Lesson 8-8

Improper Integrals

Objective

Students will…

- Be able to recognize improper integrals.
- Be able to evaluate improper integrals that have an infinite limit of integration.
- Be able to evaluate improper integrals that have an infinite discontinuity.

Improper Integrals

Integrals are said to be improper when they have an unbounded or infinite upper and/or lower bounds, i.e. $\int_a^\infty$, $\int_{-\infty}^b$, $\int_{-\infty}^\infty$

These can be evaluated by doing the following

DEFINITION OF IMPROPER INTEGRALS WITH INFINITE INTEGRATION LIMITS

1. If f is continuous on the interval $[a, \infty)$, then

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx.$$

2. If f is continuous on the interval $(-\infty, b]$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx.$$

3. If f is continuous on the interval $(-\infty, \infty)$, then

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx$$

where c is any real number (see Exercise 120).

Examples

Evaluate $\int_1^\infty \frac{dx}{x}$

Examples

Evaluate $\int_0^\infty e^{-x}\,dx$

Examples

Evaluate $\int_0^\infty \frac{1}{x^2+1}dx$

Examples

Evaluate $\int_0^\infty (1 - x)e^{-x}\, dx$

Examples

Evaluate $\int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}} dx$

Special Type of Improper Integral

Theorem 8.5- $\int_1^\infty \frac{dx}{x^p} = \begin{cases} \frac{1}{p-1}, & if\ p > 1 \\ diverges & if\ p \leq 1 \end{cases}$

Example

Evaluate $\int_1^\infty \frac{4}{\sqrt[4]{x}} dx$

Homework 4/12

8.8 #15-31 (odd)

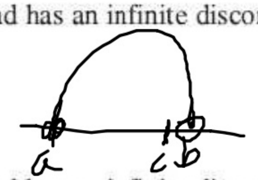
Improper Integrals with Discontinuities

The second type of improper integrals is one that has an infinite discontinuity **<u>at or between</u>** the limits of integration.

DEFINITION OF IMPROPER INTEGRALS WITH INFINITE DISCONTINUITIES

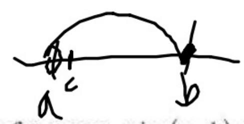
1. If f is continuous on the interval $[a, b)$ and has an infinite discontinuity at b, then

$$\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx.$$



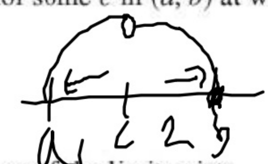
2. If f is continuous on the interval $(a, b]$ and has an infinite discontinuity at a, then

$$\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx.$$



3. If f is continuous on the interval $[a, b]$, except for some c in (a, b) at which f has an infinite discontinuity, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$



In the first two cases, the improper integral **converges** if the limit exists—otherwise, the improper integral **diverges.** In the third case, the improper integral on the left diverges if either of the improper integrals on the right diverges.

disc. @ $x=0=a$ Example

Evaluate $\int_0^1 \frac{dx}{\sqrt[3]{x}}$

$$= \lim_{c\to 0^+} \int_c^1 x^{-1/3}\,dx = \lim_{c\to 0^+}\left(\frac{3}{2}x^{2/3}\Big|_c^1 \right)$$

$$= \lim_{c\to 0^+}\left(\frac{3}{2} - \frac{3}{2}c^{2/3} \right)$$

$$= \frac{3}{2} - 0 = \boxed{\frac{3}{2}}$$

Examples

Evaluate $\int_0^2 \frac{dx}{x^3} = \lim_{c \to 0^+} \int_c^2 x^{-3}\,dx = \lim_{c \to 0^+} \left(-\frac{1}{2} x^{-2} \Big|_c^2 \right)$

$= \lim_{c \to 0^+} \left(-\frac{1}{8} + \frac{1}{2c^2} \right) = -\frac{1}{8} + \infty = \infty$

Diverges

Examples

$u = \sqrt{x}$ $\quad 2\,du = x^{-1/2}$ $\quad x+1 = (\sqrt{x})^2 + 1$

$du = \frac{1}{2}x^{-1/2}\,dx$

Evaluate $\int_0^\infty \frac{dx}{\sqrt{x}(x+1)}$ $= \int_0^1 \frac{dx}{\sqrt{x}(x+1)} + \int_1^\infty \frac{dx}{\sqrt{x}(x+1)}$

$= \lim_{c\to0^+} \int_c^1 \frac{dx}{x\sqrt{x}+\sqrt{x}} + \lim_{b\to\infty} \int_1^b \frac{dx}{x\sqrt{x}+\sqrt{x}} = \lim_{c\to0^+}\left(2\arctan\sqrt{x}\Big|_c^1\right)$

$= \lim_{c\to0^+}\left(2\left(\frac{\pi}{4}\right) - 2\arctan\sqrt{c}\right) + \lim_{b\to\infty}\left(2\arctan\sqrt{b} - 2(\pi/4)\right)$

$= \frac{\pi}{2} - 0 + \pi - \frac{\pi}{2}$

$\boxed{\pi}$

Homework 4/17

8.8 #33-47 (odd), 49, 50

Eval. $\int_{-1}^{2} \frac{dx}{x^3} = \int_{-1}^{0} \frac{dx}{x^3} + \int_{0}^{2} \frac{dx}{x^3}$

disc. @ $x=0$

$$= \lim_{c \to 0^-} \int_{-1}^{c} x^{-3}\,dx + \lim_{c \to 0^+} \int_{c}^{2} x^{-3}\,dx$$

$$= \lim_{c \to 0^-} \left(-\frac{1}{2} x^{-2} \Big|_{-1}^{c} \right) + \lim_{c \to 0^+} \left(-\frac{1}{2} x^{-2} \Big|_{c}^{2} \right) = \lim \left(\frac{-1}{2c^2} + \frac{1}{2} \right) + \lim \left(-\frac{1}{8} + \frac{1}{2c^2} \right)$$

∞ $\quad$ ∞

$\boxed{\infty}$

$$\int_{-1}^{2} x^{-3}\,dx = \left(-\frac{1}{2}x^{-2}\Big|_{-1}^{2}\right)$$

$$-\frac{1}{8} + \frac{1}{2}\ \frac{4}{8} = \boxed{\frac{3}{8}}$$