$	$-$		$+$		$-$	$+$

Since the quotient must be negative, the solution is

$$(-\infty, -1) \cup (1, 2)$$

as illustrated in Figure 7. ■

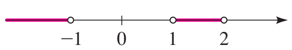


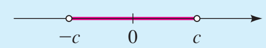
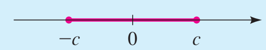
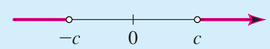
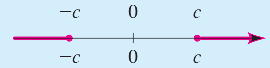
Figure 7

Absolute Value Inequalities

We use the following properties to solve inequalities that involve absolute value.

These properties hold when x is replaced by any algebraic expression. (In the figures we assume that $c > 0$.)

Properties of Absolute Value Inequalities

Inequality	Equivalent form	Graph
1. $\lvert x \rvert < c$	$-c < x < c$	
2. $\lvert x \rvert \le c$	$-c \le x \le c$	
3. $\lvert x \rvert > c$	$x < -c$ or $c < x$	
4. $\lvert x \rvert \ge c$	$x \le -c$ or $c \le x$	

These properties can be proved using the definition of absolute value. To prove Property 1, for example, note that the inequality $|x| < c$ says that the distance from x to 0 is less than c, and from Figure 8 you can see that this is true if and only if x is between $-c$ and c.

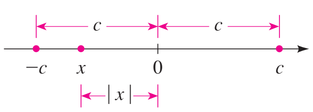


Figure 8

Example 6 Solving an Absolute Value Inequality

Solve the inequality $|x - 5| < 2$.

Solution 1 The inequality $|x - 5| < 2$ is equivalent to

$$-2 < x - 5 < 2 \qquad \text{Property 1}$$

$$3 < x < 7 \qquad \text{Add 5}$$

The solution set is the open interval $(3, 7)$.

Solution 2 Geometrically, the solution set consists of all numbers x whose distance from 5 is less than 2. From Figure 9 we see that this is the interval $(3, 7)$. ■

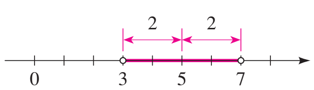


Figure 9

ALTERNATE EXAMPLE 6
Solve the inequality $|x - 9| < 8$.

ANSWER
$(1, 17)$

IN-CLASS MATERIALS

Ask students if there is an inequality that is true for every value of x *except* a single number, say 22. See if they can come up with something like $(x - 22)^2 > 0$.

ALTERNATE EXAMPLE 7
Solve the inequality
$|5x + 4| \geq 6.$
Write the solution using interval notation.

ANSWER
$(-\infty, -2] \cup \left[\frac{2}{5}, \infty\right)$

Example 7 Solving an Absolute Value Inequality

Solve the inequality $|3x + 2| \geq 4$.

Solution By Property 4 the inequality $|3x + 2| \geq 4$ is equivalent to

$$3x + 2 \geq 4 \quad \text{or} \quad 3x + 2 \leq -4$$

$$3x \geq 2 \qquad 3x \leq -6 \qquad \text{Subtract 2}$$

$$x \geq \tfrac{2}{3} \qquad x \leq -2 \qquad \text{Divide by 3}$$

So the solution set is

$$\{x \mid x \leq -2 \quad \text{or} \quad x \geq \tfrac{2}{3}\} = (-\infty, -2] \cup [\tfrac{2}{3}, \infty)$$

The set is graphed in Figure 10. ■

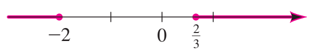


Figure 10

Modeling with Inequalities

Modeling real-life problems frequently leads to inequalities because we are often interested in determining when one quantity is more (or less) than another.

ALTERNATE EXAMPLE 8
A carnival has two plans for tickets.

Plan A: $7 entrance fee and 50¢ each ride
Plan B: $3 entrance fee and 75¢ each ride

How many rides would you have to take for plan A to be less expensive than plan B?

ANSWER
$16 < x$

Example 8 Carnival Tickets

A carnival has two plans for tickets.

Plan A: $5 entrance fee and 25¢ each ride

Plan B: $2 entrance fee and 50¢ each ride

How many rides would you have to take for plan A to be less expensive than plan B?

Solution We are asked for the number of rides for which plan A is less expensive than plan B. So let

Identify the variable

$$x = \text{number of rides}$$

The information in the problem may be organized as follows.

Express all unknown quantities in terms of the variable

In Words	In Algebra
Number of rides	x
Cost with plan A	$5 + 0.25x$
Cost with plan B	$2 + 0.50x$

Now we set up the model.

Set up the model

$$\text{cost with plan A} < \text{cost with plan B}$$

$$5 + 0.25x < 2 + 0.50x$$

Solve

$$3 + 0.25x < 0.50x \qquad \text{Subtract 2}$$

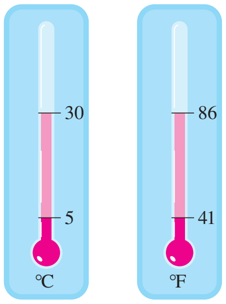
$$3 < 0.25x \qquad \text{Subtract } 0.25x$$

$$12 < x \qquad \text{Divide by 0.25}$$

So if you plan to take *more than* 12 rides, plan A is less expensive. ■

IN-CLASS MATERIALS

One of the themes of this chapter is taking verbal descriptions and translating them into algebraic equations. Assume that we know that a length of tubing (to three significant figures) is 11.6 cm. Because of significant figures, there is a range of possible exact lengths of tubing. For example, if the actual length is 11.6000323 cm, we would report the length as 11.6 cm. Have students try to express the range of values as an absolute value inequality: $|x - 11.6| \leq 0.05$.



Example 9 Fahrenheit and Celsius Scales

The instructions on a box of film indicate that the box should be stored at a temperature between 5 °C and 30 °C. What range of temperatures does this correspond to on the Fahrenheit scale?

Solution The relationship between degrees Celsius (C) and degrees Fahrenheit (F) is given by the equation $C = \frac{5}{9}(F - 32)$. Expressing the statement on the box in terms of inequalities, we have

$$5 < C < 30$$

So the corresponding Fahrenheit temperatures satisfy the inequalities

$$5 < \tfrac{5}{9}(F - 32) < 30$$

$$\tfrac{9}{5}\cdot 5 < F - 32 < \tfrac{9}{5}\cdot 30 \qquad \text{Multiply by } \tfrac{9}{5}$$

$$9 < F - 32 < 54 \qquad \text{Simplify}$$

$$9 + 32 < F < 54 + 32 \qquad \text{Add 32}$$

$$41 < F < 86 \qquad \text{Simplify}$$

The film should be stored at a temperature between 41 °F and 86 °F. ■

Example 10 Concert Tickets

A group of students decide to attend a concert. The cost of chartering a bus to take them to the concert is \$450, which is to be shared equally among the students. The concert promoters offer discounts to groups arriving by bus. Tickets normally cost \$50 each but are reduced by 10¢ per ticket for each person in the group (up to the maximum capacity of the bus). How many students must be in the group for the total cost per student to be less than \$54?

Solution We are asked for the number of students in the group. So let

Identify the variable

$$x = \text{number of students in the group}$$

The information in the problem may be organized as follows.

Express all unknown quantities in terms of the variable

In Words	In Algebra
Number of students in group	x
Bus cost per student	$\frac{450}{x}$
Ticket cost per student	$50 - 0.10x$

Now we set up the model.

Set up the model

$$\boxed{\text{bus cost per student}} + \boxed{\text{ticket cost per student}} < \boxed{54}$$

$$\frac{450}{x} + (50 - 0.10x) < 54$$

ALTERNATE EXAMPLE 9
The instructions on a box of film indicate that the box should be stored at a temperature between 5° C and 40° C. What range of temperatures does this correspond to on the Fahrenheit scale?

ANSWER
$41 < F < 104$

ALTERNATE EXAMPLE 10
A group of students decide to attend a concert. The cost of chartering a bus to take them to the concert is \$150, which is to be shared equally among the students. The concert promoters offer discounts to groups arriving by bus. Tickets normally cost \$50 each but are reduced by 10¢ per ticket for each person in the group (up to the maximum capacity of the bus). How many students must be in the group for the total cost per student to be less than \$52?

ANSWER
$(30, \infty)$

IN-CLASS MATERIALS

There is a particular type of inequality that shows up often in higher mathematics: $0 < |x - a| < \delta$. Ask students to graph $0 < |x - 3| < \frac{1}{10}$ and similar inequalities, emphasizing the idea that x is close to 3, but not equal to 3. For fun, you can now write the definition of "limit" on the blackboard, and see what you and the students can do with it at this stage. Note that $\frac{x^2 - 9}{x - 3}$ is not defined at $x = 3$, but is close to 6 when x is close to 3. Then write the formal definition:

$$\textbf{We say } \lim_{x\to 3}\frac{x^2 - 9}{x - 3} = 6 \textbf{ because for every } \varepsilon > 0\textbf{, there is a } \delta$$

$$\textbf{such that } \left|\frac{x^2 - 9}{x - 3} - 6\right| < \varepsilon \textbf{ whenever } 0 < |x - 3| < \delta.$$

Remember, putting this on the board at this point in the game is for fun only.

Solve

$$\frac{450}{x} - 4 - 0.10x < 0 \qquad \text{Subtract 54}$$

$$\frac{450 - 4x - 0.10x^2}{x} < 0 \qquad \text{Common denominator}$$

$$\frac{4500 - 40x - x^2}{x} < 0 \qquad \text{Multiply by 10}$$

$$\frac{(90 + x)(50 - x)}{x} < 0 \qquad \text{Factor numerator}$$

	$(-\infty, -90)$	$(-90, 0)$	$(0, 50)$	$(50, \infty)$
Sign of $90 + x$	−	+	+	+
Sign of $50 - x$	+	+	+	−
Sign of x	−	−	+	+
Sign of $\frac{(90 + x)(50 - x)}{x}$	+	−	+	−

The sign diagram shows that the solution of the inequality is $(-90, 0) \cup (50, \infty)$. Because we cannot have a negative number of students, it follows that the group must have more than 50 students for the total cost per person to be less than \$54. ■

1.7 Exercises

1–6 ■ Let $S = \{-2, -1, 0, \frac{1}{2}, 1, \sqrt{2}, 2, 4\}$. Determine which elements of S satisfy the inequality.

1. $3 - 2x \le \frac{1}{2}$

2. $2x - 1 \ge x$

3. $1 < 2x - 4 \le 7$

4. $-2 \le 3 - x < 2$

5. $\frac{1}{x} \le \frac{1}{2}$

6. $x^2 + 2 < 4$

7–28 ■ Solve the linear inequality. Express the solution using interval notation and graph the solution set.

7. $2x - 5 > 3$

8. $3x + 11 < 5$

9. $7 - x \ge 5$

10. $5 - 3x \le -16$

11. $2x + 1 < 0$

12. $0 < 5 - 2x$

13. $3x + 11 \le 6x + 8$

14. $6 - x \ge 2x + 9$

15. $\frac{1}{2}x - \frac{2}{3} > 2$

16. $\frac{2}{5}x + 1 < \frac{1}{5} - 2x$

17. $\frac{1}{3}x + 2 < \frac{1}{6}x - 1$

18. $\frac{2}{3} - \frac{1}{2}x \ge \frac{1}{6} + x$

19. $4 - 3x \le -(1 + 8x)$

20. $2(7x - 3) \le 12x + 16$

21. $2 \le x + 5 < 4$

22. $5 \le 3x - 4 \le 14$

23. $-1 < 2x - 5 < 7$

24. $1 < 3x + 4 \le 16$

25. $-2 < 8 - 2x \le -1$

26. $-3 \le 3x + 7 \le \frac{1}{2}$

27. $\frac{1}{6} < \frac{2x - 13}{12} \le \frac{2}{3}$

28. $-\frac{1}{2} \le \frac{4 - 3x}{5} \le \frac{1}{4}$

29–62 ■ Solve the nonlinear inequality. Express the solution using interval notation and graph the solution set.

29. $(x + 2)(x - 3) < 0$

30. $(x - 5)(x + 4) \ge 0$

31. $x(2x + 7) \ge 0$

32. $x(2 - 3x) \le 0$

33. $x^2 - 3x - 18 \le 0$

34. $x^2 + 5x + 6 > 0$

35. $2x^2 + x \ge 1$

36. $x^2 < x + 2$

37. $3x^2 - 3x < 2x^2 + 4$

38. $5x^2 + 3x \ge 3x^2 + 2$

39. $x^2 > 3(x + 6)$

40. $x^2 + 2x > 3$

41. $x^2 < 4$

42. $x^2 \ge 9$

43. $-2x^2 \le 4$

44. $(x + 2)(x - 1)(x - 3) \le 0$

45. $x^3 - 4x > 0$

46. $16x \le x^3$

47. $\frac{x - 3}{x + 1} \ge 0$

48. $\frac{2x + 6}{x - 2} < 0$

49. $\frac{4x}{2x + 3} > 2$

50. $-2 < \frac{x + 1}{x - 3}$

EXAMPLES

1. Solve $-5 < x^2 + 3x + 2 < 10$.

2. Quadratic inequalities involving absolute values:

$$|x^2 - 8x + 6| < 6 \Rightarrow x \in (0, 2) \cup (6, 8)$$

$$|x^2 - 8x + 6| \ge 6 \Rightarrow x \in (-\infty, 0] \cup [2, 6] \cup [8, \infty)$$

ANSWER

1. $-\frac{3}{2} - \frac{1}{2}\sqrt{41} < x < -\frac{3}{2} + \frac{1}{2}\sqrt{41}$

51. $\dfrac{2x+1}{x-5} \le 3$

52. $\dfrac{3+x}{3-x} \ge 1$

53. $\dfrac{4}{x} < x$

54. $\dfrac{x}{x+1} > 3x$

55. $1 + \dfrac{2}{x+1} \le \dfrac{2}{x}$

56. $\dfrac{3}{x-1} - \dfrac{4}{x} \ge 1$

57. $\dfrac{6}{x-1} - \dfrac{6}{x} \ge 1$

58. $\dfrac{x}{2} \ge \dfrac{5}{x+1} + 4$

59. $\dfrac{x+2}{x+3} < \dfrac{x-1}{x-2}$

60. $\dfrac{1}{x+1} + \dfrac{1}{x+2} \le 0$

61. $x^4 > x^2$

62. $x^5 > x^2$

63–76 ■ Solve the absolute value inequality. Express the answer using interval notation and graph the solution set.

63. $|x| \le 4$

64. $|3x| < 15$

65. $|2x| > 7$

66. $\frac{1}{2}|x| \ge 1$

67. $|x-5| \le 3$

68. $|x+1| \ge 1$

69. $|2x-3| \le 0.4$

70. $|5x-2| < 6$

71. $\left|\dfrac{x-2}{3}\right| < 2$

72. $\left|\dfrac{x+1}{2}\right| \ge 4$

73. $|x+6| < 0.001$

74. $3 - |2x+4| \le 1$

75. $8 - |2x-1| \ge 6$

76. $7|x+2| + 5 > 4$

77–80 ■ A phrase describing a set of real numbers is given. Express the phrase as an inequality involving an absolute value.

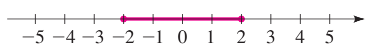
77. All real numbers x less than 3 units from 0

78. All real numbers x more than 2 units from 0

79. All real numbers x at least 5 units from 7

80. All real numbers x at most 4 units from 2

81–86 ■ A set of real numbers is graphed. Find an inequality involving an absolute value that describes the set.

81.


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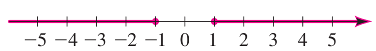

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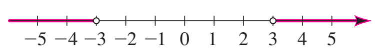
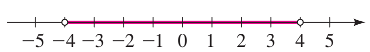
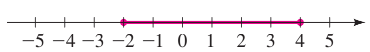
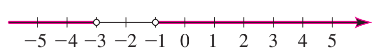


84.


85.


86.


87–90 ■ Determine the values of the variable for which the expression is defined as a real number.

87. $\sqrt{16-9x^2}$

88. $\sqrt{3x^2-5x+2}$

89. $\left(\dfrac{1}{x^2-5x-14}\right)^{1/2}$

90. $\sqrt[4]{\dfrac{1-x}{2+x}}$

91. Solve the inequality for x, assuming that a, b, and c are positive constants.

(a) $a(bx-c) \ge bc$ **(b)** $a \le bx + c < 2a$

92. Suppose that a, b, c, and d are positive numbers such that

$$\frac{a}{b} < \frac{c}{d}$$

Show that $\dfrac{a}{b} < \dfrac{a+c}{b+d} < \dfrac{c}{d}$

Applications

93. Temperature Scales Use the relationship between C and F given in Example 9 to find the interval on the Fahrenheit scale corresponding to the temperature range $20 \le C \le 30$.

94. Temperature Scales What interval on the Celsius scale corresponds to the temperature range $50 \le F \le 95$?

95. Car Rental Cost A car rental company offers two plans for renting a car.

Plan A: $30 per day and 10¢ per mile
Plan B: $50 per day with free unlimited mileage

For what range of miles will plan B save you money?

96. Long-Distance Cost A telephone company offers two long-distance plans.

Plan A: $25 per month and 5¢ per minute
Plan B: $5 per month and 12¢ per minute

For how many minutes of long-distance calls would plan B be financially advantageous?

97. Driving Cost It is estimated that the annual cost of driving a certain new car is given by the formula

$$C = 0.35m + 2200$$

where m represents the number of miles driven per year and C is the cost in dollars. Jane has purchased such a car, and decides to budget between $6400 and $7100 for next year's driving costs. What is the corresponding range of miles that she can drive her new car?

98. Gas Mileage The gas mileage g (measured in mi/gal) for a particular vehicle, driven at v mi/h, is given by the formula $g = 10 + 0.9v - 0.01v^2$, as long as v is between 10 mi/h and 75 mi/h. For what range of speeds is the vehicle's mileage 30 mi/gal or better?

99. Gravity The gravitational force F exerted by the earth on an object having a mass of 100 kg is given by the equation

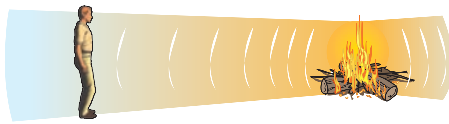
$$F = \frac{4{,}000{,}000}{d^2}$$

where d is the distance (in km) of the object from the center of the earth, and the force F is measured in newtons (N). For what distances will the gravitational force exerted by the earth on this object be between 0.0004 N and 0.01 N?

100. Bonfire Temperature In the vicinity of a bonfire, the temperature T in °C at a distance of x meters from the center of the fire was given by

$$T = \frac{600{,}000}{x^2 + 300}$$

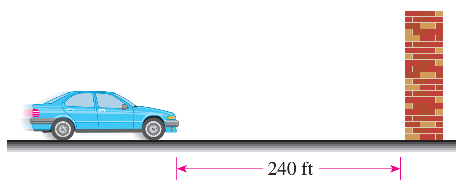
At what range of distances from the fire's center was the temperature less than 500°C?



101. Stopping Distance For a certain model of car the distance d required to stop the vehicle if it is traveling at v mi/h is given by the formula

$$d = v + \frac{v^2}{20}$$

where d is measured in feet. Kerry wants her stopping distance not to exceed 240 ft. At what range of speeds can she travel?



102. Manufacturer's Profit If a manufacturer sells x units of a certain product, his revenue R and cost C (in dollars) are given by:

$$R = 20x$$
$$C = 2000 + 8x + 0.0025x^2$$

Use the fact that

$$\text{profit} = \text{revenue} - \text{cost}$$

to determine how many units he should sell to enjoy a profit of at least $2400.

103. Air Temperature As dry air moves upward, it expands and in so doing cools at a rate of about 1 °C for each 100-meter rise, up to about 12 km.

(a) If the ground temperature is 20 °C, write a formula for the temperature at height h.

(b) What range of temperatures can be expected if a plane takes off and reaches a maximum height of 5 km?

104. Airline Ticket Price A charter airline finds that on its Saturday flights from Philadelphia to London, all 120 seats will be sold if the ticket price is $200. However, for each $3 increase in ticket price, the number of seats sold decreases by one.

(a) Find a formula for the number of seats sold if the ticket price is P dollars.

(b) Over a certain period, the number of seats sold for this flight ranged between 90 and 115. What was the corresponding range of ticket prices?

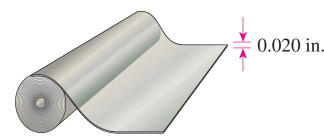
105. Theater Tour Cost A riverboat theater offers bus tours to groups on the following basis. Hiring the bus costs the group $360, to be shared equally by the group members. Theater tickets, normally $30 each, are discounted by 25¢ times the number of people in the group. How many members must be in the group so that the cost of the theater tour (bus fare plus theater ticket) is less than $39 per person?

106. Fencing a Garden A determined gardener has 120 ft of deer-resistant fence. She wants to enclose a rectangular vegetable garden in her backyard, and she wants the area enclosed to be at least 800 ft². What range of values is possible for the length of her garden?

107. Thickness of a Laminate A company manufactures industrial laminates (thin nylon-based sheets) of thickness 0.020 in, with a tolerance of 0.003 in.

(a) Find an inequality involving absolute values that describes the range of possible thickness for the laminate.

(b) Solve the inequality you found in part (a).



108. Range of Height The average height of adult males is 68.2 in, and 95% of adult males have height h that satisfies the inequality

$$\left|\frac{h - 68.2}{2.9}\right| \le 2$$

Solve the inequality to find the range of heights.

Discovery • Discussion

109. Do Powers Preserve Order? If $a < b$, is $a^2 < b^2$? (Check both positive and negative values for a and b.) If $a < b$, is $a^3 < b^3$? Based on your observations, state a general rule about the relationship between a^n and b^n when $a < b$ and n is a positive integer.

110. What's Wrong Here? It is tempting to try to solve an inequality like an equation. For instance, we might try to solve $1 < 3/x$ by multiplying both sides by x, to get $x < 3$, so the solution would be $(-\infty, 3)$. But that's wrong; for example, $x = -1$ lies in this interval but does not satisfy the original inequality. Explain why this method doesn't work (think about the *sign* of x). Then solve the inequality correctly.

111. Using Distances to Solve Absolute Value Inequalities Recall that $|a - b|$ is the distance between a and b on the number line. For any number x, what do $|x - 1|$ and $|x - 3|$ represent? Use this interpretation to solve the inequality $|x - 1| < |x - 3|$ geometrically. In general, if $a < b$, what is the solution of the inequality $|x - a| < |x - b|$?

1.8 Coordinate Geometry

The *coordinate plane* is the link between algebra and geometry. In the coordinate plane we can draw graphs of algebraic equations. The graphs, in turn, allow us to "see" the relationship between the variables in the equation. In this section we study the coordinate plane.

The Coordinate Plane

The Cartesian plane is named in honor of the French mathematician René Descartes (1596–1650), although another Frenchman, Pierre Fermat (1601–1665), also invented the principles of coordinate geometry at the same time. (See their biographies on pages 112 and 652.)

Just as points on a line can be identified with real numbers to form the coordinate line, points in a plane can be identified with ordered pairs of numbers to form the **coordinate plane** or **Cartesian plane**. To do this, we draw two perpendicular real lines that intersect at 0 on each line. Usually one line is horizontal with positive direction to the right and is called the ***x*-axis**; the other line is vertical with positive direction upward and is called the ***y*-axis**. The point of intersection of the x-axis and the y-axis is the **origin *O***, and the two axes divide the plane into four **quadrants**, labeled I, II, III, and IV in Figure 1. (The points *on* the coordinate axes are not assigned to any quadrant.)

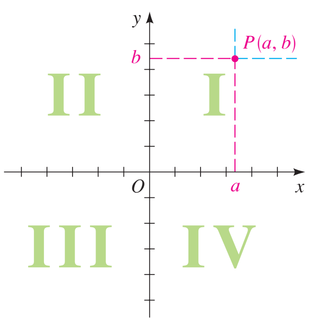


Figure 1

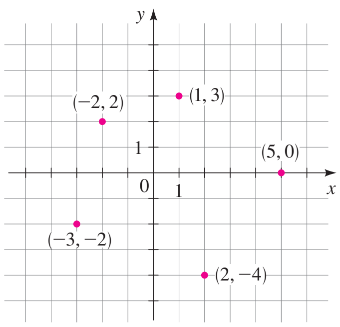


Figure 2

Any point P in the coordinate plane can be located by a unique **ordered pair** of numbers (a, b), as shown in Figure 1. The first number a is called the ***x*-coordinate** of P; the second number b is called the ***y*-coordinate** of P. We can think of the coordinates of P as its "address," because they specify its location in the plane. Several points are labeled with their coordinates in Figure 2.

Although the notation for a point (a, b) is the same as the notation for an open interval (a, b), the context should make clear which meaning is intended.

SUGGESTED TIME AND EMPHASIS

1–2 classes.
Essential material.

POINTS TO STRESS

1. The distance and midpoint formulas.
2. The relationship between a two-variable equation and its graph. (This is the most crucial concept.)
3. Sketching graphs by plotting points, using intercepts and symmetry.
4. Equations of circles and their graphs, including the technique of completing the square.

ALTERNATE EXAMPLE 1
Does the point $(1, -2)$ belong to the region given by the set: $\{(x, y) \mid -3 < y < 4\}$?

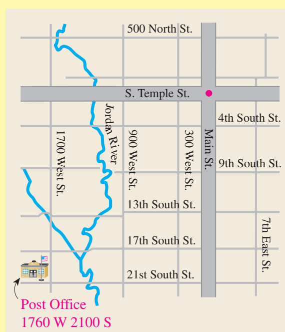
ANSWER
Yes

Coordinates as Addresses

The coordinates of a point in the xy-plane uniquely determine its location. We can think of the coordinates as the "address" of the point. In Salt Lake City, Utah, the addresses of most buildings are in fact expressed as coordinates. The city is divided into quadrants with Main Street as the vertical (North-South) axis and S. Temple Street as the horizontal (East-West) axis. An address such as

1760 W 2100 S

indicates a location 17.6 blocks west of Main Street and 21 blocks south of S. Temple Street. (This is the address of the main post office in Salt Lake City.) With this logical system it is possible for someone unfamiliar with the city to locate any address immediately, as easily as one locates a point in the coordinate plane.



Example 1 Graphing Regions in the Coordinate Plane

Describe and sketch the regions given by each set.

(a) $\{(x, y) \mid x \geq 0\}$ (b) $\{(x, y) \mid y = 1\}$ (c) $\{(x, y) \mid |y| < 1\}$

Solution

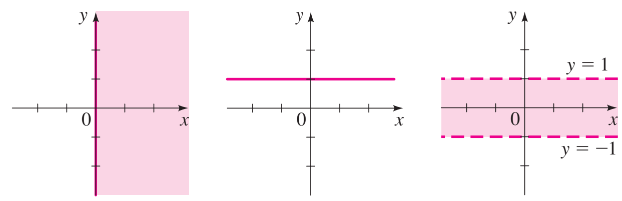
(a) The points whose x-coordinates are 0 or positive lie on the y-axis or to the right of it, as shown in Figure 3(a).

(b) The set of all points with y-coordinate 1 is a horizontal line one unit above the x-axis, as in Figure 3(b).

(c) Recall from Section 1.7 that

$$|y| < 1 \quad \text{if and only if} \quad -1 < y < 1$$

So the given region consists of those points in the plane whose y-coordinates lie between -1 and 1. Thus, the region consists of all points that lie between (but not on) the horizontal lines $y = 1$ and $y = -1$. These lines are shown as broken lines in Figure 3(c) to indicate that the points on these lines do not lie in the set.



(a) $x \geq 0$ (b) $y = 1$ (c) $|y| < 1$

Figure 3

■

The Distance and Midpoint Formulas

We now find a formula for the distance $d(A, B)$ between two points $A(x_1, y_1)$ and $B(x_2, y_2)$ in the plane. Recall from Section 1.1 that the distance between points a and b on a number line is $d(a, b) = |b - a|$. So, from Figure 4 we see that the distance between the points $A(x_1, y_1)$ and $C(x_2, y_1)$ on a horizontal line must be $|x_2 - x_1|$, and the distance between $B(x_2, y_2)$ and $C(x_2, y_1)$ on a vertical line must be $|y_2 - y_1|$.

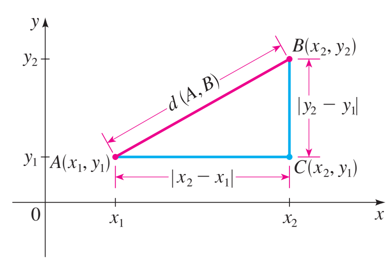


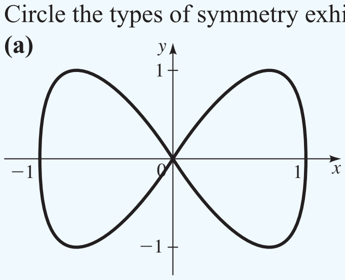
Figure 4

SAMPLE QUESTIONS

Text Questions

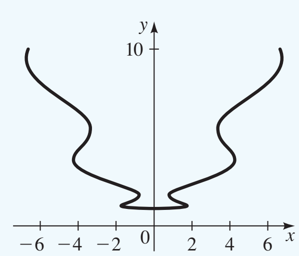
Circle the types of symmetry exhibited by each of the following figures.

(a)



(i) About the x-axis
(ii) About the y-axis
(iii) About the origin

(b)



(i) About the x-axis
(ii) About the y-axis
(iii) About the origin

Answers
(a) (i), (ii), and (iii)
(b) (ii)

Since triangle ABC is a right triangle, the Pythagorean Theorem gives

$$d(A, B) = \sqrt{|x_2 - x_1|^2 + |y_2 - y_1|^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Distance Formula

The distance between the points $A(x_1, y_1)$ and $B(x_2, y_2)$ in the plane is

$$d(A, B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

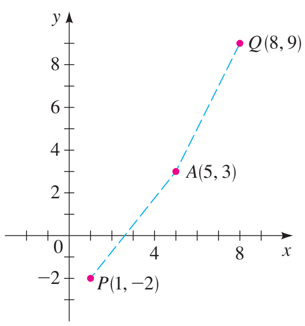


Figure 5

Example 2 Applying the Distance Formula

Which of the points $P(1, -2)$ or $Q(8, 9)$ is closer to the point $A(5, 3)$?

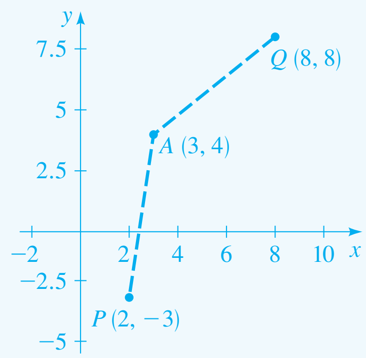
Solution By the Distance Formula, we have

$$d(P, A) = \sqrt{(5 - 1)^2 + [3 - (-2)]^2} = \sqrt{4^2 + 5^2} = \sqrt{41}$$
$$d(Q, A) = \sqrt{(5 - 8)^2 + (3 - 9)^2} = \sqrt{(-3)^2 + (-6)^2} = \sqrt{45}$$

This shows that $d(P, A) < d(Q, A)$, so P is closer to A (see Figure 5). ■

ALTERNATE EXAMPLE 2
Which of the points $P(2, -3)$ or $Q(8, 8)$ is closer to the point $A(3, 4)$?

ANSWER
Q



Now let's find the coordinates (x, y) of the midpoint M of the line segment that joins the point $A(x_1, y_1)$ to the point $B(x_2, y_2)$. In Figure 6 notice that triangles APM and MQB are congruent because $d(A, M) = d(M, B)$ and the corresponding angles are equal.

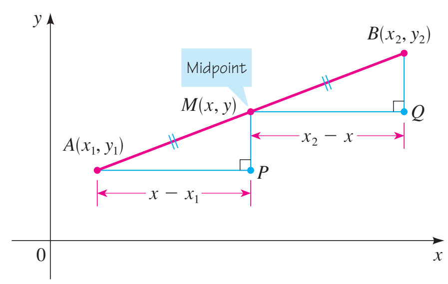


Figure 6

It follows that $d(A, P) = d(M, Q)$ and so

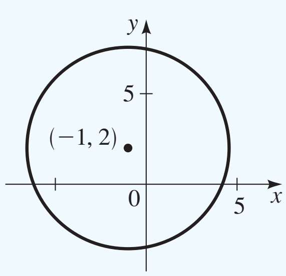
$$x - x_1 = x_2 - x$$

Solving this equation for x, we get $2x = x_1 + x_2$, and so $x = \dfrac{x_1 + x_2}{2}$. Similarly, $y = \dfrac{y_1 + y_2}{2}$.

DRILL QUESTION

Graph the circle with equation $x^2 + 2x + y^2 - 4y = 20$.

Answer



Midpoint Formula

The midpoint of the line segment from $A(x_1, y_1)$ to $B(x_2, y_2)$ is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

ALTERNATE EXAMPLE 3
Is the quadrilateral with vertices $P(1, 0)$, $Q(5, -1)$, $R(5, 5)$, and $S(1, 6)$ a parallelogram?

ANSWER
Yes

Example 3 Applying the Midpoint Formula

Show that the quadrilateral with vertices $P(1, 2)$, $Q(4, 4)$, $R(5, 9)$, and $S(2, 7)$ is a parallelogram by proving that its two diagonals bisect each other.

Solution If the two diagonals have the same midpoint, then they must bisect each other. The midpoint of the diagonal PR is

$$\left(\frac{1 + 5}{2}, \frac{2 + 9}{2}\right) = \left(3, \frac{11}{2}\right)$$

and the midpoint of the diagonal QS is

$$\left(\frac{4 + 2}{2}, \frac{4 + 7}{2}\right) = \left(3, \frac{11}{2}\right)$$

so each diagonal bisects the other, as shown in Figure 7. (A theorem from elementary geometry states that the quadrilateral is therefore a parallelogram.) ■

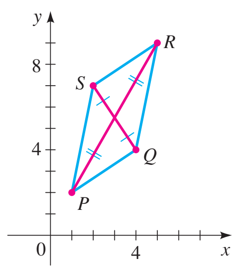


Figure 7

Graphs of Equations in Two Variables

An **equation in two variables**, such as $y = x^2 + 1$, expresses a relationship between two quantities. A point (x, y) **satisfies** the equation if it makes the equation true when the values for x and y are substituted into the equation. For example, the point $(3, 10)$ satisfies the equation $y = x^2 + 1$ because $10 = 3^2 + 1$, but the point $(1, 3)$ does not, because $3 \neq 1^2 + 1$.

Fundamental Principle of Analytic Geometry

A point (x, y) lies on the graph of an equation if and only if its coordinates satisfy the equation.

The Graph of an Equation

The **graph** of an equation in x and y is the set of all points (x, y) in the coordinate plane that satisfy the equation.

The graph of an equation is a curve, so to graph an equation we plot as many points as we can, then connect them by a smooth curve.

ALTERNATE EXAMPLE 4
For the graph of the equation $3x - y = 5$ find the coordinates of the points with $x = -1, 0, 1, 2,$ and 4.

ANSWER
$(-1, -8)$, $(0, -5)$, $(1, 2)$, $(2, 1)$, $(4, 7)$

Example 4 Sketching a Graph by Plotting Points

Sketch the graph of the equation $2x - y = 3$.

Solution We first solve the given equation for y to get

$$y = 2x - 3$$

IN-CLASS MATERIALS

Obtain a transparency of a local map, and overlay it with an appropriately-sized grid. Try to estimate the walking distance between the classroom and various points of interest. Then use the distance formula to find the distance "as the crow flies."

This helps us calculate the y-coordinates in the following table.

x	$y = 2x - 3$	(x, y)
-1	-5	$(-1, -5)$
0	-3	$(0, -3)$
1	-1	$(1, -1)$
2	1	$(2, 1)$
3	3	$(3, 3)$
4	5	$(4, 5)$

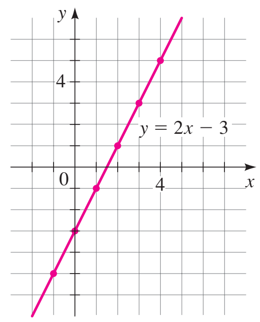


Figure 8

Of course, there are infinitely many points on the graph, and it is impossible to plot all of them. But the more points we plot, the better we can imagine what the graph represented by the equation looks like. We plot the points we found in Figure 8; they appear to lie on a line. So, we complete the graph by joining the points by a line. (In Section 1.10 we verify that the graph of this equation is indeed a line.) ■

Example 5 Sketching a Graph by Plotting Points

Sketch the graph of the equation $y = x^2 - 2$.

A detailed discussion of parabolas and their geometric properties is presented in Chapter 10.

Solution We find some of the points that satisfy the equation in the following table. In Figure 9 we plot these points and then connect them by a smooth curve. A curve with this shape is called a *parabola*.

x	$y = x^2 - 2$	(x, y)
-3	7	$(-3, 7)$
-2	2	$(-2, 2)$
-1	-1	$(-1, -1)$
0	-2	$(0, -2)$
1	-1	$(1, -1)$
2	2	$(2, 2)$
3	7	$(3, 7)$

■

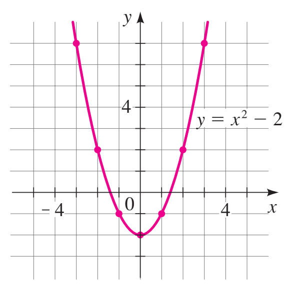


Figure 9

ALTERNATE EXAMPLE 5
For the graph of the equation $y = x^2 - 3$ find the coordinates of the points with $x = -3, -1, 0, 1$, and 3.

ANSWER
$(-3, 6), (-1, -2), (0, -3), (1, -2), (3, 6)$

Example 6 Graphing an Absolute Value Equation

Sketch the graph of the equation $y = |x|$.

Solution We make a table of values:

x	$y = \|x\|$	(x, y)
-3	3	$(-3, 3)$
-2	2	$(-2, 2)$
-1	1	$(-1, 1)$
0	0	$(0, 0)$
1	1	$(1, 1)$
2	2	$(2, 2)$
3	3	$(3, 3)$

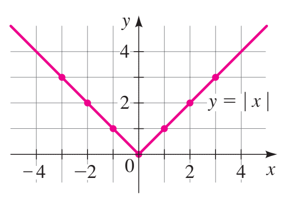


Figure 10

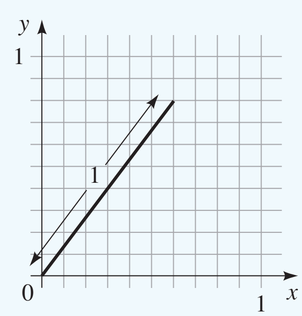
In Figure 10 we plot these points and use them to sketch the graph of the equation. ■

ALTERNATE EXAMPLE 6
For the graph of the equation $y = |-x|$ find the coordinates of the points with $x = -2, -1, 0, 1$, and 2.

ANSWER
$(-2, 2), (-1, 1), (0, 0), (1, 1), (2, 2)$

IN-CLASS MATERIALS

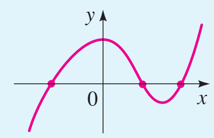
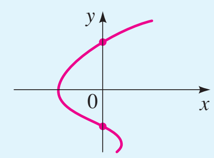
Assume that a meter stick is situated as in the diagram below. Ask the students to find the 50 cm mark, and then the 25 cm mark.



Intercepts

The x-coordinates of the points where a graph intersects the x-axis are called the **x-intercepts** of the graph and are obtained by setting $y = 0$ in the equation of the graph. The y-coordinates of the points where a graph intersects the y-axis are called the **y-intercepts** of the graph and are obtained by setting $x = 0$ in the equation of the graph.

Definition of Intercepts

Intercepts	How to find them	Where they are on the graph
x-intercepts: The x-coordinates of points where the graph of an equation intersects the x-axis	Set $y = 0$ and solve for x	
y-intercepts: The y-coordinates of points where the graph of an equation intersects the y-axis	Set $x = 0$ and solve for y	

ALTERNATE EXAMPLE 7
Find the x- and y-intercepts of the graph of the equation $y = x^2 - 3x + 2$.

ANSWER
The x-intercepts are 1 and 2. The y-intercept is 2.

Example 7 Finding Intercepts

Find the x- and y-intercepts of the graph of the equation $y = x^2 - 2$.

Solution To find the x-intercepts, we set $y = 0$ and solve for x. Thus

$$0 = x^2 - 2 \qquad \text{Set } y = 0$$

$$x^2 = 2 \qquad \text{Add 2 to each side}$$

$$x = \pm\sqrt{2} \qquad \text{Take the square root}$$

The x-intercepts are $\sqrt{2}$ and $-\sqrt{2}$.

To find the y-intercepts, we set $x = 0$ and solve for y. Thus

$$y = 0^2 - 2 \qquad \text{Set } x = 0$$

$$y = -2$$

The y-intercept is -2.

The graph of this equation was sketched in Example 5. It is repeated in Figure 11 with the x- and y-intercepts labeled. ■

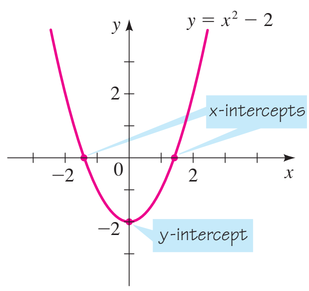


Figure 11

Circles

So far we have discussed how to find the graph of an equation in x and y. The converse problem is to find an equation of a graph, that is, an equation that represents a

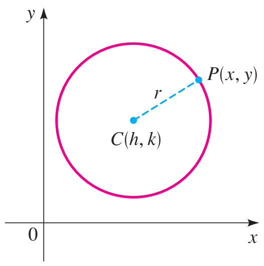


Figure 12

given curve in the xy-plane. Such an equation is satisfied by the coordinates of the points on the curve and by no other point. This is the other half of the fundamental principle of analytic geometry as formulated by Descartes and Fermat. The idea is that if a geometric curve can be represented by an algebraic equation, then the rules of algebra can be used to analyze the curve.

As an example of this type of problem, let's find the equation of a circle with radius r and center (h, k). By definition, the circle is the set of all points $P(x, y)$ whose distance from the center $C(h, k)$ is r (see Figure 12). Thus, P is on the circle if and only if $d(P, C) = r$. From the distance formula we have

$$\sqrt{(x-h)^2 + (y-k)^2} = r$$

$$(x-h)^2 + (y-k)^2 = r^2 \qquad \text{Square each side}$$

This is the desired equation.

Equation of a Circle

An equation of the circle with center (h, k) and radius r is

$$(x-h)^2 + (y-k)^2 = r^2$$

This is called the **standard form** for the equation of the circle. If the center of the circle is the origin $(0, 0)$, then the equation is

$$x^2 + y^2 = r^2$$

Example 8 Graphing a Circle

Graph each equation.

(a) $x^2 + y^2 = 25$ (b) $(x-2)^2 + (y+1)^2 = 25$

Solution

(a) Rewriting the equation as $x^2 + y^2 = 5^2$, we see that this is an equation of the circle of radius 5 centered at the origin. Its graph is shown in Figure 13.

(b) Rewriting the equation as $(x-2)^2 + (y+1)^2 = 5^2$, we see that this is an equation of the circle of radius 5 centered at $(2, -1)$. Its graph is shown in Figure 14.

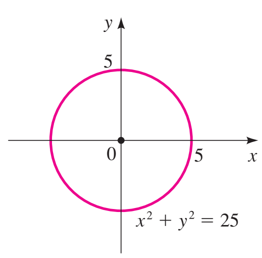


Figure 13

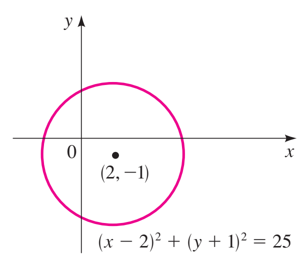


Figure 14

■

ALTERNATE EXAMPLE 8

For the graph of the equation $(x-2)^2 + (y+4)^2 = 36$, find its center and radius.

ANSWER

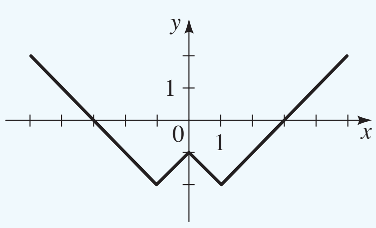
Center: $(2, -4)$

Radius: 6

IN-CLASS MATERIALS

Graph $y = ||x| - 1| - 2$ by plotting points. This is a nice curve because it is easy to get points, but students will not immediately see what the curve should look like. Have different students obtain different points, and plot them on a common graph on the board.

Answer



ALTERNATE EXAMPLE 9

Find an equation of the circle that has the points $P(1, 8)$ and $Q(3, -6)$ as the endpoints of a diameter.

ANSWER

$(x - 2)^2 + (y - 1)^2 = 50$

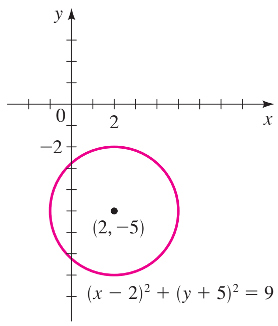


Figure 15

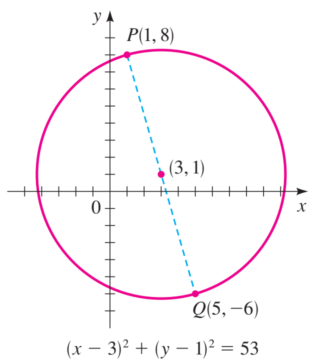


Figure 16

Example 9 Finding an Equation of a Circle

(a) Find an equation of the circle with radius 3 and center $(2, -5)$.

(b) Find an equation of the circle that has the points $P(1, 8)$ and $Q(5, -6)$ as the endpoints of a diameter.

Solution

(a) Using the equation of a circle with $r = 3$, $h = 2$, and $k = -5$, we obtain

$$(x - 2)^2 + (y + 5)^2 = 9$$

The graph is shown in Figure 15.

(b) We first observe that the center is the midpoint of the diameter PQ, so by the Midpoint Formula the center is

$$\left(\frac{1+5}{2}, \frac{8-6}{2}\right) = (3, 1)$$

The radius r is the distance from P to the center, so by the Distance Formula

$$r^2 = (3 - 1)^2 + (1 - 8)^2 = 2^2 + (-7)^2 = 53$$

Therefore, the equation of the circle is

$$(x - 3)^2 + (y - 1)^2 = 53$$

The graph is shown in Figure 16. ■

Let's expand the equation of the circle in the preceding example.

$$(x - 3)^2 + (y - 1)^2 = 53 \qquad \text{Standard form}$$

$$x^2 - 6x + 9 + y^2 - 2y + 1 = 53 \qquad \text{Expand the squares}$$

$$x^2 - 6x + y^2 - 2y = 43 \qquad \text{Subtract 10 to get expanded form}$$

Suppose we are given the equation of a circle in expanded form. Then to find its center and radius we must put the equation back in standard form. That means we must reverse the steps in the preceding calculation, and to do that we need to know what to add to an expression like $x^2 - 6x$ to make it a perfect square—that is, we need to complete the square, as in the next example.

Completing the square is used in many contexts in algebra. In Section 1.5 we used completing the square to solve quadratic equations.

ALTERNATE EXAMPLE 10

For the graph of the circle $x^2 + y^2 + 4x - 10y + 6 = 0$ find its center and radius.

ANSWER

$(-2, 5)$, $\sqrt{23}$

Example 10 Identifying an Equation of a Circle

Show that the equation $x^2 + y^2 + 2x - 6y + 7 = 0$ represents a circle, and find the center and radius of the circle.

Solution We first group the x-terms and y-terms. Then we complete the square within each grouping. That is, we complete the square for $x^2 + 2x$ by adding $\left(\frac{1}{2} \cdot 2\right)^2 = 1$, and we complete the square for $y^2 - 6y$ by adding $\left[\frac{1}{2} \cdot (-6)\right]^2 = 9$.

$$(x^2 + 2x \qquad) + (y^2 - 6y \qquad) = -7 \qquad \text{Group terms}$$

$$(x^2 + 2x + 1) + (y^2 - 6y + 9) = -7 + 1 + 9 \qquad \text{Complete the square by adding 1 and 9 to each side}$$

$$(x + 1)^2 + (y - 3)^2 = 3 \qquad \text{Factor and simplify}$$

We must add the same numbers to *each side* to maintain equality.

IN-CLASS MATERIALS

Emphasize that the equation $(x - 2)^2 + (y + 3)^2 = 100$ and the equation $x^2 - 4x + y^2 + 6y = -3$ have the same graph—the same points satisfy both equations. Perhaps test $(2, 7)$ and $(1, 1)$ in both equations and notice that $(2, 7)$ satisfies both, and $(1, 1)$ fails to satisfy either. The only difference between the two equations is the form: it is easier to work with the first than the second.

Comparing this equation with the standard equation of a circle, we see that $h = -1$, $k = 3$, and $r = \sqrt{3}$, so the given equation represents a circle with center $(-1, 3)$ and radius $\sqrt{3}$. ■

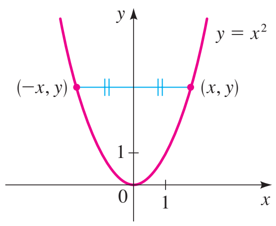
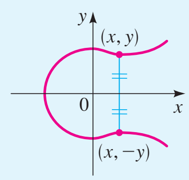
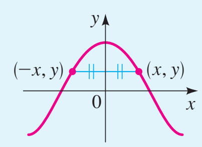
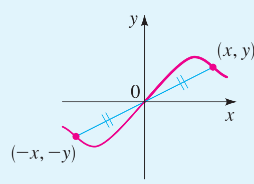


Figure 17

Symmetry

Figure 17 shows the graph of $y = x^2$. Notice that the part of the graph to the left of the y-axis is the mirror image of the part to the right of the y-axis. The reason is that if the point (x, y) is on the graph, then so is $(-x, y)$, and these points are reflections of each other about the y-axis. In this situation we say the graph is **symmetric with respect to the y-axis**. Similarly, we say a graph is **symmetric with respect to the x-axis** if whenever the point (x, y) is on the graph, then so is $(x, -y)$. A graph is **symmetric with respect to the origin** if whenever (x, y) is on the graph, so is $(-x, -y)$.

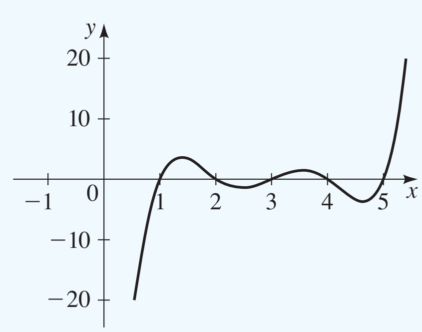
Definition of Symmetry

Type of symmetry	How to test for symmetry	What the graph looks like (figures in this section)	Geometric meaning
Symmetry with respect to the x-axis	The equation is unchanged when y is replaced by $-y$	 (Figures 13, 18)	Graph is unchanged when reflected in the x-axis
Symmetry with respect to the y-axis	The equation is unchanged when x is replaced by $-x$	 (Figures 9, 10, 11, 13, 17)	Graph is unchanged when reflected in the y-axis
Symmetry with respect to the origin	The equation is unchanged when x is replaced by $-x$ and y by $-y$	 (Figures 13, 19)	Graph is unchanged when rotated 180° about the origin

EXAMPLES

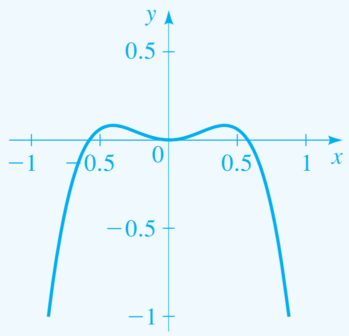
1. Investigate the trapezoidal region given by the inequalities $x > 2$, $y > \frac{3}{2}x - 3$, $y > -x + 7$, and $y < -x + 12$.

2. A curve with a lot of intercepts: $y = x^5 - 15x^4 + 85x^3 - 225x^2 + 274x - 120$ looks frightening until students are given the factored form: $y = (x-1)(x-2)(x-3)(x-4)(x-5)$.



ALTERNATE EXAMPLE 11
Test the equation $y = x^2 - 3x^4$ for symmetry and sketch the graph.

ANSWER
There is y-axis symmetry.



ALTERNATE EXAMPLE 12
What viewing rectangle gives an appropriate representation of the graph of the equation $y = x^2 + 4$?

(a) [−2, 2] by [−2, 2]
(b) [−6, 6] by [−6, 6]
(c) [−12, 12] by [−9, 33]
(d) [−70, 70] by [−100, 1400]

ANSWER
(c)

The remaining examples in this section show how symmetry helps us sketch the graphs of equations.

Example 11 Using Symmetry to Sketch a Graph

Test the equation $x = y^2$ for symmetry and sketch the graph.

Solution If y is replaced by $-y$ in the equation $x = y^2$, we get

$$x = (-y)^2 \quad \text{Replace } y \text{ by } -y$$

$$x = y^2 \quad \text{Simplify}$$

and so the equation is unchanged. Therefore, the graph is symmetric about the x-axis. But changing x to $-x$ gives the equation $-x = y^2$, which is not the same as the original equation, so the graph is not symmetric about the y-axis.

We use the symmetry about the x-axis to sketch the graph by first plotting points just for $y > 0$ and then reflecting the graph in the x-axis, as shown in Figure 18.

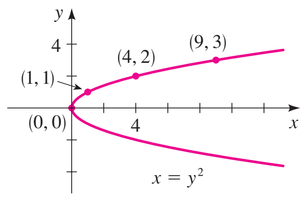


Figure 18

y	$x = y^2$	(x, y)
0	0	(0, 0)
1	1	(1, 1)
2	4	(4, 2)
3	9	(9, 3)

■

Example 12 Using Symmetry to Sketch a Graph

Test the equation $y = x^3 - 9x$ for symmetry and sketch its graph.

Solution If we replace x by $-x$ and y by $-y$ in the equation, we get

$$-y = (-x)^3 - 9(-x) \quad \text{Replace } x \text{ by } -x \text{ and } y \text{ by } -y$$

$$-y = -x^3 + 9x \quad \text{Simplify}$$

$$y = x^3 - 9x \quad \text{Multiply by } -1$$

and so the equation is unchanged. This means that the graph is symmetric with respect to the origin. We sketch it by first plotting points for $x > 0$ and then using symmetry about the origin (see Figure 19).

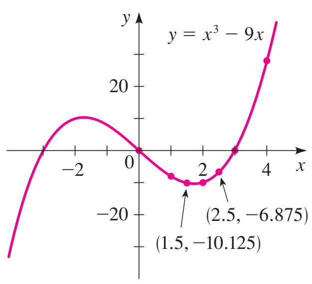


Figure 19

x	$y = x^3 - 9x$	(x, y)
0	0	(0, 0)
1	−8	(1, −8)
1.5	−10.125	(1.5, −10.125)
2	−10	(2, −10)
2.5	−6.875	(2.5, −6.875)
3	0	(3, 0)
4	28	(4, 28)

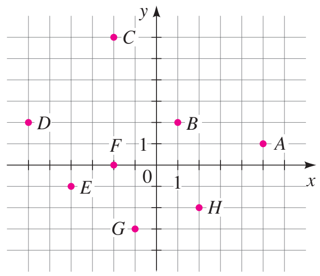
■

1.8 Exercises

1. Plot the given points in a coordinate plane:

$(2, 3), (-2, 3), (4, 5), (4, -5), (-4, 5), (-4, -5)$

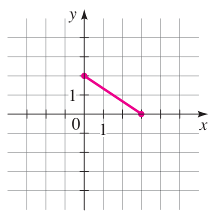
2. Find the coordinates of the points shown in the figure.



3–6 ■ A pair of points is graphed.

(a) Find the distance between them.
(b) Find the midpoint of the segment that joins them.

3.



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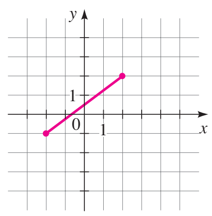

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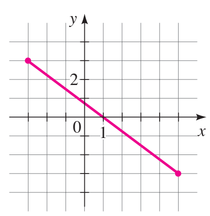

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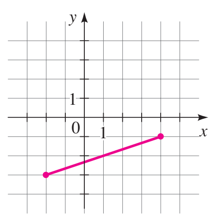


7–12 ■ A pair of points is graphed.

(a) Plot the points in a coordinate plane.
(b) Find the distance between them.
(c) Find the midpoint of the segment that joins them.

7. $(0, 8), (6, 16)$

8. $(-2, 5), (10, 0)$

9. $(-3, -6), (4, 18)$

10. $(-1, -1), (9, 9)$

11. $(6, -2), (-6, 2)$

12. $(0, -6), (5, 0)$

13. Draw the rectangle with vertices $A(1, 3)$, $B(5, 3)$, $C(1, -3)$, and $D(5, -3)$ on a coordinate plane. Find the area of the rectangle.

14. Draw the parallelogram with vertices $A(1, 2)$, $B(5, 2)$, $C(3, 6)$, and $D(7, 6)$ on a coordinate plane. Find the area of the parallelogram.

15. Plot the points $A(1, 0)$, $B(5, 0)$, $C(4, 3)$, and $D(2, 3)$, on a coordinate plane. Draw the segments AB, BC, CD, and DA. What kind of quadrilateral is $ABCD$, and what is its area?

16. Plot the points $P(5, 1)$, $Q(0, 6)$, and $R(-5, 1)$, on a coordinate plane. Where must the point S be located so that the quadrilateral $PQRS$ is a square? Find the area of this square.

17–26 ■ Sketch the region given by the set.

17. $\{(x, y) \mid x \geq 3\}$

18. $\{(x, y) \mid y < 3\}$

19. $\{(x, y) \mid y = 2\}$

20. $\{(x, y) \mid x = -1\}$

21. $\{(x, y) \mid 1 < x < 2\}$

22. $\{(x, y) \mid 0 \leq y \leq 4\}$

23. $\{(x, y) \mid |x| > 4\}$

24. $\{(x, y) \mid |y| \leq 2\}$

25. $\{(x, y) \mid x \geq 1 \text{ and } y < 3\}$

26. $\{(x, y) \mid |x| \leq 2 \text{ and } |y| \leq 3\}$

27. Which of the points $A(6, 7)$ or $B(-5, 8)$ is closer to the origin?

28. Which of the points $C(-6, 3)$ or $D(3, 0)$ is closer to the point $E(-2, 1)$?

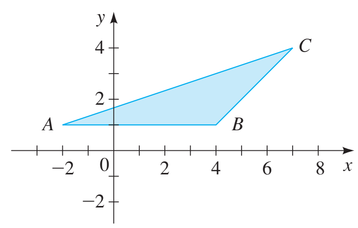
29. Which of the points $P(3, 1)$ or $Q(-1, 3)$ is closer to the point $R(-1, -1)$?

30. **(a)** Show that the points $(7, 3)$ and $(3, 7)$ are the same distance from the origin.

(b) Show that the points (a, b) and (b, a) are the same distance from the origin.

31. Show that the triangle with vertices $A(0, 2)$, $B(-3, -1)$, and $C(-4, 3)$ is isosceles.

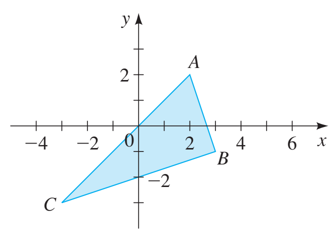
32. Find the area of the triangle shown in the figure.



33. Refer to triangle ABC in the figure.

(a) Show that triangle ABC is a right triangle by using the converse of the Pythagorean Theorem (see page 54).

(b) Find the area of triangle ABC.



34. Show that the triangle with vertices $A(6, -7)$, $B(11, -3)$, and $C(2, -2)$ is a right triangle by using the converse of the Pythagorean Theorem. Find the area of the triangle.

35. Show that the points $A(-2, 9)$, $B(4, 6)$, $C(1, 0)$, and $D(-5, 3)$ are the vertices of a square.

36. Show that the points $A(-1, 3)$, $B(3, 11)$, and $C(5, 15)$ are collinear by showing that $d(A, B) + d(B, C) = d(A, C)$.

37. Find a point on the y-axis that is equidistant from the points $(5, -5)$ and $(1, 1)$.

38. Find the lengths of the medians of the triangle with vertices $A(1, 0)$, $B(3, 6)$, and $C(8, 2)$. (A *median* is a line segment from a vertex to the midpoint of the opposite side.)

39. Plot the points $P(-1, -4)$, $Q(1, 1)$, and $R(4, 2)$, on a coordinate plane. Where should the point S be located so that the figure $PQRS$ is a parallelogram?

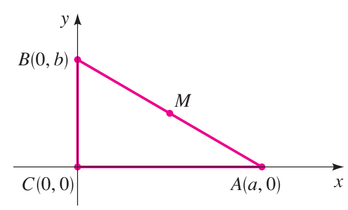
40. If $M(6, 8)$ is the midpoint of the line segment AB, and if A has coordinates $(2, 3)$, find the coordinates of B.

41. **(a)** Sketch the parallelogram with vertices $A(-2, -1)$, $B(4, 2)$, $C(7, 7)$, and $D(1, 4)$.

(b) Find the midpoints of the diagonals of this parallelogram.

(c) From part (b) show that the diagonals bisect each other.

42. The point M in the figure is the midpoint of the line segment AB. Show that M is equidistant from the vertices of triangle ABC.



43–46 ■ Determine whether the given points are on the graph of the equation.

43. $x - 2y - 1 = 0$; $(0, 0), (1, 0), (-1, -1)$

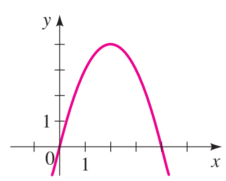
44. $y(x^2 + 1) = 1$; $(1, 1), (1, \frac{1}{2}), (-1, \frac{1}{2})$

45. $x^2 + xy + y^2 = 4$; $(0, -2), (1, -2), (2, -2)$

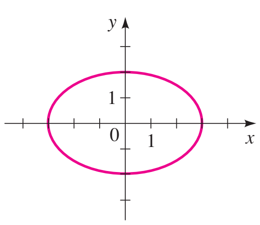
46. $x^2 + y^2 = 1$; $(0, 1), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

47–50 ■ An equation and its graph are given. Find the x- and y-intercepts.

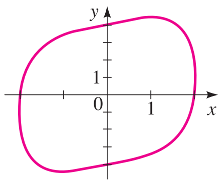
47. $y = 4x - x^2$



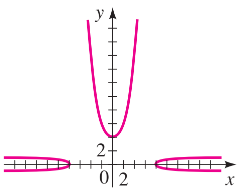
48. $\frac{x^2}{9} + \frac{y^2}{4} = 1$



49. $x^4 + y^2 - xy = 16$



50. $x^2 + y^3 - x^2y^2 = 64$



51–70 ■ Make a table of values and sketch the graph of the equation. Find the x- and y-intercepts and test for symmetry.

51. $y = -x + 4$

52. $y = 3x + 3$

53. $2x - y = 6$

54. $x + y = 3$

55. $y = 1 - x^2$

56. $y = x^2 + 2$

57. $4y = x^2$

58. $8y = x^3$

59. $y = x^2 - 9$

60. $y = 9 - x^2$

61. $xy = 2$

62. $y = \sqrt{x + 4}$

63. $y = \sqrt{4 - x^2}$

64. $y = -\sqrt{4 - x^2}$

65. $x + y^2 = 4$

66. $x = y^3$

67. $y = 16 - x^4$

68. $x = |y|$

69. $y = 4 - |x|$

70. $y = |4 - x|$

71–76 ■ Test the equation for symmetry.

71. $y = x^4 + x^2$

72. $x = y^4 - y^2$

73. $x^2y^2 + xy = 1$

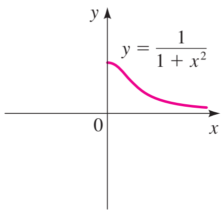
74. $x^4y^4 + x^2y^2 = 1$

75. $y = x^3 + 10x$

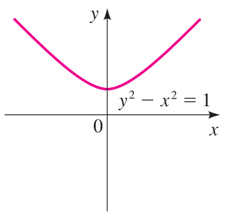
76. $y = x^2 + |x|$

77–80 ■ Complete the graph using the given symmetry property.

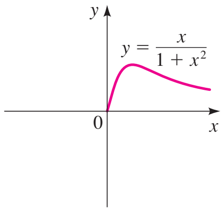
77. Symmetric with respect to the y-axis



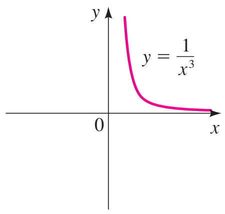
78. Symmetric with respect to the x-axis



79. Symmetric with respect to the origin



80. Symmetric with respect to the origin



81–86 ■ Find an equation of the circle that satisfies the given conditions.

81. Center $(2, -1)$; radius 3

82. Center $(-1, -4)$; radius 8

83. Center at the origin; passes through $(4, 7)$

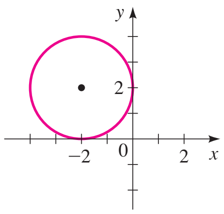
84. Endpoints of a diameter are $P(-1, 1)$ and $Q(5, 9)$

85. Center $(7, -3)$; tangent to the x-axis

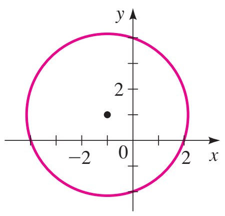
86. Circle lies in the first quadrant, tangent to both x-and y-axes; radius 5

87–88 ■ Find the equation of the circle shown in the figure.

87.



88.



89–94 ■ Show that the equation represents a circle, and find the center and radius of the circle.

89. $x^2 + y^2 - 4x + 10y + 13 = 0$

90. $x^2 + y^2 + 6y + 2 = 0$

91. $x^2 + y^2 - \frac{1}{2}x + \frac{1}{2}y = \frac{1}{8}$

92. $x^2 + y^2 + \frac{1}{2}x + 2y + \frac{1}{16} = 0$

93. $2x^2 + 2y^2 - 3x = 0$

94. $3x^2 + 3y^2 + 6x - y = 0$

95–96 ■ Sketch the region given by the set.

95. $\{(x, y) \mid x^2 + y^2 \le 1\}$

96. $\{(x, y) \mid x^2 + y^2 > 4\}$

97. Find the area of the region that lies outside the circle $x^2 + y^2 = 4$ but inside the circle

$$x^2 + y^2 - 4y - 12 = 0$$

98. Sketch the region in the coordinate plane that satisfies both the inequalities $x^2 + y^2 \le 9$ and $y \ge |x|$. What is the area of this region?

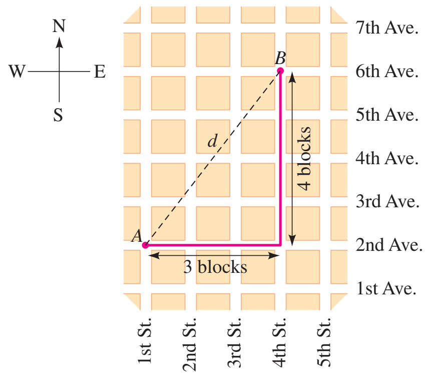
Applications

99. Distances in a City A city has streets that run north and south, and avenues that run east and west, all equally spaced. Streets and avenues are numbered sequentially, as shown in the figure. The *walking* distance between points A and B is 7 blocks—that is, 3 blocks east and 4 blocks north. To find the *straight-line* distances d, we must use the Distance Formula.

(a) Find the straight-line distance (in blocks) between A and B.

(b) Find the walking distance and the straight-line distance between the corner of 4th St. and 2nd Ave. and the corner of 11th St. and 26th Ave.

(c) What must be true about the points P and Q if the walking distance between P and Q equals the straight-line distance between P and Q?



100. Halfway Point Two friends live in the city described in Exercise 99, one at the corner of 3rd St. and 7th Ave., the other at the corner of 27th St. and 17th Ave. They frequently meet at a coffee shop halfway between their homes.

(a) At what intersection is the coffee shop located?

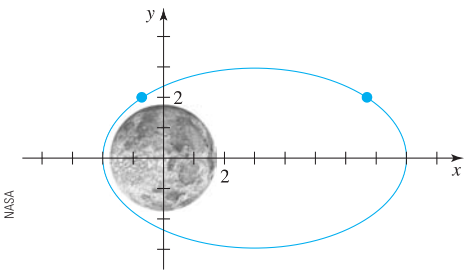
(b) How far must each of them walk to get to the coffee shop?

101. Orbit of a Satellite A satellite is in orbit around the moon. A coordinate plane containing the orbit is set up with the center of the moon at the origin, as shown in the graph, with distances measured in megameters (Mm). The equation of the satellite's orbit is

$$\frac{(x-3)^2}{25} + \frac{y^2}{16} = 1$$

(a) From the graph, determine the closest and the farthest that the satellite gets to the center of the moon.

(b) There are two points in the orbit with y-coordinates 2. Find the x-coordinates of these points, and determine their distances to the center of the moon.



Discovery • Discussion

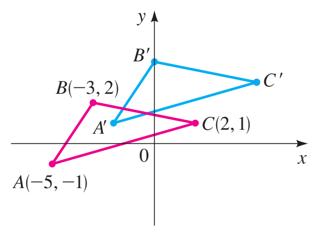
102. Shifting the Coordinate Plane Suppose that each point in the coordinate plane is shifted 3 units to the right and 2 units upward.

(a) The point $(5, 3)$ is shifted to what new point?

(b) The point (a, b) is shifted to what new point?

(c) What point is shifted to $(3, 4)$?

(d) Triangle ABC in the figure has been shifted to triangle $A'B'C'$. Find the coordinates of the points A', B', and C'.



100 CHAPTER 1 Fundamentals

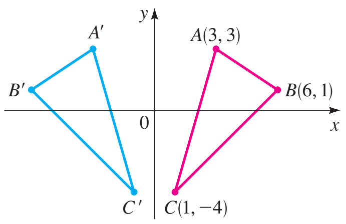
103. Reflecting in the Coordinate Plane Suppose that the y-axis acts as a mirror that reflects each point to the right of it into a point to the left of it.

(a) The point $(3, 7)$ is reflected to what point?

(b) The point (a, b) is reflected to what point?

(c) What point is reflected to $(-4, -1)$?

(d) Triangle ABC in the figure is reflected to triangle $A'B'C'$. Find the coordinates of the points A', B', and C'.



104. Completing a Line Segment Plot the points $M(6, 8)$ and $A(2, 3)$ on a coordinate plane. If M is the midpoint of the line segment AB, find the coordinates of B. Write a brief description of the steps you took to find B, and your reasons for taking them.

105. Completing a Parallelogram Plot the points $P(0, 3)$, $Q(2, 2)$, and $R(5, 3)$ on a coordinate plane. Where should the point S be located so that the figure $PQRS$ is a parallelogram? Write a brief description of the steps you took and your reasons for taking them.

106. Circle, Point, or Empty Set? Complete the squares in the general equation $x^2 + ax + y^2 + by + c = 0$ and simplify the result as much as possible. Under what conditions on the coefficients a, b, and c does this equation represent a circle? A single point? The empty set? In the case that the equation does represent a circle, find its center and radius.

107. Do the Circles Intersect?

(a) Find the radius of each circle in the pair, and the distance between their centers; then use this information to determine whether the circles intersect.

(i) $(x - 2)^2 + (y - 1)^2 = 9$;
$(x - 6)^2 + (y - 4)^2 = 16$

(ii) $x^2 + (y - 2)^2 = 4$;
$(x - 5)^2 + (y - 14)^2 = 9$

(iii) $(x - 3)^2 + (y + 1)^2 = 1$;
$(x - 2)^2 + (y - 2)^2 = 25$

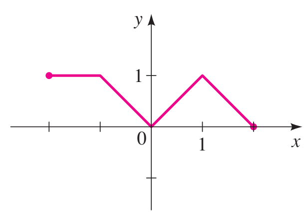
(b) How can you tell, just by knowing the radii of two circles and the distance between their centers, whether the circles intersect? Write a short paragraph describing how you would decide this and draw graphs to illustrate your answer.

108. Making a Graph Symmetric The graph shown in the figure is not symmetric about the x-axis, the y-axis, or the origin. Add more line segments to the graph so that it exhibits the indicated symmetry. In each case, add as little as possible.

(a) Symmetry about the x-axis

(b) Symmetry about the y-axis

(c) Symmetry about the origin



1.9 Graphing Calculators; Solving Equations and Inequalities Graphically

In Sections 1.5 and 1.7 we solved equations and inequalities algebraically. In the preceding section we learned how to sketch the graph of an equation in a coordinate plane. In this section we use graphs to solve equations and inequalities. To do this, we must first draw a graph using a graphing device. So, we begin by giving a few guidelines to help us use graphing devices effectively.

SUGGESTED TIME AND EMPHASIS

1 class.
Recommended material.

POINTS TO STRESS

1. Approximating the solution to an equation by finding the roots of an expression graphically.
2. Approximating the solution to an inequality by graphing both sides of the inequality.

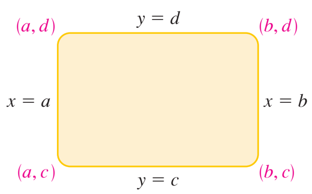


Figure 1
The viewing rectangle $[a, b]$ by $[c, d]$

Using a Graphing Calculator

A graphing calculator or computer displays a rectangular portion of the graph of an equation in a display window or viewing screen, which we call a **viewing rectangle**. The default screen often gives an incomplete or misleading picture, so it is important to choose the viewing rectangle with care. If we choose the x-values to range from a minimum value of `Xmin` $= a$ to a maximum value of `Xmax` $= b$ and the y-values to range from a minimum value of `Ymin` $= c$ to a maximum value of `Ymax` $= d$, then the displayed portion of the graph lies in the rectangle

$$[a, b] \times [c, d] = \{(x, y) \mid a \le x \le b, c \le y \le d\}$$

as shown in Figure 1. We refer to this as the $[a, b]$ by $[c, d]$ viewing rectangle.

The graphing device draws the graph of an equation much as you would. It plots points of the form (x, y) for a certain number of values of x, equally spaced between a and b. If the equation is not defined for an x-value, or if the corresponding y-value lies outside the viewing rectangle, the device ignores this value and moves on to the next x-value. The machine connects each point to the preceding plotted point to form a representation of the graph of the equation.

ALTERNATE EXAMPLE 1
Graph the equation in an appropriate viewing rectangle:
$x^3 - 6x^2 + 11x - 5$

ANSWER
Answers can vary.

[0, 4] by [−4, 6] is an example of a good viewing rectangle for this function.

Example 1 Choosing an Appropriate Viewing Rectangle

Graph the equation $y = x^2 + 3$ in an appropriate viewing rectangle.

Solution Let's experiment with different viewing rectangles. We'll start with the viewing rectangle $[-2, 2]$ by $[-2, 2]$, so we set

`Xmin` $= -2$	`Ymin` $= -2$
`Xmax` $= 2$	`Ymax` $= 2$

The resulting graph in Figure 2(a) is blank! This is because $x^2 \ge 0$, so $x^2 + 3 \ge 3$ for all x. Thus, the graph lies entirely above the viewing rectangle, so this viewing rectangle is not appropriate. If we enlarge the viewing rectangle to $[-4, 4]$ by $[-4, 4]$, as in Figure 2(b), we begin to see a portion of the graph.

Now let's try the viewing rectangle $[-10, 10]$ by $[-5, 30]$. The graph in Figure 2(c) seems to give a more complete view of the graph. If we enlarge the viewing rectangle even further, as in Figure 2(d), the graph doesn't show clearly that the y-intercept is 3.

So, the viewing rectangle $[-10, 10]$ by $[-5, 30]$ gives an appropriate representation of the graph.

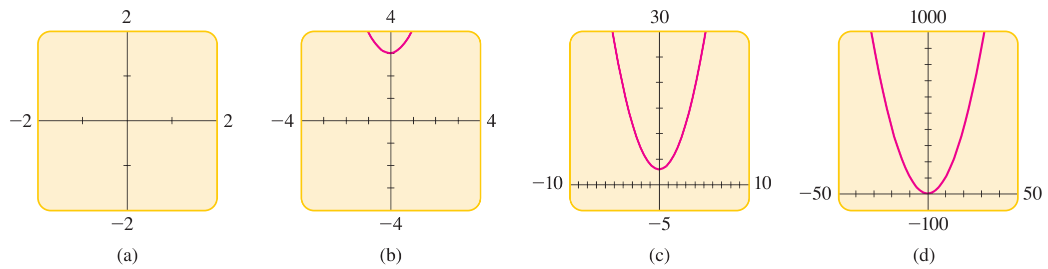


Figure 2 Graphs of $y = x^2 + 3$ ■

SAMPLE QUESTION
Text Question
Give an advantage of solving an equation graphically, and an advantage of solving an equation algebraically.

Alan Turing (1912–1954) was at the center of two pivotal events of the 20th century—World War II and the invention of computers. At the age of 23 Turing made his mark on mathematics by solving an important problem in the foundations of mathematics that was posed by David Hilbert at the 1928 International Congress of Mathematicians (see page 708). In this research he invented a theoretical machine, now called a Turing machine, which was the inspiration for modern digital computers. During World War II Turing was in charge of the British effort to decipher secret German codes. His complete success in this endeavor played a decisive role in the Allies' victory. To carry out the numerous logical steps required to break a coded message, Turing developed decision procedures similar to modern computer programs. After the war he helped develop the first electronic computers in Britain. He also did pioneering work on artificial intelligence and computer models of biological processes. At the age of 42 Turing died of poisoning after eating an apple that had mysteriously been laced with cyanide.

Example 2 Two Graphs on the Same Screen

Graph the equations $y = 3x^2 - 6x + 1$ and $y = 0.23x - 2.25$ together in the viewing rectangle $[-1, 3]$ by $[-2.5, 1.5]$. Do the graphs intersect in this viewing rectangle?

Solution Figure 3(a) shows the essential features of both graphs. One is a parabola and the other is a line. It looks as if the graphs intersect near the point $(1, -2)$. However, if we zoom in on the area around this point as shown in Figure 3(b), we see that although the graphs almost touch, they don't actually intersect.

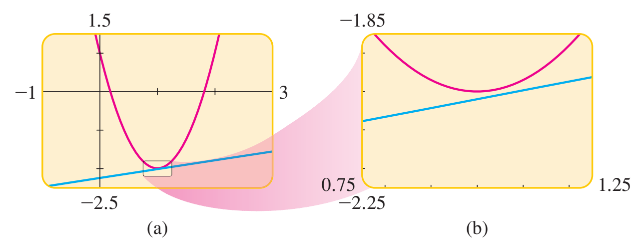


Figure 3 ■

You can see from Examples 1 and 2 that the choice of a viewing rectangle makes a big difference in the appearance of a graph. If you want an overview of the essential features of a graph, you must choose a relatively large viewing rectangle to obtain a global view of the graph. If you want to investigate the details of a graph, you must zoom in to a small viewing rectangle that shows just the feature of interest.

Most graphing calculators can only graph equations in which y is isolated on one side of the equal sign. The next example shows how to graph equations that don't have this property.

Example 3 Graphing a Circle

Graph the circle $x^2 + y^2 = 1$.

Solution We first solve for y, to isolate it on one side of the equal sign.

$$y^2 = 1 - x^2 \qquad \text{Subtract } x^2$$

$$y = \pm\sqrt{1 - x^2} \qquad \text{Take square roots}$$

Therefore, the circle is described by the graphs of *two* equations:

$$y = \sqrt{1 - x^2} \qquad \text{and} \qquad y = -\sqrt{1 - x^2}$$

The first equation represents the top half of the circle (because $y \geq 0$), and the second represents the bottom half of the circle (because $y \leq 0$). If we graph the

ALTERNATE EXAMPLE 2
Graph the equations $y = 2x^2 - 4x + 1$ and $y = 0.15x - 1.15$ using the graphing calculator together on the same screen in the viewing rectangle [−1, 3] by [−1.5, 3.5]. Do the graphs intersect in this viewing rectangle?

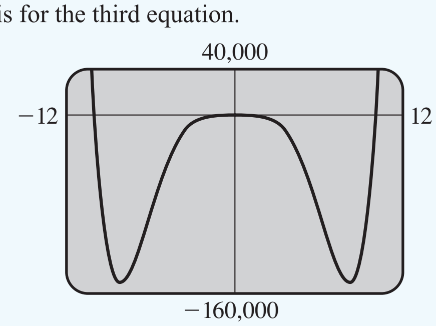
ANSWER
Yes

ALTERNATE EXAMPLE 3
Graph the circle $x^2 + y^2 = 9$ using a graphing calculator and determine the radius and the coordinates of the center.

ANSWER
3, (0, 0)

IN-CLASS MATERIALS

Have students find a viewing rectangle for $y = 3x + 1$, $y = -x^2 + 2$, and $y = x^6 - 100x^4 + 50$. Specify that the figure (below) is for the third equation.



DRILL QUESTION

Solve the equation $x^3 - \sqrt{x} = 2x - 1$ to the nearest thousandth.

Answer

$x \approx 0.2556$ or $x \approx 1.464$

first equation in the viewing rectangle $[-2, 2]$ by $[-2, 2]$, we get the semicircle shown in Figure 4(a). The graph of the second equation is the semicircle in Figure 4(b). Graphing these semicircles together on the same viewing screen, we get the full circle in Figure 4(c).

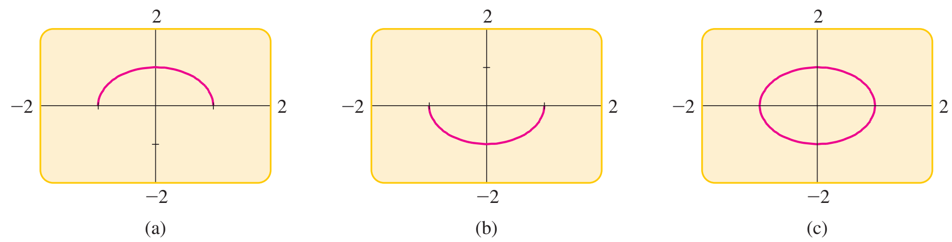


Figure 4 Graphing the equation $x^2 + y^2 = 1$ ■

The graph in Figure 4(c) looks somewhat flattened. Most graphing calculators allow you to set the scales on the axes so that circles really look like circles. On the TI-82 and TI-83, from the `ZOOM` menu, choose `ZSquare` to set the scales appropriately. (On the TI-86 the command is `Zsq`.)

Solving Equations Graphically

In Section 1.5 we learned how to solve equations. To solve an equation like

$$3x - 5 = 0$$

we used the **algebraic method**. This means we used the rules of algebra to isolate x on one side of the equation. We view x as an *unknown* and we use the rules of algebra to hunt it down. Here are the steps in the solution:

$$3x - 5 = 0$$

$$3x = 5 \qquad \text{Add 5}$$

$$x = \tfrac{5}{3} \qquad \text{Divide by 3}$$

So the solution is $x = \frac{5}{3}$.

We can also solve this equation by the **graphical method**. In this method we view x as a *variable* and sketch the graph of the equation

$$y = 3x - 5$$

Different values for x give different values for y. Our goal is to find the value of x for which $y = 0$. From the graph in Figure 5 we see that $y = 0$ when $x \approx 1.7$. Thus, the solution is $x \approx 1.7$. Note that from the graph we obtain an approximate solution.

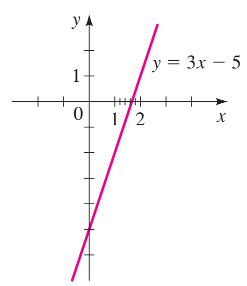


Figure 5

"Algebra is a merry science," Uncle Jakob would say. "We go hunting for a little animal whose name we don't know, so we call it *x*. When we bag our game we pounce on it and give it its right name."

ALBERT EINSTEIN

We summarize these methods in the following box.

IN-CLASS MATERIALS

If graphing calculators are going to be an important part of the course, do not hurry the concept of finding appropriate windows. Let the students play with functions such as $\sin x$, $\ln x$, $\cos^{-1}x$, and so forth. At this stage, don't ask students to worry too much about what they mean; let them play. Let them develop the idea of the variety of functions their calculators contain, and the idea that each function gives rise to its own special curve. In the unfortunate event that your students can't "play," perhaps ask them which functions are bounded or unbounded, which are symmetric about the y-axis, which are symmetric about the origin, and so on.

Solving an Equation

Algebraic method

Use the rules of algebra to isolate the unknown x on one side of the equation.

Example: $2x = 6 - x$

$3x = 6$ Add x

$x = 2$ Divide by 3

The solution is $x = 2$.

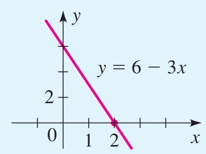
Graphical method

Move all terms to one side and set equal to y. Sketch the graph to find the value of x where $y = 0$.

Example: $2x = 6 - x$

$0 = 6 - 3x$

Set $y = 6 - 3x$ and graph.



From the graph the solution is $x \approx 2$.

The Discovery Project on page 283 describes a numerical method for solving equations.

The advantage of the algebraic method is that it gives exact answers. Also, the process of unraveling the equation to arrive at the answer helps us understand the algebraic structure of the equation. On the other hand, for many equations it is difficult or impossible to isolate x.

The graphical method gives a numerical approximation to the answer. This is an advantage when a numerical answer is desired. (For example, an engineer might find an answer expressed as $x \approx 2.6$ more immediately useful than $x = \sqrt{7}$.) Also, graphing an equation helps us visualize how the solution is related to other values of the variable.

Example 4 Solving a Quadratic Equation Algebraically and Graphically

ALTERNATE EXAMPLE 4
Solve the equation $x^2 - 14x + 49 = 0$.

ANSWER
7

Solve the quadratic equations algebraically and graphically.

(a) $x^2 - 4x + 2 = 0$ (b) $x^2 - 4x + 4 = 0$ (c) $x^2 - 4x + 6 = 0$

Solution 1: Algebraic

We use the quadratic formula to solve each equation.

The quadratic formula is discussed on page 49.

(a) $x = \dfrac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 1 \cdot 2}}{2} = \dfrac{4 \pm \sqrt{8}}{2} = 2 \pm \sqrt{2}$

There are two solutions, $x = 2 + \sqrt{2}$ and $x = 2 - \sqrt{2}$.

(b) $x = \dfrac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 1 \cdot 4}}{2} = \dfrac{4 \pm \sqrt{0}}{2} = 2$

There is just one solution, $x = 2$.

(c) $x = \dfrac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 1 \cdot 6}}{2} = \dfrac{4 \pm \sqrt{-8}}{2}$

There is no real solution.

IN-CLASS MATERIALS

Have students determine the intersection points of $y = x^2 - 2$ and $y = -x^2 + 5x + 1$ both graphically and algebraically.

Answers

$x = -\frac{1}{2}, x = 3$

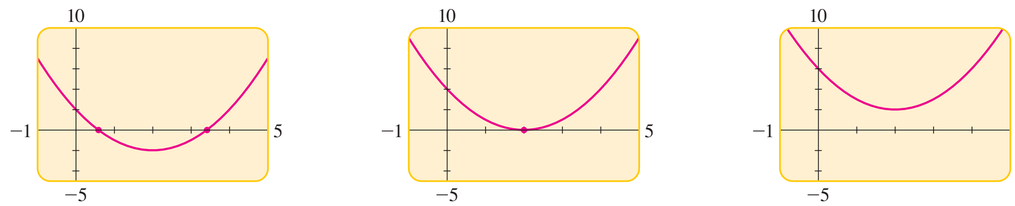
Solution 2: Graphical

We graph the equations $y = x^2 - 4x + 2$, $y = x^2 - 4x + 4$, and $y = x^2 - 4x + 6$ in Figure 6. By determining the x-intercepts of the graphs, we find the following solutions.

(a) $x \approx 0.6$ and $x \approx 3.4$

(b) $x = 2$

(c) There is no x-intercept, so the equation has no solution.



(a) $y = x^2 - 4x + 2$ (b) $y = x^2 - 4x + 4$ (c) $y = x^2 - 4x + 6$

Figure 6

■

The graphs in Figure 6 show visually why a quadratic equation may have two solutions, one solution, or no real solution. We proved this fact algebraically in Section 1.5 when we studied the discriminant.

Example 5 Another Graphical Method

Solve the equation algebraically and graphically: $5 - 3x = 8x - 20$

Solution 1: Algebraic

$$5 - 3x = 8x - 20$$
$$-3x = 8x - 25 \quad \text{Subtract 5}$$
$$-11x = -25 \quad \text{Subtract } 8x$$
$$x = \frac{-25}{-11} = 2\tfrac{3}{11} \quad \text{Divide by } -11 \text{ and simplify}$$

Solution 2: Graphical

We could move all terms to one side of the equal sign, set the result equal to y, and graph the resulting equation. But to avoid all this algebra, we graph two equations instead:

$$y_1 = 5 - 3x \quad \text{and} \quad y_2 = 8x - 20$$

The solution of the original equation will be the value of x that makes y_1 equal to y_2; that is, the solution is the x-coordinate of the intersection point of the two graphs. Using the TRACE feature or the `intersect` command on a graphing calculator, we see from Figure 7 that the solution is $x \approx 2.27$. ■

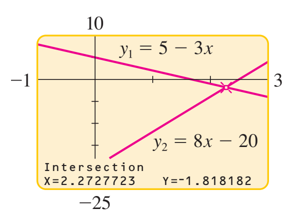


Figure 7

In the next example we use the graphical method to solve an equation that is extremely difficult to solve algebraically.

ALTERNATE EXAMPLE 5

Solve the equation algebraically and graphically:
$5 - 4x = 7x - 7$

ANSWER

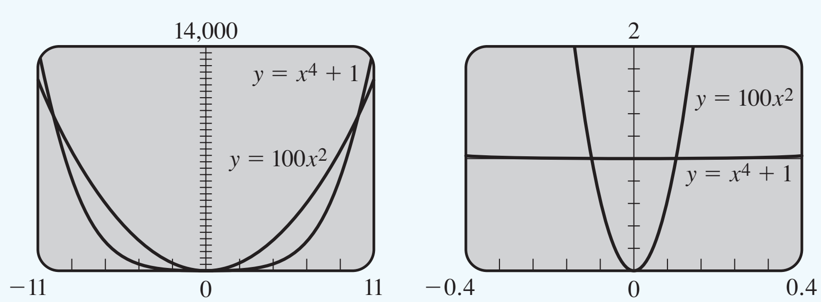
1.1

IN-CLASS MATERIALS

Have students compare the values of $100x^2$ and $x^4 + 1$ for $x = 0$, $x = \frac{1}{2}$, $x = 1$, and $x = 2$. Then ask the question: For what values of x is $100x^2 > x^4 + 1$? See if they can use their calculators to approximate the answer.

Answer

$100x^2 > x^4 + 1$ for $x \in [-10, -0.1] \cup [0.1, 10]$, approximately. Notice that it is possible to find the right windows by trial and error, but sometimes plugging a number or two into the functions can give a clue as to the best y-range.



Example 6 Solving an Equation in an Interval

Solve the equation

$$x^3 - 6x^2 + 9x = \sqrt{x}$$

in the interval $[1, 6]$.

Solution We are asked to find all solutions x that satisfy $1 \le x \le 6$, so we will graph the equation in a viewing rectangle for which the x-values are restricted to this interval.

$$x^3 - 6x^2 + 9x = \sqrt{x}$$

$$x^3 - 6x^2 + 9x - \sqrt{x} = 0 \qquad \text{Subtract } \sqrt{x}$$

We can also use the **zero** command to find the solutions, as shown in Figures 8(a) and 8(b).

Figure 8 shows the graph of the equation $y = x^3 - 6x^2 + 9x - \sqrt{x}$ in the viewing rectangle $[1, 6]$ by $[-5, 5]$. There are two x-intercepts in this viewing rectangle; zooming in we see that the solutions are $x \approx 2.18$ and $x \approx 3.72$.

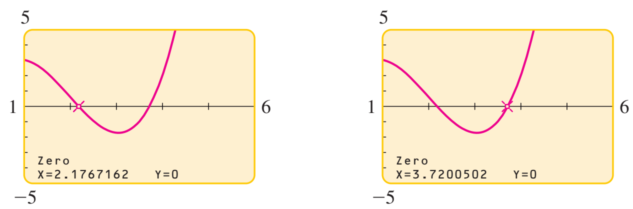


Figure 8 (a) (b)

The equation in Example 6 actually has four solutions. You are asked to find the other two in Exercise 57.

ALTERNATE EXAMPLE 6
Solve the equation in the interval [6, 9]:
$x^3 - 14x^2 + 49x = \sqrt{x}$

ANSWER
6.4, 7.6

Example 7 Intensity of Light

Two light sources are 10 m apart. One is three times as intense as the other. The light intensity L (in lux) at a point x meters from the weaker source is given by

$$L = \frac{10}{x^2} + \frac{30}{(10 - x)^2}$$

(See Figure 9.) Find the points at which the light intensity is 4 lux.

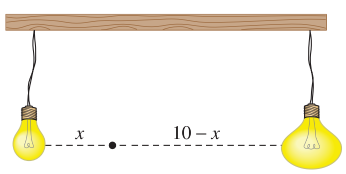


Figure 9

Solution We need to solve the equation

$$4 = \frac{10}{x^2} + \frac{30}{(10 - x)^2}$$

ALTERNATE EXAMPLE 7
Two light sources are 25 m apart. One is twice as intense as the other. The light intensity L at a point x meters from the weaker source is given by
$$L = \frac{10}{x^2} + \frac{20}{(25 - x)^2}$$
Find the points between the two lights at which the light intensity is 4 lux.

ANSWER
1.59, 22.76 m from the weaker source

EXAMPLES
Graphs that some calculators get wrong:

1. $y = \sqrt{x} - 2\sqrt{x - 4}$ Some calculators do not give a graph with the correct domain.
2. $y = \sin x + \frac{1}{100}\cos 100x$ The "standard" viewing rectangle misses the bumps that a rectangle of $[-0.1, 0.1] \times [-0.1, 0.1]$ will catch.
3. $y = x^{1/3}$ Some calculators dislike taking negative numbers to rational powers, even when it is possible.
4. $y = \dfrac{\sin(x - \sqrt{2})}{x - \sqrt{2}}$ Calculators are not good at graphing functions with holes.

The graphs of

$$y_1 = 4 \qquad \text{and} \qquad y_2 = \frac{10}{x^2} + \frac{30}{(10 - x)^2}$$

are shown in Figure 10. Zooming in (or using the `intersect` command) we find two solutions, $x \approx 1.67431$ and $x \approx 7.1927193$. So the light intensity is 4 lux at the points that are 1.67 m and 7.19 m from the weaker source. ■

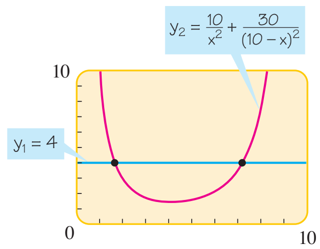


Figure 10

Solving Inequalities Graphically

Inequalities can be solved graphically. To describe the method we solve

$$x^2 - 5x + 6 \le 0$$

This inequality was solved algebraically in Section 1.7, Example 3. To solve the inequality graphically, we draw the graph of

$$y = x^2 - 5x + 6$$

Our goal is to find those values of x for which $y \le 0$. These are simply the x-values for which the graph lies below the x-axis. From Figure 11 we see that the solution of the inequality is the interval $[2, 3]$.

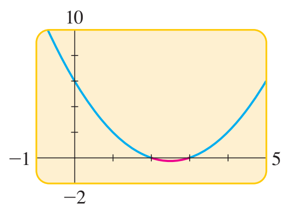


Figure 11
$x^2 - 5x + 6 \le 0$

ALTERNATE EXAMPLE 8
Solve the inequality
$x^4 - 2x^2 \le x + 10$.

ANSWER
$[-2, 2.15]$

Example 8 Solving an Inequality Graphically

Solve the inequality $3.7x^2 + 1.3x - 1.9 \le 2.0 - 1.4x$.

Solution We graph the equations

$$y_1 = 3.7x^2 + 1.3x - 1.9 \qquad \text{and} \qquad y_2 = 2.0 - 1.4x$$

in the same viewing rectangle in Figure 12. We are interested in those values of x for which $y_1 \le y_2$; these are points for which the graph of y_2 lies on or above the graph of y_1. To determine the appropriate interval, we look for the x-coordinates of points where the graphs intersect. We conclude that the solution is (approximately) the interval $[-1.45, 0.72]$. ■

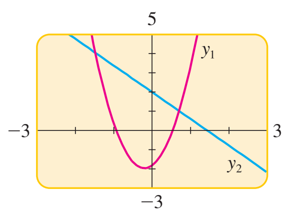


Figure 12
$y_1 = 3.7x^2 + 1.3x - 1.9$
$y_2 = 2.0 - 1.4x$

ALTERNATE EXAMPLE 9
Solve the inequality
$x^3 - 12x^2 \ge -122$ to the nearest integer.

ANSWER
$[-3, 4] \cup [11, \infty)$

Example 9 Solving an Inequality Graphically

Solve the inequality $x^3 - 5x^2 \ge -8$.

Solution We write the inequality as

$$x^3 - 5x^2 + 8 \ge 0$$

and then graph the equation

$$y = x^3 - 5x^2 + 8$$

in the viewing rectangle $[-6, 6]$ by $[-15, 15]$, as shown in Figure 13. The solution of the inequality consists of those intervals on which the graph lies on or above the x-axis. By moving the cursor to the x-intercepts we find that, correct to one decimal place, the solution is $[-1.1, 1.5] \cup [4.6, \infty)$. ■

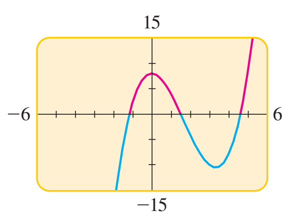


Figure 13
$x^3 - 5x^2 + 8 \ge 0$

1.9 Exercises

1–6 ■ Use a graphing calculator or computer to decide which viewing rectangle (a)–(d) produces the most appropriate graph of the equation.

1. $y = x^4 + 2$

(a) $[-2, 2]$ by $[-2, 2]$
(b) $[0, 4]$ by $[0, 4]$
(c) $[-8, 8]$ by $[-4, 40]$
(d) $[-40, 40]$ by $[-80, 800]$

2. $y = x^2 + 7x + 6$

(a) $[-5, 5]$ by $[-5, 5]$
(b) $[0, 10]$ by $[-20, 100]$
(c) $[-15, 8]$ by $[-20, 100]$
(d) $[-10, 3]$ by $[-100, 20]$

3. $y = 100 - x^2$

(a) $[-4, 4]$ by $[-4, 4]$
(b) $[-10, 10]$ by $[-10, 10]$
(c) $[-15, 15]$ by $[-30, 110]$
(d) $[-4, 4]$ by $[-30, 110]$

4. $y = 2x^2 - 1000$

(a) $[-10, 10]$ by $[-10, 10]$
(b) $[-10, 10]$ by $[-100, 100]$
(c) $[-10, 10]$ by $[-1000, 1000]$
(d) $[-25, 25]$ by $[-1200, 200]$

5. $y = 10 + 25x - x^3$

(a) $[-4, 4]$ by $[-4, 4]$
(b) $[-10, 10]$ by $[-10, 10]$
(c) $[-20, 20]$ by $[-100, 100]$
(d) $[-100, 100]$ by $[-200, 200]$

6. $y = \sqrt{8x - x^2}$

(a) $[-4, 4]$ by $[-4, 4]$
(b) $[-5, 5]$ by $[0, 100]$
(c) $[-10, 10]$ by $[-10, 40]$
(d) $[-2, 10]$ by $[-2, 6]$

7–18 ■ Determine an appropriate viewing rectangle for the equation and use it to draw the graph.

7. $y = 100x^2$
8. $y = -100x^2$
9. $y = 4 + 6x - x^2$
10. $y = 0.3x^2 + 1.7x - 3$
11. $y = \sqrt[4]{256 - x^2}$
12. $y = \sqrt{12x - 17}$
13. $y = 0.01x^3 - x^2 + 5$
14. $y = x(x + 6)(x - 9)$
15. $y = x^4 - 4x^3$
16. $y = \dfrac{x}{x^2 + 25}$
17. $y = 1 + |x - 1|$
18. $y = 2x - |x^2 - 5|$

19. Graph the circle $x^2 + y^2 = 9$ by solving for y and graphing two equations as in Example 3.

20. Graph the circle $(y - 1)^2 + x^2 = 1$ by solving for y and graphing two equations as in Example 3.

21. Graph the equation $4x^2 + 2y^2 = 1$ by solving for y and graphing two equations corresponding to the negative and positive square roots. (This graph is called an *ellipse*.)

22. Graph the equation $y^2 - 9x^2 = 1$ by solving for y and graphing the two equations corresponding to the positive and negative square roots. (This graph is called a *hyperbola*.)

23–26 ■ Do the graphs intersect in the given viewing rectangle? If they do, how many points of intersection are there?

23. $y = -3x^2 + 6x - \frac{1}{2}$, $y = \sqrt{7 - \frac{7}{12}x^2}$; $[-4, 4]$ by $[-1, 3]$

24. $y = \sqrt{49 - x^2}$, $y = \frac{1}{5}(41 - 3x)$; $[-8, 8]$ by $[-1, 8]$

25. $y = 6 - 4x - x^2$, $y = 3x + 18$; $[-6, 2]$ by $[-5, 20]$

26. $y = x^3 - 4x$, $y = x + 5$; $[-4, 4]$ by $[-15, 15]$

27–36 ■ Solve the equation both algebraically and graphically.

27. $x - 4 = 5x + 12$
28. $\frac{1}{2}x - 3 = 6 + 2x$
29. $\dfrac{2}{x} + \dfrac{1}{2x} = 7$
30. $\dfrac{4}{x + 2} - \dfrac{6}{2x} = \dfrac{5}{2x + 4}$
31. $x^2 - 32 = 0$
32. $x^3 + 16 = 0$
33. $16x^4 = 625$
34. $2x^5 - 243 = 0$
35. $(x - 5)^4 - 80 = 0$
36. $6(x + 2)^5 = 64$

37–44 ■ Solve the equation graphically in the given interval. State each answer correct to two decimals.

37. $x^2 - 7x + 12 = 0$; $[0, 6]$

38. $x^2 - 0.75x + 0.125 = 0$; $[-2, 2]$

39. $x^3 - 6x^2 + 11x - 6 = 0$; $[-1, 4]$

40. $16x^3 + 16x^2 = x + 1$; $[-2, 2]$

41. $x - \sqrt{x + 1} = 0$; $[-1, 5]$

42. $1 + \sqrt{x} = \sqrt{1 + x^2}$; $[-1, 5]$

43. $x^{1/3} - x = 0$; $[-3, 3]$

44. $x^{1/2} + x^{1/3} - x = 0$; $[-1, 5]$

45–48 ■ Find all real solutions of the equation, correct to two decimals.

45. $x^3 - 2x^2 - x - 1 = 0$
46. $x^4 - 8x^2 + 2 = 0$
47. $x(x - 1)(x + 2) = \frac{1}{6}x$
48. $x^4 = 16 - x^3$

49–56 ■ Find the solutions of the inequality by drawing appropriate graphs. State each answer correct to two decimals.

49. $x^2 - 3x - 10 \le 0$

50. $0.5x^2 + 0.875x \le 0.25$

51. $x^3 + 11x \le 6x^2 + 6$

52. $16x^3 + 24x^2 > -9x - 1$

53. $x^{1/3} < x$

54. $\sqrt{0.5x^2 + 1} \le 2|x|$

55. $(x + 1)^2 < (x - 1)^2$

56. $(x + 1)^2 \le x^3$

57. In Example 6 we found two solutions of the equation $x^3 - 6x^2 + 9x = \sqrt{x}$, the solutions that lie between 1 and 6. Find two more solutions, correct to two decimals.

Applications

58. Estimating Profit An appliance manufacturer estimates that the profit y (in dollars) generated by producing x cooktops per month is given by the equation

$$y = 10x + 0.5x^2 - 0.001x^3 - 5000$$

where $0 \le x \le 450$.

(a) Graph the equation.

(b) How many cooktops must be produced to begin generating a profit?

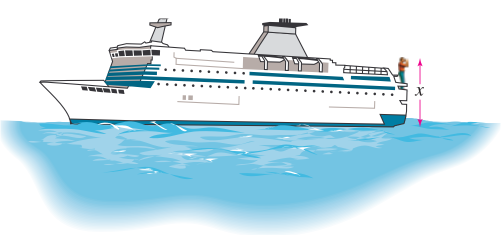
(c) For what range of values of x is the company's profit greater than $15,000?

59. How Far Can You See? If you stand on a ship in a calm sea, then your height x (in ft) above sea level is related to the farthest distance y (in mi) that you can see by the equation

$$y = \sqrt{1.5x + \left(\frac{x}{5280}\right)^2}$$

(a) Graph the equation for $0 \le x \le 100$.

(b) How high up do you have to be to be able to see 10 mi?



Discovery • Discussion

60. Equation Notation on Graphing Calculators When you enter the following equations into your calculator, how does what you see on the screen differ from the usual way of writing the equations? (Check your user's manual if you're not sure.)

(a) $y = |x|$

(b) $y = \sqrt[5]{x}$

(c) $y = \dfrac{x}{x - 1}$

(d) $y = x^3 + \sqrt[3]{x + 2}$

61. Enter Equations Carefully A student wishes to graph the equations

$$y = x^{1/3} \qquad \text{and} \qquad y = \frac{x}{x + 4}$$

on the same screen, so he enters the following information into his calculator:

$$\texttt{Y}_1 = \texttt{X^1/3} \qquad \texttt{Y}_2 = \texttt{X/X + 4}$$

The calculator graphs two lines instead of the equations he wanted. What went wrong?

62. Algebraic and Graphical Solution Methods Write a short essay comparing the algebraic and graphical methods for solving equations. Make up your own examples to illustrate the advantages and disadvantages of each method.

63. How Many Solutions? This exercise deals with the family of equations

$$x^3 - 3x = k$$

(a) Draw the graphs of

$$y_1 = x^3 - 3x \qquad \text{and} \qquad y_2 = k$$

in the same viewing rectangle, in the cases $k = -4$, -2, 0, 2, and 4. How many solutions of the equation $x^3 - 3x = k$ are there in each case? Find the solutions correct to two decimals.

(b) For what ranges of values of k does the equation have one solution? two solutions? three solutions?

1.10 Lines

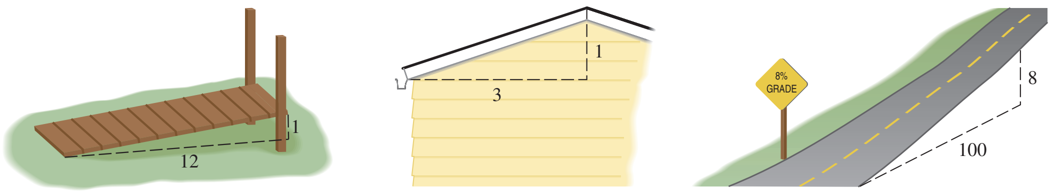
In this section we find equations for straight lines lying in a coordinate plane. The equations will depend on how the line is inclined, so we begin by discussing the concept of slope.

The Slope of a Line

We first need a way to measure the "steepness" of a line, or how quickly it rises (or falls) as we move from left to right. We define *run* to be the distance we move to the right and *rise* to be the corresponding distance that the line rises (or falls). The *slope* of a line is the ratio of rise to run:

$$\text{slope} = \frac{\text{rise}}{\text{run}}$$

Figure 1 shows situations where slope is important. Carpenters use the term *pitch* for the slope of a roof or a staircase; the term *grade* is used for the slope of a road.



Slope of a ramp
Slope $= \frac{1}{12}$

Pitch of a roof
Slope $= \frac{1}{3}$

Grade of a road
Slope $= \frac{8}{100}$

Figure 1

If a line lies in a coordinate plane, then the **run** is the change in the x-coordinate and the **rise** is the corresponding change in the y-coordinate between any two points on the line (see Figure 2). This gives us the following definition of slope.

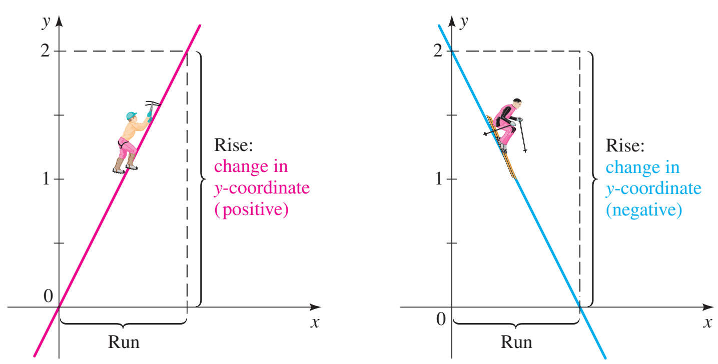


Figure 2

SUGGESTED TIME AND EMPHASIS

1 class.
Essential material.

POINTS TO STRESS

1. Computing equations of lines in their various forms, given information such as a point and a slope, or two points.
2. The concepts of parallel, perpendicular, slope, vertical, horizontal, and rate of change as reflected in equations of lines.
3. Graphing lines from their equations.

René Descartes (1596–1650) was born in the town of La Haye in southern France. From an early age Descartes liked mathematics because of "the certainty of its results and the clarity of its reasoning." He believed that in order to arrive at truth, one must begin by doubting everything, including one's own existence; this led him to formulate perhaps the most well-known sentence in all of philosophy: "I think, therefore I am." In his book *Discourse on Method* he described what is now called the Cartesian plane. This idea of combining algebra and geometry enabled mathematicians for the first time to "see" the equations they were studying. The philosopher John Stuart Mill called this invention "the greatest single step ever made in the progress of the exact sciences." Descartes liked to get up late and spend the morning in bed thinking and writing. He invented the coordinate plane while lying in bed watching a fly crawl on the ceiling, reasoning that he could describe the exact location of the fly by knowing its distance from two perpendicular walls. In 1649 Descartes became the tutor of Queen Christina of Sweden. She liked her lessons at 5 o'clock in the morning when, she said, her mind was sharpest. However, the change from his usual habits and the ice-cold library where they studied proved too much for him. In February 1650, after just two months of this, he caught pneumonia and died.

Slope of a Line

The **slope** m of a nonvertical line that passes through the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$$

The slope of a vertical line is not defined.

The slope is independent of which two points are chosen on the line. We can see that this is true from the similar triangles in Figure 3:

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_2' - y_1'}{x_2' - x_1'}$$

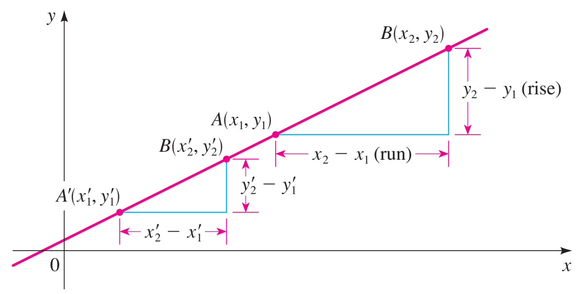


Figure 3

Figure 4 shows several lines labeled with their slopes. Notice that lines with positive slope slant upward to the right, whereas lines with negative slope slant downward to the right. The steepest lines are those for which the absolute value of the slope is the largest; a horizontal line has slope zero.

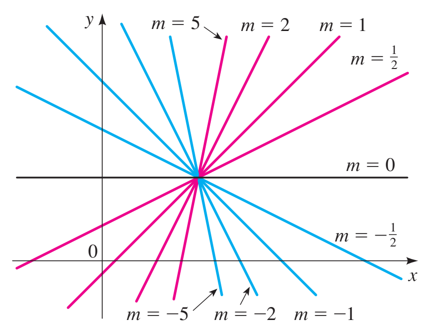
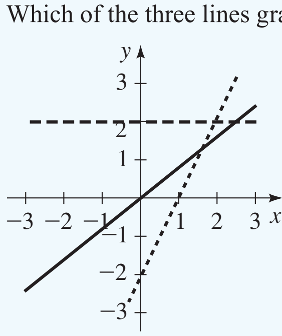


Figure 4
Lines with various slopes

SAMPLE QUESTION
Text Question
Which of the three lines graphed below has the largest slope?



Answer
The dotted line that goes through (0, −2)

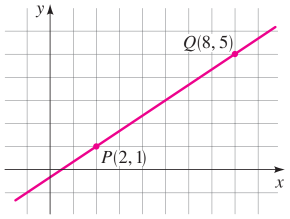


Figure 5

Example 1 Finding the Slope of a Line through Two Points

Find the slope of the line that passes through the points $P(2, 1)$ and $Q(8, 5)$.

Solution Since any two different points determine a line, only one line passes through these two points. From the definition, the slope is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - 1}{8 - 2} = \frac{4}{6} = \frac{2}{3}$$

This says that for every 3 units we move to the right, the line rises 2 units. The line is drawn in Figure 5. ■

ALTERNATE EXAMPLE 1
Find the slope of the line that passes through the points $P(2, 1)$ and $Q(23, 25)$.

ANSWER
$\frac{8}{7}$

Equations of Lines

Now let's find the equation of the line that passes through a given point $P(x_1, y_1)$ and has slope m. A point $P(x, y)$ with $x \neq x_1$ lies on this line if and only if the slope of the line through P_1 and P is equal to m (see Figure 6), that is,

$$\frac{y - y_1}{x - x_1} = m$$

This equation can be rewritten in the form $y - y_1 = m(x - x_1)$; note that the equation is also satisfied when $x = x_1$ and $y = y_1$. Therefore, it is an equation of the given line.

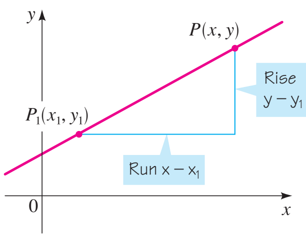


Figure 6

Point-Slope Form of the Equation of a Line

An equation of the line that passes through the point (x_1, y_1) and has slope m is

$$y - y_1 = m(x - x_1)$$

Example 2 Finding the Equation of a Line with Given Point and Slope

(a) Find an equation of the line through $(1, -3)$ with slope $-\frac{1}{2}$.

(b) Sketch the line.

Solution

(a) Using the point-slope form with $m = -\frac{1}{2}$, $x_1 = 1$, and $y_1 = -3$, we obtain an equation of the line as

$$y + 3 = -\tfrac{1}{2}(x - 1) \quad \text{From point-slope equation}$$
$$2y + 6 = -x + 1 \quad \text{Multiply by 2}$$
$$x + 2y + 5 = 0 \quad \text{Rearrange}$$

(b) The fact that the slope is $-\frac{1}{2}$ tells us that when we move to the right 2 units, the line drops 1 unit. This enables us to sketch the line in Figure 7. ■

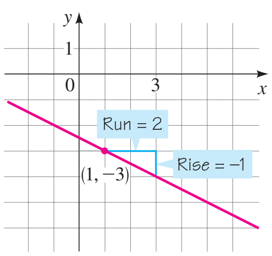


Figure 7

ALTERNATE EXAMPLE 2
Find an equation of the line through $(1, -3)$ with slope $-\frac{3}{4}$.

ANSWER
$3x + 4y + 9 = 0$

DRILL QUESTION

Find an equation of the line through the points $(2, 4)$ and $(3, -1)$.

Answer

$y = -5x + 14$

ALTERNATE EXAMPLE 3
Find an equation of the line through the points $(-4, 3)$ and $(4, -3)$.

ANSWER
$3x + 4y + 0 = 0$

Example 3 Finding the Equation of a Line through Two Given Points

Find an equation of the line through the points $(-1, 2)$ and $(3, -4)$.

Solution The slope of the line is

$$m = \frac{-4 - 2}{3 - (-1)} = -\frac{6}{4} = -\frac{3}{2}$$

We can use *either* point, $(-1, 2)$ *or* $(3, -4)$, in the point-slope equation. We will end up with the same final answer.

Using the point-slope form with $x_1 = -1$ and $y_1 = 2$, we obtain

$$y - 2 = -\tfrac{3}{2}(x + 1) \qquad \text{From point-slope equation}$$
$$2y - 4 = -3x - 3 \qquad \text{Multiply by 2}$$
$$3x + 2y - 1 = 0 \qquad \text{Rearrange}$$

■

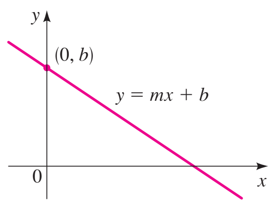


Figure 8

Suppose a nonvertical line has slope m and y-intercept b (see Figure 8). This means the line intersects the y-axis at the point $(0, b)$, so the point-slope form of the equation of the line, with $x = 0$ and $y = b$, becomes

$$y - b = m(x - 0)$$

This simplifies to $y = mx + b$, which is called the **slope-intercept form** of the equation of a line.

Slope-Intercept Form of the Equation of a Line

An equation of the line that has slope m and y-intercept b is

$$y = mx + b$$

ALTERNATE EXAMPLE 4
Find the slope and y-intercept of the line $3y - 4x = 7$.

ANSWER
$\frac{4}{3}, \frac{7}{3}$

Example 4 Lines in Slope-Intercept Form

(a) Find the equation of the line with slope 3 and y-intercept -2.

(b) Find the slope and y-intercept of the line $3y - 2x = 1$.

Solution

(a) Since $m = 3$ and $b = -2$, from the slope-intercept form of the equation of a line we get

$$y = 3x - 2$$

(b) We first write the equation in the form $y = mx + b$:

$$3y - 2x = 1$$
$$3y = 2x + 1 \qquad \text{Add } 2x$$
$$y = \tfrac{2}{3}x + \tfrac{1}{3} \qquad \text{Divide by 3}$$

Slope | y-intercept
$y = \frac{2}{3}x + \frac{1}{3}$

From the slope-intercept form of the equation of a line, we see that the slope is $m = \frac{2}{3}$ and the y-intercept is $b = \frac{1}{3}$. ■

IN-CLASS MATERIALS

One good way to get students used to slope is to have them do quick estimates. Stand in front of the class, raise your arm at various angles, and have them write down (or call out) estimates. Students should be able to tell a positive from a negative slope, and if a slope is closer to $\frac{1}{3}$ than it is to 3. Also note that if the x- and y-scales on a graph are different, the appearance of the slope can be misleading.

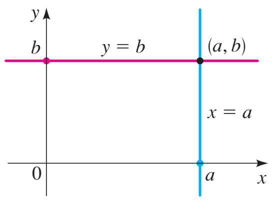


Figure 9

If a line is horizontal, its slope is $m = 0$, so its equation is $y = b$, where b is the y-intercept (see Figure 9). A vertical line does not have a slope, but we can write its equation as $x = a$, where a is the x-intercept, because the x-coordinate of every point on the line is a.

Vertical and Horizontal Lines

An equation of the vertical line through (a, b) is $x = a$.
An equation of the horizontal line through (a, b) is $y = b$.

Example 5 Vertical and Horizontal Lines

(a) The graph of the equation $x = 3$ is a vertical line with x-intercept 3.
(b) The graph of the equation $y = -2$ is a horizontal line with y-intercept -2.

The lines are graphed in Figure 10. ■

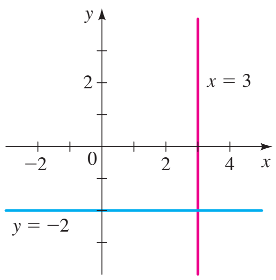


Figure 10

A **linear equation** is an equation of the form

$$Ax + By + C = 0$$

where A, B, and C are constants and A and B are not both 0. The equation of a line is a linear equation:

- A nonvertical line has the equation $y = mx + b$ or $-mx + y - b = 0$, which is a linear equation with $A = -m$, $B = 1$, and $C = -b$.
- A vertical line has the equation $x = a$ or $x - a = 0$, which is a linear equation with $A = 1$, $B = 0$, and $C = -a$.

Conversely, the graph of a linear equation is a line:

- If $B \neq 0$, the equation becomes

$$y = -\frac{A}{B}x - \frac{C}{B}$$

and this is the slope-intercept form of the equation of a line (with $m = -A/B$ and $b = -C/B$).

- If $B = 0$, the equation becomes

$$Ax + C = 0$$

or $x = -C/A$, which represents a vertical line.

We have proved the following.

General Equation of a Line

The graph of every **linear equation**

$$Ax + By + C = 0 \qquad (A, B \text{ not both zero})$$

is a line. Conversely, every line is the graph of a linear equation.

ALTERNATE EXAMPLE 5
Describe the graphs of the following equations:
(a) $x = -2$
(b) $y = 3$
(c) $y = 0$
(d) $x = 0$

ANSWERS
(a) A vertical line with x intercept -2
(b) A horizontal line with y intercept 3
(c) The x-axis
(d) The y-axis

IN-CLASS MATERIALS

A classic example is to have students come up with conversion formulas from Fahrenheit to centigrade and vice versa, using the fact that in centigrade measurement, water freezes at 0° and boils at 100° (−32°F and 212°F respectively). Other conversion formulas can be found, using facts such as the following: 3 ft ≈ 0.914 m, 1 L ≈ 0.264 gal (US), a 250-calorie snack is equivalent to 0.992 BTU.

ALTERNATE EXAMPLE 6
Find the coordinates of the x-intercept and the y-intercept of the line $4x - 3y - 12 = 0$.

ANSWER
(3, 0), (0, – 4)

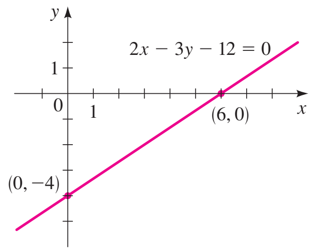


Figure 11

Example 6 Graphing a Linear Equation

Sketch the graph of the equation $2x - 3y - 12 = 0$.

Solution 1 Since the equation is linear, its graph is a line. To draw the graph, it is enough to find any two points on the line. The intercepts are the easiest points to find.

x-intercept: Substitute $y = 0$, to get $2x - 12 = 0$, so $x = 6$
y-intercept: Substitute $x = 0$, to get $-3y - 12 = 0$, so $y = -4$

With these points we can sketch the graph in Figure 11.

Solution 2 We write the equation in slope-intercept form:

$$2x - 3y - 12 = 0$$
$$2x - 3y = 12 \qquad \text{Add 12}$$
$$-3y = -2x + 12 \qquad \text{Subtract } 2x$$
$$y = \tfrac{2}{3}x - 4 \qquad \text{Divide by } -3$$

This equation is in the form $y = mx + b$, so the slope is $m = \frac{2}{3}$ and the y-intercept is $b = -4$. To sketch the graph, we plot the y-intercept, and then move 3 units to the right and 2 units up as shown in Figure 12. ■

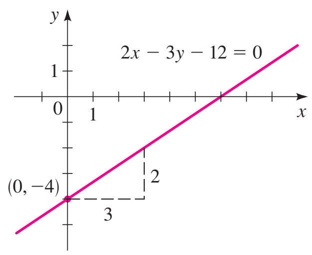


Figure 12

Parallel and Perpendicular Lines

Since slope measures the steepness of a line, it seems reasonable that parallel lines should have the same slope. In fact, we can prove this.

Parallel Lines

Two nonvertical lines are parallel if and only if they have the same slope.

■ **Proof** Let the lines l_1 and l_2 in Figure 13 have slopes m_1 and m_2. If the lines are parallel, then the right triangles ABC and DEF are similar, so

$$m_1 = \frac{d(B, C)}{d(A, C)} = \frac{d(E, F)}{d(D, F)} = m_2$$

Conversely, if the slopes are equal, then the triangles will be similar, so $\angle BAC = \angle EDF$ and the lines are parallel. ■

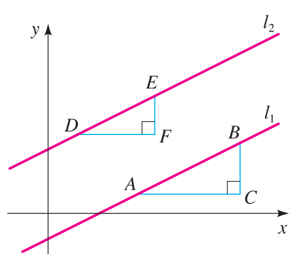


Figure 13

ALTERNATE EXAMPLE 7
Find an equation of the line through the point (4, 2) that is parallel to the line $15x + 12y + 11 = 0$.

ANSWER
$5x + 4y - 28 = 0$

Example 7 Finding the Equation of a Line Parallel to a Given Line

Find an equation of the line through the point (5, 2) that is parallel to the line $4x + 6y + 5 = 0$.

Solution First we write the equation of the given line in slope-intercept form.

$$4x + 6y + 5 = 0$$
$$6y = -4x - 5 \qquad \text{Subtract } 4x + 5$$
$$y = -\tfrac{2}{3}x - \tfrac{5}{6} \qquad \text{Divide by 6}$$

IN-CLASS MATERIALS

Go over an interpolation with students. For example, compare your school's budget this year with that of ten years ago. Find the equation of the appropriate line. Then try to predict the budget five years ago, and see if a linear model was appropriate. (Linear models will be discussed more thoroughly in the next section.) You may also touch on extrapolation, but point out that extrapolation tends to be less reliable than interpolation.

So the line has slope $m = -\frac{2}{3}$. Since the required line is parallel to the given line, it also has slope $m = -\frac{2}{3}$. From the point-slope form of the equation of a line, we get

$$y - 2 = -\tfrac{2}{3}(x - 5) \qquad \text{Slope } m = -\tfrac{2}{3},\text{ point } (5, 2)$$

$$3y - 6 = -2x + 10 \qquad \text{Multiply by 3}$$

$$2x + 3y - 16 = 0 \qquad \text{Rearrange}$$

Thus, the equation of the required line is $2x + 3y - 16 = 0$. ■

The condition for perpendicular lines is not as obvious as that for parallel lines.

Perpendicular Lines

Two lines with slopes m_1 and m_2 are perpendicular if and only if $m_1m_2 = -1$, that is, their slopes are negative reciprocals:

$$m_2 = -\frac{1}{m_1}$$

Also, a horizontal line (slope 0) is perpendicular to a vertical line (no slope).

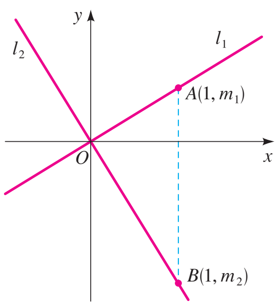


Figure 14

■ **Proof** In Figure 14 we show two lines intersecting at the origin. (If the lines intersect at some other point, we consider lines parallel to these that intersect at the origin. These lines have the same slopes as the original lines.)

If the lines l_1 and l_2 have slopes m_1 and m_2, then their equations are $y = m_1x$ and $y = m_2x$. Notice that $A(1, m_1)$ lies on l_1 and $B(1, m_2)$ lies on l_2. By the Pythagorean Theorem and its converse (see page 54), $OA \perp OB$ if and only if

$$[d(O, A)]^2 + [d(O, B)]^2 = [d(A, B)]^2$$

By the Distance Formula, this becomes

$$(1^2 + m_1^2) + (1^2 + m_2^2) = (1 - 1)^2 + (m_2 - m_1)^2$$

$$2 + m_1^2 + m_2^2 = m_2^2 - 2m_1m_2 + m_1^2$$

$$2 = -2m_1m_2$$

$$m_1m_2 = -1$$

■

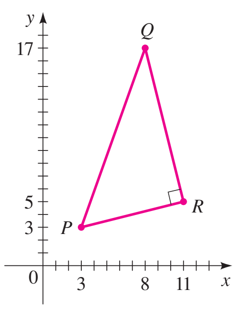


Figure 15

Example 8 Perpendicular Lines

Show that the points $P(3, 3)$, $Q(8, 17)$, and $R(11, 5)$ are the vertices of a right triangle.

Solution The slopes of the lines containing PR and QR are, respectively,

$$m_1 = \frac{5 - 3}{11 - 3} = \frac{1}{4} \qquad \text{and} \qquad m_2 = \frac{5 - 17}{11 - 8} = -4$$

Since $m_1m_2 = -1$, these lines are perpendicular and so PQR is a right triangle. It is sketched in Figure 15.

ALTERNATE EXAMPLE 8

Is it true that the points $P(2, 3)$, $Q(7, 17)$, and $R(10, 5)$ are the vertices of a right triangle.

ANSWER

Yes

IN-CLASS MATERIALS

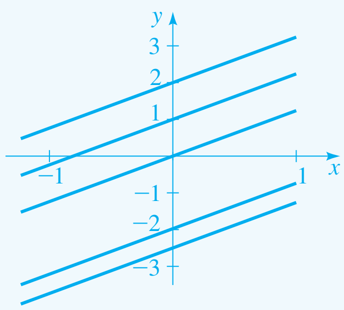
Ask students to figure out if three given points are vertices of a right triangle. They can go ahead and plot them, to get a guess going. Ask them to come up with a strategy, and reveal (or ideally elicit) the idea of writing equations of lines between each pair of points, and checking for perpendicular lines. (3, 4), (3, 12), and (6, 5) do not form a right triangle; (−2, −1), (−2, 8), and (8, −1) form a right triangle.

ALTERNATE EXAMPLE 9
Find an equation of the line that is perpendicular to the line $8x + 6y + 5 = 0$ and passes through the origin. Express your answer in slope-intercept form.

ANSWER
$y = \frac{3}{4}x$

ALTERNATE EXAMPLE 10
Use a graphing calculator to graph the family of lines $y = 6x + b$ for various values of b. What property do the lines share?

ANSWER
They all have the same slope.



Example 9 Finding an Equation of a Line Perpendicular to a Given Line

Find an equation of the line that is perpendicular to the line $4x + 6y + 5 = 0$ and passes through the origin.

Solution In Example 7 we found that the slope of the line $4x + 6y + 5 = 0$ is $-\frac{2}{3}$. Thus, the slope of a perpendicular line is the negative reciprocal, that is, $\frac{3}{2}$. Since the required line passes through $(0, 0)$, the point-slope form gives

$$y - 0 = \tfrac{3}{2}(x - 0)$$
$$y = \tfrac{3}{2}x$$ ■

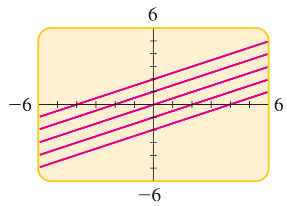


Figure 16
$y = 0.5x + b$

Example 10 Graphing a Family of Lines

Use a graphing calculator to graph the family of lines

$$y = 0.5x + b$$

for $b = -2, -1, 0, 1, 2$. What property do the lines share?

Solution The lines are graphed in Figure 16 in the viewing rectangle $[-6, 6]$ by $[-6, 6]$. The lines all have the same slope, so they are parallel. ■

Applications: Slope as Rate of Change

When a line is used to model the relationship between two quantities, the slope of the line is the **rate of change** of one quantity with respect to the other. For example, the graph in Figure 17(a) gives the amount of gas in a tank that is being filled. The slope between the indicated points is

$$m = \frac{6 \text{ gallons}}{3 \text{ minutes}} = 2 \text{ gal/min}$$

The slope is the *rate* at which the tank is being filled, 2 gallons per minute. In Figure 17(b), the tank is being drained at the *rate* of 0.03 gallon per minute, and the slope is -0.03.

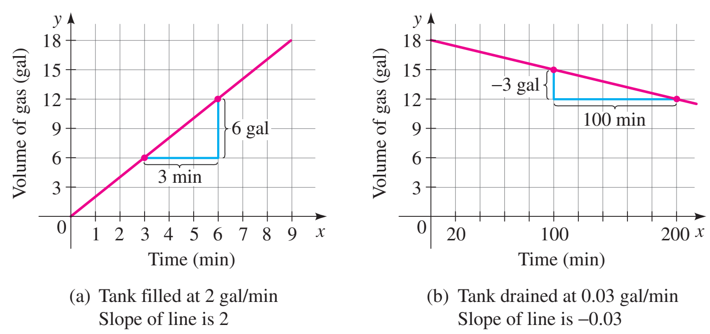


Figure 17

IN-CLASS MATERIALS

Don't neglect to give students horizontal and vertical lines to explore—they often find equations of vertical lines challenging.

The next two examples give other situations where the slope of a line is a rate of change.

Example 11 **Slope as Rate of Change**

A dam is built on a river to create a reservoir. The water level w in the reservoir is given by the equation

$$w = 4.5t + 28$$

where t is the number of years since the dam was constructed, and w is measured in feet.

(a) Sketch a graph of this equation.

(b) What do the slope and w-intercept of this graph represent?

Solution

(a) This equation is linear, so its graph is a line. Since two points determine a line, we plot two points that lie on the graph and draw a line through them.

When $t = 0$, then $w = 4.5(0) + 28 = 28$, so $(0, 28)$ is on the line.

When $t = 2$, then $w = 4.5(2) + 28 = 37$, so $(2, 37)$ is on the line.

The line determined by these points is shown in Figure 18.

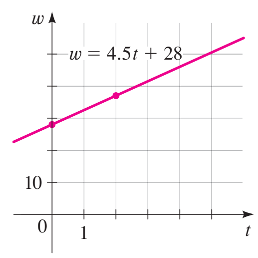
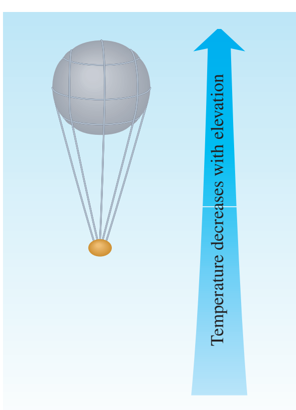


Figure 18

(b) The slope is $m = 4.5$; it represents the rate of change of water level with respect to time. This means that the water level *increases* 4.5 ft per year. The w-intercept is 28, and occurs when $t = 0$, so it represents the water level when the dam was constructed. ■

ALTERNATE EXAMPLE 11
The cost of a cup of coffee at Jag's Java is given by $c = 0.25t + 1$ where t is in years since 2005 and c is in dollars. What do the slope and c-intercept of this graph represent?

ANSWER
The slope, $m = .25$, means that the cost is going up by a quarter every year. The c-intercept, 1, means that in 2005 a cup of coffee cost one dollar.



Example 12 **Linear Relationship between Temperature and Elevation**

(a) As dry air moves upward, it expands and cools. If the ground temperature is 20°C and the temperature at a height of 1 km is 10°C, express the temperature T (in °C) in terms of the height h (in kilometers). (Assume that the relationship between T and h is linear.)

(b) Draw the graph of the linear equation. What does its slope represent?

(c) What is the temperature at a height of 2.5 km?

Solution

(a) Because we are assuming a linear relationship between T and h, the equation must be of the form

$$T = mh + b$$

ALTERNATE EXAMPLE 12
As dry air moves upward, it expands and cools. If the ground temperature is 22°C and the temperature at a height of 1 km is 10°C, express the temperature T(in °C) in terms of the height h (in kilometers). (Assume that the relationship between T and h is linear.) Also, find the temperature at a height of 1.5 km.

ANSWER
$T = -12h + 22$, 4

EXAMPLES
Equations of some standard lines:

1. Through (3, 5) and (−1, 13): $y = -2x + 11$

2. Through (6, −5) with a slope of $\frac{1}{3}$: $y = \frac{1}{3}x - 7$

where m and b are constants. When $h = 0$, we are given that $T = 20$, so

$$20 = m(0) + b$$
$$b = 20$$

Thus, we have

$$T = mh + 20$$

When $h = 1$, we have $T = 10$ and so

$$10 = m(1) + 20$$
$$m = 10 - 20 = -10$$

The required expression is

$$T = -10h + 20$$

(b) The graph is sketched in Figure 19. The slope is $m = -10°\text{C/km}$, and this represents the rate of change of temperature with respect to distance above the ground. So the temperature *decreases* 10°C per kilometer of height.

(c) At a height of $h = 2.5$ km, the temperature is

$$T = -10(2.5) + 20 = -25 + 20 = -5°\text{C}$$

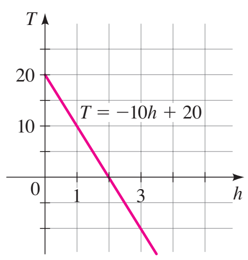


Figure 19

1.10 Exercises

1–8 ■ Find the slope of the line through P and Q.

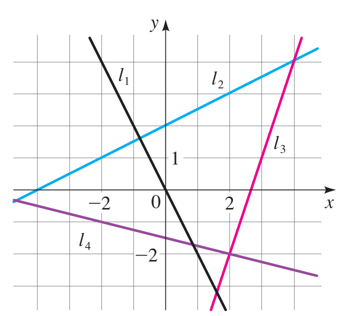
1. $P(0, 0)$, $Q(4, 2)$ **2.** $P(0, 0)$, $Q(2, -6)$

3. $P(2, 2)$, $Q(-10, 0)$ **4.** $P(1, 2)$, $Q(3, 3)$

5. $P(2, 4)$, $Q(4, 3)$ **6.** $P(2, -5)$, $Q(-4, 3)$

7. $P(1, -3)$, $Q(-1, 6)$ **8.** $P(-1, -4)$, $Q(6, 0)$

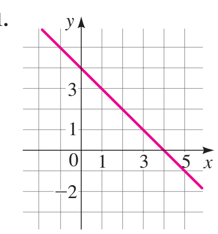
9. Find the slopes of the lines l_1, l_2, l_3, and l_4 in the figure below.



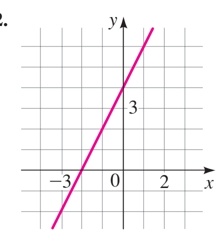
10. **(a)** Sketch lines through $(0, 0)$ with slopes $1, 0, \frac{1}{2}, 2$, and -1.
(b) Sketch lines through $(0, 0)$ with slopes $\frac{1}{3}, \frac{1}{2}, -\frac{1}{3}$, and 3.

11–14 ■ Find an equation for the line whose graph is sketched.

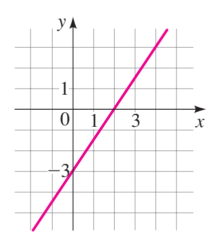
11.



12.



13.



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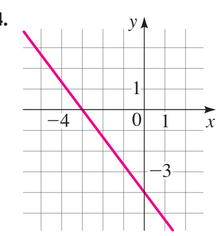


15–34 ■ Find an equation of the line that satisfies the given conditions.

15. Through $(2, 3)$; slope 1

16. Through $(-2, 4)$; slope -1

17. Through $(1, 7)$; slope $\frac{2}{3}$

18. Through $(-3, -5)$; slope $-\frac{7}{2}$

19. Through $(2, 1)$ and $(1, 6)$

20. Through $(-1, -2)$ and $(4, 3)$

21. Slope 3; y-intercept -2

22. Slope $\frac{2}{5}$; y-intercept 4

23. x-intercept 1; y-intercept -3

24. x-intercept -8; y-intercept 6

25. Through $(4, 5)$; parallel to the x-axis

26. Through $(4, 5)$; parallel to the y-axis

27. Through $(1, -6)$; parallel to the line $x + 2y = 6$

28. y-intercept 6; parallel to the line $2x + 3y + 4 = 0$

29. Through $(-1, 2)$; parallel to the line $x = 5$

30. Through $(2, 6)$; perpendicular to the line $y = 1$

31. Through $(-1, -2)$; perpendicular to the line $2x + 5y + 8 = 0$

32. Through $\left(\frac{1}{2}, -\frac{2}{3}\right)$; perpendicular to the line $4x - 8y = 1$

33. Through $(1, 7)$; parallel to the line passing through $(2, 5)$ and $(-2, 1)$

34. Through $(-2, -11)$; perpendicular to the line passing through $(1, 1)$ and $(5, -1)$

35. **(a)** Sketch the line with slope $\frac{3}{2}$ that passes through the point $(-2, 1)$.

(b) Find an equation for this line.

36. **(a)** Sketch the line with slope -2 that passes through the point $(4, -1)$.

(b) Find an equation for this line.

37–40 ■ Use a graphing device to graph the given family of lines in the same viewing rectangle. What do the lines have in common?

37. $y = -2x + b$ for $b = 0, \pm1, \pm3, \pm6$

38. $y = mx - 3$ for $m = 0, \pm0.25, \pm0.75, \pm1.5$

39. $y = m(x - 3)$ for $m = 0, \pm0.25, \pm0.75, \pm1.5$

40. $y = 2 + m(x + 3)$ for $m = 0, \pm0.5, \pm1, \pm2, \pm6$

41–52 ■ Find the slope and y-intercept of the line and draw its graph.

41. $x + y = 3$

42. $3x - 2y = 12$

43. $x + 3y = 0$

44. $2x - 5y = 0$

45. $\frac{1}{2}x - \frac{1}{3}y + 1 = 0$

46. $-3x - 5y + 30 = 0$

47. $y = 4$

48. $4y + 8 = 0$

49. $3x - 4y = 12$

50. $x = -5$

51. $3x + 4y - 1 = 0$

52. $4x + 5y = 10$

53. Use slopes to show that $A(1, 1)$, $B(7, 4)$, $C(5, 10)$, and $D(-1, 7)$ are vertices of a parallelogram.

54. Use slopes to show that $A(-3, -1)$, $B(3, 3)$, and $C(-9, 8)$ are vertices of a right triangle.

55. Use slopes to show that $A(1, 1)$, $B(11, 3)$, $C(10, 8)$, and $D(0, 6)$ are vertices of a rectangle.

56. Use slopes to determine whether the given points are collinear (lie on a line).

(a) $(1, 1), (3, 9), (6, 21)$

(b) $(-1, 3), (1, 7), (4, 15)$

57. Find an equation of the perpendicular bisector of the line segment joining the points $A(1, 4)$ and $B(7, -2)$.

58. Find the area of the triangle formed by the coordinate axes and the line

$$2y + 3x - 6 = 0$$

59. **(a)** Show that if the x- and y-intercepts of a line are nonzero numbers a and b, then the equation of the line can be written in the form

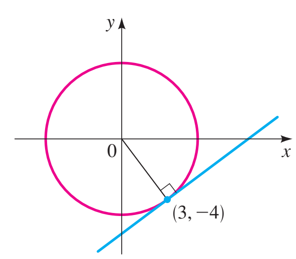
$$\frac{x}{a} + \frac{y}{b} = 1$$

This is called the **two-intercept form** of the equation of a line.

(b) Use part (a) to find an equation of the line whose x-intercept is 6 and whose y-intercept is -8.

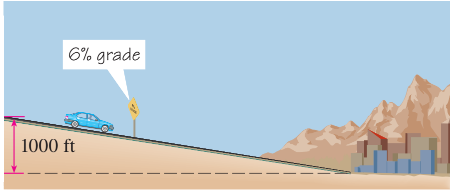
60. **(a)** Find an equation for the line tangent to the circle $x^2 + y^2 = 25$ at the point $(3, -4)$. (See the figure.)

(b) At what other point on the circle will a tangent line be parallel to the tangent line in part (a)?



Applications

61. Grade of a Road West of Albuquerque, New Mexico, Route 40 eastbound is straight and makes a steep descent toward the city. The highway has a 6% grade, which means that its slope is $-\frac{6}{100}$. Driving on this road you notice from elevation signs that you have descended a distance of 1000 ft. What is the change in your horizontal distance?



62. Global Warming Some scientists believe that the average surface temperature of the world has been rising steadily. The average surface temperature is given by

$$T = 0.02t + 8.50$$

where T is temperature in °C and t is years since 1900.

(a) What do the slope and T-intercept represent?

(b) Use the equation to predict the average global surface temperature in 2100.

63. Drug Dosages If the recommended adult dosage for a drug is D (in mg), then to determine the appropriate dosage c for a child of age a, pharmacists use the equation

$$c = 0.0417D(a + 1)$$

Suppose the dosage for an adult is 200 mg.

(a) Find the slope. What does it represent?

(b) What is the dosage for a newborn?

64. Flea Market The manager of a weekend flea market knows from past experience that if she charges x dollars for a rental space at the flea market, then the number y of spaces she can rent is given by the equation $y = 200 - 4x$.

(a) Sketch a graph of this linear equation. (Remember that the rental charge per space and the number of spaces rented must both be nonnegative quantities.)

(b) What do the slope, the y-intercept, and the x-intercept of the graph represent?

65. Production Cost A small-appliance manufacturer finds that if he produces x toaster ovens in a month his production cost is given by the equation

$$y = 6x + 3000$$

(where y is measured in dollars).

(a) Sketch a graph of this linear equation.

(b) What do the slope and y-intercept of the graph represent?

66. Temperature Scales The relationship between the Fahrenheit (F) and Celsius (C) temperature scales is given by the equation $F = \frac{9}{5}C + 32$.

(a) Complete the table to compare the two scales at the given values.

(b) Find the temperature at which the scales agree. [*Hint:* Suppose that a is the temperature at which the scales agree. Set $F = a$ and $C = a$. Then solve for a.]

C	F
−30°	
−20°	
−10°	
0°	
	50°
	68°
	86°

67. Crickets and Temperature Biologists have observed that the chirping rate of crickets of a certain species is related to temperature, and the relationship appears to be very nearly linear. A cricket produces 120 chirps per minute at 70°F and 168 chirps per minute at 80°F.

(a) Find the linear equation that relates the temperature t and the number of chirps per minute n.

(b) If the crickets are chirping at 150 chirps per minute, estimate the temperature.

68. Depreciation A small business buys a computer for \$4000. After 4 years the value of the computer is expected to be \$200. For accounting purposes, the business uses *linear depreciation* to assess the value of the computer at a given time. This means that if V is the value of the computer at time t, then a linear equation is used to relate V and t.

(a) Find a linear equation that relates V and t.

(b) Sketch a graph of this linear equation.

(c) What do the slope and V-intercept of the graph represent?

(d) Find the depreciated value of the computer 3 years from the date of purchase.

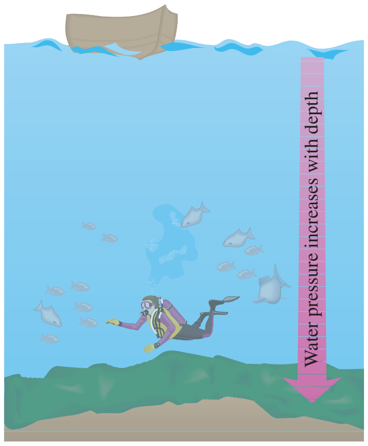
69. Pressure and Depth At the surface of the ocean, the water pressure is the same as the air pressure above the water, 15 lb/in^2. Below the surface, the water pressure increases by 4.34 lb/in^2 for every 10 ft of descent.

(a) Find an equation for the relationship between pressure and depth below the ocean surface.

(b) Sketch a graph of this linear equation.

(c) What do the slope and y-intercept of the graph represent?

(d) At what depth is the pressure 100 lb/in^2?



70. Distance, Speed, and Time Jason and Debbie leave Detroit at 2:00 P.M. and drive at a constant speed, traveling west on I-90. They pass Ann Arbor, 40 mi from Detroit, at 2:50 P.M.

(a) Express the distance traveled in terms of the time elapsed.

(b) Draw the graph of the equation in part (a).

(c) What is the slope of this line? What does it represent?

71. Cost of Driving The monthly cost of driving a car depends on the number of miles driven. Lynn found that in May her driving cost was \$380 for 480 mi and in June her cost was \$460 for 800 mi. Assume that there is a linear relationship between the monthly cost C of driving a car and the distance driven d.

(a) Find a linear equation that relates C and d.

(b) Use part (a) to predict the cost of driving 1500 mi per month.

(c) Draw the graph of the linear equation. What does the slope of the line represent?

(d) What does the y-intercept of the graph represent?

(e) Why is a linear relationship a suitable model for this situation?

72. Manufacturing Cost The manager of a furniture factory finds that it costs \$2200 to manufacture 100 chairs in one day and \$4800 to produce 300 chairs in one day.

(a) Assuming that the relationship between cost and the number of chairs produced is linear, find an equation that expresses this relationship. Then graph the equation.

(b) What is the slope of the line in part (a), and what does it represent?

(c) What is the y-intercept of this line, and what does it represent?

Discovery • Discussion

73. What Does the Slope Mean? Suppose that the graph of the outdoor temperature over a certain period of time is a line. How is the weather changing if the slope of the line is positive? If it's negative? If it's zero?

74. Collinear Points Suppose you are given the coordinates of three points in the plane, and you want to see whether they lie on the same line. How can you do this using slopes? Using the Distance Formula? Can you think of another method?

1.11 Modeling Variation

Mathematical models are discussed in more detail in *Focus on Modeling*, which begins on page 239.

When scientists talk about a mathematical model for a real-world phenomenon, they often mean an equation that describes the relationship between two quantities. For instance, the model may describe how the population of an animal species varies with time or how the pressure of a gas varies as its temperature changes. In this section we study a kind of modeling called *variation*.

Direct Variation

Two types of mathematical models occur so often that they are given special names. The first is called *direct variation* and occurs when one quantity is a constant multiple of the other, so we use an equation of the form $y = kx$ to model this dependence.

SUGGESTED TIME AND EMPHASIS

1 class.
Recommended material.

POINT TO STRESS

Direct and inverse proportionality.

ALTERNATE EXAMPLE 1a

During a thunderstorm you see the lightning before you hear the thunder because light travels much faster than sound. The distance d between you and the storm varies directly as the time interval t between the lightning and the thunder. Suppose that the thunder from a storm 6480 ft away takes 6 s to reach you.

Determine the constant of proportionality and write the equation for the variation.

ANSWER

$d = 1080t$

ALTERNATE EXAMPLE 1c

During a thunderstorm you see the lightning before you hear the thunder because light travels much faster than sound. The distance d between you and the storm varies directly as the time interval t between the lightning and the thunder. The equation for the variation is $d = 1080t$.

If the time interval between the lightning and thunder is 8 s, how far away is the storm?

ANSWER

8640

Direct Variation

If the quantities x and y are related by an equation

$$y = kx$$

for some constant $k \neq 0$, we say that y **varies directly as** x, or y is **directly proportional to** x, or simply y **is proportional to** x. The constant k is called the **constant of proportionality**.

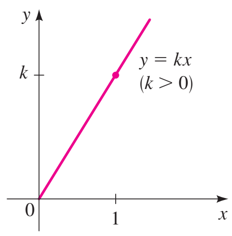
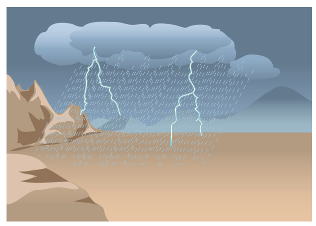


Figure 1

Recall that the graph of an equation of the form $y = mx + b$ is a line with slope m and y-intercept b. So the graph of an equation $y = kx$ that describes direct variation is a line with slope k and y-intercept 0 (see Figure 1).

Example 1 Direct Variation



During a thunderstorm you see the lightning before you hear the thunder because light travels much faster than sound. The distance between you and the storm varies directly as the time interval between the lightning and the thunder.

(a) Suppose that the thunder from a storm 5400 ft away takes 5 s to reach you. Determine the constant of proportionality and write the equation for the variation.

(b) Sketch the graph of this equation. What does the constant of proportionality represent?

(c) If the time interval between the lightning and thunder is now 8 s, how far away is the storm?

Solution

(a) Let d be the distance from you to the storm and let t be the length of the time interval. We are given that d varies directly as t, so

$$d = kt$$

where k is a constant. To find k, we use the fact that $t = 5$ when $d = 5400$. Substituting these values in the equation, we get

$$5400 = k(5) \quad \text{Substitute}$$

$$k = \frac{5400}{5} = 1080 \quad \text{Solve for } k$$

Substituting this value of k in the equation for d, we obtain

$$d = 1080t$$

as the equation for d as a function of t.

(b) The graph of the equation $d = 1080t$ is a line through the origin with slope 1080 and is shown in Figure 2. The constant $k = 1080$ is the approximate speed of sound (in ft/s).

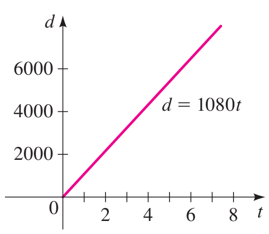


Figure 2

(c) When $t = 8$, we have

$$d = 1080 \cdot 8 = 8640$$

So, the storm is 8640 ft $\approx$ 1.6 mi away. ■

Inverse Variation

Another equation that is frequently used in mathematical modeling is $y = k/x$, where k is a constant.

Inverse Variation

If the quantities x and y are related by the equation

$$y = \frac{k}{x}$$

for some constant $k \neq 0$, we say that y **is inversely proportional to** x, or y **varies inversely as** x.

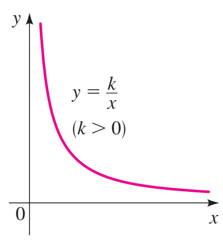


Figure 3
Inverse variation

The graph of $y = k/x$ for $x > 0$ is shown in Figure 3 for the case $k > 0$. It gives a picture of what happens when y is inversely proportional to x.

Example 2 Inverse Variation

Boyle's Law states that when a sample of gas is compressed at a constant temperature, the pressure of the gas is inversely proportional to the volume of the gas.

(a) Suppose the pressure of a sample of air that occupies 0.106 m^3 at 25 °C is 50 kPa. Find the constant of proportionality, and write the equation that expresses the inverse proportionality.

(b) If the sample expands to a volume of 0.3 m^3, find the new pressure.

Solution

(a) Let P be the pressure of the sample of gas and let V be its volume. Then, by the definition of inverse proportionality, we have

$$P = \frac{k}{V}$$

where k is a constant. To find k we use the fact that $P = 50$ when $V = 0.106$. Substituting these values in the equation, we get

$$50 = \frac{k}{0.106} \qquad \text{Substitute}$$

$$k = (50)(0.106) = 5.3 \qquad \text{Solve for } k$$

ALTERNATE EXAMPLE 2a
Boyle's law states that when a sample of gas is compressed at a constant temperature, the pressure P of the gas is inversely proportional to the volume V of the gas. Suppose the pressure of a sample of air that occupies 0.0424 m^3 at 25ºC is 125 kPa. Find the constant of proportionality, and write the equation that expresses the inverse proportionality.

ANSWER
$P = \frac{5.3}{V}$

ALTERNATE EXAMPLE 2b
Boyle's law states that when a sample of gas is compressed at a constant temperature, the pressure P of the gas is inversely proportional to the volume V of the gas. The equation that expresses the inverse proportionality is $P = \frac{5.3}{V}$. If the sample occupies a volume of 0.9 m^3, find the pressure.

ANSWER
5.9

SAMPLE QUESTIONS
Text Questions

(a) If y is proportional to x, must there be a linear equation relating y and x?
(b) If there is a linear equation relating y and x, must y be proportional to x?

Answers

(a) Yes
(b) No

DRILL QUESTION

The mass of a memo is proportional to the number of pages it contains. If a 100-page memo measures 89 g, how much mass would a 35-page memo have?

Answer

$(0.89)\,35 = 31.15$ g

Putting this value of k in the equation for P, we have

$$P = \frac{5.3}{V}$$

(b) When $V = 0.3$, we have

$$P = \frac{5.3}{0.3} \approx 17.7$$

So, the new pressure is about 17.7 kPa. ■

Joint Variation

A physical quantity often depends on more than one other quantity. If one quantity is proportional to two or more other quantities, we call this relationship *joint variation*.

Joint Variation

If the quantities x, y, and z are related by the equation

$$z = kxy$$

where k is a nonzero constant, we say that z **varies jointly as** x and y, or z **is jointly proportional to** x and y.

In the sciences, relationships between three or more variables are common, and any combination of the different types of proportionality that we have discussed is possible. For example, if

$$z = k\frac{x}{y}$$

we say that z **is proportional to** x and **inversely proportional to** y.

Example 3 Newton's Law of Gravitation

Newton's Law of Gravitation says that two objects with masses m_1 and m_2 attract each other with a force F that is jointly proportional to their masses and inversely proportional to the square of the distance r between the objects. Express Newton's Law of Gravitation as an equation.

Solution Using the definitions of joint and inverse variation, and the traditional notation G for the gravitational constant of proportionality, we have

$$F = G\frac{m_1 m_2}{r^2}$$ ■

If m_1 and m_2 are fixed masses, then the gravitational force between them is $F = C/r^2$ (where $C = Gm_1m_2$ is a constant). Figure 4 shows the graph of this equation for $r > 0$ with $C = 1$. Observe how the gravitational attraction decreases with increasing distance.

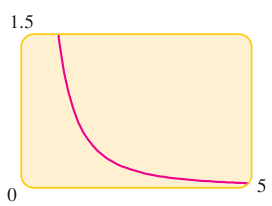


Figure 4
Graph of $F = \frac{1}{r^2}$

ALTERNATE EXAMPLE 3

The volume of a cone is jointly proportional to its height and the square of its radius. Express the volume of a cone as an equation.

ANSWER

$V = kr^2h$

$\left(k \text{ turns out to be } \frac{\pi}{3}\right)$

IN-CLASS MATERIALS

Have students come up with as many examples of proportionality as they can. One quick, effective way would be to give them a minute to write down as many as they can think of, then another couple of minutes to discuss their list with a neighbor, generating more, and finally have them write down their answers on the board. With luck, some will accidentally give examples that are not proportional or that are inversely proportional. If these don't come up, you can ask, "What about the height of a person and that person's average rent?" or "What about the cost of a computer and its weight, all other things being equal?"

1.11 Exercises

1–12 ■ Write an equation that expresses the statement.

1. T varies directly as x.

2. P is directly proportional to w.

3. v is inversely proportional to z.

4. w is jointly proportional to m and n.

5. y is proportional to s and inversely proportional to t.

6. P varies inversely as T.

7. z is proportional to the square root of y.

8. A is proportional to the square of t and inversely proportional to the cube of x.

9. V is jointly proportional to l, w, and h.

10. S is jointly proportional to the squares of r and θ.

11. R is proportional to i and inversely proportional to P and t.

12. A is jointly proportional to the square roots of x and y.

13–22 ■ Express the statement as an equation. Use the given information to find the constant of proportionality.

13. y is directly proportional to x. If $x = 6$, then $y = 42$.

14. z varies inversely as t. If $t = 3$, then $z = 5$.

15. M varies directly as x and inversely as y. If $x = 2$ and $y = 6$, then $M = 5$.

16. S varies jointly as p and q. If $p = 4$ and $q = 5$, then $S = 180$.

17. W is inversely proportional to the square of r. If $r = 6$, then $W = 10$.

18. t is jointly proportional to x and y and inversely proportional to r. If $x = 2$, $y = 3$, and $r = 12$, then $t = 25$.

19. C is jointly proportional to l, w, and h. If $l = w = h = 2$, then $C = 128$.

20. H is jointly proportional to the squares of l and w. If $l = 2$ and $w = \frac{1}{3}$, then $H = 36$.

21. s is inversely proportional to the square root of t. If $s = 100$, then $t = 25$.

22. M is jointly proportional to a, b, and c, and inversely proportional to d. If a and d have the same value, and if b and c are both 2, then $M = 128$.

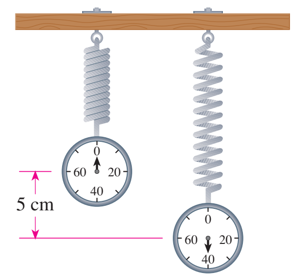
Applications

23. Hooke's Law Hooke's Law states that the force needed to keep a spring stretched x units beyond its natural length is directly proportional to x. Here the constant of proportionality is called the **spring constant**.

(a) Write Hooke's Law as an equation.

(b) If a spring has a natural length of 10 cm and a force of 40 N is required to maintain the spring stretched to a length of 15 cm, find the spring constant.

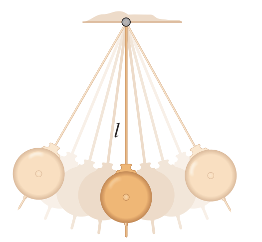
(c) What force is needed to keep the spring stretched to a length of 14 cm?



24. Law of the Pendulum The period of a pendulum (the time elapsed during one complete swing of the pendulum) varies directly with the square root of the length of the pendulum.

(a) Express this relationship by writing an equation.

(b) In order to double the period, how would we have to change the length l?



25. Printing Costs The cost C of printing a magazine is jointly proportional to the number of pages p in the magazine and the number of magazines printed m.

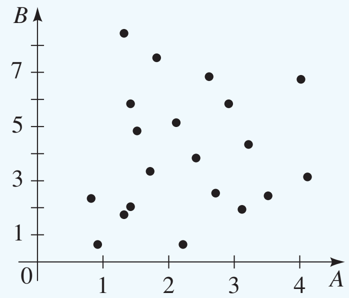
(a) Write an equation that expresses this joint variation.

(b) Find the constant of proportionality if the printing cost is $60,000 for 4000 copies of a 120-page magazine.

(c) How much would the printing cost be for 5000 copies of a 92-page magazine?

IN-CLASS MATERIALS

Point out that proportionality is a model, just like a linear model or any other kind of model. For example, show them an example of a scatter plot such as the one below.



Clearly there is no straight line that will go through the points, but we can model the relationship by a straight line to try to make predictions. In this case, the straight-line model would not be very good.

26. Boyle's Law The pressure P of a sample of gas is directly proportional to the temperature T and inversely proportional to the volume V.

(a) Write an equation that expresses this variation.

(b) Find the constant of proportionality if 100 L of gas exerts a pressure of 33.2 kPa at a temperature of 400 K (absolute temperature measured on the Kelvin scale).

(c) If the temperature is increased to 500 K and the volume is decreased to 80 L, what is the pressure of the gas?

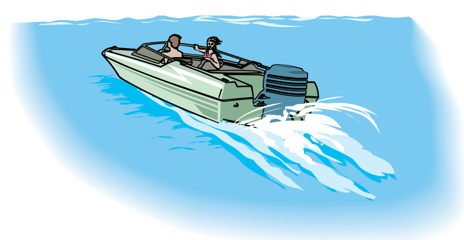
27. Power from a Windmill The power P that can be obtained from a windmill is directly proportional to the cube of the wind speed s.

(a) Write an equation that expresses this variation.

(b) Find the constant of proportionality for a windmill that produces 96 watts of power when the wind is blowing at 20 mi/h.

(c) How much power will this windmill produce if the wind speed increases to 30 mi/h?

28. Power Needed to Propel a Boat The power P (measured in horse power, hp) needed to propel a boat is directly proportional to the cube of the speed s. An 80-hp engine is needed to propel a certain boat at 10 knots. Find the power needed to drive the boat at 15 knots.

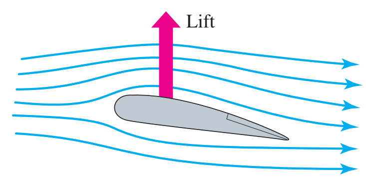


29. Loudness of Sound The loudness L of a sound (measured in decibels, dB) is inversely proportional to the square of the distance d from the source of the sound. A person 10 ft from a lawn mower experiences a sound level of 70 dB; how loud is the lawn mower when the person is 100 ft away?

30. Stopping Distance The stopping distance D of a car after the brakes have been applied varies directly as the square of the speed s. A certain car traveling at 50 mi/h can stop in 240 ft. What is the maximum speed it can be traveling if it needs to stop in 160 ft?

31. A Jet of Water The power P of a jet of water is jointly proportional to the cross-sectional area A of the jet and to the cube of the velocity v. If the velocity is doubled and the cross-sectional area is halved, by what factor will the power increase?

32. Aerodynamic Lift The lift L on an airplane wing at takeoff varies jointly as the square of the speed s of the plane and the area A of its wings. A plane with a wing area of 500 ft^2 traveling at 50 mi/h experiences a lift of 1700 lb. How much lift would a plane with a wing area of 600 ft^2 traveling at 40 mi/h experience?



33. Drag Force on a Boat The drag force F on a boat is jointly proportional to the wetted surface area A on the hull and the square of the speed s of the boat. A boat experiences a drag force of 220 lb when traveling at 5 mi/h with a wetted surface area of 40 ft^2. How fast must a boat be traveling if it has 28 ft^2 of wetted surface area and is experiencing a drag force of 175 lb?

34. Skidding in a Curve A car is traveling on a curve that forms a circular arc. The force F needed to keep the car from skidding is jointly proportional to the weight w of the car and the square of its speed s, and is inversely proportional to the radius r of the curve.

(a) Write an equation that expresses this variation.

(b) A car weighing 1600 lb travels around a curve at 60 mi/h. The next car to round this curve weighs 2500 lb

IN-CLASS MATERIALS

Discuss the difference between a general linear relationship and direct variation: in a directly proportional relationship, the origin is a data point. For example, the number of cans of beans you buy and the price you pay are in direct proportion—zero cans costs zero dollars. If you have to pay a fee to get into the store (as in some discount stores) the two quantities are no longer in direct proportion.

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and requires the same force as the first car to keep from skidding. How fast is the second car traveling?

35. Electrical Resistance The resistance R of a wire varies directly as its length L and inversely as the square of its diameter d.

(a) Write an equation that expresses this joint variation.

(b) Find the constant of proportionality if a wire 1.2 m long and 0.005 m in diameter has a resistance of 140 ohms.

(c) Find the resistance of a wire made of the same material that is 3 m long and has a diameter of 0.008 m.

36. Kepler's Third Law Kepler's Third Law of planetary motion states that the square of the period T of a planet (the time it takes for the planet to make a complete revolution about the sun) is directly proportional to the cube of its average distance d from the sun.

(a) Express Kepler's Third Law as an equation.

(b) Find the constant of proportionality by using the fact that for our planet the period is about 365 days and the average distance is about 93 million miles.

(c) The planet Neptune is about 2.79×10^9 mi from the sun. Find the period of Neptune.

37. Radiation Energy The total radiation energy E emitted by a heated surface per unit area varies as the fourth power of its absolute temperature T. The temperature is 6000 K at the surface of the sun and 300 K at the surface of the earth.

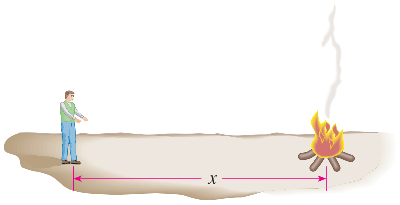
(a) How many times more radiation energy per unit area is produced by the sun than by the earth?

(b) The radius of the earth is 3960 mi and the radius of the sun is 435,000 mi. How many times more total radiation does the sun emit than the earth?

38. Value of a Lot The value of a building lot on Galiano Island is jointly proportional to its area and the quantity of water produced by a well on the property. A 200 ft by 300 ft lot has a well producing 10 gallons of water per minute, and is valued at $48,000. What is the value of a 400 ft by 400 ft lot if the well on the lot produces 4 gallons of water per minute?

39. Growing Cabbages In the short growing season of the Canadian arctic territory of Nunavut, some gardeners find it possible to grow gigantic cabbages in the midnight sun. Assume that the final size of a cabbage is proportional to the amount of nutrients it receives, and inversely proportional to the number of other cabbages surrounding it. A cabbage that received 20 oz of nutrients and had 12 other cabbages around it grew to 30 lb. What size would it grow to if it received 10 oz of nutrients and had only 5 cabbage "neighbors"?

40. Heat of a Campfire The heat experienced by a hiker at a campfire is proportional to the amount of wood on the fire, and inversely proportional to the cube of his distance from the fire. If he is 20 ft from the fire, and someone doubles the amount of wood burning, how far from the fire would he have to be so that he feels the same heat as before?



41. Frequency of Vibration The frequency f of vibration of a violin string is inversely proportional to its length L. The constant of proportionality k is positive and depends on the tension and density of the string.

(a) Write an equation that represents this variation.

(b) What effect does doubling the length of the string have on the frequency of its vibration?

42. Spread of a Disease The rate r at which a disease spreads in a population of size P is jointly proportional to the number x of infected people and the number $P - x$ who are not infected. An infection erupts in a small town with population $P = 5000$.

(a) Write an equation that expresses r as a function of x.

(b) Compare the rate of spread of this infection when 10 people are infected to the rate of spread when 1000 people are infected. Which rate is larger? By what factor?

(c) Calculate the rate of spread when the entire population is infected. Why does this answer make intuitive sense?

Discovery • Discussion

43. Is Proportionality Everything? A great many laws of physics and chemistry are expressible as proportionalities. Give at least one example of a function that occurs in the sciences that is *not* a proportionality.

IN-CLASS MATERIALS

There are many proportional and inversely proportional relationships that students already understand, but students may not have thought about using that vocabulary. For example, the area of a circle is directly proportional to the square of its radius. The area of a rectangle is jointly proportional to its length and width. The time it takes to make a trip (of fixed distance) is inversely proportional to the average speed.

1 Review

Concept Check

1. Define each term in your own words. (Check by referring to the definition in the text.)
   (a) An integer **(b)** A rational number
   (c) An irrational number **(d)** A real number
2. State each of these properties of real numbers.
   (a) Commutative Property
   (b) Associative Property
   (c) Distributive Property
3. What is an open interval? What is a closed interval? What notation is used for these intervals?
4. What is the absolute value of a number?
5. **(a)** In the expression a^x, which is the base and which is the exponent?
   (b) What does a^x mean if $x = n$, a positive integer?
   (c) What if $x = 0$?
   (d) What if x is a negative integer: $x = -n$, where n is a positive integer?
   (e) What if $x = m/n$, a rational number?
   (f) State the Laws of Exponents.
6. **(a)** What does $\sqrt[n]{a} = b$ mean?
   (b) Why is $\sqrt{a^2} = |a|$?
   (c) How many real nth roots does a positive real number have if n is odd? If n is even?
7. Explain how the procedure of rationalizing the denominator works.
8. State the Special Product Formulas for $(a + b)^2$, $(a - b)^2$, $(a + b)^3$, and $(a - b)^3$.
9. State each Special Factoring Formula.
   (a) Difference of squares **(b)** Difference of cubes
   (c) Sum of cubes
10. What is a solution of an equation?
11. How do you solve an equation involving radicals? Why is it important to check your answers when solving equations of this type?
12. How do you solve an equation
    (a) algebraically? **(b)** graphically?
13. Write the general form of each type of equation.
    (a) A linear equation **(b)** A quadratic equation
14. What are the three ways to solve a quadratic equation?
15. State the Zero-Product Property.
16. Describe the process of completing the square.
17. State the quadratic formula.
18. What is the discriminant of a quadratic equation?
19. State the rules for working with inequalities.
20. How do you solve
    (a) a linear inequality?
    (b) a nonlinear inequality?
21. **(a)** How do you solve an equation involving an absolute value?
    (b) How do you solve an inequality involving an absolute value?
22. **(a)** Describe the coordinate plane.
    (b) How do you locate points in the coordinate plane?
23. State each formula.
    (a) The Distance Formula
    (b) The Midpoint Formula
24. Given an equation, what is its graph?
25. How do you find the x-intercepts and y-intercepts of a graph?
26. Write an equation of the circle with center (h, k) and radius r.
27. Explain the meaning of each type of symmetry. How do you test for it?
    (a) Symmetry with respect to the x-axis
    (b) Symmetry with respect to the y-axis
    (c) Symmetry with respect to the origin

EXAMPLES

1. Newton's Law: The rate of change of temperature of an object is proportional to the difference between the current temperature of the object and the temperature of its surroundings.
2. Torricelli's Law: The velocity at which liquid pours out of a cylindrical container (like orange juice out of a can with a hole at the bottom) is proportional to the square root of its height in the container.
3. Einstein's Law: The energy of a photon is directly proportional to its frequency.
4. Einstein's Theory of Relativity: The energy of a particle is proportional to its mass. (The constant of proportionality being the speed of light, squared.)

28. Define the slope of a line.

29. Write each form of the equation of a line.

(a) The point-slope form

(b) The slope-intercept form

30. (a) What is the equation of a vertical line?

(b) What is the equation of a horizontal line?

31. What is the general equation of a line?

32. Given lines with slopes m_1 and m_2, explain how you can tell if the lines are

(a) parallel **(b)** perpendicular

33. Write an equation that expresses each relationship.

(a) y is directly proportional to x.

(b) y is inversely proportional to x.

(c) z is jointly proportional to x and y.

Exercises

1–4 ■ State the property of real numbers being used.

1. $3x + 2y = 2y + 3x$

2. $(a + b)(a - b) = (a - b)(a + b)$

3. $4(a + b) = 4a + 4b$

4. $(A + 1)(x + y) = (A + 1)x + (A + 1)y$

5–6 ■ Express the interval in terms of inequalities, and then graph the interval.

5. $[-2, 6)$ **6.** $(-\infty, 4]$

7–8 ■ Express the inequality in interval notation, and then graph the corresponding interval.

7. $x \geq 5$ **8.** $-1 < x \leq 5$

9–18 ■ Evaluate the expression.

9. $|3 - |-9||$ **10.** $1 - |1 - |-1||$

11. $2^{-3} - 3^{-2}$ **12.** $\sqrt[3]{-125}$

13. $216^{-1/3}$ **14.** $64^{2/3}$

15. $\dfrac{\sqrt{242}}{\sqrt{2}}$ **16.** $\sqrt[4]{4}\sqrt[4]{324}$

17. $2^{1/2}8^{1/2}$ **18.** $\sqrt{2}\sqrt{50}$

19–28 ■ Simplify the expression.

19. $\dfrac{x^2(2x)^4}{x^3}$ **20.** $(a^2)^{-3}(a^3b)^2(b^3)^4$

21. $(3xy^2)^3(\frac{2}{3}x^{-1}y)^2$ **22.** $\left(\dfrac{r^2s^{4/3}}{r^{1/3}s}\right)^6$

23. $\sqrt[3]{(x^3y)^2y^4}$ **24.** $\sqrt{x^2y^4}$

25. $\left(\dfrac{9x^3y}{y^{-3}}\right)^{1/2}$ **26.** $\left(\dfrac{x^{-2}y^3}{x^2y}\right)^{-1/2}\left(\dfrac{x^3y}{y^{1/2}}\right)^2$

27. $\dfrac{8r^{1/2}s^{-3}}{2r^{-2}s^4}$ **28.** $\left(\dfrac{ab^2c^{-3}}{2a^3b^{-4}}\right)^{-2}$

29. Write the number 78,250,000,000 in scientific notation.

30. Write the number 2.08×10^{-8} in ordinary decimal notation.

31. If $a \approx 0.00000293$, $b \approx 1.582 \times 10^{-14}$, and $c \approx 2.8064 \times 10^{12}$, use a calculator to approximate the number ab/c.

32. If your heart beats 80 times per minute and you live to be 90 years old, estimate the number of times your heart beats during your lifetime. State your answer in scientific notation.

33–48 ■ Factor the expression completely.

33. $12x^2y^4 - 3xy^5 + 9x^3y^2$ **34.** $x^2 - 9x + 18$

35. $x^2 + 3x - 10$ **36.** $6x^2 + x - 12$

37. $4t^2 - 13t - 12$ **38.** $x^4 - 2x^2 + 1$

39. $25 - 16t^2$ **40.** $2y^6 - 32y^2$

41. $x^6 - 1$ **42.** $y^3 - 2y^2 - y + 2$

43. $x^{-1/2} - 2x^{1/2} + x^{3/2}$ **44.** $a^4b^2 + ab^5$

45. $4x^3 - 8x^2 + 3x - 6$ **46.** $8x^3 + y^6$

47. $(x^2 + 2)^{5/2} + 2x(x^2 + 2)^{3/2} + x^2\sqrt{x^2 + 2}$

48. $3x^3 - 2x^2 + 18x - 12$

49–64 ■ Perform the indicated operations and simplify.

49. $(2x + 1)(3x - 2) - 5(4x - 1)$

50. $(2y - 7)(2y + 7)$

51. $(1 + x)(2 - x) - (3 - x)(3 + x)$

52. $\sqrt{x}(\sqrt{x}+1)(2\sqrt{x}-1)$

53. $x^2(x-2)+x(x-2)^2$

54. $\dfrac{x^2-2x-3}{2x^2+5x+3}$

55. $\dfrac{x^2+2x-3}{x^2+8x+16}\cdot\dfrac{3x+12}{x-1}$

56. $\dfrac{t^3-1}{t^2-1}$

57. $\dfrac{x^2-2x-15}{x^2-6x+5}\div\dfrac{x^2-x-12}{x^2-1}$

58. $\dfrac{2}{x}+\dfrac{1}{x-2}+\dfrac{3}{(x-2)^2}$

59. $\dfrac{1}{x-1}-\dfrac{2}{x^2-1}$

60. $\dfrac{1}{x+2}+\dfrac{1}{x^2-4}-\dfrac{2}{x^2-x-2}$

61. $\dfrac{\dfrac{1}{x}-\dfrac{1}{2}}{x-2}$

62. $\dfrac{\dfrac{1}{x}-\dfrac{1}{x+1}}{\dfrac{1}{x}+\dfrac{1}{x+1}}$

63. $\dfrac{\sqrt{6}}{\sqrt{3}+\sqrt{2}}$ (rationalize the denominator)

64. $\dfrac{\sqrt{x+h}-\sqrt{x}}{h}$ (rationalize the numerator)

65–80 ■ Find all real solutions of the equation.

65. $7x-6=4x+9$

66. $8-2x=14+x$

67. $\dfrac{x+1}{x-1}=\dfrac{3x}{3x-6}$

68. $(x+2)^2=(x-4)^2$

69. $x^2-9x+14=0$

70. $x^2+24x+144=0$

71. $2x^2+x=1$

72. $3x^2+5x-2=0$

73. $4x^3-25x=0$

74. $x^3-2x^2-5x+10=0$

75. $3x^2+4x-1=0$

76. $\dfrac{1}{x}+\dfrac{2}{x-1}=3$

77. $\dfrac{x}{x-2}+\dfrac{1}{x+2}=\dfrac{8}{x^2-4}$

78. $x^4-8x^2-9=0$

79. $|x-7|=4$

80. $|2x-5|=9$

81. The owner of a store sells raisins for \$3.20 per pound and nuts for \$2.40 per pound. He decides to mix the raisins and nuts and sell 50 lb of the mixture for \$2.72 per pound. What quantities of raisins and nuts should he use?

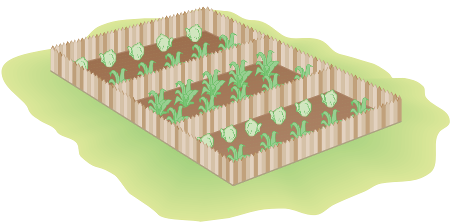
82. Anthony leaves Kingstown at 2:00 P.M. and drives to Queensville, 160 mi distant, at 45 mi/h. At 2:15 P.M. Helen leaves Queensville and drives to Kingstown at 40 mi/h. At what time do they pass each other on the road?

83. A woman cycles 8 mi/h faster than she runs. Every morning she cycles 4 mi and runs $2\frac{1}{2}$ mi, for a total of one hour of exercise. How fast does she run?

84. The hypotenuse of a right triangle has length 20 cm. The sum of the lengths of the other two sides is 28 cm. Find the lengths of the other two sides of the triangle.

85. Abbie paints twice as fast as Beth and three times as fast as Cathie. If it takes them 60 min to paint a living room with all three working together, how long would it take Abbie if she works alone?

86. A homeowner wishes to fence in three adjoining garden plots, one for each of her children, as shown in the figure. If each plot is to be 80 ft^2 in area, and she has 88 ft of fencing material at hand, what dimensions should each plot have?



87–94 ■ Solve the inequality. Express the solution using interval notation and graph the solution set on the real number line.

87. $3x-2>-11$

88. $-1<2x+5\le 3$

89. $x^2+4x-12>0$

90. $x^2\le 1$

91. $\dfrac{x-4}{x^2-4}\le 0$

92. $\dfrac{5}{x^3-x^2-4x+4}<0$

93. $|x-5|\le 3$

94. $|x-4|<0.02$

95–98 ■ Solve the equation or inequality graphically.

95. $x^2-4x=2x+7$

96. $\sqrt{x+4}=x^2-5$

97. $4x - 3 \geq x^2$

98. $x^3 - 4x^2 - 5x > 2$

99–100 ■ Two points P and Q are given.

(a) Plot P and Q on a coordinate plane.
(b) Find the distance from P to Q.
(c) Find the midpoint of the segment PQ.
(d) Sketch the line determined by P and Q, and find its equation in slope-intercept form.
(e) Sketch the circle that passes through Q and has center P, and find the equation of this circle.

99. $P(2, 0)$, $Q(-5, 12)$ **100.** $P(7, -1)$, $Q(2, -11)$

101–102 ■ Sketch the region given by the set.

101. $\{(x, y) \mid -4 < x < 4 \text{ and } -2 < y < 2\}$

102. $\{(x, y) \mid x \geq 4 \text{ or } y \geq 2\}$

103. Which of the points $A(4, 4)$ or $B(5, 3)$ is closer to the point $C(-1, -3)$?

104. Find an equation of the circle that has center $(2, -5)$ and radius $\sqrt{2}$.

105. Find an equation of the circle that has center $(-5, -1)$ and passes through the origin.

106. Find an equation of the circle that contains the points $P(2, 3)$ and $Q(-1, 8)$ and has the midpoint of the segment PQ as its center.

107–110 ■ Determine whether the equation represents a circle, a point, or has no graph. If the equation is that of a circle, find its center and radius.

107. $x^2 + y^2 + 2x - 6y + 9 = 0$

108. $2x^2 + 2y^2 - 2x + 8y = \frac{1}{2}$

109. $x^2 + y^2 + 72 = 12x$

110. $x^2 + y^2 - 6x - 10y + 34 = 0$

111–118 ■ Test the equation for symmetry and sketch its graph.

111. $y = 2 - 3x$

112. $2x - y + 1 = 0$

113. $x + 3y = 21$

114. $x = 2y + 12$

115. $y = 16 - x^2$

116. $8x + y^2 = 0$

117. $x = \sqrt{y}$

118. $y = -\sqrt{1 - x^2}$

119–122 ■ Use a graphing device to graph the equation in an appropriate viewing rectangle.

119. $y = x^2 - 6x$

120. $y = \sqrt{5 - x}$

121. $y = x^3 - 4x^2 - 5x$

122. $\frac{x^2}{4} + y^2 = 1$

123. Find an equation for the line that passes through the points $(-1, -6)$ and $(2, -4)$.

124. Find an equation for the line that passes through the point $(6, -3)$ and has slope $-\frac{1}{2}$.

125. Find an equation for the line that has x-intercept 4 and y-intercept 12.

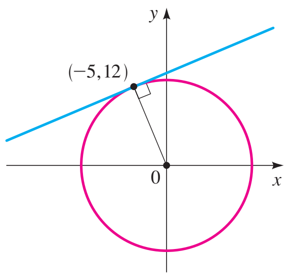
126. Find an equation for the line that passes through the point $(1, 7)$ and is perpendicular to the line $x - 3y + 16 = 0$.

127. Find an equation for the line that passes through the origin and is parallel to the line $3x + 15y = 22$.

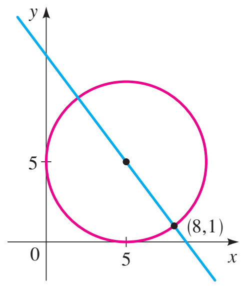
128. Find an equation for the line that passes through the point $(5, 2)$ and is parallel to the line passing through $(-1, -3)$ and $(3, 2)$.

129–130 ■ Find equations for the circle and the line in the figure.

129.



130.



131. Hooke's Law states that if a weight w is attached to a hanging spring, then the stretched length s of the spring is linearly related to w. For a particular spring we have

$$s = 0.3w + 2.5$$

where s is measured in inches and w in pounds.

(a) What do the slope and s-intercept in this equation represent?

(b) How long is the spring when a 5-lb weight is attached?

132. Margarita is hired by an accounting firm at a salary of \$60,000 per year. Three years later her annual salary has increased to \$70,500. Assume her salary increases linearly.

(a) Find an equation that relates her annual salary S and the number of years t that she has worked for the firm.

(b) What do the slope and S-intercept of her salary equation represent?

(c) What will her salary be after 12 years with the firm?

133. Suppose that M varies directly as z, and $M = 120$ when $z = 15$. Write an equation that expresses this variation.

134. Suppose that z is inversely proportional to y, and that $z = 12$ when $y = 16$. Write an equation that expresses z in terms of y.

135. The intensity of illumination I from a light varies inversely as the square of the distance d from the light.

(a) Write this statement as an equation.

(b) Determine the constant of proportionality if it is known that a lamp has an intensity of 1000 candles at a distance of 8 m.

(c) What is the intensity of this lamp at a distance of 20 m?

136. The frequency of a vibrating string under constant tension is inversely proportional to its length. If a violin string 12 inches long vibrates 440 times per second, to what length must it be shortened to vibrate 660 times per second?

137. The terminal velocity of a parachutist is directly proportional to the square root of his weight. A 160-lb parachutist attains a terminal velocity of 9 mi/h. What is the terminal velocity for a parachutist weighing 240 lb?

138. The maximum range of a projectile is directly proportional to the square of its velocity. A baseball pitcher throws a ball at 60 mi/h, with a maximum range of 242 ft. What is his maximum range if he throws the ball at 70 mi/h?

1 Test

1. (a) Graph the intervals $(-5, 3]$ and $(2, \infty)$ on the real number line.
(b) Express the inequalities $x \le 3$ and $-1 \le x < 4$ in interval notation.
(c) Find the distance between -7 and 9 on the real number line.

2. Evaluate each expression.
(a) $(-3)^4$ (b) -3^4 (c) 3^{-4} (d) $\dfrac{5^{23}}{5^{21}}$ (e) $\left(\dfrac{2}{3}\right)^{-2}$ (f) $16^{-3/4}$

3. Write each number in scientific notation.
(a) 186,000,000,000 (b) 0.0000003965

4. Simplify each expression. Write your final answer without negative exponents.
(a) $\sqrt{200} - \sqrt{32}$ (b) $(3a^3b^3)(4ab^2)^2$ (c) $\left(\dfrac{3x^{3/2}y^3}{x^2y^{-1/2}}\right)^{-2}$
(d) $\dfrac{x^2 + 3x + 2}{x^2 - x - 2}$ (e) $\dfrac{x^2}{x^2 - 4} - \dfrac{x + 1}{x + 2}$ (f) $\dfrac{\dfrac{y}{x} - \dfrac{x}{y}}{\dfrac{1}{y} - \dfrac{1}{x}}$

5. Rationalize the denominator and simplify: $\dfrac{\sqrt{10}}{\sqrt{5} - 2}$

6. Perform the indicated operations and simplify.
(a) $3(x + 6) + 4(2x - 5)$ (b) $(x + 3)(4x - 5)$ (c) $(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b})$
(d) $(2x + 3)^2$ (e) $(x + 2)^3$

7. Factor each expression completely.
(a) $4x^2 - 25$ (b) $2x^2 + 5x - 12$ (c) $x^3 - 3x^2 - 4x + 12$
(d) $x^4 + 27x$ (e) $3x^{3/2} - 9x^{1/2} + 6x^{-1/2}$ (f) $x^3y - 4xy$

8. Find all real solutions.
(a) $x + 5 = 14 - \frac{1}{2}x$ (b) $\dfrac{2x}{x + 1} = \dfrac{2x - 1}{x}$ (c) $x^2 - x - 12 = 0$
(d) $2x^2 + 4x + 1 = 0$ (e) $\sqrt{3 - \sqrt{x + 5}} = 2$ (f) $x^4 - 3x^2 + 2 = 0$
(g) $3|x - 4| = 10$

9. Mary drove from Amity to Belleville at a speed of 50 mi/h. On the way back, she drove at 60 mi/h. The total trip took $4\frac{2}{5}$ h of driving time. Find the distance between these two cities.

10. A rectangular parcel of land is 70 ft longer than it is wide. Each diagonal between opposite corners is 130 ft. What are the dimensions of the parcel?

11. Solve each inequality. Write the answer using interval notation, and sketch the solution on the real number line.
(a) $-4 < 5 - 3x \le 17$ (b) $x(x - 1)(x + 2) > 0$
(c) $|x - 4| < 3$ (d) $\dfrac{2x - 3}{x + 1} \le 1$

12. A bottle of medicine is to be stored at a temperature between 5 °C and 10 °C. What range does this correspond to on the Fahrenheit scale? [*Note:* Fahrenheit (F) and Celsius (C) temperatures satisfy the relation $C = \frac{5}{9}(F - 32)$.]

13. For what values of x is the expression $\sqrt{6x - x^2}$ defined as a real number?

14. Solve the equation and the inequality graphically.

(a) $x^3 - 9x - 1 = 0$ (b) $x^2 - 1 \le |x + 1|$

15. (a) Plot the points $P(0, 3)$, $Q(3, 0)$, and $R(6, 3)$ in the coordinate plane. Where must the point S be located so that $PQRS$ is a square?

(b) Find the area of $PQRS$.

16. (a) Sketch the graph of $y = x^2 - 4$.

(b) Find the x- and y-intercepts of the graph.

(c) Is the graph symmetric about the x-axis, the y-axis, or the origin?

17. Let $P(-3, 1)$ and $Q(5, 6)$ be two points in the coordinate plane.

(a) Plot P and Q in the coordinate plane.

(b) Find the distance between P and Q.

(c) Find the midpoint of the segment PQ.

(d) Find the slope of the line that contains P and Q.

(e) Find the perpendicular bisector of the line that contains P and Q.

(f) Find an equation for the circle for which the segment PQ is a diameter.

18. Find the center and radius of each circle and sketch its graph.

(a) $x^2 + y^2 = 25$ (b) $(x - 2)^2 + (y + 1)^2 = 9$ (c) $x^2 + 6x + y^2 - 2y + 6 = 0$

19. Write the linear equation $2x - 3y = 15$ in slope-intercept form, and sketch its graph. What are the slope and y-intercept?

20. Find an equation for the line with the given property.

(a) It passes through the point $(3, -6)$ and is parallel to the line $3x + y - 10 = 0$.

(b) It has x-intercept 6 and y-intercept 4.

21. A geologist uses a probe to measure the temperature T (in °C) of the soil at various depths below the surface, and finds that at a depth of x cm, the temperature is given by the linear equation $T = 0.08x - 4$.

(a) What is the temperature at a depth of one meter (100 cm)?

(b) Sketch a graph of the linear equation.

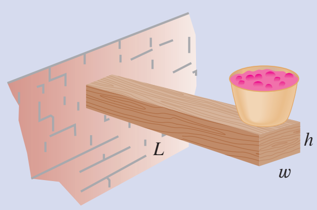
(c) What do the slope, the x-intercept, and T-intercept of the graph of this equation represent?

22. The maximum weight M that can be supported by a beam is jointly proportional to its width w and the square of its height h, and inversely proportional to its length L.

(a) Write an equation that expresses this proportionality.

(b) Determine the constant of proportionality if a beam 4 in. wide, 6 in. high, and 12 ft long can support a weight of 4800 lb.

(c) If a 10-ft beam made of the same material is 3 in. wide and 10 in. high, what is the maximum weight it can support?

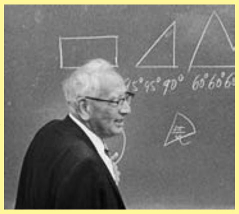


If you had difficulty with any of these problems, you may wish to review the section of this chapter indicated below.

If you had trouble with this test problem	Review this section
1	Section 1.1
2, 3, 4(a), 4(b), 4(c)	Section 1.2
4(d), 4(e), 4(f), 5	Section 1.4
6, 7	Section 1.3
8	Section 1.5
9, 10	Section 1.6
11, 12, 13	Section 1.7
14	Section 1.9
15, 16, 17(a), 17(b)	Section 1.8
17(c), 17(d)	Section 1.10
17(e), 17(f), 18	Section 1.8
19, 20, 21	Section 1.10
22	Section 1.11

Focus on Problem Solving

General Principles



George Polya (1887–1985) is famous among mathematicians for his ideas on problem solving. His lectures on problem solving at Stanford University attracted overflow crowds whom he held on the edges of their seats, leading them to discover solutions for themselves. He was able to do this because of his deep insight into the psychology of problem solving. His well-known book *How To Solve It* has been translated into 15 languages. He said that Euler (see page 288) was unique among great mathematicians because he explained *how* he found his results. Polya often said to his students and colleagues, "Yes, I see that your proof is correct, but how did you discover it?" In the preface to *How To Solve It*, Polya writes, "A great discovery solves a great problem but there is a grain of discovery in the solution of any problem. Your problem may be modest; but if it challenges your curiosity and brings into play your inventive faculties, and if you solve it by your own means, you may experience the tension and enjoy the triumph of discovery."

There are no hard and fast rules that will ensure success in solving problems. However, it is possible to outline some general steps in the problem-solving process and to give principles that are useful in solving certain problems. These steps and principles are just common sense made explicit. They have been adapted from George Polya's insightful book *How To Solve It*.

1. Understand the Problem

The first step is to read the problem and make sure that you understand it. Ask yourself the following questions:

What is the unknown?
What are the given quantities?
What are the given conditions?

For many problems it is useful to

draw a diagram

and identify the given and required quantities on the diagram.

Usually it is necessary to

introduce suitable notation

In choosing symbols for the unknown quantities, we often use letters such as a, b, c, m, n, x, and y, but in some cases it helps to use initials as suggestive symbols, for instance, V for volume or t for time.

2. Think of a Plan

Find a connection between the given information and the unknown that enables you to calculate the unknown. It often helps to ask yourself explicitly: "How can I relate the given to the unknown?" If you don't see a connection immediately, the following ideas may be helpful in devising a plan.

- **Try to recognize something familiar**

Relate the given situation to previous knowledge. Look at the unknown and try to recall a more familiar problem that has a similar unknown.

- **Try to recognize patterns**

Certain problems are solved by recognizing that some kind of pattern is occurring. The pattern could be geometric, or numerical, or algebraic. If you can see regularity or repetition in a problem, then you might be able to guess what the pattern is and then prove it.

- **Use analogy**

Try to think of an analogous problem, that is, a similar or related problem, but one that is easier than the original. If you can solve the similar, simpler problem, then it might give you the clues you need to solve the original, more difficult one. For instance, if a problem involves very large numbers, you could first try a similar problem with smaller numbers. Or if the problem is in three-dimensional geometry, you could look for something similar in two-dimensional geometry. Or if the problem you start with is a general one, you could first try a special case.

- **Introduce something extra**

You may sometimes need to introduce something new—an auxiliary aid—to make the connection between the given and the unknown. For instance, in a problem for which a diagram is useful, the auxiliary aid could be a new line drawn in the diagram. In a more algebraic problem the aid could be a new unknown that relates to the original unknown.

- **Take cases**

You may sometimes have to split a problem into several cases and give a different argument for each case. For instance, we often have to use this strategy in dealing with absolute value.

- **Work backward**

Sometimes it is useful to imagine that your problem is solved and work backward, step by step, until you arrive at the given data. Then you may be able to reverse your steps and thereby construct a solution to the original problem. This procedure is commonly used in solving equations. For instance, in solving the equation $3x - 5 = 7$, we suppose that x is a number that satisfies $3x - 5 = 7$ and work backward. We add 5 to each side of the equation and then divide each side by 3 to get $x = 4$. Since each of these steps can be reversed, we have solved the problem.

- **Establish subgoals**

In a complex problem it is often useful to set subgoals (in which the desired situation is only partially fulfilled). If you can attain or accomplish these subgoals, then you may be able to build on them to reach your final goal.

- **Indirect reasoning**

Sometimes it is appropriate to attack a problem indirectly. In using **proof by contradiction** to prove that P implies Q, we assume that P is true and Q is false and try to see why this cannot happen. Somehow we have to use this information and arrive at a contradiction to what we absolutely know is true.

- **Mathematical induction**

In proving statements that involve a positive integer n, it is frequently helpful to use the Principle of Mathematical Induction, which is discussed in Section 11.5.

3. Carry Out the Plan

In Step 2, a plan was devised. In carrying out that plan, you must check each stage of the plan and write the details that prove each stage is correct.

4. Look Back

Having completed your solution, it is wise to look back over it, partly to see if any errors have been made and partly to see if you can discover an easier way to solve the problem. Looking back also familiarizes you with the method of solution, and this may be useful for solving a future problem. Descartes said, "Every problem that I solved became a rule which served afterwards to solve other problems."

We illustrate some of these principles of problem solving with an example. Further illustrations of these principles will be presented at the end of selected chapters.

Problem Average Speed

A driver sets out on a journey. For the first half of the distance she drives at the leisurely pace of 30 mi/h; during the second half she drives 60 mi/h. What is her average speed on this trip?

Thinking about the problem

It is tempting to take the average of the speeds and say that the average speed for the entire trip is

$$\frac{30 + 60}{2} = 45 \text{ mi/h}$$

But is this simple-minded approach really correct?

Try a special case

Let's look at an easily calculated special case. Suppose that the total distance traveled is 120 mi. Since the first 60 mi is traveled at 30 mi/h, it takes 2 h. The second 60 mi is traveled at 60 mi/h, so it takes one hour. Thus, the total time is 2 + 1 = 3 hours and the average speed is

$$\frac{120}{3} = 40 \text{ mi/h}$$

So our guess of 45 mi/h was wrong.

Understand the problem

Solution We need to look more carefully at the meaning of average speed. It is defined as

$$\text{average speed} = \frac{\text{distance traveled}}{\text{time elapsed}}$$

Introduce notation

Let d be the distance traveled on each half of the trip. Let t_1 and t_2 be the times taken for the first and second halves of the trip. Now we can write down the information we have been given. For the first half of the trip, we have

State what is given

$$30 = \frac{d}{t_1} \qquad (1)$$

and for the second half, we have

$$60 = \frac{d}{t_2} \qquad (2)$$

Identify the unknown

Now we identify the quantity we are asked to find:

$$\text{average speed for entire trip} = \frac{\text{total distance}}{\text{total time}} = \frac{2d}{t_1 + t_2}$$

Connect the given with the unknown

To calculate this quantity, we need to know t_1 and t_2, so we solve Equations 1 and 2 for these times:

$$t_1 = \frac{d}{30} \qquad t_2 = \frac{d}{60}$$

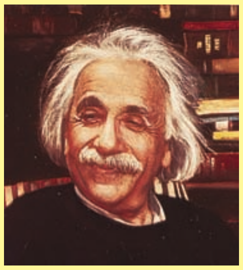
Now we have the ingredients needed to calculate the desired quantity:

$$\text{average speed} = \frac{2d}{t_1 + t_2} = \frac{2d}{\dfrac{d}{30} + \dfrac{d}{60}}$$

$$= \frac{60(2d)}{60\left(\dfrac{d}{30} + \dfrac{d}{60}\right)} \qquad \text{Multiply numerator and denominator by 60}$$

$$= \frac{120d}{2d + d} = \frac{120d}{3d} = 40$$

So, the average speed for the entire trip is 40 mi/h. ■



Bettmann/Corbis

Don't feel bad if you don't solve these problems right away. Problems 2 and 6 were sent to Albert Einstein by his friend Wertheimer. Einstein (and his friend Bucky) enjoyed the problems and wrote back to Wertheimer. Here is part of his reply:

> Your letter gave us a lot of amusement. The first intelligence test fooled both of us (Bucky and me). Only on working it out did I notice that no time is available for the downhill run! Mr. Bucky was also taken in by the second example, but I was not. Such drolleries show us how stupid we are!

(See *Mathematical Intelligencer*, Spring 1990, page 41.)

Problems

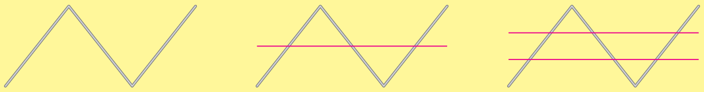
1. **Distance, Time, and Speed** A man drives from home to work at a speed of 50 mi/h. The return trip from work to home is traveled at the more leisurely pace of 30 mi/h. What is the man's average speed for the round-trip?

2. **Distance, Time, and Speed** An old car has to travel a 2-mile route, uphill and down. Because it is so old, the car can climb the first mile—the ascent—no faster than an average speed of 15 mi/h. How fast does the car have to travel the second mile—on the descent it can go faster, of course—in order to achieve an average speed of 30 mi/h for the trip?

3. **A Speeding Fly** A car and a van are parked 120 mi apart on a straight road. The drivers start driving toward each other at noon, each at a speed of 40 mi/h. A fly starts from the front bumper of the van at noon and flies to the bumper of the car, then immediately back to the bumper of the van, back to the car, and so on, until the car and the van meet. If the fly flies at a speed of 100 mi/h, what is the total distance it travels?

4. **Comparing Discounts** Which price is better for the buyer, a 40% discount or two successive discounts of 20%?

5. **Cutting up a Wire** A piece of wire is bent as shown in the figure. You can see that one cut through the wire produces four pieces and two parallel cuts produce seven pieces. How many pieces will be produced by 142 parallel cuts? Write a formula for the number of pieces produced by n parallel cuts.



6. **Amoeba Propagation** An amoeba propagates by simple division; each split takes 3 minutes to complete. When such an amoeba is put into a glass container with a nutrient fluid, the container is full of amoebas in one hour. How long would it take for the container to be filled if we start with not one amoeba, but two?

7. **Running Laps** Two runners start running laps at the same time, from the same starting position. George runs a lap in 50 s; Sue runs a lap in 30 s. When will the runners next be side by side?

8. **Batting Averages** Player A has a higher batting average than player B for the first half of the baseball season. Player A also has a higher batting average than player B for the second half of the season. Is it necessarily true that player A has a higher batting average than player B for the entire season?

9. **Coffee and Cream** A spoonful of cream is taken from a pitcher of cream and put into a cup of coffee. The coffee is stirred. Then a spoonful of this mixture is put into the pitcher of cream. Is there now more cream in the coffee cup or more coffee in the pitcher of cream?

10. **A Melting Ice Cube** An ice cube is floating in a cup of water, full to the brim, as shown in the sketch. As the ice melts, what happens? Does the cup overflow, or does the water level drop, or does it remain the same? (You need to know Archimedes' Principle: A floating object displaces a volume of water whose weight equals the weight of the object.)

11. **Wrapping the World** A red ribbon is tied tightly around the earth at the equator. How much more ribbon would you need if you raised the ribbon 1 ft above the equator everywhere? (You don't need to know the radius of the earth to solve this problem.)

12. **Irrational Powers** Prove that it's possible to raise an irrational number to an irrational power and get a rational result. [*Hint:* The number $a = \sqrt{2}^{\sqrt{2}}$ is either rational or irrational. If a is rational, you are done. If a is irrational, consider $a^{\sqrt{2}}$.]

13. **Babylonian Square Roots** The ancient Babylonians developed the following process for finding the square root of a number N. First they made a guess at the square root—let's call this first guess r_1. Noting that

$$r_1 \cdot \left(\frac{N}{r_1}\right) = N$$

they concluded that the actual square root must be somewhere between r_1 and N/r_1, so their next guess for the square root, r_2, was the average of these two numbers:

$$r_2 = \frac{1}{2}\left(r_1 + \frac{N}{r_1}\right)$$

Continuing in this way, their next approximation was given by

$$r_3 = \frac{1}{2}\left(r_2 + \frac{N}{r_2}\right)$$

and so on. In general, once we have the nth approximation to the square root of N, we find the $(n + 1)$st using

$$r_{n+1} = \frac{1}{2}\left(r_n + \frac{N}{r_n}\right)$$

Use this procedure to find $\sqrt{72}$, correct to two decimal places.

14. A Perfect Cube Show that if you multiply three consecutive integers and then add the middle integer to the result, you get a perfect cube.

15. Number Patterns Find the last digit in the number 3^{459}. [*Hint:* Calculate the first few powers of 3, and look for a pattern.]

16. Number Patterns Use the techniques of solving a simpler problem and looking for a pattern to evaluate the number

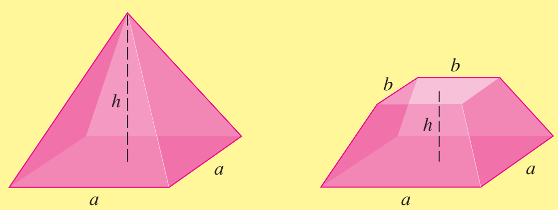
$$3999999999999^2$$

17. Right Triangles and Primes Prove that every prime number is the leg of exactly one right triangle with integer sides. (This problem was first stated by Fermat; see page 652.)

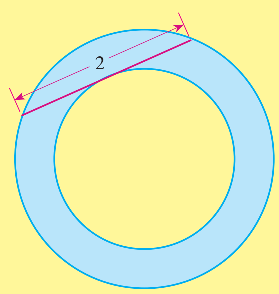
18. An Equation with No Solution Show that the equation $x^2 + y^2 = 4z + 3$ has no solution in integers. [*Hint:* Recall that an even number is of the form $2n$ and an odd number is of the form $2n + 1$. Consider all possible cases for x and y even or odd.]

19. Ending Up Where You Started A woman starts at a point P on the earth's surface and walks 1 mi south, then 1 mi east, then 1 mi north, and finds herself back at P, the starting point. Describe all points P for which this is possible (there are infinitely many).

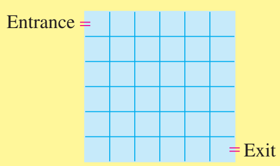
20. Volume of a Truncated Pyramid The ancient Egyptians, as a result of their pyramid-building, knew that the volume of a pyramid with height h and square base of side length a is $V = \frac{1}{3}ha^2$. They were able to use this fact to prove that the volume of a truncated pyramid is $V = \frac{1}{3}h(a^2 + ab + b^2)$, where h is the height and b and a are the lengths of the sides of the square top and bottom, as shown in the figure. Prove the truncated pyramid volume formula.



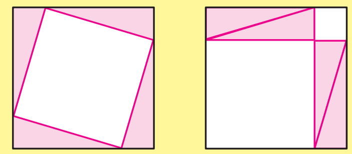
21. Area of a Ring Find the area of the region between the two concentric circles shown in the figure.



Bhaskara (born 1114) was an Indian mathematician, astronomer, and astrologer. Among his many accomplishments was an ingenious proof of the Pythagorean Theorem (see Problem 22). His important mathematical book *Lilavati* [*The Beautiful*] consists of algebra problems posed in the form of stories to his daughter Lilavati. Many of the problems begin "Oh beautiful maiden, suppose . . ." The story is told that using astrology, Bhaskara had determined that great misfortune would befall his daughter if she married at any time other than at a certain hour of a certain day. On her wedding day, as she was anxiously watching the water clock, a pearl fell unnoticed from her headdress. It stopped the flow of water in the clock, causing her to miss the opportune moment for marriage. Bhaskara's *Lilavati* was written to console her.



22. Bhaskara's Proof The Indian mathematician Bhaskara sketched the two figures shown here and wrote below them, "Behold!" Explain how his sketches prove the Pythagorean Theorem.



23. An Interesting Integer The number 1729 is the smallest positive integer that can be represented in two different ways as the sum of two cubes. What are the two ways?

24. Simple Numbers

(a) Use a calculator to find the value of the expression

$$\sqrt{3 + 2\sqrt{2}} - \sqrt{3 - 2\sqrt{2}}$$

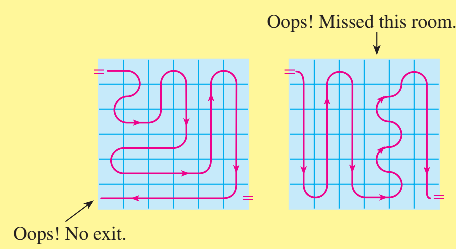
The number looks very simple. Show that the calculated value is correct.

(b) Use a calculator to evaluate

$$\frac{\sqrt{2} + \sqrt{6}}{\sqrt{2 + \sqrt{3}}}$$

Show that the calculated value is correct.

25. The Impossible Museum Tour A museum is in the shape of a square with six rooms to a side; the entrance and exit are at diagonally opposite corners, as shown in the figure to the left. Each pair of adjacent rooms is joined by a door. Some very efficient tourists would like to tour the museum by visiting each room *exactly* once. Can you find a path for such a tour? Here are examples of attempts that failed.



Here is how you can prove that the museum tour is not possible. Imagine that the rooms are colored black and white like a checkerboard.

(a) Show that the room colors alternate between white and black as the tourists walk through the museum.

(b) Use part (a) and the fact that there are an even number of rooms in the museum to conclude that the tour cannot end at the exit.

26. Coloring the Coordinate Plane Suppose that each point in the coordinate plane is colored either red or blue. Show that there must always be two points of the same color that are exactly one unit apart.

27. The Rational Coordinate Forest Suppose that each point (x, y) in the plane, both of whose coordinates are rational numbers, represents a tree. If you are standing at the point $(0, 0)$, how far could you see in this forest?

28. A Thousand Points A thousand points are graphed in the coordinate plane. Explain why it is possible to draw a straight line in the plane so that half of the points are on one side of the line and half are on the other. [*Hint:* Consider the slopes of the lines determined by each *pair* of points.]

29. Graphing a Region in the Plane Sketch the region in the plane consisting of all points (x, y) such that

$$|x| + |y| \leq 1$$

30. The Graph of an Equation Graph the equation

$$x^2y - y^3 - 5x^2 + 5y^2 = 0$$

[*Hint:* Factor.]