Lesson 8-8

Improper Integrals

Objective

Students will…

- Be able to recognize improper integrals.
- Be able to evaluate improper integrals that have an infinite limit of integration.
- Be able to evaluate improper integrals that have an infinite discontinuity.

Improper Integrals

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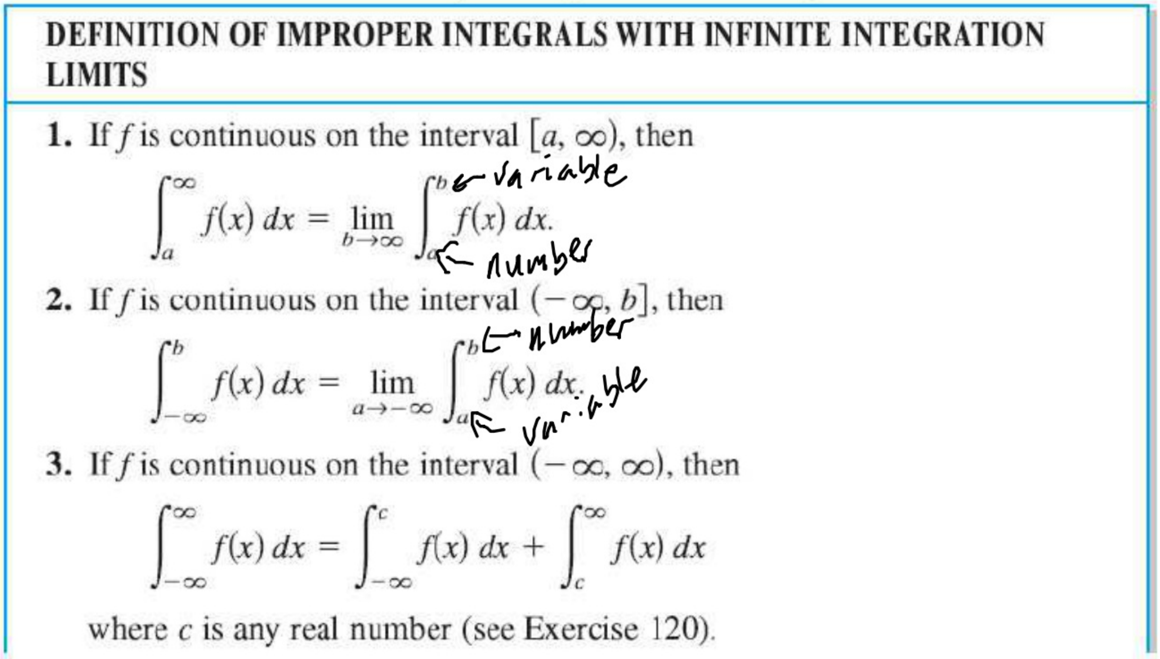
Integrals are said to be improper when they have an unbounded or infinite upper and/or lower bounds, i.e. $\int_a^\infty$, $\int_{-\infty}^b$, $\int_{-\infty}^\infty$

These can be evaluated by doing the following



DEFINITION OF IMPROPER INTEGRALS WITH INFINITE INTEGRATION LIMITS

1. If f is continuous on the interval $[a, \infty)$, then

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_{a\,\leftarrow\text{number}}^{b\,\leftarrow\text{variable}} f(x)\,dx.$$

2. If f is continuous on the interval $(-\infty, b]$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty}\int_{a\,\leftarrow\text{variable}}^{b\,\leftarrow\text{number}} f(x)\,dx.$$

3. If f is continuous on the interval $(-\infty, \infty)$, then

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx$$

where c is any real number (see Exercise 120).

Examples

Evaluate $\int_1^\infty \frac{dx}{x} = \int_1^\infty \frac{1}{x}\,dx$

$$= \lim_{b \to \infty} \int_1^b \frac{1}{x}\,dx = \lim_{b \to \infty} \left(\ln|x| \Big|_1^b \right) = \lim_{b \to \infty} \left(\ln|b| - \overset{0}{\cancel{\ln|1|}} \right)$$

$$= \lim_{b \to \infty} \ln|b| = \infty$$

⊛ Whenever limit $= \infty$ or $-\infty$, we say that the function <u>diverges</u>.

Examples

$u = -x$
$du = -1\,dx$
$-du = dx$

Evaluate $\int_0^\infty e^{-x}\,dx$ $=$

$$= \lim_{b \to \infty} \int_0^b e^{-x}\,dx = \lim_{b \to \infty} \left(-e^{-x} \Big|_0^b \right) = \lim_{b \to \infty} \left(-e^{-b} + 1 \right)$$

$$= \lim_{b \to \infty} \left(-\frac{1}{e^b} + 1 \right)$$

$$0 + 1 = \boxed{+1}$$

(*) Whenever the limit = real number, then the function <u>converges.</u>

$\tan x = 0$

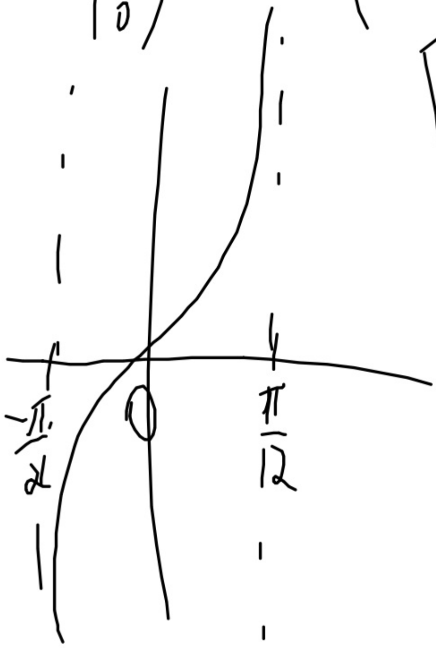
$\arctan 0 = x$

Examples

Evaluate $\int_0^\infty \frac{1}{x^2+1}dx$

$= \lim_{b\to\infty} \int_0^b \frac{1}{x^2+1}dx = \lim_{b\to\infty}\left(\arctan x \Big|_0^b\right) = \lim_{b\to\infty}\left(\arctan b - 0\right)$

$\boxed{\frac{\pi}{2}}$



Examples

$u = 1-x \quad du = -1\,dx$

$V = -e^{-x} \quad dv = e^{-x}\,dx$

Evaluate $\int_0^\infty (1-x)e^{-x}\,dx$

$$= \lim_{b\to\infty} \int_0^b (1-x)e^{-x}\,dx = \lim_{b\to\infty} \left(\underset{\llcorner u}{-e^{-x}}(1-x) - \int_0^b e^{-x}\,dx \right)$$

$$= \lim_{b\to\infty} \left(-e^{-x}(1-x) - \left(-e^{-x} \Big|_0^b \right) \right)$$

$$= \lim_{b\to\infty} \left(-\frac{1}{e^b}(1-b) - \cancel{(1(0))} - \left(-\frac{1}{e^b} + 1 \right) \right)$$

$$\lim_{b\to\infty} \left(\cancel{-\frac{1}{e^b}} + \frac{b}{e^b} + \cancel{\frac{1}{e^b}} - 1 \right) = \lim_{b\to\infty} \left(\frac{b}{e^b} - 1 \right)$$

$$\lim_{b\to\infty} \frac{b}{e^b} - \lim_{b\to\infty} 1$$

$$\overset{lh}{=} \left(\lim_{b\to\infty} \frac{1}{e^b} \right) - 1$$

$$0 - 1 = \boxed{-1}$$

$1+e^{2x}=1+(e^x)^2$

Examples

Evaluate $\int_{-\infty}^{\infty}\frac{e^x}{1+e^{2x}}dx = \int_{-\infty}^{0}\frac{e^x}{1+e^{2x}}dx + \int_{0}^{\infty}\frac{e^x}{1+e^{2x}}dx.$

$$=\lim_{a\to-\infty}\int_a^0\frac{e^x}{1+e^{2x}}dx+\lim_{b\to\infty}\int_0^b\frac{e^x}{1+e^{2x}}dx$$

$$=\lim_{a\to-\infty}\left(\arctan(e^x)\Big|_a^0\right)+\lim_{b\to\infty}\left(\arctan(e^x)\Big|_0^b\right)=\lim_{a\to-\infty}(\arctan(1)-\arctan e^a$$

$$=\lim_{a\to-\infty}\left(\arctan(1)-\arctan(ea)\right)+\lim_{b\to\infty}\left(\arctan(e^b)-\arctan(1)\right)$$

$$=\lim_{a\to-\infty}\left(\pi/4-\arctan(e^{\overset{0}{a}})\right)+\lim_{b\to\infty}\left(\arctan(\overset{\infty}{e^b})-\pi/4\right)$$

$$\cancel{\pi/4}-0\quad+\quad\pi/2-\cancel{\pi/4}$$

$$\boxed{\pi/2}$$

Special Type of Improper Integral

$\frac{dx}{x^p} = \frac{1}{x^p} dx$

Theorem 8.5- $\boxed{\int_1^\infty} \frac{dx}{x^p} = \begin{cases} \frac{1}{p-1}, & if\ p > 1 \\ diverges & if\ p \leq 1 \end{cases}$

Example

Evaluate $\int_1^\infty \frac{4}{\sqrt[4]{x}}\,dx$ $= 4\int_1^\infty \frac{1}{\sqrt[4]{x}}\,dx = 4\int_1^\infty \frac{1}{x^{1/4}}\,dx$

$= \boxed{\infty}$

diverges.

Homework 4/12

8.8 #15-31 (odd)

L'Hopital's Rule

<u>L'Hopital's Rule</u>- Let f and g be functions that are differentiable on an open interval (a, b) containing c, except possibly at c itself. Assume that $g'(x) \neq 0$ for all x in (a, b), except possibly at c itself. If the limit of $\frac{f(x)}{g(x)}$ as x approaches c produces the indeterminate form $\pm\frac{0}{0}$ or $\pm\frac{\infty}{\infty}$, then

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}$$

Provided the limit on the right exists (or is infinite).

Example

$\lim_{x \to 0} \frac{e^{2x}-1}{x}$

Examples

$$\lim_{x \to \infty} \frac{\ln x}{x}$$

Examples

$$\lim_{x \to -\infty} \frac{x^2}{e^{-x}}$$

Examples

$$\lim_{x\to\infty} e^{-x}\sqrt{x}$$

Examples

$$\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^{x}$$

Examples

$\lim_{x \to 0^+} (\sin x)^x$

Examples

$$\lim_{x \to 1^+} \left(\frac{1}{\ln x} - \frac{1}{x-1} \right)$$

Homework 4/10

8.7 #11-51 (e.o.o)