Lesson 8-7

Indeterminate Forms And L'Hopital's Rule

Objective

Students will…

- Be able to recognize indeterminate forms for limits.
- Be able to apply L'Hopital's Rule to evaluate the limit of functions in indeterminate forms.

$$e^{2x} = e^{2\cdot x} = (e^x)^2$$

Indeterminate Forms

What is $\frac{0}{0}$? What about $\frac{\infty}{\infty}$?

Consider the following: $\lim_{x\to -1} \frac{2x^2-2}{x+1} = \frac{2(-1)^2-2}{-1+1} = \frac{0}{0}$

$$= \frac{2(x^2-1)}{x+1} = \frac{2\cancel{(x+1)}(x-1)}{\cancel{x+1}} = \lim_{x\to -1} 2(x-1) = \boxed{-4}$$

What about $\lim_{x\to 0} \frac{e^{2x}-1}{e^x-1}$? $= \frac{e^0-1}{e^0-1} = \frac{1-1}{1-1} = \frac{0}{0}$

$$\frac{(e^x)^2-1}{e^x-1} = \frac{(e^x+1)\cancel{(e^x-1)}}{\cancel{e^x-1}} = \lim_{x\to 0} e^x+1 = e^0+1 = 1+1 = \boxed{2}$$

Indeterminate Forms

For the same reasons the following are all in **indeterminate forms**:

$$\frac{0}{0}, \frac{\infty}{\infty}, \boxed{0 \cdot \infty, 1^{\infty}, \infty^{0}, 0^{0}}, \infty - \infty$$

We were able to find the limits of the previous problems using algebra. However, some problems are not possible to use algebra to evaluate the limit of indeterminate forms. For example, $\lim_{x \to 0}(\frac{e^{2x}-1}{x})$.

When algebra fails us, we can use the **L'Hopital's Rule**.

L'Hopital's Rule

<u>L'Hopital's Rule</u>- Let f and g be functions that are differentiable on an open interval (a, b) containing c, except possibly at c itself. Assume that $g'(x) \neq 0$ for all x in (a, b), except possibly at c itself. If the limit of $\frac{f(x)}{g(x)}$ as x approaches c produces the indeterminate form $\boxed{\pm\frac{0}{0} \text{ or } \pm\frac{\infty}{\infty}}$, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

Provided the limit on the right exists (or is infinite).

Example

$$\lim_{x\to 0}\frac{e^{2x}-1}{x} \;\; \frac{0}{0} \;\; \stackrel{lh}{=} \lim_{x\to 0}\frac{2e^{2x}-0}{1} = \lim_{x\to 0} 2e^{2x} = 2e^{0} = \boxed{2}$$

Examples

$$\lim_{x\to\infty} \frac{\ln x}{x} \overset{\frac{\infty}{\infty}}{\underset{\text{LH}}{=}} \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{1}{x} = \frac{1}{\infty} = \boxed{0}$$

Examples

$x^2 e^x$ $\quad \frac{\infty}{\infty}$

$$\lim_{x\to-\infty} \frac{x^2}{e^{-x}} \overset{lh}{=} \lim_{x\to-\infty} \frac{2x}{-e^{-x}} \overset{lh}{=} \lim_{x\to-\infty} \frac{2}{e^{-x}} = \frac{2}{e^{\infty}} = \boxed{0}$$

Examples

$$\lim_{x\to\infty} e^{-x}\sqrt{x} = \lim_{x\to\infty} \frac{\sqrt{x}}{e^x} \overset{\frac{\infty}{\infty}}{\underset{}{\overset{LH}{=}}} \lim_{x\to\infty} \frac{\frac{1}{2}x^{-1/2}}{e^x} = \lim_{x\to\infty} \frac{\frac{1}{2}}{e^x x^{1/2}} = \boxed{0}$$

Examples

$1/x = x^{-1}$ $\quad \frac{d}{dx} x^{-1} = -x^{-2}$

$$y = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} \Rightarrow \ln(y) = \ln\left(\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}\right) = \lim_{x\to\infty} \ln\left(1+\frac{1}{x}\right)^{x}$$

$$\ln(y) = \lim_{x\to\infty} x \ln\left(1+\frac{1}{x}\right) = \lim_{x\to\infty}^{\frac{0}{0}} \frac{\ln(1+\frac{1}{x})}{1/x} \overset{lh}{=} \frac{\frac{1}{1+1/x}\cdot \cancel{-x^{-2}}}{\cancel{-x^{-2}}}$$

$$= \lim_{x\to\infty}\left(\frac{1}{1+1/x}\right) = \frac{1}{1+1/\infty} = \frac{1}{1+0} = 1$$

$$\ln y = 1$$

$$e^{\ln y} = e^{1}$$

$$y = e^{1} = \boxed{e} \checkmark$$

$\frac{0}{0}$ or $\frac{\infty}{\infty}$

Examples

$$y = \lim_{x\to 0^+} (\sin x)^x \quad 0^0 \Rightarrow \ln y = \ln\left(\lim_{x\to 0^+} (\sin x)^x\right) = \lim_{x\to 0^+}\left(x \ln(\sin x)\right)$$

$$\ln y = \lim_{x\to 0^+}\left(x \ln(\sin x)\right) = \lim_{x\to 0^+}\left(\frac{\ln(\sin x)}{\frac{1}{x}}\right) \frac{\infty}{\infty}$$

$$\frac{1}{x} = x^{-1}$$

$$\stackrel{lh}{=} \lim_{x\to 0^+}\left(\frac{\frac{1}{\sin x}\cdot \cos x}{-x^{-2}}\right) = \lim_{x\to 0^+}\left(\frac{\cot x}{-x^{-2}}\right) = \lim_{x\to 0^+}\left(\frac{-x^2}{\tan x}\right) \frac{0}{0}$$

$$\stackrel{lh}{=} \lim_{x\to 0^+}\left(\frac{-2x}{\sec^2 x}\right) = \frac{0}{1} = 0$$

$$\ln y = 0$$

$$e^0$$

$$y = \boxed{1}$$

Examples

$\frac{0}{0}$ or $\frac{\infty}{\infty}$

$\infty - \infty$

$$\lim_{x\to 1^+}\left(\frac{1}{\ln x}(x-1) - \frac{1}{x-1}(\ln x)\right) = \lim_{x\to 1^+}\left(\frac{x-1-\ln x}{\underbrace{(\ln x)}_{f}\underbrace{(x-1)}_{g}}\right)\frac{0}{0} \overset{lh}{=} \lim_{x\to 1^+}\left(\frac{1-\frac{1}{x}}{\frac{x-1}{x}+\frac{\ln x}{1}}\right)\frac{0}{0}$$

$$= \lim_{x\to 1^+}\left(\frac{\frac{x-1}{\cancel{x}}}{\frac{x-1+x\ln x}{\cancel{x}}}\right) = \lim_{x\to 1^+}\left(\frac{x-1}{\underset{f\ \ g}{x-1+x\ln x}}\right) \overset{lh}{=} \lim_{x\to 1^+}\left(\frac{1}{1+\ln x+1}\right)$$

$$= \lim_{x\to 1^+}\left(\frac{1}{\ln x+2}\right) = \boxed{\frac{1}{2}}$$

Homework 4/10

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