Lesson 9-1, 9-2

Sequences and Series

Objective

Students will…

- Be able to differentiate between a sequence and a series.
- Be able to understand convergence and divergence of a sequence and series.
- Be able to define monotonic and bounded sequences.

Sequences

Mathematically, a sequence is defined as a function whose domain is the set of positive integers. Think of a sequence as a horizontal table of values of a function with the use of a subscript.

$a_n = n^2$

$f(x) = x^2$

Ex.

x	$f(x)$
1	1
2	4
3	9
4	16
5	25
$\vdots$	$\vdots$

n	1	2	3	4	5	$\ldots$
a_n	1,	4,	9,	16,	25,	$\ldots$

Limit of a Sequence

If a sequence approaches a certain number, it is said to converge. If it does not (i.e. ∞ or −∞), then it is said to diverge.

Ex. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots \frac{1}{64}, \frac{1}{128} \; - - .$ 0 C

0

∞

Ex. $1, 4, 9, 16, 25, 36, \ldots$

Limit of a Sequence

Consider the previous two examples again:

Ex. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots$ $a_n = \frac{1}{2^n}$ $\lim_{n \to \infty} \frac{1}{2^n} = \boxed{0}$

Ex. $1, 4, 9, 16, 25, 36, \ldots$ $a_n = n^2$ $\lim_{n \to \infty} n^2 \; \infty$

We see that the limit of the associated function of these sequences match the limit of the sequences.

Examples

Determine whether the associated sequence converges or diverges.

$a_n = \frac{n}{1-2n}$

$\lim_{n\to\infty} \frac{n}{1-2n} \stackrel{LH}{=} \lim_{n\to\infty} \frac{1}{-2} = \boxed{-\frac{1}{2}}$

$\frac{\infty}{\infty}$

$\frac{d}{dx}a^x = \ln a\,(a^x)$

Examples

Determine whether the associated sequence converges or diverges.

$a_n = \frac{n^2}{2^n - 1}$

$$\lim_{n\to\infty} \frac{n^2}{2^n-1} \overset{lh}{=} \lim_{n\to\infty} \frac{2n}{\ln 2\,(2^n)} \overset{lh}{=} \lim_{n\to\infty} \frac{2}{(\ln 2)^2\, 2^n} = \boxed{0}$$

$\frac{\infty}{\infty}$ $\quad \frac{\infty}{\infty}$ $\quad \frac{2}{\infty}$

Conv

$4! = 4 \times 3 \times 2 \times 1$

Squeeze Theorem for Sequences

If $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} b_n$ and there exists an integer N such that $a_n \leq c_n \leq b_n$ for all $n > N$, then $\lim_{n\to\infty} c_n = L$.

alternating sequence.

Ex. Show that the sequence $c_n = \boxed{(-1)^n} \frac{1}{n!}$ converges, and find its limit.

$c_n = -1, \frac{1}{2}, -\frac{1}{6}, \frac{1}{24}, \ldots$

$-\frac{1}{2^n} \leq (-1)^n \frac{1}{n!} \leq \frac{1}{2^n}$

$0 \leq 0 \leq 0$

$b_n = \frac{1}{2^n} = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

$a_n = -\frac{1}{2^n} = -\frac{1}{2}, -\frac{1}{4}, -\frac{1}{8}, \ldots$

Absolute Value Theorem

For the sequence a_n, if $\lim\limits_{n\to\infty} |a_n| = 0$, then $\lim\limits_{n\to\infty} a_n = 0$

Ex. Find the limit of the sequence $a_n = (-1)^n (n)^{-2}$

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} n^{-2} = \lim_{n\to\infty} \frac{1}{n^2} = 0$$

$$\lim_{n\to\infty} (-1)^n (n)^{-2} = \boxed{0}$$

by AVT

Monotonic Sequences

1, 1, 1, 1, 1, 1, 1

A sequence a_n is monotonic if its terms are either nondecreasing, or if its terms are nonincreasing. (always inc, or always dec, or constant).

Ex. $a_n = -n$

$-1, -2, -3, -4, \ldots$ mono.

Ex. $b_n = 3 + (-1)^n$

$2, 4, 2, 4, 2, 4, 2, 4, \ldots$ not mono

Examples

Determine whether each sequence having the given with nth term is monotonic.

a. $b_n = \frac{2n}{1+n}$ $\begin{matrix} f \\ g \end{matrix}$

$$(b_n)' = \frac{2(1+n) - 2n}{(1+n)^2} = \frac{2 + \cancel{2n} - \cancel{2n}}{(1+n)^2} = \frac{2}{(1+n)^2} \; \frac{+}{+}$$

Since the derivative of b_n is always positive, b_n is always increasing. Therefore, b_n is monotonic.

Examples

Determine whether each sequence having the given with nth term is monotonic.

b. $b_n = \frac{n^2}{2^n - 1}$ $\frac{f}{g}$

$$(b_n)' = \frac{2n(2^n-1) - \ln 2(2^n)n^2}{(2^n-1)^2} = \text{pos or neg.}$$

b_n is not monotonic, since derivative can be + or −.

Bounded Sequences

1. A sequence is bounded above if there is a biggest number in the sequence. The biggest number is called the upper bound. "Ceiling"
2. A sequence is bounded below if there is a smallest number in the sequence. The smallest number is called the lower bound. "floor"
3. A sequence is deemed bounded if it is bounded both above and below. Ceiling & floor

Theorem 9.5- If a sequence is bounded and monotonic, then it converges.

Examples

Determine whether the following sequence converges.

$a_n = 4 - \frac{3}{n}$

Examples

Determine whether the following sequence converges.

$a_n = 4 + \frac{1}{2^n}$

Series

A series is the sum of all of the numbers in the sequence, denoted by the Σ (summation) notation.

Ex. For the sequence $a_n = a_1, a_2, a_3, \ldots$

$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$ is its series.

$\sum_{n=1}^{\infty}$

A series that never ends is called an infinite series.

Geometric Series

A series given by $\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \cdots + ar^n + \cdots, a \neq 0$, is known as a geometric series with a common ratio of $\cancel{r}$.

ex. 2^n $r=2$ ex. $r=4$ $3(4^n)$ $\cancel{= 12^n}$

Theorem 9.6- A geometric series with ratio r diverges if $|r| \geq 1$. If $0 < |r| < 1$, then the series converges to the sum,

$$\sum_{n=0}^{\infty} ar^n = \boxed{\frac{a}{1-r}}, 0 < |r|$$

$\left(\frac{1}{2^n}\right) = \left(\frac{1}{2}\right)^n = \frac{1^n}{2^n}$

Examples

$\frac{3}{2^\infty} = 0$

Determine if the following series converges, and if so, find its limit.

$r = \frac{1}{2} < 1$

$$\sum_{n=0}^{\infty} \frac{3}{2^n} = \sum_{n=0}^{\infty} 3\left(\frac{1}{2^n}\right) = \sum_{n=0}^{\infty} \underset{a}{3}\,\underset{r^n}{\left(\frac{1}{2}\right)^n}$$

$\frac{a}{1-r}$

By theorem the series converges to 6.

$\frac{3}{1-\frac{1}{2}} = \frac{3}{\frac{1}{2}} = 6$

Examples r^n

Determine if the following series converges, and if so, find its limit.

$$\sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n$$

$r = \frac{3}{2} > 1$

By theorem the series diverges.

$\frac{1}{n^2}$ conv.

harmonic series.. $\boxed{\sum_{n=1}^{\infty} \frac{1}{n}} = \infty$

Theorem 9.8 and 9.9

Theorem 9.8-If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$

$\lim_{n\to\infty} \frac{1}{n} = 0$

Theorem 9.9- If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$

Ex. Verify that the infinite series diverges.

$$\sum_{n=1}^{\infty} \frac{n}{2n+3}$$

3.99999942[illegible] 0.00001

2.999 + 0.001

3.

Properties of Infinite Series

THEOREM 9.7 PROPERTIES OF INFINITE SERIES

Let $\Sigma\, a_n$ and $\Sigma\, b_n$ be convergent series, and let A, B, and c be real numbers. If $\Sigma\, a_n = A$ and $\Sigma\, b_n = B$, then the following series converge to the indicated sums.

1. $\displaystyle\sum_{n=1}^{\infty} ca_n = cA$

2. $\displaystyle\sum_{n=1}^{\infty} (a_n + b_n) = A + B$

3. $\displaystyle\sum_{n=1}^{\infty} (a_n - b_n) = A - B$

Homework 4/27

9.1 #31-41 (odd), 47-67 (e.o.o), 83, 87, 91, 95, 97

9.2 #7-15 (odd), 23-27 (odd), 57-69 (e.o.o)