

Probability is the mathematical study of chance and random processes. The laws of probability are essential for understanding genetics, opinion polls, pricing stock options, setting odds in horseracing and games of


Many questions in mathematics involve counting. For example, in how many ways can a committee of two men and three women be chosen from a group of 35 men and 40 women? How many different license plates can be made using three letters followed by three digits? How many different poker hands are possible?

Closely related to the problem of counting is that of probability. We consider questions such as these: If a committee of five people is chosen randomly from a group of 35 men and 40 women, what are the chances that no women will be chosen for the committee? What is the likelihood of getting a straight flush in a poker game? In studying probability, we give precise mathematical meaning to phrases such as "what are the chances . . . ?" and "what is the likelihood . . . ?"

### 11.1 COUNTING PRINCIPLES



FIGURE 1
Tree diagram

Suppose that three towns, Ashbury, Brampton, and Carmichael, are located in such a way that two roads connect Ashbury to Brampton and three roads connect Brampton to Carmichael. How many different routes can one take to travel from Ashbury to Carmichael via Brampton? The key idea in answering this question is to consider the problem in stages. At the first stage-from Ashbury to Bramptonthere are two choices. For each of these choices, there are three choices to make at the second stage-from Brampton to Carmichael. Thus, the number of different routes is $2 \times 3=6$. These routes are conveniently enumerated by a tree diagram as in Figure 1.

The method used to solve this problem leads to the following principle.

## FUNDAMENTAL COUNTING PRINCIPLE

Suppose that two events occur in order. If the first can occur in $m$ ways and the second in $n$ ways (after the first has occurred), then the two events can occur in order in $m \times n$ ways.

There is an immediate consequence of this principle for any number of events: If $E_{1}, E_{2}, \ldots, E_{k}$ are events that occur in order and if $E_{1}$ can occur in $n_{1}$ ways, $E_{2}$ in $n_{2}$ ways, and so on, then the events can occur in order in $n_{1} \times n_{2} \times \cdots \times n_{k}$ ways.

## EXAMPLE 1 - Using the Fundamental Counting Principle

An ice-cream store offers three types of cones and 31 flavors. How many different single-scoop ice-cream cones is it possible to buy at this store?


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## SOLUTION

There are two choices: type of cone and flavor of ice cream. At the first stage we choose a type of cone, and at the second stage we choose a flavor. We can think of the different stages as boxes:


The first box can be filled in three ways and the second in 31 ways:


Thus, by the Fundamental Counting Principle, there are $3 \times 31=93$ ways of choosing a single-scoop ice-cream cone at this store.

EXAMPLE 2 - Using the Fundamental Counting Principle


In a certain state, automobile license plates display three letters followed by three digits. How many such plates are possible if repetition of the letters
(a) is allowed?
(b) is not allowed?

## SOLUTION

(a) There are six choices, one for each letter or digit on the license plate. As in the preceding example, we sketch a box for each stage:


At the first stage, we choose a letter (from 26 possible choices); at the second stage, another letter (again from 26 choices); at the third stage, another letter ( 26 choices); at the fourth stage, a digit (from 10 possible choices); at the fifth stage, a digit (again from 10 choices); and at the sixth stage, another digit (10 choices). By the Fundamental Counting Principle, the number of possible license plates is

$$
26 \times 26 \times 26 \times 10 \times 10 \times 10=17,576,000
$$

(b) If repetition of letters is not allowed, then we can arrange the choices as follows:


At the first stage, we have 26 letters to choose from, but once the first letter is chosen, there are only 25 letters to choose from at the second stage. Once the first two letters are chosen, 24 letters are left to choose from for the third stage. The digits are chosen as before. Thus, the number of possible license plates in this case is

$$
26 \times 25 \times 24 \times 10 \times 10 \times 10=15,600,000
$$



Factorial notation is explained on page 851.

## EXAMPLE 3 ■ Using Factorial Notation

In how many different ways can a race with six runners be completed? Assume there is no tie.

## SOLUTION

There are six possible choices for first place, five choices for second place (since only five runners are left after first place has been decided), four choices for third place, and so on. So, by the Fundamental Counting Principle, the number of different ways this race can be completed is

$$
6 \times 5 \times 4 \times 3 \times 2 \times 1=6!=720
$$

### 11.1 EXERCISES

1. A vendor sells ice cream from a cart on the boardwalk. He offers vanilla, chocolate, strawberry, and pistachio ice cream, served on either a waffle, sugar, or plain cone. How many different single-scoop ice-cream cones can you buy from this vendor?
2. How many three-letter "words" (strings of letters) can be formed using the 26 letters of the alphabet if repetition of letters
(a) is allowed?
(b) is not allowed?
3. How many three-letter "words" (strings of letters) can be formed using the letters $W X Y Z$ if repetition of letters
(a) is allowed?
(b) is not allowed?
4. Eight horses are entered in a race.
(a) How many different orders are possible for completing the race?
(b) In how many different ways can first, second, and third places be decided? (Assume there is no tie.)
5. A multiple-choice test has five questions with four choices for each question. In how many different ways can the test be completed?
6. Telephone numbers consist of seven digits; the first digit cannot be 0 or 1 . How many telephone numbers are possible?
7. In how many different ways can a race with five runners be completed? (Assume there is no tie.)
8. In how many ways can five people be seated in a row of five seats?
9. A restaurant offers six different main courses, eight types of drinks, and three kinds of desserts. How many different meals consisting of a main course, a drink, and a dessert does the restaurant offer?
10. In how many ways can five different mathematics books be placed next to each other on a shelf?
11. Towns A, B, C, and D are located in such a way that there are four roads from $A$ to $B$, five roads from $B$ to $C$, and six roads from C to D . How many routes are there from town A to town D via towns B and C?
12. In a family of four children, how many different boy-girl birth-order combinations are possible? (The birth orders $B B B G$ and $B B G B$ are different.)
13. A coin is flipped five times, and the resulting sequence of heads and tails is recorded. How many such sequences are possible?
14. A red die and a white die are rolled, and the numbers showing are recorded. How many different outcomes are possible? (The singular form of the word dice is die.)

15. A red die, a blue die, and a white die are rolled, and the numbers showing are recorded. How many different outcomes are possible?
16. Two cards are chosen in order from a deck. In how many ways can this be done if
(a) the first card must be a spade and the second must be a heart?
(b) both cards must be spades?
17. A girl has 5 skirts, 8 blouses, and 12 pairs of shoes. How many different skirt-blouse-shoe outfits can she wear? (Assume that each item matches all the others, so she is willing to wear any combination.)
18. A company's employee ID number system consists of one letter followed by three digits. How many different ID numbers are possible with this system?
19. A company has 2844 employees. Each employee is to be given an ID number that consists of one letter followed by two digits. Is it possible to give each employee a different ID number using this scheme? Explain.
20. An all-star baseball team has a roster of seven pitchers and three catchers. How many pitcher-catcher pairs can the manager select from this roster?
21. Standard automobile license plates in California display a nonzero digit, followed by three letters, followed by three digits. How many different standard plates are possible in this system?

22. A combination lock has 60 different positions. To open the lock, the dial is turned to a certain number in the clockwise direction, then to a number in the counterclockwise direction, and finally to a third number in the clockwise direction. If successive numbers in the combination cannot be the same, how many different combinations are possible?

23. A true-false test contains ten questions. In how many different ways can this test be completed?
24. An automobile dealer offers five models. Each model comes in a choice of four colors, three types of stereo equipment, with or without air conditioning, and with or without a sunroof. In how many different ways can a customer order an auto from this dealer?
25. The registrar at a certain university classifies students according to a major, minor, year ( $1,2,3,4$ ), and sex (M, F). Each student must choose one major and either one or no minor from the 32 fields taught at this university. How many different student classifications are possible?
26. How many monograms consisting of three initials are possible?
27. A state has registered 8 million automobiles. To simplify the license plate system, a state employee suggests that each plate display only two letters followed by three digits. Will this system create enough different license plates for all the vehicles registered?
28. A state license plate design has six places. Each plate begins with a fixed number of letters, and the remaining places are filled with digits. (For example, one letter followed by five digits, two letters followed by four digits, and so on.) The state has 17 million registered vehicles.
(a) The state decides to change to a system consisting of one letter followed by five digits. Will this design allow for enough different plates to accommodate all the vehicles registered?
(b) Find a system that will be sufficient if the smallest possible number of letters is to be used.
29. In how many ways can a president, vice president, and secretary be chosen from a class of 30 students?
30. In how many ways can a president, vice president, and secretary be chosen from a class of 20 females and 30 males if the president must be a female and the vice president a male?
31. A senate subcommittee consists of ten Democrats and seven Republicans. In how many ways can a chairman, vice chairman, and secretary be chosen if the chairman must be a Democrat and the vice chairman must be a Republican?
32. Social Security numbers consist of nine digits, with the first digit between 0 and 6, inclusive. How many Social Security numbers are possible?
33. Five-letter "words" are formed using the letters $A, B, C, D$, $E, F, G$. How many such words are possible for each of the following conditions?
(a) No condition is imposed.
(b) No letter can be repeated in a word.
(c) Each word must begin with the letter $A$.
(d) The letter $C$ must be in the middle.
(e) The middle letter must be a vowel.
34. How many five-letter palindromes are possible? (A palindrome is a string of letters that reads the same backward and forward, such as the string $X C Z C X$.)
35. A certain computer programming language allows names of variables to consist of two characters, the first being any letter and the second any letter or digit. How many names of variables are possible?
36. How many different three-character code words consisting of letters or digits are possible for the following code designs?
(a) The first entry must be a letter.
(b) The first entry cannot be zero.
37. In how many ways can four men and four women be seated in a row of eight seats for the following situations?
(a) The women are to be seated together, and the men are to be seated together.
(b) They are to be seated alternately by gender.
38. In how many ways can five different mathematics books be placed on a shelf if the two algebra books are to be placed next to each other?

39. Eight mathematics books and three chemistry books are to be placed on a shelf. In how many ways can this be done if the mathematics books are next to each other and the chemistry books are next to each other?
40. Three-digit numbers are formed using the digits $2,4,5$, and 7 , with repetition of digits allowed. How many such numbers can be formed if
(a) the numbers are less than 700 ?
(b) the numbers are even?
(c) the numbers are divisible by 5 ?
41. How many three-digit odd numbers can be formed using the digits $1,2,4$, and 6 if repetition of digits is not allowed?

## DISCOVERY•DISCUSSION

42. Pairs of Initials Explain why in any group of 677 people, at least two people must have the same pair of initials.
43. Area Codes Until recently, telephone area codes in the United States, Canada, and the Caribbean islands were chosen according to the following rules: (i) The first digit cannot be 0 or a 1 , and (ii) the second digit must be a 0 or a 1. But in 1995, the second rule was abandoned when the area code 360 was introduced in parts of western Washington State. Since then, many other new area codes that violate Rule (ii) have come into use, although Rule (i) still remains in effect.
(a) How many area code + telephone number combinations were possible under the old rules? (See Exercise 6 for a description of local telephone numbers.)
(b) How many area code + telephone number combinations are now possible under the new rules?
(c) Why do you think it was necessary to make this change?
(d) How many area codes that violate Rule (ii) are you personally familiar with?

### 11.2 PERMUTATIONS AND COMBINATIONS

Permutations of three colored squares


In this section we single out two important special cases of the Fundamental Counting Principle-permutations and combinations.

## Permutations

A permutation of a set of distinct objects is an ordering of these objects. For example, some permutations of the letters $A B C D W X Y Z$ are

XAYBZWCD ZAYBCDWX DBWAZXYC YDXAWCZB
How many such permutations are possible? Since there are eight choices for the first position, seven for the second (after the first has been chosen), six for the third (after the first two have been chosen), and so on, the Fundamental Counting Principle tells us that the number of possible permutations is

$$
8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1=40,320
$$

This same reasoning with 8 replaced by $n$ leads to the following observation.

The number of permutations of $n$ objects is $n!$.

How many permutations consisting of five letters can be made from these same eight letters? Some of these permutations are

$$
\begin{array}{llll}
X Y Z W C & A Z D W X & A Z X Y B & W D X Z B
\end{array}
$$

Again, there are eight choices for the first position, seven for the second, six for the third, five for the fourth, and four for the fifth. By the Fundamental Counting Principle, the number of such permutations is

$$
8 \times 7 \times 6 \times 5 \times 4=6720
$$

In general, if a set has $n$ elements, then the number of ways of ordering $r$ elements from the set is denoted by $\boldsymbol{P}(\boldsymbol{n}, \boldsymbol{r})$ and is called the number of permutations of $\boldsymbol{n}$ objects taken $\boldsymbol{r}$ at a time.

We have just shown that $P(8,5)=6720$. The same reasoning used to find $P(8,5)$ will help us find a general formula for $P(n, r)$. Indeed, there are $n$ objects and $r$ positions to place them in. Thus, there are $n$ choices for the first position, $n-1$ choices for the second, $n-2$ choices for the third, and so on. The last position can be filled in $n-r+1$ ways. By the Fundamental Counting Principle,

$$
P(n, r)=n(n-1)(n-2) \cdots(n-r+1)
$$

This formula can be written more compactly using factorial notation:

$$
\begin{aligned}
P(n, r) & =n(n-1)(n-2) \cdots(n-r+1) \\
& =\frac{n(n-1)(n-2) \cdots(n-r+1)(n-r) \cdots 3 \cdot 2 \cdot 1}{(n-r) \cdots 3 \cdot 2 \cdot 1}=\frac{n!}{(n-r)!}
\end{aligned}
$$

## PERMUTATIONS OF $n$ OBJECTS TAKEN $r$ AT A TIME

The number of permutations of $n$ objects taken $r$ at a time is

$$
P(n, r)=\frac{n!}{(n-r)!}
$$

## EXAMPLE 1 - Finding the Number of Permutations

A club has nine members. In how many ways can a president, vice president, and secretary be chosen from the members of this club?

## SOLUTION

We need the number of ways of selecting three members, in order, for the positions of president, vice president, and secretary from the nine club members. This number is

$$
P(9,3)=\frac{9!}{(9-3)!}=\frac{9!}{6!}=9 \times 8 \times 7=504
$$

## EXAMPLE 2 - Finding the Number of Permutations



From 20 raffle tickets in a hat, four tickets are to be selected in order. The holder of the first ticket wins a car, the second a motorcycle, the third a bicycle, and the fourth a skateboard. In how many different ways can these prizes be awarded?

## SOLUTION

The order in which the tickets are chosen determines who wins each prize. So, we need to find the number of ways of selecting four objects, in order, from 20 objects (the tickets). This number is

$$
P(20,4)=\frac{20!}{(20-4)!}=\frac{20!}{16!}=20 \times 19 \times 18 \times 17=116,280
$$

## Distinguishable Permutations

If we have a collection of ten balls, each a different color, then the number of permutations of these balls is $P(10,10)=10$ !. If all ten balls are red, then we have just one distinguishable permutation because all the ways of ordering these balls look
exactly the same. In general, when considering a set of objects, some of which are of the same kind, then two permutations are distinguishable if one cannot be obtained from the other by interchanging the positions of elements of the same kind. For example, if we have ten balls, of which six are red and the other four are each a different color, then how many distinguishable permutations are possible? The key point here is that balls of the same color are not distinguishable. So each rearrangement of the red balls, keeping all the other balls fixed, gives essentially the same permutation. Since there are 6 ! rearrangements of the red balls for each fixed position of the other balls, the total number of distinguishable permutations is $10!/ 6!$. The same type of reasoning gives the following general rule.

## DISTINGUISHABLE PERMUTATIONS

If a set of $n$ objects consists of $k$ different kinds of objects with $n_{1}$ objects of the first kind, $n_{2}$ objects of the second kind, $n_{3}$ objects of the third kind, and so on, where $n_{1}+n_{2}+\cdots+n_{k}=n$, then the number of distinguishable permutations of these objects is

$$
\frac{n!}{n_{1}!n_{2}!n_{3}!\cdots n_{k}!}
$$

EXAMPLE 3 - Finding the Number of Distinguishable Permutations
Find the number of different ways of placing 15 balls in a row given that 4 are red, 3 are yellow, 6 are black, and 2 are blue.

## SOLUTION

We want to find the number of distinguishable permutations of these balls. By the formula, this number is

$$
\frac{15!}{4!3!6!2!}=6,306,300
$$

Suppose we have 15 wooden balls in a row and four colors of paint: red, yellow, black, and blue. In how many different ways can the 15 balls be painted in such a way that we have 4 red, 3 yellow, 6 black, and 2 blue balls? A little thought will show that this number is exactly the same as that calculated in Example 3. This way of looking at the problem is somewhat different, however. Here we think of the number of ways to partition the balls into four groups, each containing 4, 3, 6, and 2 balls to be painted red, yellow, black, and blue, respectively. The next example shows how this reasoning is used.

EXAMPLE 4 - Finding the Number of Partitions
Fourteen construction workers are to be assigned to three different tasks. Seven workers are needed for mixing cement, five for laying bricks, and two for carrying
the bricks to the brick layers. In how many different ways can the workers be assigned to these tasks?

SOLUTION
We need to partition the workers into three groups containing 7, 5, and 2 workers, respectively. This number is

$$
\frac{14!}{7!5!2!}=72,072
$$

## Combinations

When finding permutations, we are interested in the number of ways of ordering elements of a set. In many counting problems, however, order is not important. For example, a poker hand is the same hand, regardless of how it is ordered. A poker player interested in the number of possible hands wants to know the number of ways of drawing five cards from 52 cards, without regard to the order in which the cards of a given hand are dealt. In this section we develop a formula for counting in situations such as this, where order doesn't matter.

A combination of $r$ elements of a set is any subset of $r$ elements from the set (without regard to order). If the set has $n$ elements, then the number of combinations of $r$ elements is denoted by $\boldsymbol{C}(\boldsymbol{n}, \boldsymbol{r})$ and is called the number of combinations of $\boldsymbol{n}$ elements taken $r$ at a time.

For example, consider a set with the four elements, $A, B, C$, and $D$. The combinations of these four elements taken three at a time are

$$
A B C \quad A B D \quad A C D \quad B C D
$$

The permutations of these elements taken three at a time are

| $A B C$ | $A B D$ | $A C D$ | $B C D$ |
| :--- | :--- | :--- | :--- |
| $A C B$ | $A D B$ | $A D C$ | $B D C$ |
| $B A C$ | $B A D$ | $C A D$ | $C B D$ |
| $B C A$ | $B D A$ | $C D A$ | $C D B$ |
| $C A B$ | $D A B$ | $D A C$ | $D B C$ |
| $C B A$ | $D B A$ | $D C A$ | $D C B$ |

We notice that the number of combinations is a lot fewer than the number of permutations. In fact, each combination of three elements generates 3! permutations. So

$$
C(4,3)=\frac{P(4,3)}{3!}=\frac{4!}{3!(4-3)!}=4
$$

In general, each combination of $r$ objects gives rise to $r$ ! permutations of these objects.

In Section 10.6 we denoted $C(n, r)$ by
$\binom{n}{r}$, but it is customary to use the notation $C(n, r)$ in the context of counting. For an explanation of why these are the same, see Exercise 76.

Thus

$$
C(n, r)=\frac{P(n, r)}{r!}=\frac{n!}{r!(n-r)!}
$$

## COMBINATIONS OF $n$ OBJECTS TAKEN $r$ AT A TIME

The number of combinations of $n$ objects taken $r$ at a time is

$$
C(n, r)=\frac{n!}{r!(n-r)!}
$$

The key difference between permutations and combinations is order. If we are interested in ordered arrangements, then we are counting permutations; but if we are concerned with subsets without regard to order, then we are counting combinations. Compare Examples 5 and 6 below (where order doesn't matter) with Examples 1 and 2 (where order does matter).

EXAMPLE 5 ■ Finding the Number of Combinations
A club has nine members. In how many ways can a committee of three be chosen from the members of this club?

## SOLUTION

We need the number of ways of choosing three of the nine members. Order is not important here, because the committee is the same no matter how its members are ordered. So, we want the number of combinations of nine objects (the club members) taken three at a time. This number is

$$
C(9,3)=\frac{9!}{3!(9-3)!}=\frac{9!}{3!6!}=\frac{9 \times 8 \times 7}{3 \times 2 \times 1}=84
$$

## EXAMPLE 6 - Finding the Number of Combinations

From 20 raffle tickets in a hat, four tickets are to be chosen at random. The holders of the winning tickets are to be awarded free trips to the Bahamas. In how many ways can the four winners be chosen?

## SOLUTION

We need to find the number of ways of choosing four winners from 20 entries. The order in which the tickets are chosen doesn't matter, because the same prize is awarded to each of the four winners. So, we want the number of combinations of 20 objects (the tickets) taken four at a time. This number is

$$
C(20,4)=\frac{20!}{4!(20-4)!}=\frac{20!}{4!16!}=\frac{20 \times 19 \times 18 \times 17}{4 \times 3 \times 2 \times 1}=4845
$$



Ronald Graham, born in Taft, California, in 1935, is considered the world's leading mathematician in the field of combinatorics, the branch of mathematics that deals with counting. For many years Graham headed the Mathematical Studies Center at Bell Laboratories in Murray Hill, New Jersey, where he solved key problems for the telephone industry. During the Apollo program, NASA needed to evaluate mission schedules so that the three astronauts aboard the spacecraft could find the time to perform all the necessary tasks. The number of ways to allot these tasks were astronomical-too vast for even a computer to sort out. Graham, using his knowledge of combinatorics, was able to reassure NASA that there were easy ways of solving their problem that were not too far from the theoretically best possible solution. Besides being a prolific mathematician, Graham is an accomplished juggler (he has been on stage with the Cirque du Soleil and is a past president of the International Jugglers Association). Several of his research papers address the mathematical aspects of juggling. He is also fluent in Mandarin Chinese and Japanese, and once spoke with President Jiang in his native language.

If a set $S$ has $n$ elements, then $C(n, k)$ is the number of ways of choosing $k$ elements from $S$, that is, the number of $k$-element subsets of $S$. Thus, the number of subsets of $S$ of all possible sizes is given by the sum

$$
C(n, 0)+C(n, 1)+C(n, 2)+\cdots+C(n, n)=2^{n}
$$

(See Section 10.6, Exercise 52, where this sum is discussed.)

A set with $n$ elements has $2^{n}$ subsets.

## EXAMPLE 7 - Finding the Number of Subsets of a Set

A pizza parlor offers the basic cheese pizza and a choice of 16 toppings. How many different kinds of pizza can be ordered at this pizza parlor?

## SOLUTION

We need the number of possible subsets of the 16 toppings (including the empty set, which corresponds to a plain cheese pizza). Thus, $2^{16}=65,536$ different pizzas can be ordered.

## Problem Solving with Permutations and Combinations

The crucial step in solving counting problems is deciding whether to use permutations, combinations, or the Fundamental Counting Principle. In some cases, the solution of a problem may require using more than one of these principles. Here are some general guidelines to help us decide how to apply these principles.

## GUIDELINES FOR SOLVING COUNTING PROBLEMS

1. FUNDAMENTAL COUNTING PRINCIPLE. When consecutive choices are being made, use the Fundamental Counting Principal.
2. DOES THE ORDER MATTER? When we want to find the number of ways of picking $r$ objects from $n$ objects, we need to ask ourselves: Does the order in which we pick the objects matter?

If the order matters, we use permutations.
If the order doesn't matter, we use combinations.

## EXAMPLE 8 - A Problem Involving Combinations

A group of 25 campers contains 15 women and 10 men. In how many ways can a scouting party of 5 be chosen if it must consist of 3 women and 2 men?

We could have first chosen the four unordered members of the commit-tee-in $C(20,4)$ ways-and then the three officers from the remaining 16 members, in $P(16,3)$ ways. Check that this gives the same answer.

## SOLUTION

Three women can be chosen from the 15 women in the group in $C(15,3)$ ways, and two men can be chosen from the 10 men in the group in $C(10,2)$ ways. Thus, by the Fundamental Counting Principle the number of ways of choosing the scouting party is

$$
C(15,3) \times C(10,2)=455 \times 45=20,475
$$

## EXAMPLE 9 - A Problem Involving Permutations and Combinations

A committee of seven-consisting of a chairman, a vice chairman, a secretary, and four other members-is to be chosen from a class of 20 students. In how many ways can this committee be chosen?

## SOLUTION

In choosing the three officers, order is important. So, the number of ways of choosing them is

$$
P(20,3)=6840
$$

Next, we need to choose four other students from the 17 remaining. Since order doesn't matter in this case, the number of ways of doing this is

$$
C(17,4)=2380
$$

Thus, by the Fundamental Counting Principle the number of ways of choosing this committee is

$$
P(20,3) \times C(17,4)=6840 \times 2380=16,279,200
$$

EXAMPLE 10 - A Group Photograph
[1]
Twelve employees at a company picnic are to stand in a row for a group photograph. In how many ways can this be done if
(a) Jane and John insist on standing next to each other?
(b) Jane and John refuse to stand next to each other?


## SOLUTION

(a) Since the order in which the people stand is important, we use permutations. But we can't use the formula for permutations directly. Since Jane and John
insist on standing together, let's think of them as one object. Thus, we have 11 objects to arrange in a row and there are $P(11,11)$ ways of doing this. For each of these arrangements, there are two ways of having Jane and John stand together-Jane-John or John-Jane. Thus, by the Fundamental Counting Principle the total number of arrangements is

$$
2 \times P(11,11)=2 \times 11!=79,833,600
$$

(b) There are $P(12,12)$ ways of arranging the 12 people. Of these, $2 \times P(11,11)$ have Jane and John standing together [by part (a)]. All the rest have Jane and John standing apart. So the number of arrangements with Jane and John apart is

$$
P(12,12)-2 \times P(11,11)=12!-2 \times 11!=399,168,000
$$

### 11.2 EXERCISES

1-6 ■ Evaluate the expression.

1. $P(8,3)$
2. $P(9,2)$
3. $P(11,4)$
4. $P(10,5)$
5. $P(100,1)$
6. $P(99,3)$
7. In how many different ways can a president, vice president, and secretary be chosen from a class of 15 students?
8. In how many different ways can first, second, and third prizes be awarded in a game with eight contestants?
9. In how many different ways can six of ten people be seated in a row of six chairs?
10. In how many different ways can six people be seated in a row of six chairs?
11. How many three-letter "words" can be made from the letters $F G H I J K$ ? (Letters may not be repeated.)
12. How many permutations are possible from the letters of the word LOVE?
13. How many different three-digit whole numbers can be formed using the digits $1,3,5$, and 7 if no repetition of digits is allowed?
14. A pianist plans to play eight pieces at a recital. In how many ways can she arrange these pieces in the program?
15. In how many different ways can a race with nine runners be completed, assuming there is no tie?
16. A ship carries five signal flags of different colors. How many different signals can be sent by hoisting exactly three of the five flags on the ship's flagpole in different orders?
17. In how many ways can first, second, and third prizes be awarded in a contest with 1000 contestants?
18. In how many ways can a president, vice president, secretary, and treasurer be chosen from a class of 30 students?
19. In how many ways can five students be seated in a row of five chairs if Jack insists on sitting in the first chair?

20. In how many ways can the students in Exercise 19 be seated if Jack insists on sitting in the middle chair?

21-24 ■ Find the number of distinguishable permutations of the given letters.
21. $A A A B B C$
22. $A A A B B B C C C$
23. $A A B C D$
24. $A B C D D D E E$
25. In how many ways can two blue marbles and four red marbles be arranged in a row?
26. In how many different ways can five red balls, two white balls, and seven blue balls be arranged in a row?
27. In how many different ways can four pennies, three nickels, two dimes, and three quarters be arranged in a row?
28. In how many different ways can the letters of the word ELEEMOSYNARY be arranged?
29. A man bought three vanilla ice-cream cones, two chocolate cones, four strawberry cones, and five butterscotch cones for his 14 chidren. In how many ways can he distribute the cones among his children?
30. When seven students take a trip, they find a hotel with three rooms available-a room for one person, a room for two people, and a room for three people. In how many different ways can the students be assigned to these rooms? (One student has to sleep in the car.)
31. Eight workers are cleaning a large house. Five are needed to clean windows, two to clean the carpets, and one to clean the rest of the house. In how many different ways can these tasks be assigned to the eight workers?
32. A jogger jogs every morning to his health club, which is eight blocks east and five blocks north of his home. He always takes a route that is as short as possible, but he likes to vary it (see the figure). How many different routes can he take? [Hint: The route shown can be thought of as ENNEEENENEENE, where $E$ is East and $N$ is North.]


33-38 ■ Evaluate the expression.
33. $C(8,3)$
34. $C(9,2)$
35. $C(11,4)$
36. $C(10,5)$
37. $C(100,1)$
38. $C(99,3)$
39. In how many ways can three books be chosen from a group of six?
40. In how many ways can three pizza toppings be chosen from 12 available toppings?
41. In how many ways can six people be chosen from a group of ten?
42. In how many ways can a committee of three members be chosen from a club of 25 members?
43. How many different fivecard hands can be dealt from a deck of 52 cards?

44. How many different seven-card hands can be picked from a deck of 52 cards?
45. A student must answer seven of the ten questions on an exam. In how many ways can she choose the seven questions?
46. A pizza parlor offers a choice of 16 different toppings. How many three-topping pizzas are possible?
47. A violinist has practiced 12 pieces. In how many ways can he choose eight of these pieces for a recital?
48. If a woman has eight skirts, in how many ways can she choose five of these to take on a weekend trip?
49. In how many ways can seven students from a class of 30 be chosen for a field trip?
50. In how many ways can the seven students in Exercise 49 be chosen if Jack must go on the field trip?
51. In how many ways can the seven students in Exercise 49 be chosen if Jack is not allowed to go on the field trip?
52. In the $6 / 49$ lottery game, a player picks six numbers from 1 to 49 . How many different choices does the player have?
53. In the California Lotto game, a player chooses six numbers from 1 to 53 . It costs $\$ 1$ to play this game. How much would it cost to buy every possible combination of six numbers to ensure picking the winning six numbers?

## YOU CAN WIN \$1,000,000 $\bullet \bullet \bullet$ LOTTO•••

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 |
| 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 |
| 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 50 |
| 51 | 52 | 53 |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |  |

54. A class has 20 students, of which 12 are females and 8 are males. In how many ways can a committee of five students be picked from this class under each condition?
(a) No restriction is placed on the number of males or females on the committee.
(b) No males are to be included on the committee.
(c) The committee must have three females and two males.
55. A set has eight elements.
(a) How many subsets containing five elements does this set have?
(b) How many subsets does this set have?
56. A travel agency has limited numbers of eight different free brochures about Australia. The agent tells you to take any that you like, but no more than one of any kind. How many different ways can you choose brochures (including not choosing any)?
57. A hamburger chain gives their customers a choice of ten different hamburger toppings. In how many different ways can a customer order a hamburger?
58. Each of 20 shoppers in a shopping mall chooses to enter or not to enter the Dressfastic clothing store. How many different outcomes of their decisions are possible?
59. From a group of ten male and ten female tennis players, two men and two women are to face each other in a men-versus-women doubles match. In how many different ways can this match be arranged?
60. A school dance committee is to consist of two freshmen, three sophomores, four juniors, and five seniors. If six freshmen, eight sophomores, twelve juniors, and ten seniors are eligible to be on the committee, in how many ways can the committee be chosen?
61. A group of 22 aspiring thespians contains ten men and twelve women. For the next play the director wants to choose a leading man, a leading lady, a supporting male role, a supporting female role, and eight extras-three women and five men. In how many ways can the cast be chosen?
62. A hockey team has 20 players of which twelve play forward, six play defense, and two are goalies. In how many ways can the coach pick a starting lineup consisting of three forwards, two defense players, and one goalie?
63. A pizza parlor offers four sizes of pizza (small, medium, large, and colossus), two types of crust (thick and thin), and 14 different toppings. How many different pizzas can be made with these choices?
64. Sixteen boys and nine girls go on a camping trip. In how many ways can a group of six be selected to gather firewood, given the following conditions?
(a) The group consists of two girls and four boys.
(b) The group contains at least two girls.
65. In how many ways can ten students be arranged in a row for a class picture if John and Jane want to stand next to each other, and Mike and Molly also insist on standing next to each other?
66. In how many ways can the ten students in Exercise 65 be arranged if Mike and Molly insist on standing together but John and Jane refuse to stand next to each other?

67-68 ■ In how many ways can four men and four women be seated in a row of eight seats for each of the following arrangements?
67. (a) The first seat is to be occupied by a man.
(b) The first and last seats are to be occupied by women.
68. (a) The women are to be seated together.
(b) The men and women are to be seated alternately by gender.
69. From a group of 30 contestants, six are to be chosen as semifinalists, then two of those are chosen as finalists, and then the top prize is awarded to one of the finalists. In how many ways can these choices be made in sequence?
70. Three delegates are to be chosen from a group of four lawyers, a priest, and three professors. In how many ways can the delegation be chosen if it must include at least one professor?
71. In how many ways can a committee of four be chosen from a group of ten if two people refuse to serve together on the same committee?
72. Twelve dots are drawn on a page in such a way that no three are collinear. How many straight lines can be formed by joining the dots?
73. A five-person committee consisting of students and teachers is being formed to study the issue of student parking privileges. Of those who have expressed an interest in serving on the committee, 12 are teachers and 14 are students. In how many ways can the committee be formed if at least one student and one teacher must be included?

## DISCOVERY•DISCUSSION

74. Complementary Combinations Without performing any calculations, explain in words why the number of ways of choosing two objects from ten objects is the same as the number of ways of choosing eight objects from ten objects. In general, explain why $C(n, r)=C(n, n-r)$.
75. An Identity Involving Combinations Kevin has ten different marbles, and he wants to give three of them to Luke and two to Mark. How many ways can he choose to do this? There are two ways of analyzing this problem: He could first pick three for Luke and then two for Mark, or he could first pick two for Mark and then three for Luke. Explain how these two viewpoints show that $C(10,3) \cdot C(7,2)=C(10,2) \cdot C(8,3)$. In general, explain why

$$
C(n, r) \cdot C(n-r, k)=C(n, k) \cdot C(n-k, r)
$$

76. Why Is $\binom{n}{r}$ the Same as $C(n, r)$ ? This exercise explains why the binomial coefficients $\binom{n}{r}$ that appear in the expansion of $(x+y)^{n}$ are the same as $C(n, r)$, the number of ways of choosing $r$ objects from $n$ objects. First, note that expanding a binomial using only the Distributive Property
gives

$$
\begin{aligned}
(x+y)^{2}= & (x+y)(x+y) \\
= & (x+y) x+(x+y) y \\
= & x x+x y+y x+y y \\
(x+y)^{3}= & (x+y)(x x+x y+y x+y y) \\
= & x x x+x x y+x y x+x y y+y x x \\
& +y x y+y y x+y y y
\end{aligned}
$$

(a) Expand $(x+y)^{5}$ using only the Distributive Property.
(b) Write all the terms that represent $x^{2} y^{3}$ together. These are all the terms that contain two $x$ 's and three $y$ 's.
(c) Note that the two $x$ 's appear in all possible positions. Conclude that the number of terms that represent $x^{2} y^{3}$ is $C(5,2)$.
(d) In general, explain why $\binom{n}{r}$ in the Binomial Theorem is the same as $C(n, r)$.

### 11.3 PROBABILITY

The mathematical theory of probability was first discussed in 1654 in a series of letters between Pascal (see page 845) and Fermat (see page 532 ). Their correspondence was prompted by a question raised by the experienced gambler the Chevalier de Méré. The Chevalier was interested in the equitable distribution of the stakes of an interrupted gambling game (see Problem 3, page 904).

If you roll a pair of dice, what are the chances of rolling a double six? What is the likelihood of winning a state lottery? The subject of probability was invented to give precise answers to questions like these. It is now an indispensable tool for making decisions in such diverse areas as business, manufacturing, psychology, genetics, and in many of the sciences. Probability is used to determine the effectiveness of new medicines, assess a fair price for an insurance policy, decide on the likelihood of a candidate winning an election, determine the opinion of many people on a certain topic (without interviewing everyone), and answer many other questions that involve a measure of uncertainty.

To discuss probability, let's begin by defining some terms. An experiment is a process, such as tossing a coin or rolling a die, that gives definite results, called the outcomes of the experiment. For tossing a coin, the possible outcomes are "heads" and "tails"; for rolling a die, the outcomes are $1,2,3,4,5$, and 6 . The sample space of an experiment is the set of all possible outcomes. If we let $H$ stand for heads and $T$ for tails, then the sample space of the coin-tossing experiment is

$$
S=\{H, T\}
$$

The sample space for rolling a die is

$$
S=\{1,2,3,4,5,6\}
$$

We will be concerned only with experiments for which all the outcomes are "equally likely." We already have an intuitive feeling for what this means. When tossing a perfectly balanced coin, heads and tails are equally likely outcomes in the sense that if this experiment is repeated many times, we expect that about half the results will be heads and half will be tails.

In any given experiment we are often concerned with a particular set of outcomes. We might be interested in a die showing an even number or in picking an ace from a deck of cards. Any particular set of outcomes is a subset of the sample space. This leads to the following definition.

## DEFINITION OF AN EVENT

If $S$ is the sample space of an experiment, then an event is any subset of the sample space.

## EXAMPLE 1 - Events in a Sample Space

If an experiment consists of tossing a coin three times and recording the results in order, the sample space is

$$
S=\{H H H, H H T, H T H, T H H, T T H, T H T, H T T, T T T\}
$$

The event $E$ of showing "exactly two heads" is the subset of $S$ that consists of all outcomes with two heads. Thus

$$
E=\{H H T, H T H, T H H\}
$$

The event $F$ of showing "at least two heads" is

$$
F=\{H H H, H H T, H T H, T H H\}
$$

and the event of showing "no heads" is $G=\{T T T\}$.

We are now ready to define the notion of probability. Intuitively, we know that rolling a die may result in any of six equally likely outcomes, so the chance of any particular outcome occurring is $\frac{1}{6}$. What is the chance of showing an even number? Of the six equally likely outcomes possible, three are even numbers. So, it's reasonable to say that the chance of showing an even number is $\frac{3}{6}=\frac{1}{2}$. This reasoning is the intuitive basis for the following definition of probability.

## DEFINITION OF PROBABILITY

Let $S$ be the sample space of an experiment in which all outcomes are equally likely, and let $E$ be an event. The probability of $E$, written $P(E)$, is

$$
P(E)=\frac{n(E)}{n(S)}=\frac{\text { number of elements in } E}{\text { number of elements in } S}
$$

Notice that $0 \leqslant n(E) \leqslant n(S)$, so the probability $P(E)$ of an event is a number between 0 and 1 , that is,

$$
0 \leqslant P(E) \leqslant 1
$$

## MATHEMATICS IN <br> THE MODERN WORLD

## Fair Voting Methods

The methods of mathematics have recently been applied to problems in the social sciences. For example, how do we find fair voting methods? You may ask, what is the problem with how we vote in elections? Well, suppose candidates A, B , and C are running for president. The final vote tally is as follows: A gets $40 \%$, B gets $39 \%$, and C gets $21 \%$. So candidate A wins. But $60 \%$ of the voters didn't want A. Moreover, you voted for C, but you dislike A so much that you would have been willing to change your vote to $B$ to avoid having $A$ win. Most of the voters who voted for C feel the same way you do, so we have a situation where most of the voters prefer B over A, but A wins. So is that fair?
In the 1950s Kenneth Arrow showed mathematically that no democratic method of voting can be completely fair, and later won a Nobel Prize for his work. Mathematicians continue to work on finding fairer voting systems. The system most often used in federal, state, and local elections is called plurality voting (the candidate with the most votes wins). Other systems include majority voting (if no candidate gets a majority, a runoff is held between the top two vote-getters), approval voting (each voter can vote for as many candidates as he or she approves of), preference voting (each voter orders the candidates according to his or her preference), and cumulative voting (each voter gets as many votes as
(continued)

The closer the probability of an event is to 1 , the more likely the event is to happen; the closer to 0 , the less likely. If $P(E)=1$, then $E$ is called the certain event and if $P(E)=0$, then $E$ is called the impossible event.

EXAMPLE 2 - Finding the Probability of an Event
A coin is tossed three times and the results are recorded. What is the probability of getting exactly two heads? At least two heads? No heads?

## SOLUTION

By the results of Example 1, the sample space $S$ of this experiment contains eight outcomes and the event $E$ of getting "exactly two heads" contains three outcomes, $\{H H T H T H, T H H\}$, so by the definition of probability,

$$
P(E)=\frac{n(E)}{n(S)}=\frac{3}{8}
$$

Similarly, the event $F$ of getting "at least two heads" has four outcomes, $\{H H H, H H T, H T H, T H H\}$, and so

$$
P(F)=\frac{n(F)}{n(S)}=\frac{4}{8}=\frac{1}{2}
$$

The event $G$ of getting "no heads" has one element, so

$$
P(G)=\frac{n(G)}{n(S)}=\frac{1}{8}
$$

To find the probability of an event, we do not need to list all the elements in the sample space and the event. What we do need is the number of elements in these sets. The counting techniques we've learned in the preceding sections will be very useful here.

## EXAMPLE 3 - Finding the Probability of an Event

A five-card poker hand is drawn from a standard deck of 52 cards. What is the probability that all five cards are spades?

## SOLUTION

The experiment here consists of choosing five cards from the deck, and the sample space $S$ consists of all possible five-card hands. Thus, the number of elements in the sample space is

$$
n(S)=C(52,5)=\frac{52!}{5!(52-5)!}=2,598,960
$$

there are candidates and can give all of his or her votes to one candidate or distribute them among the candidates as he or she sees fit). This last system is often used to select corporate boards of directors. Each system of voting has both advantages and disadvantages.


The event $E$ we are interested in consists of choosing five spades. Since the deck contains only 13 spades, the number of ways of choosing five spades is

$$
n(E)=C(13,5)=\frac{13!}{5!(13-5)!}=1287
$$

Thus, the probability of drawing five spades is

$$
P(E)=\frac{n(E)}{n(S)}=\frac{1287}{2,598,960} \approx 0.0005
$$

What does the answer to Example 3 tell us? Since $0.0005=\frac{1}{2000}$, this means that if you play poker many, many times, on average you will be dealt a hand consisting of only spades about once in every 2000 hands.

## EXAMPLE 4 ■ Finding the Probability of an Event

A bag contains 20 tennis balls, of which four are defective. If two balls are selected at random from the bag, what is the probability that both are defective?

## SOLUTION

The experiment consists of choosing two balls from 20, so the number of elements in the sample space $S$ is $C(20,2)$. Since there are four defective balls, the number of ways of picking two defective balls is $C(4,2)$. Thus, the probability of the event $E$ of picking two defective balls is

$$
P(E)=\frac{n(E)}{n(S)}=\frac{C(4,2)}{C(20,2)}=\frac{6}{190} \approx 0.032
$$

The complement of an event $E$ is the set of outcomes in the sample space that is not in $E$. We denote the complement of an event $E$ by $E^{\prime}$. We can calculate the probability of $E^{\prime}$ using the definition and the fact that $n\left(E^{\prime}\right)=n(S)-n(E)$ :

$$
P\left(E^{\prime}\right)=\frac{n\left(E^{\prime}\right)}{n(S)}=\frac{n(S)-n(E)}{n(S)}=\frac{n(S)}{n(S)}-\frac{n(E)}{n(S)}=1-P(E)
$$

## PROBABILITY OF THE COMPLEMENT OF AN EVENT

Let $S$ be the sample space of an experiment and $E$ an event. Then

$$
P\left(E^{\prime}\right)=1-P(E)
$$

This is an extremely useful result, since it is often difficult to calculate the probability of an event $E$ but easy to find the probability of $E^{\prime}$, from which $P(E)$ can be calculated immediately using this formula.


Since

$$
P\left(E^{\prime}\right)=1-P(E)
$$

we have

$$
P(E)=1-P\left(E^{\prime}\right)
$$



FIGURE 1

EXAMPLE 5 - Finding the Probability of the Complement of an Event
An urn contains 10 red balls and 15 blue balls. Six balls are drawn at random from the urn. What is the probability that at least one ball is red?

## SOLUTION

Let $E$ be the event that at least one red ball is drawn. It is tedious to count all the possible ways in which one or more of the balls drawn are red. So, let's consider $E^{\prime}$, the complement of this event-namely, that none of the balls chosen is red. The number of ways of choosing 6 blue balls from the 15 blue balls is $C(15,6)$; the number of ways of choosing 6 balls from the 25 balls is $C(25,6)$. Thus

$$
P\left(E^{\prime}\right)=\frac{n\left(E^{\prime}\right)}{n(S)}=\frac{C(15,6)}{C(25,6)}=\frac{5005}{177,100}=\frac{13}{460}
$$

By the formula for the complement of an event, we have

$$
P(E)=1-P\left(E^{\prime}\right)=1-\frac{13}{460}=\frac{447}{460} \approx 0.97
$$

## Mutually Exclusive Events

Two events that have no outcome in common are said to be mutually exclusive (see Figure 1). For example, in drawing a card from a deck, the events
$E$ : The card is an ace
$F$ : The card is a queen
are mutually exclusive, because a card cannot be both an ace and a queen.
If $E$ and $F$ are mutually exclusive events, what is the probability that $E$ or $F$ occurs? The word or indicates that we want the probability of the union of these events, that is, $E \cup F$. Since $E$ and $F$ have no element in common,

$$
n(E \cup F)=n(E)+n(F)
$$

Thus

$$
P(E \cup F)=\frac{n(E \cup F)}{n(S)}=\frac{n(E)+n(F)}{n(S)}=\frac{n(E)}{n(S)}+\frac{n(F)}{n(S)}=P(E)+P(F)
$$

We have proved the following formula.

## PROBABILITY OF THE UNION OF MUTUALLY EXCLUSIVE EVENTS

If $E$ and $F$ are mutually exclusive events in a sample space $S$, then the probability of $E$ or $F$ is

$$
P(E \cup F)=P(E)+P(F)
$$

There is a natural extension of this formula for any number of mutually exclusive events: If $E_{1}, E_{2}, \ldots, E_{n}$ are pairwise mutually exclusive, then

$$
P\left(E_{1} \cup E_{2} \cup \cdots \cup E_{n}\right)=P\left(E_{1}\right)+P\left(E_{2}\right)+\cdots+P\left(E_{n}\right)
$$

EXAMPLE 6 - The Probability of Mutually Exclusive Events


A card is drawn at random from a standard deck of 52 cards. What is the probability that the card is either a seven or a face card?

## SOLUTION

Let $E$ and $F$ denote the following events.

$$
\begin{array}{ll}
E: & \text { The card is a seven } \\
F: & \text { The card is a face card }
\end{array}
$$

Since a card cannot be both a seven and a face card, the events are mutually exclusive. We want the probability of $E$ or $F$; in other words, the probability of $E \cup F$. By the formula,

$$
P(E \cup F)=P(E)+P(F)=\frac{4}{52}+\frac{12}{52}=\frac{4}{13}
$$

## The Probability of the Union of Two Events

If two events $E$ and $F$ are not mutually exclusive, then they share outcomes in common. The situation is described graphically in Figure 2. The overlap of the two sets is their intersection, that is, $E \cap F$. Again, we are interested in the event $E$ or $F$, so we must count the elements in $E \cup F$. If we simply added the number of elements in $E$ to the number of elements in $F$, then we would be counting the elements in the overlap twice-once in $E$ and once in $F$. So, to get the correct total, we must subtract the number of elements in $E \cap F$. Thus

$$
n(E \cup F)=n(E)+n(F)-n(E \cap F)
$$

Using the formula for probability, we get

$$
\begin{aligned}
P(E \cup F) & =\frac{n(E \cup F)}{n(S)}=\frac{n(E)+n(F)-n(E \cap F)}{n(S)} \\
& =\frac{n(E)}{n(S)}+\frac{n(F)}{n(S)}-\frac{n(E \cap F)}{n(S)} \\
& =P(E)+P(F)-P(E \cap F)
\end{aligned}
$$

We have proved the following.

## PROBABILITY OF THE UNION OF TWO EVENTS

If $E$ and $F$ are events in a sample space $S$, then the probability of $E$ or $F$ is

$$
P(E \cup F)=P(E)+P(F)-P(E \cap F)
$$

EXAMPLE 7 - The Probability of the Union of Events


What is the probability that a card drawn at random from a standard 52-card deck is either a face card or a spade?

## SOLUTION



We let $E$ and $F$ denote the following events:
$E$ : The card is a face card
$F$ : The card is a spade

There are 12 face cards and 13 spades in a 52 -card deck, so

$$
P(E)=\frac{12}{52} \quad \text { and } \quad P(F)=\frac{13}{52}
$$

Since 3 cards are simultaneously face cards and spades, we have

$$
P(E \cap F)=\frac{3}{52}
$$

Thus, by the formula for the probability of the union of two events, we have

$$
\begin{aligned}
P(E \cup F) & =P(E)+P(F)-P(E \cap F) \\
& =\frac{12}{52}+\frac{13}{52}-\frac{3}{52}=\frac{11}{26}
\end{aligned}
$$

## The Intersection of Independent Events

We have considered the probability of events joined by the word or, that is, the union of events. Now we study the probability of events joined by the word andin other words, the intersection of events.

When the occurrence of one event does not affect the probability of another event, we say that the events are independent. For instance, if a balanced coin is tossed, the probability of showing heads on the second toss is $\frac{1}{2}$, regardless of the
outcome of the first toss. So, any two tosses of a coin are independent.

## PROBABILITY OF THE INTERSECTION OF INDEPENDENT EVENTS

If $E$ and $F$ are independent events in a sample space $S$, then the probability of $E$ and $F$ is

$$
P(E \cap F)=P(E) P(F)
$$

## EXAMPLE 8 ■ The Probability of Independent Events

A jar contains five red balls and four black balls. A ball is drawn at random from the jar and then replaced; then another ball is picked. What is the probability that both balls are red?

SOLUTION
The events are independent. The probability that the first ball is red is $\frac{5}{9}$. The probability that the second is red is also $\frac{5}{9}$. Thus, the probability that both balls are red is

$$
\frac{5}{9} \times \frac{5}{9}=\frac{25}{81} \approx 0.31
$$

## EXAMPLE 9 - The Birthday Problem

What is the probability that in a class of 35 students, at least two have the sar birthday?

## SOLUTION

It's reasonable to assume that the 35 birthdays are independent and that each day of the 365 days in a year is equally likely as a date of birth. (We ignore February 29.)

Let $E$ be the event that two of the students have the same birthday. It is tedious to list all the possible ways in which at least two of the students have matching birthdays. So, we consider the complementary event $E^{\prime}$, that is, that no two students have the same birthday. To find this probability, we consider the students one at a time. The probability that the first student has a birthday is 1 , the probability that the second has a birthday different from the first is $\frac{364}{365}$, the probability that the third has a birthday different from the first two is $\frac{363}{365}$, and so on. Thus

$$
\begin{array}{ll} 
& P\left(E^{\prime}\right)=1 \cdot \frac{364}{365} \cdot \frac{363}{365} \cdot \frac{362}{365} \cdots \cdots \cdot \frac{331}{365} \approx 0.186 \\
\text { So } & P(E)=1-P\left(E^{\prime}\right) \approx 1-0.186=0.814
\end{array}
$$

Most people are surprised that the probability in Example 9 is so high. For this reason, this problem is sometimes called the "birthday paradox." The table in the margin gives the probability that two people in a group will share the same birthday for groups of various sizes.

## 11.3 exercises

1. An experiment consists of tossing a coin twice.
(a) Find the sample space.
(b) Find the probability of getting heads exactly two times.
(c) Find the probability of getting heads at least one time.
(d) Find the probability of getting heads exactly one time.
2. An experiment consists of tossing a coin and rolling a die.
(a) Find the sample space.
(b) Find the probability of getting heads and an even number.
(c) Find the probability of getting heads and a number greater than 4.
(d) Find the probability of getting tails and an odd number.

3-4 - A die is rolled. Find the probability of the given event.
3. (a) The number showing is a six.
(b) The number showing is an even number.
(c) The number showing is greater than 5 .
4. (a) The number showing is a two or a three.
(b) The number showing is an odd number.
(c) The number showing is a number divisible by 3 .

5-6 - A card is drawn randomly from a standard 52-card deck. Find the probability of the given event.
5. (a) The card drawn is a king.
(b) The card drawn is a face card.
(c) The card drawn is not a face card.
6. (a) The card drawn is a heart.
(b) The card drawn is either a heart or a spade.
(c) The card drawn is a heart, a diamond, or a spade.

7-8 - A ball is drawn randomly from a jar that contains five red balls, two white balls, and one yellow ball. Find the probability of the given event.
7. (a) A red ball is drawn.
(b) The ball drawn is not yellow.
(c) A black ball is drawn.
8. (a) Neither a white nor yellow ball is drawn.
(b) A red, white, or yellow ball is drawn.
(c) The ball drawn is not white.
9. A drawer contains an unorganized collection of 18 socksthree pairs are red, two pairs are white, and four pairs are black.
(a) If one sock is drawn at random from the drawer, what is the probability that it is red?
(b) Once a sock is drawn and discovered to be red, what is the probability of drawing another red sock to make a matching pair?
10. A child's game has a spinner as shown in the figure. Find the probability of the given event.
(a) The spinner stops on an even number.
(b) The spinner stops on an odd number or a number greater than 3 .

11. A letter is chosen at random from the word EXTRATERRESTRIAL. Find the probability of the given event.
(a) The letter $T$ is chosen.
(b) The letter chosen is a vowel.
(c) The letter chosen is a consonant.

12-15 - A poker hand, consisting of five cards, is dealt from a standard deck of 52 cards. Find the probability that the hand contains the cards described.
12. Five hearts
13. Five cards of the same suit
14. Five face cards
15. An ace, king, queen, jack, and 10 of the same suit (royal flush)
16. A pair of dice is rolled, and the numbers showing are observed.
(a) List the sample space of this experiment.
(b) Find the probability of getting a sum of 7 .
(c) Find the probability of getting a sum of 9 .
(d) Find the probability that the two dice show doubles (the same number).
(e) Find the probability that the two dice show different numbers.
(f) Find the probability of getting a sum of 9 or higher.
17. A couple intends to have four children. Assume that having a boy or a girl is an equally likely event.
(a) List the sample space of this experiment.
(b) Find the probability that the couple has only boys.
(c) Find the probability that the couple has two boys and two girls.
(d) Find the probability that the couple has four children of the same sex.
(e) Find the probability that the couple has at least two girls.
18. What is the probability that a 13 -card bridge hand consists of all cards from the same suit?
19. An American roulette wheel has 38 slots; two slots are numbered 0 and 00 , and the remaining slots are numbered from 1 to 36 . Find the probability that the ball lands in an odd-numbered slot.
20. A toddler has wooden blocks showing the letters $C, E, F$, $H, N$, and $R$. Find the probability that the child arranges the letters in the indicated order.
(a) In the order FRENCH
(b) In alphabetical order
21. In the $6 / 49$ lottery game, a player selects six numbers from 1 to 49 . What is the probability of picking the six winning numbers?
22. The president of a large company selects six employees to receive a special bonus. He claims that the six employees are chosen randomly from among the 30 employees, of which 19 are women and 11 are men. What is the probability that no woman is chosen?
23. An exam has ten true-false questions. A student who has not studied answers all ten questions by just guessing. Find the probability that the student correctly answers the given number of questions.
(a) All ten questions
(b) Exactly seven questions
24. To control the quality of their product, the Bright-Light Company inspects three light bulbs out of each batch of ten bulbs manufactured. If a defective bulb is found, the batch is discarded. Suppose a batch contains two defective bulbs. What is the probability that the batch will be discarded?
25. An often-quoted example of an event of extremely low probability is that a monkey types Shakespeare's entire play Hamlet by randomly striking keys on a typewriter. Assume that the typewriter has 48 keys (including the space bar) and that the monkey is equally likely to hit any key.
(a) Find the probability that such a monkey will actually correctly type just the title of the play as his first word.
(b) What is the probability that the monkey will type the phrase "To be or not to be" as his first words?
26. A monkey is trained to arrange wooden blocks in a straight line. He is then given six blocks showing the letters $A, E$,
$H, L, M, T$. What is the probability that he will arrange them to spell the word HAMLET?
27. A monkey is trained to arrange wooden blocks in a straight line. He is then given 11 blocks showing the letters $A, B, B, I, I, L, O, P, R, T, Y$. What is the probability that the monkey will arrange the blocks to spell the word PROBABILITY?
28. Eight horses are entered in a race. You randomly predict a particular order for the horses to complete the race. What is the probability that your prediction is correct?

29. Many genetic traits are controlled by two genes, one dominant and one recessive. In Gregor Mendel's original experiments with peas, the genes controlling the height of the plant are denoted by T (tall) and t (short). The gene T is dominant, so a plant with the genotype (genetic makeup) TT or Tt is tall, whereas one with genotype tt is short. By a statistical analysis of the offspring in his experiments, Mendel concluded that offspring inherit one gene from each parent, and each possible combination of the two genes is equally likely. If each parent has the genotype Tt , then the following chart gives the possible genotypes of the offspring:


Find the probability that a given offspring of these parents will be (a) tall or (b) short.
30. Refer to Exercise 29. Make a chart of the possible genotypes of the offspring if one parent has genotype Tt and the other tt . Find the probability that a given offspring will be (a) tall or (b) short.

31-32 - Determine whether the events $E$ and $F$ in the given experiment are mutually exclusive.
31. The experiment consists of selecting a person at random.
(a) $E$ : The person is male
$F$ : The person is female
(b) $E$ : The person is tall
$F$ : The person is blond
32. The experiment consists of choosing at random a student from your class.
(a) $E$ : The student is female
$F$ : The student wears glasses
(b) E: The student has long hair
$F$ : The student is male
33-34 - A die is rolled and the number showing is observed. Determine whether the events $E$ and $F$ are mutually exclusive. Then find the probability of the event $E \cup F$.
33. (a) $E$ : The number is even
$F$ : The number is odd
(b) $E$ : The number is even
$F$ : The number is greater than 4
34. (a) $E$ : The number is greater than 3
$F$ : The number is less than 5
(b) $E$ : The number is divisible by 3
$F$ : The number is less than 3
35-36 - A card is drawn at random from a standard 52 -card deck. Determine whether the events $E$ and $F$ are mutually exclusive. Then find the probability of the event $E \cup F$.
35. (a) $E$ : The card is a face card
$F$ : The card is a spade
(b) $E$ : The card is a heart
$F$ : The card is a spade
36. (a) $E$ : The card is a club
$F$ : The card is a king
(b) $E$ : The card is an ace
$F$ : The card is a spade
37-38 ■ Refer to the spinner shown in the figure. Find the probability of the given event.

37. (a) The spinner stops on red.
(b) The spinner stops on an even number.
(c) The spinner stops on red or an even number.
38. (a) The spinner stops on blue.
(b) The spinner stops on an odd number.
(c) The spinner stops on blue or an odd number.
39. An American roulette wheel has 38 slots: two of the slots are numbered 0 and 00 , and the rest are numbered from 1 to 36 . Find the probability that the ball lands in an oddnumbered slot or in a slot with a number higher than 31 .
40. A toddler has eight wooden blocks showing the letters $A, E, I, G, L, N, T$, and $R$. What is the probability that the child will arrange the letters to spell one of the words TRIANGLE or INTEGRAL?
41. A committee of five is chosen randomly from a group of six males and eight females. What is the probability that the committee includes either all males or all females?
42. In the $6 / 49$ lottery game a player selects six numbers from 1 to 49 . What is the probability of selecting at least five of the six winning numbers?
43. A jar contains six red marbles numbered 1 to 6 and ten blue marbles numbered 1 to 10 . A marble is drawn at random from the jar. Find the probability that the given event occurs.
(a) The marble is red.
(b) The marble is odd-numbered.
(c) The marble is red or odd-numbered.
(d) The marble is blue or even-numbered.
44. A coin is tossed twice. Let $E$ be the event "the first toss shows heads" and $F$ the event "the second toss shows heads."
(a) Are the events $E$ and $F$ independent?
(b) Find the probability of showing heads on both tosses.
45. A die is rolled twice. Let $E$ be the event "the first roll shows a six" and $F$ the event "the second roll shows a six."
(a) Are the events $E$ and $F$ independent?
(b) Find the probability of showing a six on both rolls.

46-47 - Spinners A and B shown in the figure are spun at the same time.


Spinner A


Spinner B
46. (a) Are the events "spinner A stops on red" and "spinner B stops on yellow" independent?
(b) Find the probability that spinner A stops on red and spinner B stops on yellow.
47. (a) Find the probability that both spinners stop on purple.
(b) Find the probability that both spinners stop on blue.
48. A die is rolled twice. What is the probability of showing a one on both rolls?
49. A die is rolled twice. What is the probability of showing a one on the first roll and an even number on the second roll?
50. A card is drawn from a deck and replaced, and then a second card is drawn.
(a) What is the probability that both cards are aces?
(b) What is the probability that the first is an ace and the second a spade?
51. A roulette wheel has 38 slots: Two slots are numbered 0 and 00 , and the rest are numbered 1 to 36 . A player places a bet on a number between 1 and 36 and wins if a ball thrown into the spinning roulette wheel lands in the slot with the same number. Find the probability of winning on two consecutive spins of the roulette wheel.
52. A researcher claims that she has taught a monkey to spell the word MONKEY using the five wooden letters $E, O, K, M, N, Y$. If the monkey has not actually learned anything and is merely arranging the blocks randomly, what is the probability that he will spell the word correctly three consecutive times?
53. What is the probability of rolling "snake eyes" (double ones) three times in a row with a pair of dice?

54. In the $6 / 49$ lottery game, a player selects six numbers from 1 to 49 and wins if he selects the winning six numbers. What is the probability of winning the lottery two times in a row?
55. Jar A contains three red balls and four white balls. Jar B contains five red balls and two white balls. Which one of the following ways of randomly selecting balls gives the greatest probability of drawing two red balls?
(i) Draw two balls from jar B.
(ii) Draw one ball from each jar.
(iii) Put all the balls in one jar, and then draw two balls.
56. A slot machine has three wheels: Each wheel has 11 posi-tions-a bar and the digits $0,1,2, \ldots, 9$. When the handle is pulled, the three wheels spin independently before coming to rest. Find the probability that the wheels stop on the following positions.
(a) Three bars
(b) The same number on each wheel
(c) At least one bar

57. Find the probability that in a group of eight students at least two people have the same birthday.
58. What is the probability that in a group of six students at least two have birthdays in the same month?
59. A student has locked her locker with a combination lock, showing numbers from 1 to 40 , but she has forgotten the three-number combination that opens the lock. In order to open the lock, she decides to try all possible combinations. If she can try ten different combinations every minute, what is the probability that she will open the lock within one hour?
60. A mathematics department consists of ten men and eight women. Six mathematics faculty members are to be selected at random for the curriculum committee.
(a) What is the probability that two women and four men are selected?
(b) What is the probability that two or fewer women are selected?
(c) What is the probability that more than two women are selected?
61. Twenty students are arranged randomly in a row for a class picture. Paul wants to stand next to Phyllis. Find the probability that he gets his wish.
62. Eight boys and 12 girls are arranged in a row. What is the probability that all the boys will be standing at one end of the row and all the girls at the other end?

## DISCOVERY•DISCUSSION

63. The "Second Son" Paradox Mrs. Smith says, "I have two children-the older one is named William." Mrs. Jones
replies, "One of my two children is also named William." For each woman, list the sample space for the genders of her children, and calculate the probability that her other child is also a son. Explain why these two probabilities are different.
64. The "Oldest Son or Daughter" Phenomenon Poll your class to determine how many of your male classmates
are the oldest sons in their families, and how many of your female classmates are the oldest daughters in their families. You will most likely find that they form a majority of the class. Explain why a randomly selected individual has a high probability of being the oldest son or daughter in his or her family.

## Small Samples, Big Results

A national poll finds that voter preference for presidential candidates is as follows:

Candidate A $57 \%$
Candidate B 43\%
In the poll 1600 adults were surveyed. Since over 100 million voters participate in a national election, it hardly seems possible that surveying only 1600 adults would be of any value. But it is, and it can be proved mathematically that if the sample of 1600 adults is selected at random, then the results are accurate to within $\pm 3 \%$ more than $95 \%$ of the time. Scientists use these methods to determine properties of a big population by testing a small sample. For example, a small sample of fish from a lake is tested to determine the proportion that is diseased, or a small sample of a manufactured product is tested to determine the proportion that is defective.

We can get a feeling for how this works through a simple experiment. Put 1000 white beans and 1000 black beans in a bag, and mix them thoroughly. (It takes a lot of time to count out 1000 beans, so get several friends to count out 100 or so each.) Take a small cup and scoop up a small sample from the beans.

1. Record the proportion of black (or white) beans in the sample. How closely does the proportion in the sample compare with the actual proportion in the bag?
2. Take several samples and record the proportion of black (or white) beans in each sample.
(a) Graph your results.
(b) Average your results. How close is your average to 0.5 ?
3. Try the experiment again but with 500 black beans and 1500 white beans. What proportion of black (or white) beans would you expect in this sample?

PROGRAM: SAMPLE
$: 100 \rightarrow N$
$: 0 \rightarrow$ C
:For (J, 1, N)
:rand $\rightarrow B$
$:(B \leqslant 0.25)+C \rightarrow C$
:End
:Disp "PROPORTION=" :Disp C/N

4. Have some friends mix black and white beans without telling you the number of each. Estimate the proportion of black (or white) beans by taking a few small samples.
5. This bean experiment can be simulated on your graphing calculator. Let's designate the numbers in the interval [0, 0.25] as "black beans" and those in $(0.25,1]$ as "white beans." Now use the random number generator in your calculator to randomly pick a sample of numbers between 0 and 1 . Try samples of size $100,400,1600$, and larger. Determine the proportion of these that are "black beans." What would you expect this proportion to be? Do your results improve when you use a larger sample? (The TI-83 program in the margin takes a sample of 100 random numbers.)
6. Biologists use sampling techniques to estimate fish populations. For example, to estimate the number of trout in a lake, some of the trout are collected, tagged, and released. Later, a sample is taken and the number of tagged trout in the sample is recorded. The total number of trout in the lake can be estimated from the following proportionality:

$$
\frac{\text { number of tagged trout in lake }}{\text { number of trout in lake }} \approx \frac{\text { number of tagged trout in sample }}{\text { number of trout in sample }}
$$

Model this process using beans as follows. Start with a bag containing an unknown number of white beans. Remove a handful of the beans, count them, then "tag" them by marking them with a felt tip pen. Return the tagged beans to the bag and mix thoroughly. Now take a sample from the bag and use the number of tagged beans in the sample to estimate the total number of beans in the bag.

### 11.4 EXPECTED VALUE



FIGURE 1

In the game shown in Figure 1, you pay $\$ 1$ to spin the arrow. If the arrow stops in a red region, you get $\$ 3$ (the dollar you paid plus $\$ 2$ ); otherwise, you lose the dollar you paid. If you play this game many times, how much would you expect to win? Or lose? To answer these questions, let's consider the probabilities of winning and losing. Since three of the regions are red, the probability of winning is $\frac{3}{10}=0.3$ and that of losing is $\frac{7}{10}=0.7$. Remember, this means that if you play this game many times, you expect to win "on average" three out of ten times. So, suppose you play the game 1000 times. Then you would expect to win 300 times and lose 700 times. Since we win $\$ 2$ or lose $\$ 1$ in each game, our expected payoff in 1000 games is

$$
2(300)+(-1)(700)=-100
$$

So the average expected return per game is $\frac{-100}{1000}=-0.1$. In other words, we expect to lose, on average, 10 cents per game. Another way to view this average is to divide each side of the preceding equation by 1000 . Writing $E$ for the result, we get

$$
\begin{aligned}
E & =\frac{2(300)+(-1)(700)}{1000} \\
& =2\left(\frac{300}{1000}\right)+(-1) \frac{700}{1000} \\
& =2(0.3)+(-1)(0.7)
\end{aligned}
$$

Thus, the expected return, or expected value, per game is

$$
E=a_{1} p_{1}+a_{2} p_{2}
$$

where $a_{1}$ is the payoff that occurs with probability $p_{1}$ and $a_{2}$ is the payoff that occurs with probability $p_{2}$. This example leads us to the following definition of expected value.

## DEFINITION OF EXPECTED VALUE

A game gives payoffs $a_{1}, a_{2}, \ldots, a_{n}$ with probabilities $p_{1}, p_{2}, \ldots, p_{n}$. The expected value (or expectation) $E$ of this game is

$$
E=a_{1} p_{1}+a_{2} p_{2}+\cdots+a_{n} p_{n}
$$

The expected value is an average expectation per game if the game is played many times. In general, $E$ need not be one of the possible payoffs. In the preceding example the expected value is -10 cents, but it's impossible to lose exactly 10 cents in any given trial of the game.

EXAMPLE 1 - Finding an Expected Value
A die is rolled, and you receive $\$ 1$ for each point that shows. What is your expectation?

## SOLUTION

Each face of the die has probability $\frac{1}{6}$ of showing. So you get $\$ 1$ with probability $\frac{1}{6}$, $\$ 2$ with probability $\frac{1}{6}, \$ 3$ with probability $\frac{1}{6}$, and so on. Thus, the expected value is

$$
E=1\left(\frac{1}{6}\right)+2\left(\frac{1}{6}\right)+3\left(\frac{1}{6}\right)+4\left(\frac{1}{6}\right)+5\left(\frac{1}{6}\right)+6\left(\frac{1}{6}\right)=\frac{21}{6}=3.5
$$

This means that if you play this game many times, you will make, on average, $\$ 3.50$ per game.

## EXAMPLE 2 - Finding an Expected Value



In Monte Carlo, the game of roulette is played on a wheel with slots numbered 0,1 , $2, \ldots, 36$. The wheel is spun, and a ball dropped in the wheel is equally likely to end up in any one of the slots. To play the game, you bet $\$ 1$ on any number other than zero. (For example, you may bet $\$ 1$ on number 23.) If the ball stops in your slot, you get $\$ 36$ (the $\$ 1$ you bet plus $\$ 35$ ). Find the expected value of this game.

## sOLUTION

You gain $\$ 35$ with probability $\frac{1}{37}$, and you lose $\$ 1$ with probability $\frac{36}{37}$. Thus

$$
E=(35) \frac{1}{37}+(-1) \frac{36}{37} \approx-0.027
$$

In other words, if you play this game many times, you would expect to lose 2.7 cents on every dollar you bet (on average). Consequently, the house expects to gain 2.7 cents on every dollar that is bet. This expected value is what makes gambling very profitable for the gaming house and very unprofitable for the gambler.

### 11.4 EXERCISES

1-10 ■ Find the expected value (or expectation) of the games described.

1. Mike wins $\$ 2$ if a coin toss shows heads and $\$ 1$ if it shows tails.
2. Jane wins $\$ 10$ if a die roll shows a six, and she loses $\$ 1$ otherwise.
3. The game consists of drawing a card from a deck. You win $\$ 100$ if you draw the ace of spades or lose $\$ 1$ if you draw any other card.
4. Tim wins $\$ 3$ if a coin toss shows heads or $\$ 2$ if it shows tails.
5. Carol wins $\$ 3$ if a die roll shows a six, and she wins $\$ 0.50$ otherwise.
6. A coin is tossed twice. Albert wins $\$ 2$ for each heads and must pay $\$ 1$ for each tails.
7. A die is rolled. Tom wins $\$ 2$ if the die shows an even number and he pays $\$ 2$ otherwise.
8. A card is drawn from a deck. You win $\$ 104$ if the card is an ace, $\$ 26$ if it is a face card, and $\$ 13$ if it is the 8 of clubs.
9. A bag contains two silver dollars and eight slugs. You pay 50 cents to reach into the bag and take a coin, which you get to keep.
10. A bag contains eight white balls and two black balls. John picks two balls at random from the bag, and he wins $\$ 5$ if he does not pick a black ball.
11. In the game of roulette as played in Las Vegas, the wheel has 38 slots: Two slots are numbered 0 and 00 , and the rest are numbered 1 to 36 . A $\$ 1$ bet on any number other than 0 or 00 wins $\$ 36$ ( $\$ 35$ plus the $\$ 1$ bet). Find the expected value of this game.
12. A sweepstakes offers a first prize of $\$ 1,000,000$, second prize of $\$ 100,000$, and third prize of $\$ 10,000$. Suppose that two million people enter the contest and three names are drawn randomly for the three prizes.
(a) Find the expected winnings for a person participating in this contest.
(b) Is it worth paying a dollar to enter this sweepstakes?
13. A box contains 100 envelopes. Ten envelopes contain $\$ 10$ each, ten contain $\$ 5$ each, two are "unlucky," and the rest are empty. A player draws an envelope from the box and keeps whatever is in it. If a person draws an unlucky envelope, however, he must pay $\$ 100$. What is the expectation of a person playing this game?
14. A safe containing $\$ 1,000,000$ is locked with a combination lock. You pay $\$ 1$ for one guess at the six-digit combination. If you open the lock, you get to keep the million dollars. What is your expectation?
15. An investor buys 1000 shares of a risky stock for $\$ 5$ a share. She estimates that the probability the stock will rise in value to $\$ 20$ a share is 0.1 and the probability that it will fall to $\$ 1$ a share is 0.9 . If the only criterion for her decision to buy this stock was the expected value of her profit, did she make a wise investment?
16. A slot machine has three wheels, and each wheel has 11 positions-the digits $0,1,2, \ldots, 9$ and the picture of a watermelon. When a quarter is placed in the machine and the handle is pulled, the three wheels spin independently and come to rest. When three watermelons show, the payout is $\$ 5$; otherwise, nothing is paid. What is the expected value of this game?
17. In a $6 / 49$ lottery game, a player pays $\$ 1$ and selects six numbers from 1 to 49 . Any player who has chosen the six winning numbers wins $\$ 1,000,000$. Assuming this is the only way to win, what is the expected value of this game?
18. A bag contains two silver dollars and six slugs. A game consists of reaching into the bag and drawing a coin, which
you get to keep. Determine the "fair price" of playing this game, that is, the price at which the player can be expected to break even if he plays the game many times (in other words, the price at which his expectation is zero).
19. A game consists of drawing a card from a deck. You win $\$ 13$ if you draw an ace. What is a "fair price" to pay to play this game? (See Exercise 18.)

## DISCOVERY•DISCUSSION

20. The Expected Value of a Sweepstakes Contest A magazine clearinghouse holds a sweepstakes contest to sell subscriptions. If you return the winning number, you win $\$ 1,000,000$. You have a 1 -in-20-million chance of winning, but your only cost to enter the contest is a first-class stamp to mail the entry. Use the current price of a first-class stamp to calculate your expected net winnings if you enter this contest. Is it worth entering the sweepstakes?

## 11 REVIEW

## CONCEPT CHECK

1. What does the Fundamental Counting Principle say?
2. (a) What is a permutation of a set of distinct objects?
(b) How many permutations are there of $n$ objects?
(c) How many permutations are there of $n$ objects taken $r$ at a time?
(d) What is the number of distinguishable permutations of $n$ objects if there are $k$ different kinds of objects with $n_{1}$ objects of the first kind, $n_{2}$ objects of the second kind, and so on?
3. (a) What is a combination of $r$ elements of a set?
(b) How many combinations are there of $n$ elements taken $r$ at a time?
(c) How many subsets does a set with $n$ elements have?
4. In solving a problem involving picking $r$ objects from $n$ objects, how do you know whether to use permutations or combinations?
5. (a) What is meant by the sample space of an experiment?
(b) What is an event?
(c) Define the probability of an event $E$ in a sample space $S$
(d) What is the probability of the complement of $E$ ?
6. (a) What are mutually exclusive events?
(b) If $E$ and $F$ are mutually exclusive events, what is the probability of the union of $E$ and $F$ ? What if $E$ and $F$ are not mutually exclusive?
7. (a) What are independent events?
(b) If $E$ and $F$ are independent events, what is the probability of the intersection of $E$ and $F$ ?
8. Suppose that a game gives payoffs $a_{1}, a_{2}, \ldots, a_{n}$ with probabilities $p_{1}, p_{2}, \ldots, p_{n}$. What is the expected value of the game? What is the significance of the expected value?

## EXERCISES

1. A coin is tossed, a die is rolled, and a card is drawn from a deck. How many possible outcomes does this experiment have?
2. How many three-digit numbers can be formed using the digits $1,2,3,4,5,6$ if repetition of digits
(a) is allowed?
(b) is not allowed?
3. (a) How many different two-element subsets does the set $\{A, E, I, O, U\}$ have?
(b) How many different two-letter "words" can be made using the letters from the set in part (a)?
4. An airline company overbooks a particular flight and seven passengers are "bumped" from the flight. If 120 passengers are booked on this flight, in how many ways can the airline choose the seven passengers to be bumped?
5. A quiz has ten true-false questions. How many different ways is it possible to earn a score of exactly $70 \%$ on this quiz?
6. A test has ten true-false questions and five multiple-choice questions with four choices for each. In how many ways can this test be completed?
7. If you must answer only eight of ten questions on a test, how many ways do you have of choosing the questions you will omit?
8. An ice-cream store offers 15 flavors of ice cream. The specialty is a banana split with four scoops of ice cream. If each scoop must be a different flavor, how many different banana splits may be ordered?
9. A company uses a different three-letter security code for each of its employees. What is the maximum number of codes this security system can generate?
10. A group of students determines that they can stand in a row for their class picture in 120 different ways. How many students are in this class?
11. A coin is tossed ten times. In how many different ways can the result be three heads and seven tails?
12. The Yukon Territory in Canada uses a license-plate system for automobiles that consists of two letters followed by three numbers. Explain how we can know that fewer than 700,000 autos are licensed in the Yukon.
13. A group of friends have reserved a tennis court. They find that there are ten different ways in which two of them can play a singles game on this court. How many friends are in this group?
14. A pizza parlor advertises that they prepare 2048 different types of pizza. How many toppings does this parlor offer?
15. In Morse code, each letter is represented by a sequence of dots and dashes, with repetition allowed. How many letters can be represented using Morse code if three or fewer symbols are used?
16. The genetic code is based on the four nucleotides adenine (A), cytosine (C), guanine (G), and thymine (T). These are connected in long strings to form DNA molecules. For example, a sequence in the DNA may look like
CAGTGGTACC . . . . The code uses "words," all the same
length, that are composed of the nucleotides A, C, G, and T. It is known that at least 20 different words exist. What is the minimum word length necessary to generate 20 words?
17. Given 16 subjects from which to choose, in how many ways can a student select fields of study as follows?
(a) A major and a minor
(b) A major, a first minor, and a second minor
(c) A major and two minors
18. (a) How many three-digit numbers can be formed using the digits $0,1, \ldots, 9$ ? (Remember, a three-digit number cannot have 0 as the leftmost digit.)
(b) If a number is chosen randomly from the set $\{0,1,2, \ldots, 1000\}$, what is the probability that the number chosen is a three-digit number?

19-20 ■ An anagram of a word is a permutation of the letters of that word. For example, anagrams of the word triangle include griantle, integral, and tenalgir.
19. How many anagrams of the word TRIANGLE are possible?
20. How many anagrams are possible from the word MISSISSIPPI?
21. A committee of seven is to be chosen from a group of ten men and eight women. In how many ways can the committee be chosen using each of the following selection requirements?
(a) No restriction is placed on the number of men and women on the committee.
(b) The committee must have exactly four men and three women.
(c) Susie refuses to serve on the committee.
(d) At least five women must serve on the committee.
(e) At most two men can serve on the committee.
(f) The committee is to have a chairman, a vice chairman, a secretary, and four other members.
22. A jar contains ten red balls labeled $0,1,2, \ldots, 9$ and five white balls labeled $0,1,2,3,4$. If a ball is drawn from the jar, find the probability of the given event.
(a) The ball is red.
(b) The ball is even-numbered.
(c) The ball is white and odd-numbered.
(d) The ball is red or odd-numbered.
23. If two balls are drawn from the jar in Exercise 22, find the probability of the given event.
(a) Both balls are red.
(b) One ball is white and the other is red.
(c) At least one ball is red.
(d) Both balls are red and even-numbered.
(e) Both balls are white and odd-numbered.
24. A coin is tossed three times in a row, and the outcomes of each toss are observed.
(a) Find the sample space for this experiment.
(b) Find the probability of getting three heads.
(c) Find the probability of getting two or more heads.
(d) Find the probability of getting tails on the first toss.
25. A shelf has ten books: two mysteries, four romance novels, and four mathematics textbooks. If you select a book at random to take to the beach, what is the probability that it turns out to be a mathematics text?
26. A die is rolled and a card is selected from a standard 52 -card deck. What is the probability that both the die and the card show a six?
27. Find the probability that the indicated card is drawn at random from a 52-card deck.
(a) An ace
(b) An ace or a jack
(c) An ace or a spade
(d) A red ace
28. A card is drawn from a 52-card deck, a die is rolled, and a coin is tossed. Find the probability of each outcome.
(a) The ace of spades, a six, and heads
(b) A spade, a six, and heads
(c) A face card, a number greater than 3, and heads
29. Two dice are rolled. Find the probability of each outcome.
(a) The dice show the same number.
(b) The dice show different numbers.
30. Four cards are dealt from a standard 52-card deck. Find the probability that the cards are
(a) all kings
(b) all spades
(c) all the same color
31. In the "numbers game" lottery, a player picks a three-digit number (from 000 to 999 ), and if the number is selected in the drawing, the player wins $\$ 500$. If another number with the same digits (in any order) is drawn, the player wins $\$ 50$. John plays the number 159.
(a) What is the probability that he will win $\$ 500$ ?
(b) What is the probability that he will win $\$ 50$ ?
32. In a television game show, a contestant is given five cards with a different digit on each and is asked to arrange them to match the price of a brand-new car. If she gets the price right, she wins the car. What is the probability that she wins, assuming that she knows the first digit but must guess the remaining four?
33. Two dice are rolled. John gets $\$ 5$ if they show the same number, or he pays $\$ 1$ if they show different numbers. What is the expected value of this game?
34. Three dice are rolled. John gets $\$ 5$ if they all show the same number; he pays $\$ 1$ otherwise. What is the expected value of this game?
35. Mary will win $\$ 1,000,000$ if she can name the 13 original states in the order in which they ratified the U.S. Constitution. Mary has no knowledge of this order, so she makes a guess. What is her expectation?
36. A pizza parlor offers 12 different toppings, one of which is anchovies. If a pizza is ordered at random, what is the probability that anchovies is one of the toppings?
37. A drawer contains an unorganized collection of 50 socks20 are red and 30 are blue. Suppose the lights go out so Kathy can't distinguish the color of the socks.
(a) What is the minimum number of socks Kathy must take out of the drawer to be sure of getting a matching pair?
(b) If two socks are taken at random from the drawer, what is the probability that they make a matching pair?
38. A volleyball team has nine players. In how many ways can a starting lineup be chosen if it consists of two forward players and three defense players?
39. Zip codes consist of five digits.
(a) How many different zip codes are possible?
(b) How many different zip codes can be read when the envelope is turned upside down? (An upside down 9 is a 6 ; and 0,1 , and 8 are the same when read upside down.)
(c) What is the probability that a randomly chosen zip code can be read upside down?
(d) How many zip codes read the same upside down as right side up?
40. In the $\mathrm{Zip}+4$ postal code system, zip codes consist of nine digits.
(a) How many different $\mathrm{Zip}+4$ codes are possible?
(b) How many different $\mathrm{Zip}+4$ codes are palindromes? (A palindrome is a number that reads the same from left to right as right to left.)
(c) What is the probability that a randomly chosen $\mathrm{Zip}+4$ code is a palindrome?
41. Let $N=3,600,000$. (Note that $N=2^{7} 3^{2} 5^{5}$.)
(a) How many divisors does $N$ have?
(b) How many even divisors does $N$ have?
(c) How many divisors of $N$ are multiples of 6 ?
(d) What is the probability that a randomly chosen divisor of $N$ is even?
42. The U.S. Senate has two senators from each of the 50 states. In how many ways can a committee of five senators be chosen if no state is to have two members on the committee?

1. How many five-letter "words" can be made from the letters $A, B, C, D, E, F, G, H, I, J$ if repetition (a) is allowed or (b) is not allowed?
2. A restaurant offers five main courses, three types of desserts, and four kinds of drinks. In how many ways can a customer order a meal consisting of one choice from each category?
3. Over the past year, John has purchased 30 books.
(a) In how many ways can he pick four of these books and arrange them, in order, on his nightstand bookshelf?
(b) In how many ways can he choose four of these books to take with him on his vacation at the shore?
4. A commuter must travel from Ajax to Barrie and back every day. Four roads join the two cities. The commuter likes to vary the trip as much as possible, so he always leaves and returns by different roads. In how many different ways can he make the round-trip?
5. A pizza parlor offers four sizes of pizza and 14 different toppings. A customer may choose any number of toppings (or no topping at all). How many different pizzas does this parlor offer?
6. An anagram of a word is a rearrangement of the letters of the word.
(a) How many anagrams of the word LOVE are possible?
(b) How many different anagrams of the word KISSES are possible?
7. A board of directors consisting of eight members is to be chosen from a pool of 30 candidates. The board is to have a chairman, a treasurer, a secretary, and five other members. In how many ways can the board of directors be chosen?
8. One card is drawn from a deck. Find the probability of the given event.
(a) The card is red.
(b) The card is a king.
(c) The card is a red king.
9. A jar contains five red balls, numbered 1 to 5 , and eight white balls, numbered 1 to 8 . A ball is chosen at random from the jar. Find the probability of the given event.
(a) The ball is red.
(b) The ball is even-numbered.
(c) The ball is red or even-numbered.
10. Three people are chosen at random from a group of five men and ten women. What is the probability that all three are men?
11. Two dice are rolled. What is the probability of getting doubles?
12. In a group of four students, what is the probability that at least two have the same astrological sign?
13. You are to draw one card from a deck. If it is an ace, you win $\$ 10$; if it is a face card, you win $\$ 1$; otherwise, you lose $\$ 0.50$. What is the expected value of this game?

## Focus on Modeling The Monte Carlo Method

A good way to familiarize ourselves with a fact is to experiment with it. For instance, to convince ourselves that the earth is a sphere (which was considered a major paradox at one time), we could go up in a space shuttle to see that it is so; to see whether a given equation is an identity, we might try some special cases to make sure there are no obvious counterexamples. In problems involving probability, we can perform an experiment many times and use the results to estimate the probability in question. In fact, we often model the experiment on a computer, thereby making it feasible to perform the experiment a large number of times. This technique is called the Monte Carlo method, named after the famous gambling casino in Monaco.

## EXAMPLE 1 ■ The Contestant's Dilemma

In a TV game show, a contestant chooses one of three doors. Behind one of them is a valuable prize-the other two doors have nothing behind them. After the contestant has made her choice, the host opens one of the other two doors, one that he knows does not conceal a prize, and then gives her the opportunity to change her choice.

Should the contestant switch, stay, or does it matter? In other words, by switching doors, does she increase, decrease, or leave unchanged her probability of winning? At first, it may seem that switching doors doesn't make any difference. After all, two doors are left-one with the prize and one without-so it seems reasonable that the contestant has an equal chance of winning or losing. But if you play this game many times, you will find that by switching doors you actually win about $\frac{2}{3}$ of the time.

The authors modeled this game on a computer and found that in one million games the simulated contestant (who always switches) won 667,049 times-very close to $\frac{2}{3}$ of the time. Thus, it seems that switching doors does make a difference: Switching increases the contestant's chances of winning. This experiment forces us to reexamine our reasoning. Here is why switching doors is the correct strategy:

1. When the contestant first made her choice, she had a $\frac{1}{3}$ chance of winning. If she doesn't switch, no matter what the host does, her probability of winning remains $\frac{1}{3}$.
2. If the contestant decides to switch, she will switch to the winning door if she had initially chosen a losing one, or to a losing door if she had initially chosen the winning one. Since the probability of having initially selected a losing door is $\frac{2}{3}$, by switching the probability of winning then becomes $\frac{2}{3}$.

We conclude that the contestant should switch, because her probability of winning is $\frac{2}{3}$ if she switches and $\frac{1}{3}$ if she doesn't. Put simply, there is a much greater chance that she initially chose a losing door (since there are more of these), so she should switch.

The "contestant's dilemma" problem discussed here is an example of how subtle probability can be. This problem was posed in a nationally syndicated column in Parade magazine in 1990. The correct solution was presented in the column, but it generated considerable controversy, with thousands of letters arguing that the solution was wrong. This shows how problems in probability can be quite tricky. Without a lot of experience in probabilistic thinking, it's easy to make a mistake. Even great mathematicians like D'Alembert and Roberval (see Problems 2 and 3) made mistakes in probability. Professor David Burton writes in his book The History of Mathematics, "Probability theory abounds in paradoxes that wrench the common sense and trip the unwary."

```
PROGRAM: HEADTAIL
\(: 0 \rightarrow \mathrm{~J}: \mathrm{O} \rightarrow \mathrm{K}\)
:For(N, 1, 100)
:rand \(\rightarrow X\)
\(: \operatorname{int}(2 X) \rightarrow Y\)
\(: J+(1-Y) \rightarrow J\)
\(: K+Y \rightarrow K\)
:END
:Disp "HEADS=",Ј
:Disp "TAILS=",K
```

FIGURE 1
Relative frequency of "heads"

An experiment can be modeled using any computer language or programmable calculator that has a random-number generator. This is a command or function (usually called Rnd or Rand) that returns a randomly chosen number $x$ with $0 \leqslant x<1$. In the next example we see how to use this to model a simple experiment.

## EXAMPLE 2 - Monte Carlo Model of a Coin Toss

When a balanced coin is tossed, each outcome-"heads" or "tails"-has probability $\frac{1}{2}$. This doesn't mean that if we toss a coin several times, we will necessarily get exactly half heads and half tails. We would expect, however, the proportion of heads and of tails to get closer and closer to $\frac{1}{2}$ as the number of tosses increases. To test this hypothesis, we could toss a coin a very large number of times and keep track of the results. But this is a very tedious process, so we will use the Monte Carlo method to model this process.

To model a coin toss with a calculator or computer, we use the random-number generator to get a random number $x$ such that $0 \leqslant x<1$. Because the number is chosen randomly, the probability that it lies in the first half of this interval $\left(0 \leqslant x<\frac{1}{2}\right)$ is the same as the probability that it lies in the second half $\left(\frac{1}{2} \leqslant x<1\right)$. Thus, we could model the outcome "heads" by the event that $0 \leqslant x<\frac{1}{2}$ and the outcome "tails" by the event that $\frac{1}{2} \leqslant x<1$.

An easier way to keep track of heads and tails is to note that if $0 \leqslant x<1$, then $0 \leqslant 2 x<2$, and so $\llbracket 2 x \rrbracket$, the integer part of $2 x$, is either 0 or 1 , each with probability $\frac{1}{2}$. (On most programmable calculators, the function Int gives the integer part of a number.) Thus, we could model "heads" with the outcome " 0 " and "tails" with the outcome " 1 " when we take the integer part of $2 x$. The program in the margin models 100 tosses of a coin on the TI-83 calculator. The graph in Figure 1 shows what proportion $p$ of the tosses have come up "heads" after $n$ tosses. As you can see, this proportion settles down near 0.5 as the number $n$ of tosses increases-just as we hypothesized.


In general, if a process has $n$ equally likely outcomes, then we can model the process using a random-number generator as follows: If our program or calculator produces the random number $x$, with $0 \leqslant x<1$, then the integer part of $n x$ will be a random choice from the $n$ integers $0,1,2, \ldots, n-1$. Thus, we can use the outcomes $0,1,2, \ldots, n-1$ as models for the outcomes of the actual experiment.

## Problems

1. In a game show like the one described in Example 1, a prize is concealed behind one of ten doors. After the contestant chooses a door, the host opens eight losing doors and then gives the contestant the opportunity to switch to the other unopened door.
(a) Play this game with a friend 30 or more times, using the strategy of switching doors each time. Count the number of times you win, and estimate the probability of winning with this strategy.
(b) Calculate the probability of winning with the "switching" strategy using reasoning similar to that in Example 1. Compare with your result from part (a).
2. A couple intend to have two children. What is the probability that they will have one child of each sex? The French mathematician D'Alembert analyzed this problem (incorrectly) by reasoning that three outcomes are possible: two boys, or two girls, or one child of each sex. He concluded that the probability of having one of each sex is $\frac{1}{3}$, mistakenly assuming that the three outcomes are "equally likely."
(a) Model this problem with a pair of coins (using "heads" for boys and "tails" for girls), or write a program to model the problem. Perform the experiment 40 or more times, counting the number of boy-girl combinations. Estimate the probability of having one child of each sex.
(b) Calculate the correct probability of having one child of each sex, and compare this with your result from part (a).
3. A game between two players consists of tossing a coin. Player A gets a point if the coin shows heads, and player B gets a point if it shows tails. The first player to get six points wins an $\$ 8000$ jackpot. As it happens, the police raid the place when player A has five points and B has three points. After everyone has calmed down, how should the jackpot be divided between the two players? In other words, what is the probability of A winning (and that of B winning) if the game were to continue?

The French mathematicians Pascal and Fermat corresponded about this problem, and both came to the same correct conclusion (though by very different reasonings). Their friend Roberval disagreed with both of them. He argued that player A has probability $\frac{3}{4}$ of winning, because the game can end in the four ways $H, T H, T T H, T T T$, and in three of these, A wins. Roberval's reasoning was wrong.
(a) Continue the game from the point at which it was interrupted, using either a coin or a modeling program. Perform this experiment 80 or more times, and estimate the probability that player A wins.
(b) Calculate the probability that player A wins. Compare with your estimate from part (a).
4. In the World Series, the top teams in the National League and the American League play a best-of-seven series; that is, they play until one team has won four games. (No tie is allowed, so this results in a maximum of seven games.) Suppose the teams are evenly matched, so that the probability that either team wins a given game is $\frac{1}{2}$.
(a) Use a coin or a modeling program to model a World Series, where "heads" represents a win by Team A and "tails" a win by Team B. Perform this experiment at least 80 times, keeping track of how many games are needed to decide each series. Estimate the probability that an evenly matched series will end in four games. Do the same for five, six, and seven games.
(b) What is the probability that the series will end in four games? Five games? Six games? Seven games? Compare with your estimates from part (a).
(c) Find the expected value for the number of games until the series ends. [Hint: This will be $P($ four games $) \times 4+P($ five $) \times 5+P($ six $) \times 6+P($ seven $) \times 7$.]


PROGRAM: PI
$: 0 \rightarrow P$
:For(N, 1, 1000)
$:$ rand $\rightarrow X:$ rand $\rightarrow Y$
$: P+\left(\left(X^{2}+Y^{2}\right)<1\right) \rightarrow P$
:End
:Disp "PI IS APPROX",4*P/N

5. In this problem we use the Monte Carlo method to estimate the value of $\pi$. The circle in the figure has radius 1 , so its area is $\pi$ and the square has area 4 . If we choose a point at random from the square, the probability that it lies inside the circle will be

$$
\frac{\text { area of circle }}{\text { area of square }}=\frac{\pi}{4}
$$

The Monte Carlo method involves choosing many points inside the square. Then we have

$$
\frac{\text { number of hits inside circle }}{\text { number of hits inside square }} \approx \frac{\pi}{4}
$$

Thus, 4 times this ratio will give us an approximation for $\pi$.
To implement this method, we use a random-number generator to obtain the coordinates $(x, y)$ of a random point in the square, and then check to see if it lies inside the circle (that is, we check if $x^{2}+y^{2}<1$ ). Note that we only need to use points in the first quadrant, since the ratio of areas is the same in each quadrant. The program in the margin shows a way of doing this on the TI-83 calculator for 1000 randomly selected points.

Carry out this Monte Carlo simulation for as many points as you can. How do your results compare with the actual value of $\pi$ ? Do you think this is a reasonable way to get a good approximation for $\pi$ ?
6. The Monte Carlo method can be used to estimate the area under the graph of a function. The figure shows the region under the graph of $f(x)=x^{2}$, above the $x$-axis, between $x=0$ and $x=1$. If we choose a point in the square at random, the probability that it lies under the graph of $f(x)=x^{2}$ is the area under the graph divided by the area of the square. So if we randomly select a large number of points in the square, we have

$$
\frac{\text { number of hits under graph }}{\text { number of hits in square }} \approx \frac{\text { area under graph }}{\text { area of square }}
$$

Modify the program from Problem 5 to carry out this Monte Carlo simulation and approximate the required area. [Note: The exact value of this area will be found in Chapter 12.]
7. Choose two numbers at random from the interval $[0,1)$. What is the probability that the sum of the two numbers is less than 1 ?
(a) Use a Monte Carlo model to estimate the probability.
(b) Calculate the exact value of the probability. [Hint: Call the numbers $x$ and $y$. Choosing these numbers is the same as choosing an ordered pair $(x, y)$ in the unit square $\{(x, y) \mid 0 \leqslant x<1,0 \leqslant y<1\}$. What proportion of the points in this square corresponds to $x+y$ being less than 1?]

