Lesson 10-4b, 10-5

Polar Equations and Their Graphs

Objective

Students will…

- Be able to graph polar graphs using a graphing device.
- Be able to identify equations of special polar graphs.
- Be able to find the tangent line of polar graphs at a point.
- Be able to find area of polar regions, and arc length of polar graphs.

Polar Graphs

To graph polar equations, the first step is to write them into parametric forms, using $x = r\cos\theta$ and $y = r\sin\theta$.

Example: Sketch the graph of $r = 2\cos 3\theta$

$x = 2\cos 3\theta \cos\theta$

$y = 2\cos 3\theta \sin\theta$

"rose curve"

Special Polar Graphs

There are several important types of graphs that have equations that are simpler in polar form than in rectangular form. (Refer to the attached notes)

$x = r\cos\theta = f(\theta)\cos\theta$

$y = r\sin\theta = f(\theta)\sin\theta$

Slope in Polar Form (Derivatives)

If f is differentiable function of θ, then the slope of the tangent line to the graph of $r = f(\theta)$ at the point (r, θ) is

g'f f'g

$$\boxed{\frac{y'}{x'}} = \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{f(\theta)\cos\theta + f'(\theta)\sin\theta}{-f(\theta)\sin\theta + f'(\theta)\cos\theta}$$

g'f f'g

, provided that $\frac{dx}{d\theta} \neq 0$ at (r, θ).

Moreover:

$y' = 0$

1. Solutions to $\frac{dy}{d\theta} = 0$ yield horizontal tangents, provided that $\frac{dx}{d\theta} \neq 0$.
2. Solutions to $\frac{dx}{d\theta} = 0$ yield vertical tangents, provided that $\frac{dy}{d\theta} \neq 0$.

$x' = 0$

Examples

Find the horizontal and vertical tangent lines of $r = \sin\theta$, $0 \le \theta \le \pi$

$y = \sin^2\theta \Rightarrow y' = 2\sin\theta\cos\theta \overset{?}{=} 0 \Rightarrow \boxed{\theta = 0, \pi/2, \pi} \leftarrow$ horiz.

$x = \underset{f}{\sin\theta}\,\underset{g}{\cos\theta} \Rightarrow x' = \cos^2\theta - \sin^2\theta \overset{?}{=} 0 \Rightarrow \boxed{\theta = \pi/4, 3\pi/4}$

$\cos^2\theta = \sin^2\theta$

vertical

Horiz: $(0,0)$, $(0,1)$, $(0,0)$

Vert: $\left(\frac{1}{2}, \frac{1}{2}\right)$, $\left(-\frac{1}{2}, \frac{1}{2}\right)$

Examples

$\sin^2\theta = (1-\cos^2\theta)$

$\cos^2\theta + \sin^2\theta = 1$

3×4 $\quad 2x^2 + x - 1$

4×3

Find the horizontal and vertical tangents to the graph of

$r = 2(1-\cos\theta)$

$y = 2(1-\cos\theta)\sin\theta = 2\sin\theta - 2\sin\theta\cos\theta$

$x = 2(1-\cos\theta)\cos\theta = 2\cos\theta - 2\cos^2\theta$

$y' = 2\cos\theta - 2(\cos^2\theta - \sin^2\theta)$

$x' = -2\sin\theta + 4(\cos\theta\sin\theta$

$2\cos\theta - 2(\sin^2\theta - \cos^2\theta) = 0$

$2\cos\theta - 2\sin^2\theta + 2\cos^2\theta = 0$

$2\cos\theta - 2 + 2\cos^2\theta + 2\cos^2\theta = 0$

$4\cos^2\theta + 2\cos\theta - 2 = 0$

$\cancel{2}(2\cos^2\theta + \cos\theta - 1) = 0$

$(2\cos\theta - 1)(\cos\theta + 1) = 0$

$\cos\theta = \frac{1}{2}$ or $\cos\theta = -1$

$\theta = \pi/3, 5\pi/3$

$\theta = \pi$

$-2\sin\theta + 4\sin\theta\cos\theta = 0$

$4\sin\theta\cos\theta = 2\sin\theta$

$2\sin\theta\cos\theta = \sin\theta$

~~$\cos\theta = 1/2$ or $\sin\theta = 2$~~

$\theta = 0, \pi$

$\theta = \pi/3, 5\pi/3$ or

Area of Polar Region

$\psi \quad \rho \quad \Omega \quad \phi \quad \varepsilon \quad \gamma \quad \lambda \quad \pi \quad \eta \quad \varsigma \quad \sigma \quad \theta$

If f is continuous and nonnegative on the interval $[\alpha, \beta]$, $0 < \beta - \alpha \leq 2\pi$, then the area of the region bounded by the graph of $r = f(\theta)$ between the radial lines $\theta = \alpha$ and $\theta = \beta$ is given by

$$A = \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2 \, d\theta = \frac{1}{2}\int_{\alpha}^{\beta} r^2 \, d\theta, \qquad 0 < \beta - \alpha \leq 2\pi$$

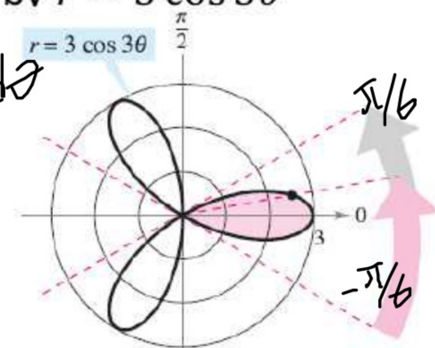
$u = 6\theta$
$du = 6\,d\theta$
$\frac{1}{6}du = d\theta$

$\cancel{(\cos u)^2}$

$\cos\frac{6\theta}{2} = \sqrt{\frac{1+\cos 6\theta}{2}}$

Examples

Find the area of one petal of the rose curve given by $r = 3\cos 3\theta$



$$A = \frac{9}{2}\int_{-\pi/6}^{\pi/6} \cos^2 3\theta \, d\theta = \frac{9}{4}\int_{-\pi/6}^{\pi/6} 1+\cos 6\theta \, d\theta$$

$$= \frac{9}{24}\int_{-\pi/6}^{\pi/6} 1+\cos u \, du = \frac{9}{24}\left(u + \sin u\right)\Big|_{-\pi}^{\pi}$$

$$= \frac{9}{24}\left(\pi + 0 - (-\pi + 0)\right) = \frac{18\pi}{24} = \boxed{\frac{3\pi}{4}}$$

Examples

$u = 2\theta$

$du = 2\,d\theta$ $\quad 4\sin^2\theta = 4\left(\frac{1-\cos 2\theta}{2}\right)$

$\frac{1}{2}du = d\theta$

Find the area of the region lying between the inner and outer loop of the limacon $r = 1 - 2\sin\theta$ $\Rightarrow r^2 = 1 - 4\sin\theta + 4\sin^2\theta$

$$A = \frac{1}{2}\int_{\pi/6}^{5\pi/6} 1 - 4\sin\theta + 4\sin^2\theta \, d\theta = \frac{1}{2}\int_{\pi/6}^{5\pi/6} 1 - 4\sin\theta \, d\theta + \int_{\pi/6}^{5\pi/6} 1 - \cos 2\theta \, d\theta$$

$$= \frac{1}{2}\int_{\pi/6}^{5\pi/6} 1 - 4\sin\theta \, d\theta + \frac{1}{2}\int_{\pi/3}^{5\pi/3} 1 - \cos u \, du = \frac{1}{2}\left(\theta - 4\cos\theta\right)\Big|_{\pi/6}^{5\pi/6} + \frac{1}{2}\left(u - \sin u\right)\Big|_{\pi/3}^{5\pi/3}$$

$$= \frac{1}{2}\left(\frac{5\pi}{6} + 2\sqrt{3} - \left(\frac{\pi}{6} - 2\sqrt{3}\right)\right) + \frac{1}{2}\left(\frac{5\pi}{3} + \frac{\sqrt{3}}{2} - \left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right)\right)$$

$$\boxed{\pi - \frac{3\sqrt{3}}{2}}$$

Arc Length in Polar Form

Let f be a function whose derivative is continuous on the interval $\alpha \leq \theta \leq \beta$. The length of the graph of $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ is $s = \int_{\alpha}^{\beta} \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\underbrace{\frac{dr}{d\theta}}_{r'}\right)^2}\, d\theta$

Examples

Find the length of the arc from $\theta = 0$ to $\theta = 2\pi$ for the cardioid $r = f(\theta) = 2 - 2\cos\theta$

$f'(\theta) = 2\sin\theta$

Note: $1-\cos\theta = 2\frac{(1-\cos\theta)}{2}$

$\Rightarrow \sqrt{2}\sqrt{\frac{1-\cos\theta}{2}}$

$u = \frac{1}{2}\theta$

$du = \frac{1}{2}\,d\theta$

$2\,du = d\theta$

$$s = \int_0^{2\pi}\sqrt{4 - 8\cos\theta + 4\cos^2\theta + 4\sin^2\theta}\,d\theta \Rightarrow s = \int_0^{2\pi}\sqrt{4(1 - 2\cos\theta + \underbrace{\cos^2\theta + \sin^2\theta}_{1})}\,d\theta$$

$$s = 2\int_0^{2\pi}\sqrt{2 - 2\cos\theta}\,d\theta \Rightarrow 2\int_0^{2\pi}\sqrt{2(1-\cos\theta)}\,d\theta = 2\sqrt{2}\int_0^{2\pi}\sqrt{1-\cos\theta}\,d\theta$$

$$s = 4\int_0^{2\pi}\sin\frac{\theta}{2}\,d\theta = 8\int_0^{\pi}\sin u\,du = 8\left(-\cos u\Big|_0^{\pi}\right)$$

$$= 8\left(1 - (-1)\right) = \boxed{16}$$

Homework 4/24

10.4 # 7 – 15 pg. π = 3

10.5 #