Lesson 5-3, 5-5, 5-7

Inverse, Inverse Trig, and Non-e Bases

Objective

Students will…

- Be able to test for existence of an inverse function, and differentiate/integrate them.
- Be able to know the derivatives and integrals of inverse trig functions.
- Be able to find the derivatives and integrals of exponential functions with a non-e base.

One-to-One Functions

Function is defined as a relation having one output, per input. This only deals with the **number** of outputs, not necessarily the **type** of outputs. A one-to-one function is a function where no input shares a same output with another input. In other words,

$f(x_1) = f(x_2)$ if and **only** if $x_1 = x_2$

or

$f(x_1) \neq f(x_2)$ if and **only** if $x_1 \neq x_2$

Again, the definition of a function only deals with the number of outputs. Two different inputs could share the same output, as long as they both have **one** single output.

Inverse Functions

The whole point of finding out whether a function is one-to-one or not has to do with inverse functions. For any one-to-one function, an inverse function must exist.

Inverse functions is the "opposite" function. By definition, for a function f, let $f(x) = y$. Then, the inverse function f^{-1},

$f^{-1}(y) = x$, for any *y*.

You can also think inverse function as the function that "undo's" the its original function.

How to find the inverse function

1. Write “y =“ instead of “f(x) =“

2. Replace the switch the “y” and the “x”

3. Solve the equation for “y”

4. The resulting equation is the inverse function, $f^{-1}(x)$

Showing One-to-One-ness (Existence of Inverse)

One of the ways to visually identify one-to-one functions is using the horizontal line test, which simply states that a function is one-to-one if and only if no horizontal line intersects its graph more than once.

But using the differentiation, we can prove the one-to-one-ness in another way.

Theorem 5.7- If f is strictly monotonic (only increasing or only decreasing) on its entire domain, then it is one-to-one and therefore has an inverse function.

We can use **derivative** to determine whether a function is monotonic or not.

Examples

Determine whether the following functions has an inverse.

a. $f(x) = x^3 + x - 1$

b. $f(x) = x^3 - x + 1$

Example

Find the inverse function of $f(x) = \sqrt{2x - 3}$

Derivative of an inverse function

What do we know about the differentiation (derivatives) of inverse functions?

Let f be a function that is differentiable on an interval I. Let g be the inverse function of f.

$g'(x) = \frac{1}{f'(g(x))}$, where $f'(g(x)) \neq 0$ (cannot divide by zero)

Example

Let $f(x) = \frac{1}{4}x^3 + x - 1$.

a. What is the value of $f^{-1}(x)$ when $x = 3$?

b. What is the value of $(f^{-1})'(x)$ when $x = 3$?

Bases other than e

We learned how to differentiate and integrate exponential or logarithmic functions with base e. What about non-e bases? For these, we need to introduce a little trick or a different approach. Consider…

If a is a positive real number $(a \neq 1)$ and x is any real number, then the exponential function to the base a is denoted by a^x and is defined by

$$a^x = e^{\ln a^x} = e^{(\ln a)x}$$

If a is a positive real number $(a \neq 1)$ and x is any real number, then the logarithmic function to the base a is denoted by $\log_a x$ and is defined by

$$\log_a x = \frac{1}{\ln a} \ln x = \frac{\ln x}{\ln a} \text{ (change of base!)}$$

Example

Find $\frac{d}{dx} a^x$

Example

Find $\frac{d}{dx}\log_a x$

Example

Find $\int a^x \, dx$

General Rule

THEOREM 5.13 DERIVATIVES FOR BASES OTHER THAN e

Let a be a positive real number $(a \neq 1)$ and let u be a differentiable function of x.

1. $\frac{d}{dx}[a^x] = (\ln a)a^x$
2. $\frac{d}{dx}[a^u] = (\ln a)a^u \frac{du}{dx}$
3. $\frac{d}{dx}[\log_a x] = \frac{1}{(\ln a)x}$
4. $\frac{d}{dx}[\log_a u] = \frac{1}{(\ln a)u} \frac{du}{dx}$

$$\int a^x \, dx = \left(\frac{1}{\ln a}\right)a^x + C$$

Derivatives of Inverse Trig

THEOREM 5.16 DERIVATIVES OF INVERSE TRIGONOMETRIC FUNCTIONS

Let u be a differentiable function of x.

$$\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1-u^2}} \qquad \frac{d}{dx}[\arccos u] = \frac{-u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx}[\arctan u] = \frac{u'}{1+u^2} \qquad \frac{d}{dx}[\text{arccot } u] = \frac{-u'}{1+u^2}$$

$$\frac{d}{dx}[\text{arcsec } u] = \frac{u'}{|u|\sqrt{u^2-1}} \qquad \frac{d}{dx}[\text{arccsc } u] = \frac{-u'}{|u|\sqrt{u^2-1}}$$

Integrals of Inverse Trig

THEOREM 5.17 INTEGRALS INVOLVING INVERSE TRIGONOMETRIC FUNCTIONS

Let u be a differentiable function of x, and let $a > 0$.

1. $\displaystyle\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin \frac{u}{a} + C$

2. $\displaystyle\int \frac{du}{a^2 + u^2} = \frac{1}{a} \arctan \frac{u}{a} + C$

3. $\displaystyle\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \operatorname{arcsec} \frac{|u|}{a} + C$

Homework 2/12

5.3 #23-28, 29-35 (odd), 47-51 (odd)

5.5 #37-47 (odd), 61-69 (odd)