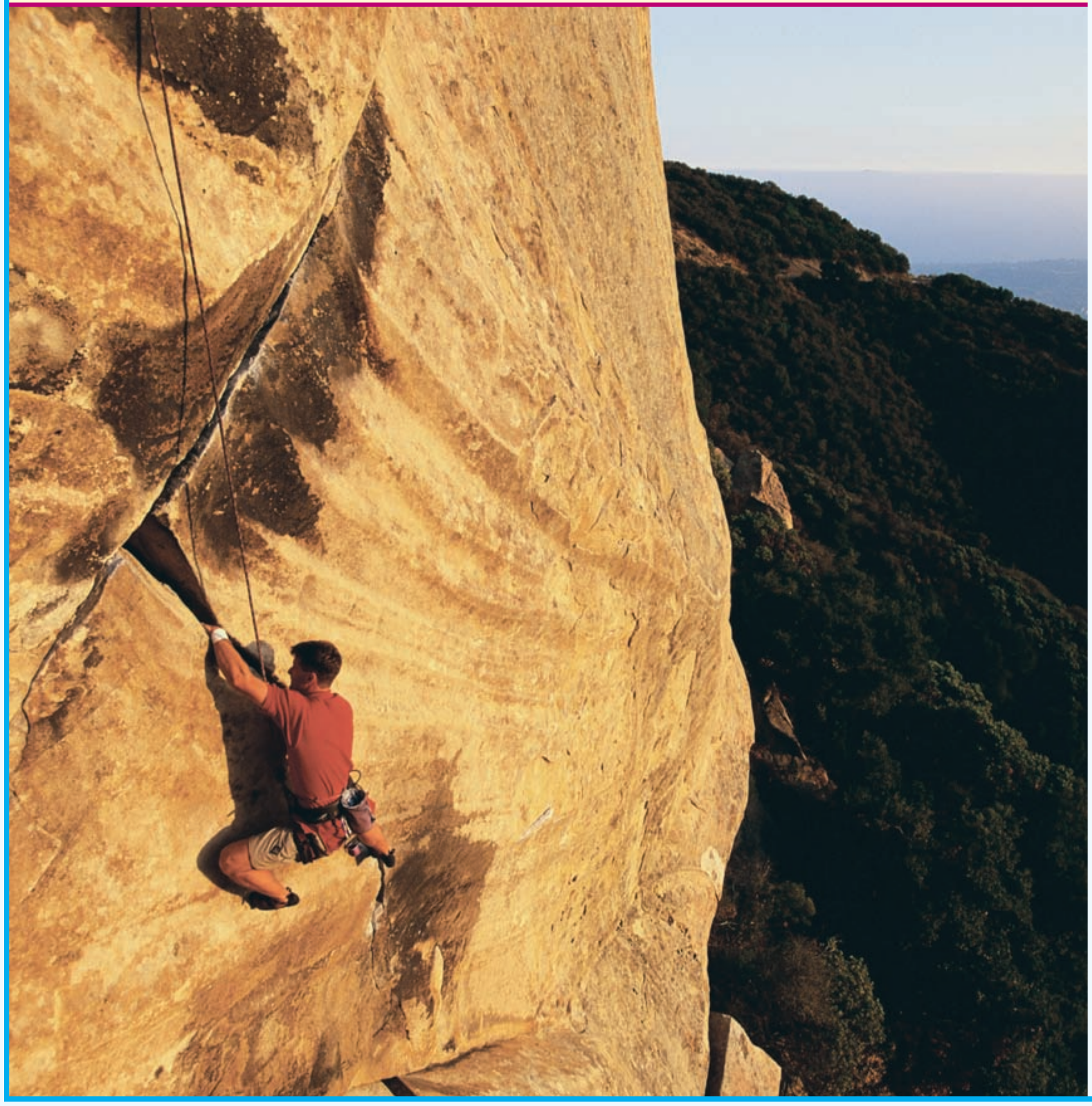
57050_02_ch02_p146-247.qxd 07/04/2008 04:58 PM Page 146

2 Functions

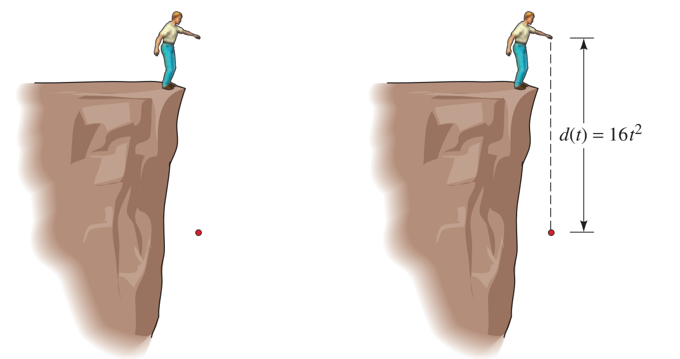


57050_02_ch02_p146-247.qxd 07/04/2008 04:58 PM Page 147

Chapter Overview

Perhaps the most useful mathematical idea for modeling the real world is the concept of *function*, which we study in this chapter. To understand what a function is, let's look at an example.

If a rock climber drops a stone from a high cliff, what happens to the stone? Of course the stone falls; how far it has fallen at any given moment depends upon how long it has been falling. That's a general description, but it doesn't tell us exactly when the stone will hit the ground.



General description: The stone falls. **Function:** In t seconds the stone falls $16t^2$ ft.

What we need is a *rule* that relates the position of the stone to the time it has fallen. Physicists know that the rule is: In t seconds the stone falls $16t^2$ feet. If we let $d(t)$ stand for the distance the stone has fallen at time t, then we can express this rule as

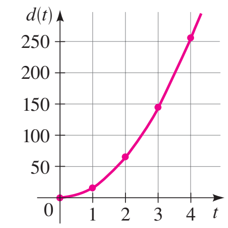
$$d(t) = 16t^2$$

This "rule" for finding the distance in terms of the time is called a *function*. We say that distance is a *function* of time. To understand this rule or function better, we can make a table of values or draw a graph. The graph allows us to easily visualize how far and how fast the stone falls.

Galen Rowell/Corbis

57050_02_ch02_p146-247.qxd 07/04/2008 04:58 PM Page 148

Time t	Distance $d(t)$
0	0
1	16
2	64
3	144
4	256



You can see why functions are important. For example, if a physicist finds the "rule" or function that relates distance fallen to elapsed time, then she can predict when a missile will hit the ground. If a biologist finds the function or "rule" that relates the number of bacteria in a culture to the time, then he can predict the number of bacteria for some future time. If a farmer knows the function or "rule" that relates the yield of apples to the number of trees per acre, then he can decide how many trees per acre to plant to maximize the yield.

In this chapter we will learn how functions are used to model real-world situations and how to find such functions.

SUGGESTED TIME AND EMPHASIS

$\frac{1}{2}$–1 class.
Essential material.

2.1 What Is a Function?

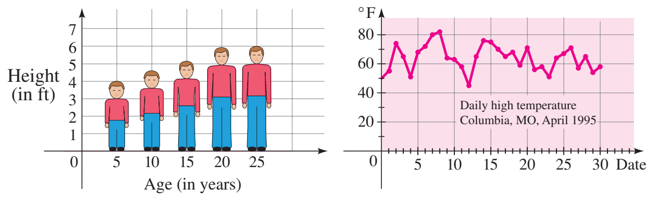
In this section we explore the idea of a function and then give the mathematical definition of function.

Functions All Around Us

In nearly every physical phenomenon we observe that one quantity depends on another. For example, your height depends on your age, the temperature depends on the date, the cost of mailing a package depends on its weight (see Figure 1). We use the term *function* to describe this dependence of one quantity on another. That is, we say the following:

- Height is a function of age.
- Temperature is a function of date.
- Cost of mailing a package is a function of weight.

The U.S. Post Office uses a simple rule to determine the cost of mailing a package based on its weight. But it's not so easy to describe the rule that relates height to age or temperature to date.



w (ounces)	Postage (dollars)
$0 < w \le 1$	0.37
$1 < w \le 2$	0.60
$2 < w \le 3$	0.83
$3 < w \le 4$	1.06
$4 < w \le 5$	1.29
$5 < w \le 6$	1.52

Height is a function of age. Temperature is a function of date. Postage is a function of weight.

Figure 1

POINTS TO STRESS

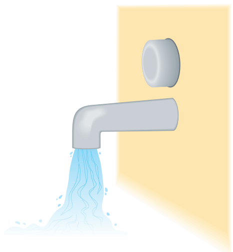
1. The idea of function, viewed as the dependence of one quantity on a different quantity.
2. The notation associated with numeric functions, including piecewise-defined functions.
3. Domains and ranges from an algebraic perspective.
4. Four different representations of functions (verbal, algebraic, visual, and numeric).

57050_02_ch02_p146-247.qxd 07/04/2008 04:58 PM Page 149

Can you think of other functions? Here are some more examples:

- The area of a circle is a function of its radius.
- The number of bacteria in a culture is a function of time.
- The weight of an astronaut is a function of her elevation.
- The price of a commodity is a function of the demand for that commodity.

The rule that describes how the area A of a circle depends on its radius r is given by the formula $A = \pi r^2$. Even when a precise rule or formula describing a function is not available, we can still describe the function by a graph. For example, when you turn on a hot water faucet, the temperature of the water depends on how long the water has been running. So we can say



- Temperature of water from the faucet is a function of time.

Figure 2 shows a rough graph of the temperature T of the water as a function of the time t that has elapsed since the faucet was turned on. The graph shows that the initial temperature of the water is close to room temperature. When the water from the hot water tank reaches the faucet, the water's temperature T increases quickly. In the next phase, T is constant at the temperature of the water in the tank. When the tank is drained, T decreases to the temperature of the cold water supply.

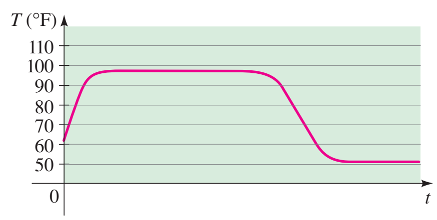


Figure 2
Graph of water temperature T as a function of time t

Definition of Function

We have previously used letters to stand for numbers. Here we do something quite different. We use letters to represent *rules*.

A function is a rule. In order to talk about a function, we need to give it a name. We will use letters such as $f, g, h, \ldots$ to represent functions. For example, we can use the letter f to represent a rule as follows:

"f" is the rule "square the number"

When we write $f(2)$, we mean "apply the rule f to the number 2." Applying the rule gives $f(2) = 2^2 = 4$. Similarly, $f(3) = 3^2 = 9$, $f(4) = 4^2 = 16$, and in general $f(x) = x^2$.

Definition of Function

A **function** f is a rule that assigns to each element x in a set A exactly one element, called $f(x)$, in a set B.

SAMPLE QUESTION
Text Question
What is a function?

Answer
Answers will vary. Anything that gets at the idea of assigning an element in one set to an element in another set should be given full credit.

IN-CLASS MATERIALS

If students are using calculators, discuss the ties between the idea of a function and a calculator key. The keys such as sin, cos, tan, and $\sqrt{\ }$ represent functions. It is easy to compute and graph functions on a calculator. Contrast this with equations such as $y^3 - x^3 = 2xy$ which have graphs, but are not easy to work with, even with a calculator. (Even symbolic algebra calculators such as the *TI*–89 do not do well with all general relations.) Point out that the calculator often gives approximations to function values—applying the square root function key to the number 2 gives 1.4142136 which is close to, but not equal to, $\sqrt{2}$.

57050_02_ch02_p146-247.qxd 07/04/2008 04:58 PM Page 150

We usually consider functions for which the sets A and B are sets of real numbers. The symbol $f(x)$ is read "f of x" or "f at x" and is called the **value of f at x**, or the **image of x under f**. The set A is called the **domain** of the function. The **range** of f is the set of all possible values of $f(x)$ as x varies throughout the domain, that is,

$$\text{range of } f = \{f(x) \mid x \in A\}$$

The symbol that represents an arbitrary number in the domain of a function f is called an **independent variable**. The symbol that represents a number in the range of f is called a **dependent variable**. So if we write $y = f(x)$, then x is the independent variable and y is the dependent variable.

It's helpful to think of a function as a **machine** (see Figure 3). If x is in the domain of the function f, then when x enters the machine, it is accepted as an **input** and the machine produces an **output** $f(x)$ according to the rule of the function. Thus, we can think of the domain as the set of all possible inputs and the range as the set of all possible outputs.

The $\sqrt{\ }$ key on your calculator is a good example of a function as a machine. First you input x into the display. Then you press the key labeled $\sqrt{\ }$. (On most *graphing* calculators, the order of these operations is reversed.) If $x < 0$, then x is not in the domain of this function; that is, x is not an acceptable input and the calculator will indicate an error. If $x \geq 0$, then an approximation to $\sqrt{x}$ appears in the display, correct to a certain number of decimal places. (Thus, the $\sqrt{\ }$ key on your calculator is not quite the same as the exact mathematical function f defined by $f(x) = \sqrt{x}$.)

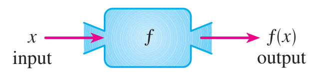


Figure 3
Machine diagram of f

Another way to picture a function is by an **arrow diagram** as in Figure 4. Each arrow connects an element of A to an element of B. The arrow indicates that $f(x)$ is associated with x, $f(a)$ is associated with a, and so on.

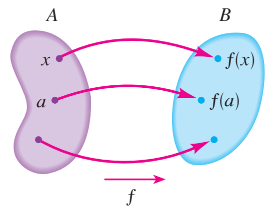


Figure 4
Arrow diagram of f

Example 1 The Squaring Function

The squaring function assigns to each real number x its square x^2. It is defined by

$$f(x) = x^2$$

(a) Evaluate $f(3)$, $f(-2)$, and $f(\sqrt{5})$.

(b) Find the domain and range of f.

(c) Draw a machine diagram for f.

Solution

(a) The values of f are found by substituting for x in $f(x) = x^2$.

$$f(3) = 3^2 = 9 \qquad f(-2) = (-2)^2 = 4 \qquad f(\sqrt{5}) = (\sqrt{5})^2 = 5$$

(b) The domain of f is the set $\mathbb{R}$ of all real numbers. The range of f consists of all values of $f(x)$, that is, all numbers of the form x^2. Since $x^2 \geq 0$ for all real numbers x, we can see that the range of f is $\{y \mid y \geq 0\} = [0, \infty)$.

(c) A machine diagram for this function is shown in Figure 5. ■

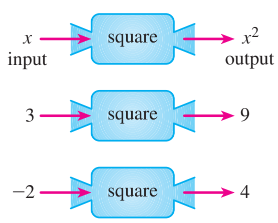


Figure 5
Machine diagram

ALTERNATE EXAMPLE 1a
The squaring function assigns to each real number x its square x^2. It is defined by $f(x) = x^2$. Evaluate $f(7)$.

ANSWER
$f(7) = 49$

DRILL QUESTION

Let $f(x) = x + \sqrt{x}$. Find $f(0)$ and $f(4)$.

Answer

$f(0) = 0$, $f(4) = 6$

IN-CLASS MATERIALS

This course emphasizes functions where both the domain and range sets are numeric. One could give a more abstract definition of function, where D and R can be any set. For example, there is a function mapping each student in the class to his or her birthplace. A nice thing about this point of view is that it can be pointed out that the map from each student to his or her telephone number may *not* be a function, since a student may have more than one telephone number, or none at all.

57050_02_ch02_p146-247.qxd 07/04/2008 04:58 PM Page 151

Evaluating a Function

In the definition of a function the independent variable x plays the role of a "placeholder." For example, the function $f(x) = 3x^2 + x - 5$ can be thought of as

$$f(\square) = 3 \cdot \square^2 + \square - 5$$

To evaluate f at a number, we substitute the number for the placeholder.

Example 2 Evaluating a Function

Let $f(x) = 3x^2 + x - 5$. Evaluate each function value.

(a) $f(-2)$ (b) $f(0)$ (c) $f(4)$ (d) $f(\frac{1}{2})$

Solution To evaluate f at a number, we substitute the number for x in the definition of f.

(a) $f(-2) = 3 \cdot (-2)^2 + (-2) - 5 = 5$

(b) $f(0) = 3 \cdot 0^2 + 0 - 5 = -5$

(c) $f(4) = 3 \cdot 4^2 + 4 - 5 = 47$

(d) $f(\frac{1}{2}) = 3 \cdot (\frac{1}{2})^2 + \frac{1}{2} - 5 = -\frac{15}{4}$ ■

Example 3 A Piecewise Defined Function

A cell phone plan costs \$39 a month. The plan includes 400 free minutes and charges 20¢ for each additional minute of usage. The monthly charges are a function of the number of minutes used, given by

$$C(x) = \begin{cases} 39 & \text{if } 0 \le x \le 400 \\ 39 + 0.2(x - 400) & \text{if } x > 400 \end{cases}$$

Find $C(100)$, $C(400)$, and $C(480)$.

A piecewise-defined function is defined by different formulas on different parts of its domain. The function C of Example 3 is piecewise defined.

Solution Remember that a function is a rule. Here is how we apply the rule for this function. First we look at the value of the input x. If $0 \le x \le 400$, then the value of $C(x)$ is 39. On the other hand, if $x > 400$, then the value of $C(x)$ is $39 + 0.2(x - 400)$.

Since $100 \le 400$, we have $C(100) = 39$.

Since $400 \le 400$, we have $C(400) = 39$.

Since $480 > 400$, we have $C(480) = 39 + 0.2(480 - 400) = 55$.

Thus, the plan charges \$39 for 100 minutes, \$39 for 400 minutes, and \$55 for 480 minutes. ■

Example 4 Evaluating a Function

Expressions like the one in part (d) of Example 4 occur frequently in calculus; they are called *difference quotients*, and they represent the average change in the value of f between $x = a$ and $x = a + h$.

If $f(x) = 2x^2 + 3x - 1$, evaluate the following.

(a) $f(a)$ (b) $f(-a)$

(c) $f(a + h)$ (d) $\dfrac{f(a+h) - f(a)}{h}$, $h \ne 0$

ALTERNATE EXAMPLE 2a
Let $f(x) = 5x^2 + x - 6$. Evaluate the function value $f(-5)$.

ANSWER
$f(-5) = 114$

ALTERNATE EXAMPLE 2c
Let $f(x) = 3x^2 + x - 6$. Evaluate the function value $f(5)$.

ANSWER
$f(5) = 74$

ALTERNATE EXAMPLE 2d
Let $f(x) = 3x^2 + x - 7$. Evaluate the function value $f\left(\frac{1}{6}\right)$.

ANSWER
$f\left(\frac{1}{6}\right) = -\frac{27}{4}$

ALTERNATE EXAMPLE 3
Evaluate the following function at $x = 1$.

$$f(x) = \begin{cases} 3 - x & \text{if } x \le 3 \\ x^2 & \text{if } x > 3 \end{cases}$$

ANSWER
$f(1) = 2$

ALTERNATE EXAMPLE 4c
If $f(x) = 3x^2 + 4x - 1$, evaluate $f(a + h)$.

ANSWER
$3a^2 + 6a \cdot h + 3h^2 + 4a + 4h - 1$

ALTERNATE EXAMPLE 4d
If $f(x) = 2x^2 + 4x - 1$, evaluate $\dfrac{f(a+h) - f(a)}{h}$, $h \ne 0$.

ANSWER
$4a + 2h + 4$

IN-CLASS MATERIALS

Function notation can trip students up. Start with a function such as $f(x) = x^2 - x$ and have your students find $f(0)$, $f(1)$, $f(\sqrt{3})$, and $f(-1)$. Then have them find $f(\pi)$, $f(y)$, and (of course) $f(x + h)$. Some students will invariably, some day, assume that $f(a + b) = f(a) + f(b)$ for all functions, but this can be minimized if plenty of examples such as $f(2 + 3)$ are done at the outset.

57050_02_ch02_p146-247.qxd 07/04/2008 04:58 PM Page 152

Solution

(a) $f(a) = 2a^2 + 3a - 1$

(b) $f(-a) = 2(-a)^2 + 3(-a) - 1 = 2a^2 - 3a - 1$

(c) $f(a + h) = 2(a + h)^2 + 3(a + h) - 1$

$= 2(a^2 + 2ah + h^2) + 3(a + h) - 1$

$= 2a^2 + 4ah + 2h^2 + 3a + 3h - 1$

(d) Using the results from parts (c) and (a), we have

$$\frac{f(a+h) - f(a)}{h} = \frac{(2a^2 + 4ah + 2h^2 + 3a + 3h - 1) - (2a^2 + 3a - 1)}{h}$$

$$= \frac{4ah + 2h^2 + 3h}{h} = 4a + 2h + 3$$ ■

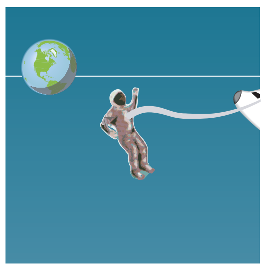
Example 5 The Weight of an Astronaut

If an astronaut weighs 130 pounds on the surface of the earth, then her weight when she is h miles above the earth is given by the function

$$w(h) = 130\left(\frac{3960}{3960 + h}\right)^2$$

(a) What is her weight when she is 100 mi above the earth?

(b) Construct a table of values for the function w that gives her weight at heights from 0 to 500 mi. What do you conclude from the table?



The weight of an object on or near the earth is the gravitational force that the earth exerts on it. When in orbit around the earth, an astronaut experiences the sensation of "weightlessness" because the centripetal force that keeps her in orbit is exactly the same as the gravitational pull of the earth.

Solution

(a) We want the value of the function w when $h = 100$; that is, we must calculate $w(100)$.

$$w(100) = 130\left(\frac{3960}{3960 + 100}\right)^2 \approx 123.67$$

So at a height of 100 mi, she weighs about 124 lb.

(b) The table gives the astronaut's weight, rounded to the nearest pound, at 100-mile increments. The values in the table are calculated as in part (a).

h	$w(h)$
0	130
100	124
200	118
300	112
400	107
500	102

The table indicates that the higher the astronaut travels, the less she weighs. ■

ALTERNATE EXAMPLE 5

If the speed limit on a 100-mile stretch of road is 75 miles per hour, then the amount of time it takes a car going x miles per hour over the limit to traverse the stretch is given by $t(x) = \dfrac{100}{75 + x}$.

(a) How long does it take the car to travel the stretch if the car is going 10 miles per hour over the limit?

(b) How long does it take the car to travel the stretch if the car is not speeding at all?

(c) Construct a table of values giving the travel time for x values from 0 to 40-miles per hour. What do you conclude from the table?

ANSWERS

(a) We want the value of the function t when $x = 10$.

$t(10) = \dfrac{100}{75 + 10}$

≈ 1.18 hours.

(b) We want the value of the function t when $x = 0$.

$t(10) = \dfrac{100}{75 + 0}$

≈ 1.33 hours.

(c) The table gives the travel time, rounded to two decimal places, in 5-mile increments. The values of the table are calculated as in parts (a) and (b).

x	$t(x)$
0	1.33
5	1.25
10	1.18
15	1.11
20	1.05
25	1.00
30	0.95
35	0.90
40	0.87

We can conclude that the more the car speeds, the less time it takes to make the trip, but after a certain point, the time saved isn't all that much.

IN-CLASS MATERIALS

Discuss the straightforward things to look for when trying to find the domain of a function: zero denominators and negative even roots. Discuss the domain and range of a function such as

$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is an integer} \\ 0 & \text{if } x \text{ is not an integer} \end{cases}$$

If the class seems interested, perhaps let them think about

$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

Let $f(x) = \dfrac{x(x-2)}{(x-2)}$ and $g(x) = x$. Ask students if the functions are the same function. If they say "yes," ask them to compare the domains, or to compute $g(2)$ and $f(2)$. If they say "no" ask them to find a value such that $f(x) \neq g(x)$.

The Domain of a Function

Recall that the *domain* of a function is the set of all inputs for the function. The domain of a function may be stated explicitly. For example, if we write

$$f(x) = x^2, \qquad 0 \le x \le 5$$

then the domain is the set of all real numbers x for which $0 \le x \le 5$. If the function is given by an algebraic expression and the domain is not stated explicitly, then by convention *the domain of the function is the domain of the algebraic expression—that is, the set of all real numbers for which the expression is defined as a real number*. For example, consider the functions

Domains of algebraic expressions are discussed on page 35.

$$f(x) = \frac{1}{x-4} \qquad\qquad g(x) = \sqrt{x}$$

The function f is not defined at $x = 4$, so its domain is $\{x \mid x \ne 4\}$. The function g is not defined for negative x, so its domain is $\{x \mid x \ne 0\}$.

Example 6 Finding Domains of Functions

Find the domain of each function.

(a) $f(x) = \dfrac{1}{x^2 - x}$ (b) $g(x) = \sqrt{9 - x^2}$ (c) $h(t) = \dfrac{t}{\sqrt{t+1}}$

Solution

(a) The function is not defined when the denominator is 0. Since

$$f(x) = \frac{1}{x^2 - x} = \frac{1}{x(x-1)}$$

we see that $f(x)$ is not defined when $x = 0$ or $x = 1$. Thus, the domain of f is

$$\{x \mid x \ne 0, x \ne 1\}$$

The domain may also be written in interval notation as

$$(\infty, 0) \cup (0, 1) \cup (1, \infty)$$

(b) We can't take the square root of a negative number, so we must have $9 - x^2 \ge 0$. Using the methods of Section 1.7, we can solve this inequality to find that $-3 \le x \le 3$. Thus, the domain of g is

$$\{x \mid -3 \le x \le 3\} = [-3, 3]$$

(c) We can't take the square root of a negative number, and we can't divide by 0, so we must have $t + 1 > 0$, that is, $t > -1$. So the domain of h is

$$\{t \mid t > -1\} = (-1, \infty)$$

■

Four Ways to Represent a Function

To help us understand what a function is, we have used machine and arrow diagrams. We can describe a specific function in the following four ways:

- verbally (by a description in words)
- algebraically (by an explicit formula)

ALTERNATE EXAMPLE 6a
Find the domain of the function $f(x) = \dfrac{4}{x^2 + x}$.

ANSWER
$(-\infty, -1) \cup (-1, 0) \cup (0, \infty)$

ALTERNATE EXAMPLE 6b
Find the domain of the function $g(x) = \sqrt{16 - x^2}$.

ANSWER
$[-4, 4]$

EXAMPLE
A function with a non-trivial domain: $\dfrac{\sqrt{x^2 - 5x + 6}}{x^2 - 2x + 1}$ has domain $(-\infty, 1) \cup (1, 2] \cup [3, \infty)$.

IN-CLASS MATERIALS

Real-world piecewise functions:

1. The cost of mailing a letter that weighs x ounces (see Exercise 76 in Section 2.2)
2. The cost of making x photocopies (given that there is usually a bulk discount)
3. The cost of printing x pages from a computer (at some point the toner cartridge must be replaced)

57050_02_ch02_p146-247.qxd 07/04/2008 04:59 PM Page 154

- visually (by a graph)
- numerically (by a table of values)

A single function may be represented in all four ways, and it is often useful to go from one representation to another to gain insight into the function. However, certain functions are described more naturally by one method than by the others. An example of a verbal description is

$$P(t) \qquad \text{is} \qquad \text{"the population of the world at time } t\text{"}$$

The function P can also be described numerically by giving a table of values (see Table 1 on page 386). A useful representation of the area of a circle as a function of its radius is the algebraic formula

$$A(r) = \pi r^2$$

The graph produced by a seismograph (see the box) is a visual representation of the vertical acceleration function $a(t)$ of the ground during an earthquake. As a final example, consider the function $C(w)$, which is described verbally as "the cost of mailing a first-class letter with weight w." The most convenient way of describing this function is numerically—that is, using a table of values.

We will be using all four representations of functions throughout this book. We summarize them in the following box.

Four Ways to Represent a Function

Verbal

Using words:

$P(t)$ is "the population of the world at time t"

Relation of population P and time t

Algebraic

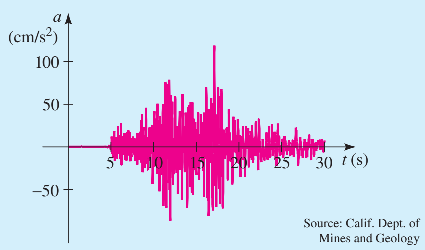
Using a formula:

$$A(r) = \pi r^2$$

Area of a circle

Visual

Using a graph:



Vertical acceleration during an earthquake

Numerical

Using a table of values:

w (ounces)	$C(w)$ (dollars)
$0 < w \le 1$	0.37
$1 < w \le 2$	0.60
$2 < w \le 3$	0.83
$3 < w \le 4$	1.06
$4 < w \le 5$	1.29
⋮	⋮

Cost of mailing a first-class letter

2.1 Exercises

1–4 ■ Express the rule in function notation. (For example, the rule "square, then subtract 5" is expressed as the function $f(x) = x^2 - 5$.)

1. Add 3, then multiply by 2
2. Divide by 7, then subtract 4
3. Subtract 5, then square
4. Take the square root, add 8, then multiply by $\frac{1}{3}$

5–8 ■ Express the function (or rule) in words.

5. $f(x) = \dfrac{x-4}{3}$
6. $g(x) = \dfrac{x}{3} - 4$
7. $h(x) = x^2 + 2$
8. $k(x) = \sqrt{x+2}$

9–10 ■ Draw a machine diagram for the function.

9. $f(x) = \sqrt{x-1}$
10. $f(x) = \dfrac{3}{x-2}$

11–12 ■ Complete the table.

11. $f(x) = 2(x-1)^2$

x	$f(x)$
-1	
0	
1	
2	
3	

12. $g(x) = |2x + 3|$

x	$g(x)$
-3	
-2	
0	
1	
3	

13–20 ■ Evaluate the function at the indicated values.

13. $f(x) = 2x + 1$;
$f(1), f(-2), f(\frac{1}{2}), f(a), f(-a), f(a+b)$

14. $f(x) = x^2 + 2x$;
$f(0), f(3), f(-3), f(a), f(-x), f\left(\dfrac{1}{a}\right)$

15. $g(x) = \dfrac{1-x}{1+x}$;
$g(2), g(-2), g(\frac{1}{2}), g(a), g(a-1), g(-1)$

16. $h(t) = t + \dfrac{1}{t}$;
$h(1), h(-1), h(2), h(\frac{1}{2}), h(x), h\left(\dfrac{1}{x}\right)$

17. $f(x) = 2x^2 + 3x - 4$;
$f(0), f(2), f(-2), f(\sqrt{2}), f(x+1), f(-x)$

18. $f(x) = x^3 - 4x^2$;
$f(0), f(1), f(-1), f(\frac{3}{2}), f\left(\dfrac{x}{2}\right), f(x^2)$

19. $f(x) = 2|x-1|$;
$f(-2), f(0), f(\frac{1}{2}), f(2), f(x+1), f(x^2+2)$

20. $f(x) = \dfrac{|x|}{x}$;
$f(-2), f(-1), f(0), f(5), f(x^2), f\left(\dfrac{1}{x}\right)$

21–24 ■ Evaluate the piecewise defined function at the indicated values.

21. $f(x) = \begin{cases} x^2 & \text{if } x < 0 \\ x+1 & \text{if } x \ge 0 \end{cases}$
$f(-2), f(-1), f(0), f(1), f(2)$

22. $f(x) = \begin{cases} 5 & \text{if } x \le 2 \\ 2x-3 & \text{if } x > 2 \end{cases}$
$f(-3), f(0), f(2), f(3), f(5)$

23. $f(x) = \begin{cases} x^2 + 2x & \text{if } x \le -1 \\ x & \text{if } -1 < x \le 1 \\ -1 & \text{if } x > 1 \end{cases}$
$f(-4), f(-\frac{3}{2}), f(-1), f(0), f(25)$

24. $f(x) = \begin{cases} 3x & \text{if } x < 0 \\ x+1 & \text{if } 0 \le x \le 2 \\ (x-2)^2 & \text{if } x > 2 \end{cases}$
$f(-5), f(0), f(1), f(2), f(5)$

25–28 ■ Use the function to evaluate the indicated expressions and simplify.

25. $f(x) = x^2 + 1$; $f(x+2), f(x) + f(2)$
26. $f(x) = 3x - 1$; $f(2x), 2f(x)$
27. $f(x) = x + 4$; $f(x^2), (f(x))^2$
28. $f(x) = 6x - 18$; $f\left(\dfrac{x}{3}\right), \dfrac{f(x)}{3}$

29–36 ■ Find $f(a)$, $f(a+h)$, and the difference quotient $\dfrac{f(a+h) - f(a)}{h}$, where $h \neq 0$.

29. $f(x) = 3x + 2$
30. $f(x) = x^2 + 1$

57050_02_ch02_p146-247.qxd 07/04/2008 04:59 PM Page 156

31. $f(x) = 5$

32. $f(x) = \dfrac{1}{x+1}$

33. $f(x) = \dfrac{x}{x+1}$

34. $f(x) = \dfrac{2x}{x-1}$

35. $f(x) = 3 - 5x + 4x^2$

36. $f(x) = x^3$

37–58 ■ Find the domain of the function.

37. $f(x) = 2x$

38. $f(x) = x^2 + 1$

39. $f(x) = 2x, \quad -1 \le x \le 5$

40. $f(x) = x^2 + 1, \quad 0 \le x \le 5$

41. $f(x) = \dfrac{1}{x-3}$

42. $f(x) = \dfrac{1}{3x-6}$

43. $f(x) = \dfrac{x+2}{x^2-1}$

44. $f(x) = \dfrac{x^4}{x^2+x-6}$

45. $f(x) = \sqrt{x-5}$

46. $f(x) = \sqrt[4]{x+9}$

47. $f(t) = \sqrt[3]{t-1}$

48. $g(x) = \sqrt{7-3x}$

49. $h(x) = \sqrt{2x-5}$

50. $G(x) = \sqrt{x^2-9}$

51. $g(x) = \dfrac{\sqrt{2+x}}{3-x}$

52. $g(x) = \dfrac{\sqrt{x}}{2x^2+x-1}$

53. $g(x) = \sqrt[4]{x^2-6x}$

54. $g(x) = \sqrt{x^2-2x-8}$

55. $f(x) = \dfrac{3}{\sqrt{x-4}}$

56. $f(x) = \dfrac{x^2}{\sqrt{6-x}}$

57. $f(x) = \dfrac{(x+1)^2}{\sqrt{2x-1}}$

58. $f(x) = \dfrac{x}{\sqrt[4]{9-x^2}}$

Applications

59. Production Cost The cost C in dollars of producing x yards of a certain fabric is given by the function

$$C(x) = 1500 + 3x + 0.02x^2 + 0.0001x^3$$

(a) Find $C(10)$ and $C(100)$.

(b) What do your answers in part (a) represent?

(c) Find $C(0)$. (This number represents the *fixed costs*.)

60. Area of a Sphere The surface area S of a sphere is a function of its radius r given by

$$S(r) = 4\pi r^2$$

(a) Find $S(2)$ and $S(3)$.

(b) What do your answers in part (a) represent?

61. How Far Can You See? Due to the curvature of the earth, the maximum distance D that you can see from the top of a tall building or from an airplane at height h is given by the function

$$D(h) = \sqrt{2rh + h^2}$$

where $r = 3960$ mi is the radius of the earth and D and h are measured in miles.

(a) Find $D(0.1)$ and $D(0.2)$.

(b) How far can you see from the observation deck of Toronto's CN Tower, 1135 ft above the ground?

(c) Commercial aircraft fly at an altitude of about 7 mi. How far can the pilot see?

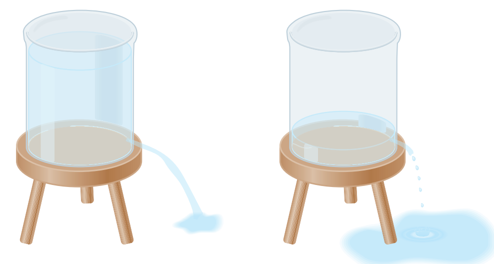
62. Torricelli's Law A tank holds 50 gallons of water, which drains from a leak at the bottom, causing the tank to empty in 20 minutes. The tank drains faster when it is nearly full because the pressure on the leak is greater. **Torricelli's Law** gives the volume of water remaining in the tank after t minutes as

$$V(t) = 50\left(1 - \frac{t}{20}\right)^2 \qquad 0 \le t \le 20$$

(a) Find $V(0)$ and $V(20)$.

(b) What do your answers to part (a) represent?

(c) Make a table of values of $V(t)$ for $t = 0, 5, 10, 15, 20$.



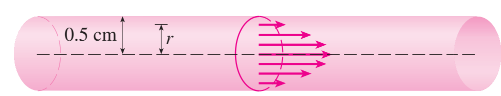
63. Blood Flow As blood moves through a vein or an artery, its velocity v is greatest along the central axis and decreases as the distance r from the central axis increases (see the figure). The formula that gives v as a function of r is called the **law of laminar flow**. For an artery with radius 0.5 cm, we have

$$v(r) = 18{,}500(0.25 - r^2) \qquad 0 \le r \le 0.5$$

(a) Find $v(0.1)$ and $v(0.4)$.

(b) What do your answers to part (a) tell you about the flow of blood in this artery?

(c) Make a table of values of $v(r)$ for $r = 0, 0.1, 0.2, 0.3, 0.4, 0.5$.



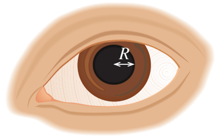
57050_02_ch02_p146-247.qxd 07/04/2008 04:59 PM Page 157

64. Pupil Size When the brightness x of a light source is increased, the eye reacts by decreasing the radius R of the pupil. The dependence of R on x is given by the function

$$R(x) = \sqrt{\frac{13 + 7x^{0.4}}{1 + 4x^{0.4}}}$$

(a) Find $R(1)$, $R(10)$, and $R(100)$.

(b) Make a table of values of $R(x)$.



65. Relativity According to the Theory of Relativity, the length L of an object is a function of its velocity v with respect to an observer. For an object whose length at rest is 10 m, the function is given by

$$L(v) = 10\sqrt{1 - \frac{v^2}{c^2}}$$

where c is the speed of light.

(a) Find $L(0.5c)$, $L(0.75c)$, and $L(0.9c)$.

(b) How does the length of an object change as its velocity increases?

66. Income Tax In a certain country, income tax T is assessed according to the following function of income x:

$$T(x) = \begin{cases} 0 & \text{if } 0 \le x \le 10{,}000 \\ 0.08x & \text{if } 10{,}000 < x \le 20{,}000 \\ 1600 + 0.15x & \text{if } 20{,}000 < x \end{cases}$$

(a) Find $T(5{,}000)$, $T(12{,}000)$, and $T(25{,}000)$.

(b) What do your answers in part (a) represent?

67. Internet Purchases An Internet bookstore charges \$15 shipping for orders under \$100, but provides free shipping for orders of \$100 or more. The cost C of an order is a function of the total price x of the books purchased, given by

$$C(x) = \begin{cases} x + 15 & \text{if } x < 100 \\ x & \text{if } x \ge 100 \end{cases}$$

(a) Find $C(75)$, $C(90)$, $C(100)$, and $C(105)$.

(b) What do your answers in part (a) represent?

68. Cost of a Hotel Stay A hotel chain charges \$75 each night for the first two nights and \$50 for each additional night's stay. The total cost T is a function of the number of nights x that a guest stays.

(a) Complete the expressions in the following piecewise defined function.

$$T(x) = \begin{cases} \underline{\qquad\quad} & \text{if } 0 \le x \le 2 \\ \underline{\qquad\quad} & \text{if } x > 2 \end{cases}$$

(b) Find $T(2)$, $T(3)$, and $T(5)$.

(c) What do your answers in part (b) represent?

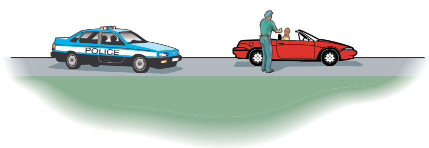
69. Speeding Tickets In a certain state the maximum speed permitted on freeways is 65 mi/h and the minimum is 40. The fine F for violating these limits is \$15 for every mile above the maximum or below the minimum.

(a) Complete the expressions in the following piecewise defined function, where x is the speed at which you are driving.

$$F(x) = \begin{cases} \underline{\qquad\quad} & \text{if } 0 < x < 40 \\ \underline{\qquad\quad} & \text{if } 40 \le x \le 65 \\ \underline{\qquad\quad} & \text{if } x > 65 \end{cases}$$

(b) Find $F(30)$, $F(50)$, and $F(75)$.

(c) What do your answers in part (b) represent?



70. Height of Grass A home owner mows the lawn every Wednesday afternoon. Sketch a rough graph of the height of the grass as a function of time over the course of a four-week period beginning on a Sunday.

71. Temperature Change You place a frozen pie in an oven and bake it for an hour. Then you take it out and let it cool before eating it. Sketch a rough graph of the temperature of the pie as a function of time.

72. Daily Temperature Change Temperature readings T (in °F) were recorded every 2 hours from midnight to noon in Atlanta, Georgia, on March 18, 1996. The time t was measured in hours from midnight. Sketch a rough graph of T as a function of t.

t	T
0	58
2	57
4	53
6	50
8	51
10	57
12	61

57050_02_ch02_p146-247.qxd 07/04/2008 04:59 PM Page 158

73. Population Growth The population P (in thousands) of San Jose, California, from 1988 to 2000 is shown in the table. (Midyear estimates are given.) Draw a rough graph of P as a function of time t.

t	P
1988	733
1990	782
1992	800
1994	817
1996	838
1998	861
2000	895

Discovery • Discussion

74. Examples of Functions At the beginning of this section we discussed three examples of everyday, ordinary functions: Height is a function of age, temperature is a function of date, and postage cost is a function of weight. Give three other examples of functions from everyday life.

75. Four Ways to Represent a Function In the box on page 154 we represented four different functions verbally, algebraically, visually, and numerically. Think of a function that can be represented in all four ways, and write the four representations.

SUGGESTED TIME AND EMPHASIS

1 class.
Essential material.

2.2 Graphs of Functions

The most important way to visualize a function is through its graph. In this section we investigate in more detail the concept of graphing functions.

Graphing Functions

The Graph of a Function

If f is a function with domain A, then the **graph** of f is the set of ordered pairs

$$\{(x, f(x)) \mid x \in A\}$$

In other words, the graph of f is the set of all points (x, y) such that $y = f(x)$; that is, the graph of f is the graph of the equation $y = f(x)$.

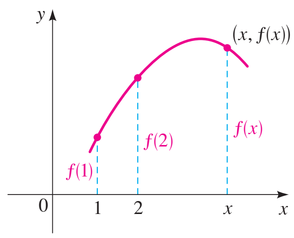


Figure 1
The height of the graph above the point x is the value of $f(x)$.

The graph of a function f gives a picture of the behavior or "life history" of the function. We can read the value of $f(x)$ from the graph as being the height of the graph above the point x (see Figure 1).

A function f of the form $f(x) = mx + b$ is called a **linear function** because its graph is the graph of the equation $y = mx + b$, which represents a line with slope m and y-intercept b. A special case of a linear function occurs when the slope is $m = 0$. The function $f(x) = b$, where b is a given number, is called a **constant function** because all its values are the same number, namely, b. Its graph is the horizontal line $y = b$. Figure 2 shows the graphs of the constant function $f(x) = 3$ and the linear function $f(x) = 2x + 1$.

POINTS TO STRESS

1. Gaining information about a function from its graph, including finding function values, domain, and range.
2. The Vertical Line Test.
3. Graphs of piecewise-defined functions.
4. The greatest integer function.

57050_02_ch02_p146-247.qxd 4/10/08 10:01 PM Page 159

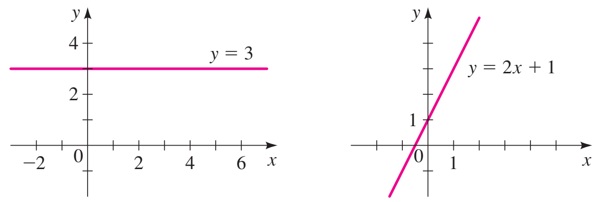


Figure 2 The constant function $f(x) = 3$ The linear function $f(x) = 2x + 1$

Example 1 Graphing Functions

Sketch the graphs of the following functions.
(a) $f(x) = x^2$ (b) $g(x) = x^3$ (c) $h(x) = \sqrt{x}$

Solution We first make a table of values. Then we plot the points given by the table and join them by a smooth curve to obtain the graph. The graphs are sketched in Figure 3.

x	$f(x) = x^2$
0	0
$\pm\frac{1}{2}$	$\frac{1}{4}$
± 1	1
± 2	4
± 3	9

x	$g(x) = x^3$
0	0
$\frac{1}{2}$	$\frac{1}{8}$
1	1
2	8
$-\frac{1}{2}$	$-\frac{1}{8}$
-1	-1
-2	-8

x	$h(x) = \sqrt{x}$
0	0
1	1
2	$\sqrt{2}$
3	$\sqrt{3}$
4	2
5	$\sqrt{5}$

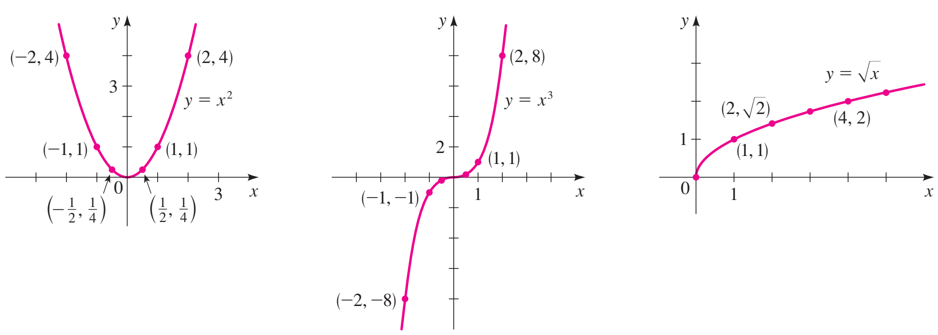


Figure 3 (a) $f(x) = x^2$ (b) $g(x) = x^3$ (c) $h(x) = \sqrt{x}$ ■

A convenient way to graph a function is to use a graphing calculator, as in the next example.

ALTERNATE EXAMPLE 1a
For the graph of the function $f(x) = 3x^2$ find the coordinates of the points with $x = 0$, $x = \pm\frac{1}{2}$, $x = \pm 1$, $x = \pm 2$, $x = \pm 3$.

ANSWER
$(0, 0)$, $\left(\frac{1}{2}, \frac{3}{4}\right)$, $\left(-\frac{1}{2}, \frac{3}{4}\right)$, $(1, 3)$, $(-1, 3)$, $(2, 12)$, $(-2, 12)$, $(3, 27)$, $(-3, 27)$

ALTERNATE EXAMPLE 1b
For the graph of the function $f(x) = 4x^3$ find the coordinates of the points with $x = 0$, $x = \pm\frac{1}{2}$, $x = \pm 1$, $x = \pm 2$.

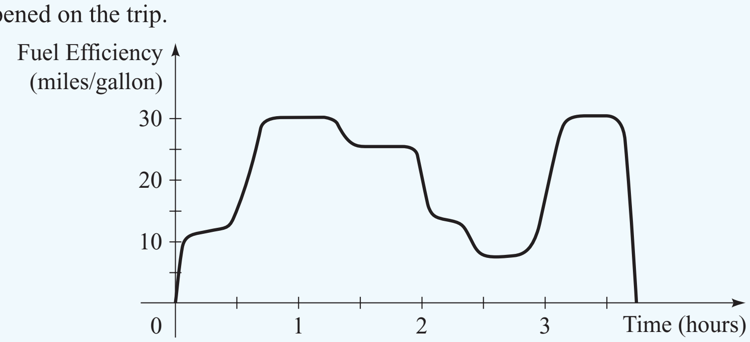
ANSWER
$(0, 0)$, $\left(\frac{1}{2}, \frac{1}{2}\right)$, $\left(-\frac{1}{2}, -\frac{1}{2}\right)$, $(1, 4)$, $(-1, -4)$, $(2, 32)$, $(-2, -32)$

SAMPLE QUESTION
Text Question
Your text discusses the greatest integer function $[\![x]\!]$. Compute $[\![2.6]\!]$, $[\![2]\!]$, $[\![-2.6]\!]$, and $[\![-2]\!]$.

Answer
$[\![2.6]\!] = 2$, $[\![2]\!] = 2$, $[\![-2.6]\!] = -3$, $[\![-2]\!] = -2$

IN-CLASS MATERIALS

Draw a graph of fuel efficiency versus time on a trip, such as the one below. Lead a discussion of what could have happened on the trip.



57050_02_ch02_p146-247.qxd 07/04/2008 04:59 PM Page 160

ALTERNATE EXAMPLE 2

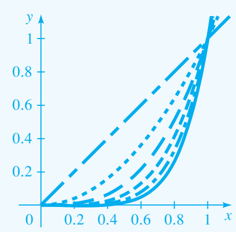
(a) Graph the functions $f(x) = x^n$ for $n = 1, 2, 3, 4, 5, 6$ in the viewing rectangle $[0, 1.1]$ by $[0, 1.1]$.

(b) Graph the functions $f(x) = x^n$ for $n = 1, 2, 3, 4, 5, 6$ in the viewing rectangle $[0, 2.6]$ by $[0, 300]$.

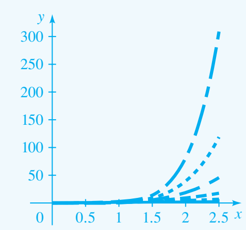
(c) What conclusions can you draw?

ANSWERS

(a)



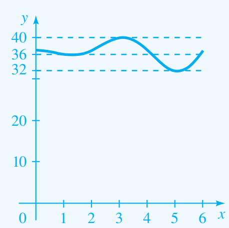
(b)



(c) For values of x between 0 and 1, as n increases, the curve grows more slowly. All the curves go through the points (0, 0) and (1, 1). For values of x larger than 1, the curves grow more quickly as n increases, dramatically so.

ALTERNATE EXAMPLE 3

The function T graphed in the figure below gives the temperature in degrees Fahrenheit between noon and 6 P.M. at a certain weather station.



(a) Find $T(1)$, $T(3)$, and $T(5)$.

(b) What is the highest temperature achieved in this time interval? At what time is the temperature at or below freezing?

ANSWERS

(a) 36 degrees, 40 degrees, 32 degrees.

(b) 40 degrees, 5:00 P.M.

Example 2 A Family of Power Functions

(a) Graph the functions $f(x) = x^n$ for $n = 2, 4$, and 6 in the viewing rectangle $[-2, 2]$ by $[-1, 3]$.

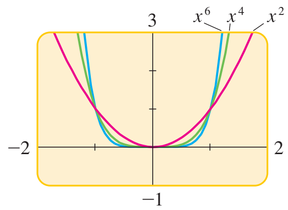
(b) Graph the functions $f(x) = x^n$ for $n = 1, 3$, and 5 in the viewing rectangle $[-2, 2]$ by $[-2, 2]$.

(c) What conclusions can you draw from these graphs?

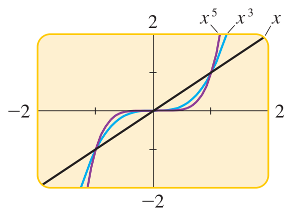
Solution The graphs for parts (a) and (b) are shown in Figure 4.

(c) We see that the general shape of the graph of $f(x) = x^n$ depends on whether n is even or odd.

If n is even, the graph of $f(x) = x^n$ is similar to the parabola $y = x^2$.
If n is odd, the graph of $f(x) = x^n$ is similar to that of $y = x^3$. ■



(a) Even powers of x



(b) Odd powers of x

Figure 4

A family of power functions $f(x) = x^n$

Notice from Figure 4 that as n increases the graph of $y = x^n$ becomes flatter near 0 and steeper when $x > 1$. When $0 < x < 1$, the lower powers of x are the "bigger" functions. But when $x > 1$, the higher powers of x are the dominant functions.

Getting Information from the Graph of a Function

The values of a function are represented by the height of its graph above the x-axis. So, we can read off the values of a function from its graph.

Example 3 Find the Values of a Function from a Graph

The function T graphed in Figure 5 gives the temperature between noon and 6 P.M. at a certain weather station.

(a) Find $T(1)$, $T(3)$, and $T(5)$.

(b) Which is larger, $T(2)$ or $T(4)$?

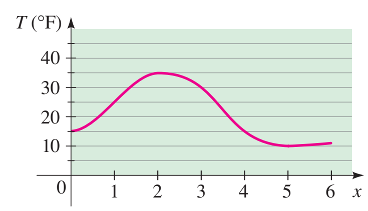


Figure 5

Temperature function

Solution

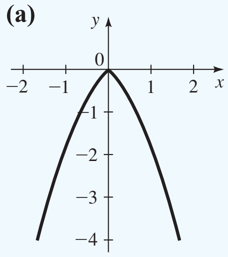
(a) $T(1)$ is the temperature at 1:00 P.M. It is represented by the height of the graph above the x-axis at $x = 1$. Thus, $T(1) = 25$. Similarly, $T(3) = 30$ and $T(5) = 10$.

(b) Since the graph is higher at $x = 2$ than at $x = 4$, it follows that $T(2)$ is larger than $T(4)$. ■

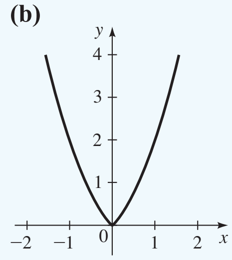
DRILL QUESTION

Let $f(x) = x^2 + |x|$. Which of the following is the graph of f? How do you know?

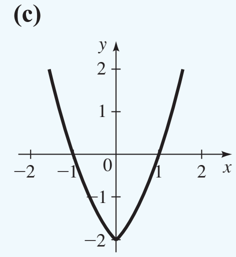
(a)



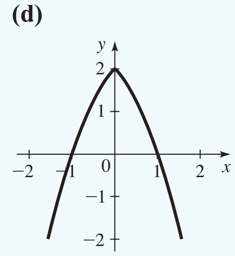
(b)



(c)



(d)



Answer

(b) is the graph of f, because $f(x) \geq 0$ for all x.

57050_02_ch02_p146-247.qxd 07/04/2008 04:59 PM Page 161

The graph of a function helps us picture the domain and range of the function on the x-axis and y-axis as shown in Figure 6.

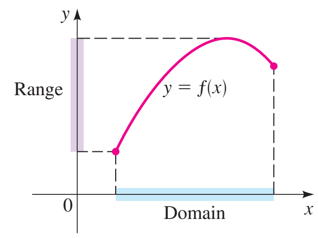


Figure 6
Domain and range of f

Example 4 Finding the Domain and Range from a Graph

(a) Use a graphing calculator to draw the graph of $f(x) = \sqrt{4 - x^2}$.
(b) Find the domain and range of f.

Solution
(a) The graph is shown in Figure 7.

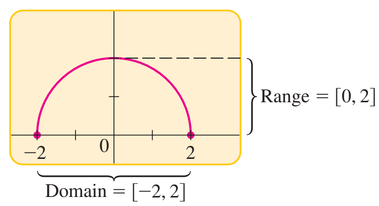


Figure 7
Graph of $f(x) = \sqrt{4 - x^2}$

(b) From the graph in Figure 7 we see that the domain is $[-2, 2]$ and the range is $[0, 2]$. ■

Graphing Piecewise Defined Functions

A piecewise defined function is defined by different formulas on different parts of its domain. As you might expect, the graph of such a function consists of separate pieces.

Example 5 Graph of a Piecewise Defined Function

Sketch the graph of the function.

$$f(x) = \begin{cases} x^2 & \text{if } x \le 1 \\ 2x + 1 & \text{if } x > 1 \end{cases}$$

Solution If $x \le 1$, then $f(x) = x^2$, so the part of the graph to the left of $x = 1$ coincides with the graph of $y = x^2$, which we sketched in Figure 3. If $x > 1$, then $f(x) = 2x + 1$, so the part of the graph to the right of $x = 1$ coincides with the

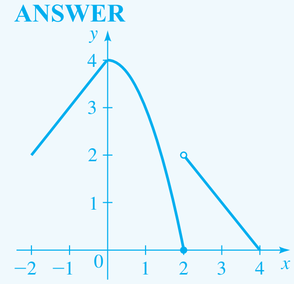
ALTERNATE EXAMPLE 4
For the graph of the function $f(x) = 4 \| x \|$ find $f(x)$ for $-2 \le x < -1$, $-1 \le x < 0$, $0 \le x < 1$, $1 \le x < 2$, $2 \le x < 3$, $3 \le x < 4$.

ANSWER
−8, −4, 0, 4, 8, 12

ALTERNATE EXAMPLE 5
Sketch a graph of the function:

$$f(x) = \begin{cases} x + 4 & \text{if } x \le 0 \\ 4 - x^2 & \text{if } 0 < x \le 2 \\ -x + 4 & \text{if } 2 > x \end{cases}$$

ANSWER



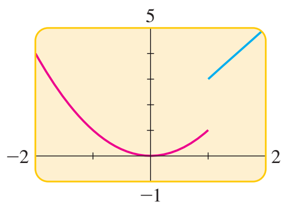
IN-CLASS MATERIALS

In 1984, United States President Ronald Reagan proposed a plan to change the United States personal income tax system. According to his plan, the income tax would be 15% on the first \$19,300 earned, 25% on the next \$18,800, and 35% on all income above and beyond that. Describe this situation to the class, and have them graph (marginal) tax rate and tax owed versus income for incomes ranging from \$0 to \$80,000. Then have them try to come up with equations describing this situation.

57050_02_ch02_p146-247.qxd 07/04/2008 04:59 PM Page 162

On many graphing calculators the graph in Figure 8 can be produced by using the logical functions in the calculator. For example, on the TI-83 the following equation gives the required graph:

$Y_1 = (X \le 1)X\text{^}2 + (X > 1)(2X + 1)$



(To avoid the extraneous vertical line between the two parts of the graph, put the calculator in `Dot` mode.)

line $y = 2x + 1$, which we graphed in Figure 2. This enables us to sketch the graph in Figure 8.

The solid dot at $(1, 1)$ indicates that this point is included in the graph; the open dot at $(1, 3)$ indicates that this point is excluded from the graph.

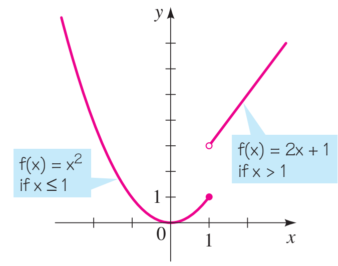


Figure 8

$$f(x) = \begin{cases} x^2 & \text{if } x \le 1 \\ 2x + 1 & \text{if } x > 1 \end{cases}$$

■

Example 6 Graph of the Absolute Value Function

Sketch the graph of the absolute value function $f(x) = |x|$.

Solution Recall that

$$|x| = \begin{cases} x & \text{if } x \ge 0 \\ -x & \text{if } x < 0 \end{cases}$$

Using the same method as in Example 5, we note that the graph of f coincides with the line $y = x$ to the right of the y-axis and coincides with the line $y = -x$ to the left of the y-axis (see Figure 9).

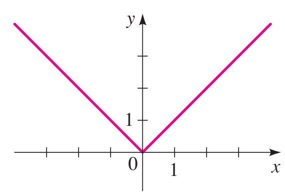


Figure 9
Graph of $f(x) = |x|$

■

The **greatest integer function** is defined by

$$[\![x]\!] = \text{greatest integer less than or equal to } x$$

For example, $[\![2]\!] = 2$, $[\![2.3]\!] = 2$, $[\![1.999]\!] = 1$, $[\![0.002]\!] = 0$, $[\![-3.5]\!] = -4$, $[\![-0.5]\!] = -1$.

Example 7 Graph of the Greatest Integer Function

Sketch the graph of $f(x) = [\![x]\!]$.

Solution The table shows the values of f for some values of x. Note that $f(x)$ is constant between consecutive integers so the graph between integers is a horizontal

ALTERNATE EXAMPLE 6
Write a piecewise-defined function whose combined graph is identical to the one defined by $f(x) = 6|x|$.

ANSWER

$$f(x) = \begin{cases} 6x & \text{if } x \ge 0 \\ -6x & \text{if } x < 0 \end{cases}$$

ALTERNATE EXAMPLE 7
Write two functions whose combined graphs are identical to the one defined by

$$f(x) = \begin{cases} x^2 & \text{if } x \le 1 \\ 2x + 3 & \text{if } x > 1x \le 2 \end{cases}$$

for the interval $(-\infty, 1]$, and for the interval $(1, \infty)$.

ANSWER
$f(x) = x^2$, $f(x) = 2x + 3$

IN-CLASS MATERIALS

In the year 2000, presidential candidate Steve Forbes proposed a "flat tax" model: 0% on the first \$36,000 and 17% on the rest. Have your students do the same analysis, and compare the two models. As an extension, perhaps have them look at a current tax table and draw similar graphs.

57050_02_ch02_p146-247.qxd 07/04/2008 04:59 PM Page 163

line segment as shown in Figure 10.

x	$[\![x]\!]$
$\vdots$	$\vdots$
$-2 \le x < -1$	-2
$-1 \le x < 0$	-1
$0 \le x < 1$	0
$1 \le x < 2$	1
$2 \le x < 3$	2
$\vdots$	$\vdots$

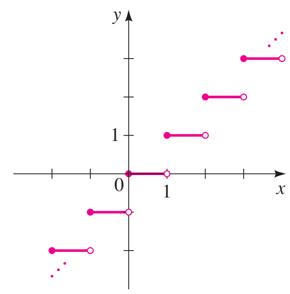


Figure 10
The greatest integer function, $y = [\![x]\!]$ ■

The greatest integer function is an example of a **step function**. The next example gives a real-world example of a step function.

Example 8 The Cost Function for Long-Distance Phone Calls

The cost of a long-distance daytime phone call from Toronto to Mumbai, India, is 69 cents for the first minute and 58 cents for each additional minute (or part of a minute). Draw the graph of the cost C (in dollars) of the phone call as a function of time t (in minutes).

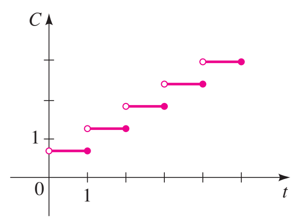


Figure 11
Cost of a long-distance call

Solution Let $C(t)$ be the cost for t minutes. Since $t > 0$, the domain of the function is $(0, \infty)$. From the given information, we have

$$C(t) = 0.69 \qquad \text{if } 0 < t \le 1$$
$$C(t) = 0.69 + 0.58 = 1.27 \qquad \text{if } 1 < t \le 2$$
$$C(t) = 0.69 + 2(0.58) = 1.85 \qquad \text{if } 2 < t \le 3$$
$$C(t) = 0.69 + 3(0.58) = 2.43 \qquad \text{if } 3 < t \le 4$$

and so on. The graph is shown in Figure 11. ■

The Vertical Line Test

The graph of a function is a curve in the xy-plane. But the question arises: Which curves in the xy-plane are graphs of functions? This is answered by the following test.

The Vertical Line Test

A curve in the coordinate plane is the graph of a function if and only if no vertical line intersects the curve more than once.

ALTERNATE EXAMPLE 8
The cost of a long-distance daytime phone call from Toronto to New York City is 20 cents for the first minute and 15 cents for each additional minute (or part of a minute). For the graph of the function C (in dollars) of the phone call as a function of time t (in minutes) find $C(t)$ for $0 < t \le 1$, $1 < t \le 2$, $2 < t \le 3$, $3 < t \le 4$.

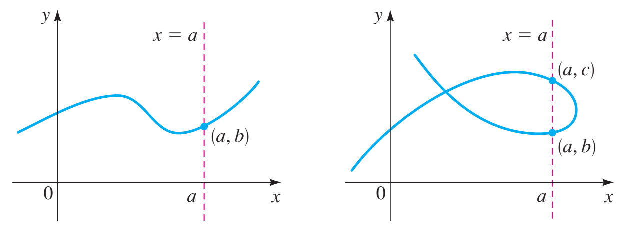
ANSWER
0.71, 1.37, 2.03, 2.69

IN-CLASS MATERIALS

Discuss the shape, symmetries, and general "flatness" near 0 of the power functions x^n for various values of n. Similarly discuss $\sqrt[n]{x}$ for n even and n odd.

57050_02_ch02_p146-247.qxd 07/04/2008 04:59 PM Page 164

We can see from Figure 12 why the Vertical Line Test is true. If each vertical line $x = a$ intersects a curve only once at (a, b), then exactly one functional value is defined by $f(a) = b$. But if a line $x = a$ intersects the curve twice, at (a, b) and at (a, c), then the curve can't represent a function because a function cannot assign two different values to a.



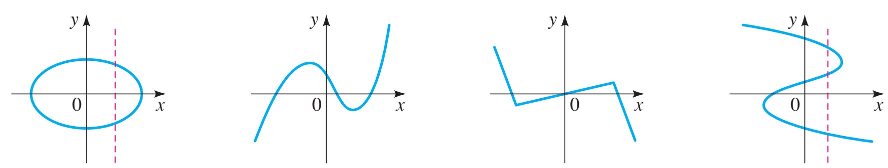
Graph of a function

Not a graph of a function

Figure 12
Vertical Line Test

Example 9 Using the Vertical Line Test

Using the Vertical Line Test, we see that the curves in parts (b) and (c) of Figure 13 represent functions, whereas those in parts (a) and (d) do not.



(a) (b) (c) (d)

Figure 13

■

Equations That Define Functions

Any equation in the variables x and y defines a relationship between these variables. For example, the equation

$$y - x^2 = 0$$

defines a relationship between y and x. Does this equation define y as a *function* of x? To find out, we solve for y and get

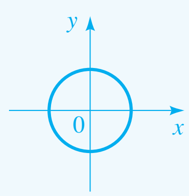
$$y = x^2$$

We see that the equation defines a rule, or function, that gives one value of y for each

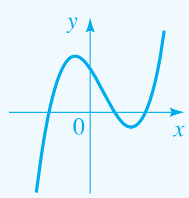
ALTERNATE EXAMPLE 9

Which curves in the xy-plane given below are graphs of functions?

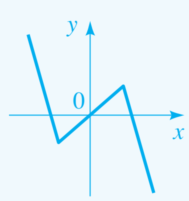
(a)



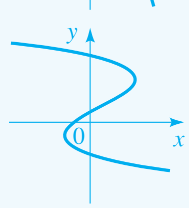
(b)



(c)



(d)

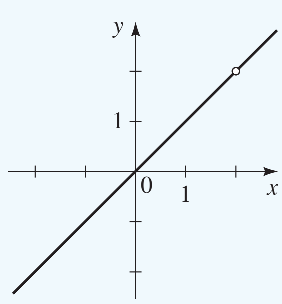


ANSWERS

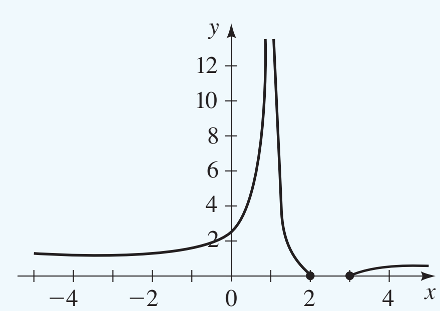
(b), (c)

IN-CLASS MATERIALS

Explore domain and range with some graphs that have holes.



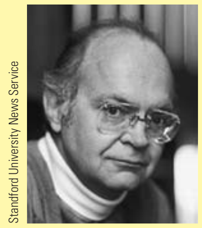
$$f(x) = \frac{x(x-2)}{x-2}$$



$$g(x) = \sqrt{\frac{x^2 - 5x + 6}{x^2 - 2x + 1}}$$

164 CHAPTER 2 Functions

57050_02_ch02_p146-247.qxd 07/04/2008 04:59 PM Page 165



Donald Knuth was born in Milwaukee in 1938 and is Professor Emeritus of Computer Science at Stanford University. While still a graduate student at Caltech, he started writing a monumental series of books entitled *The Art of Computer Programming*. President Carter awarded him the National Medal of Science in 1979. When Knuth was a high school student, he became fascinated with graphs of functions and laboriously drew many hundreds of them because he wanted to see the behavior of a great variety of functions. (Today, of course, it is far easier to use computers and graphing calculators to do this.) Knuth is famous for his invention of TEX, a system of computer-assisted typesetting. This system was used in the preparation of the manuscript for this textbook. He has also written a novel entitled *Surreal Numbers: How Two Ex-Students Turned On to Pure Mathematics and Found Total Happiness*.

Dr. Knuth has received numerous honors, among them election as an associate of the French Academy of Sciences, and as a Fellow of the Royal Society.

value of x. We can express this rule in function notation as

$$f(x) = x^2$$

But not every equation defines y as a function of x, as the following example shows.

Example 10 Equations That Define Functions

Does the equation define y as a function of x?

(a) $y - x^2 = 2$

(b) $x^2 + y^2 = 4$

Solution

(a) Solving for y in terms of x gives

$$y - x^2 = 2$$
$$y = x^2 + 2 \qquad \text{Add } x^2$$

The last equation is a rule that gives one value of y for each value of x, so it defines y as a function of x. We can write the function as $f(x) = x^2 + 2$.

(b) We try to solve for y in terms of x:

$$x^2 + y^2 = 4$$
$$y^2 = 4 - x^2 \qquad \text{Subtract } x^2$$
$$y = \pm\sqrt{4 - x^2} \qquad \text{Take square roots}$$

The last equation gives two values of y for a given value of x. Thus, the equation does not define y as a function of x. ■

The graphs of the equations in Example 10 are shown in Figure 14. The Vertical Line Test shows graphically that the equation in Example 10(a) defines a function but the equation in Example 10(b) does not.

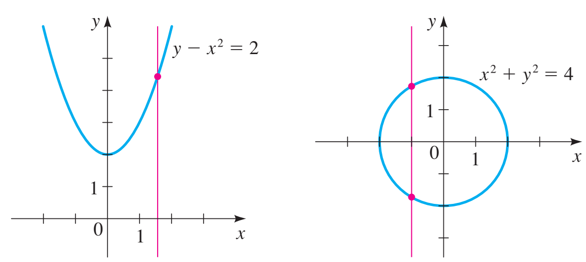


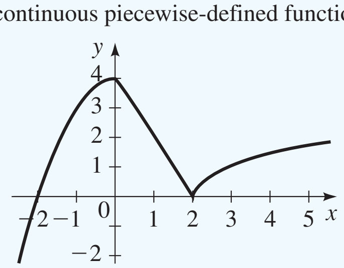
Figure 14

ALTERNATE EXAMPLE 10b
Does the equation define y as a function of x?
$x^2 + y^2 = 8$

ANSWER
No

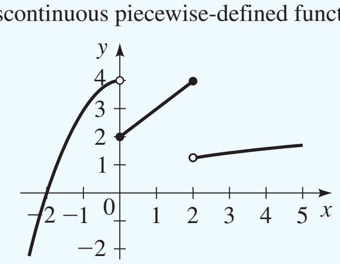
EXAMPLE

A continuous piecewise-defined function



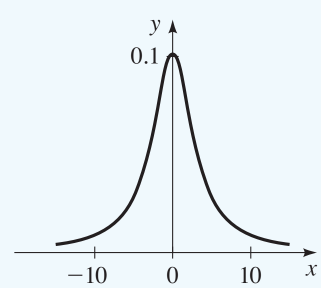
$$f(x) = \begin{cases} 4 - x^2 & \text{if } x < 0 \\ 4 - 2x & \text{if } 0 \le x \le 2 \\ \sqrt{x - 2} & \text{if } x > 2 \end{cases}$$

A discontinuous piecewise-defined function

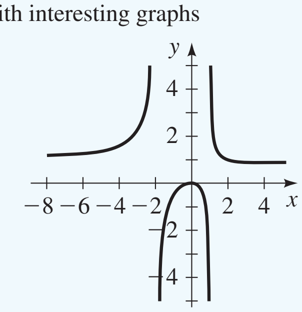


$$f(x) = \begin{cases} 4 - x^2 & \text{if } x < 0 \\ x + 2 & \text{if } 0 \le x \le 2 \\ \sqrt[3]{x} & \text{if } x > 2 \end{cases}$$

Classic rational functions with interesting graphs



$$h(x) = \frac{1}{x^2 + \pi^2}$$



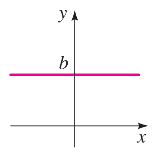
$$i(x) = \frac{x^2}{x^2 + x - 2}$$

57050_02_ch02_p146-247.qxd 07/04/2008 04:59 PM Page 166

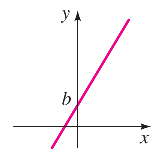
The following table shows the graphs of some functions that you will see frequently in this book.

Some Functions and Their Graphs

Linear functions
$f(x) = mx + b$

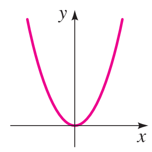


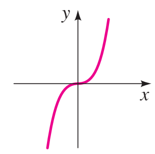
$f(x) = b$

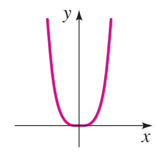


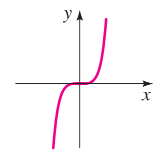
$f(x) = mx + b$

Power functions
$f(x) = x^n$

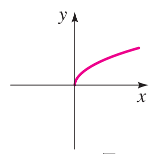

$f(x) = x^2$

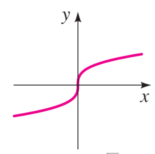

$f(x) = x^3$

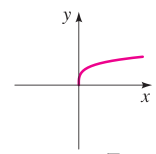

$f(x) = x^4$

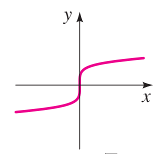

$f(x) = x^5$

Root functions
$f(x) = \sqrt[n]{x}$

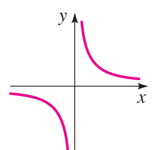

$f(x) = \sqrt{x}$

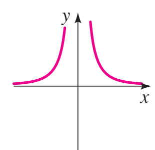

$f(x) = \sqrt[3]{x}$


$f(x) = \sqrt[4]{x}$

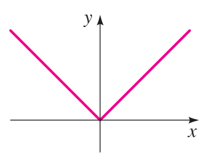

$f(x) = \sqrt[5]{x}$

Reciprocal functions
$f(x) = 1/x^n$

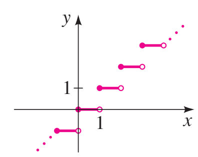

$f(x) = \frac{1}{x}$


$f(x) = \frac{1}{x^2}$

Absolute value function
$f(x) = |x|$


$f(x) = |x|$

Greatest integer function
$f(x) = [\![x]\!]$



$f(x) = [\![x]\!]$

166 CHAPTER 2 Functions

57050_02_ch02_p146-247.qxd 07/04/2008 04:59 PM Page 167

2.2 Exercises

1–22 ■ Sketch the graph of the function by first making a table of values.

1. $f(x) = 2$

2. $f(x) = -3$

3. $f(x) = 2x - 4$

4. $f(x) = 6 - 3x$

5. $f(x) = -x + 3, \quad -3 \le x \le 3$

6. $f(x) = \dfrac{x - 3}{2}, \quad 0 \le x \le 5$

7. $f(x) = -x^2$

8. $f(x) = x^2 - 4$

9. $g(x) = x^3 - 8$

10. $g(x) = 4x^2 - x^4$

11. $g(x) = \sqrt{x + 4}$

12. $g(x) = \sqrt{-x}$

13. $F(x) = \dfrac{1}{x}$

14. $F(x) = \dfrac{1}{x + 4}$

15. $H(x) = |2x|$

16. $H(x) = |x + 1|$

17. $G(x) = |x| + x$

18. $G(x) = |x| - x$

19. $f(x) = |2x - 2|$

20. $f(x) = \dfrac{x}{|x|}$

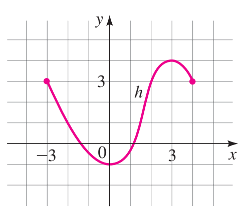
21. $g(x) = \dfrac{2}{x^2}$

22. $g(x) = \dfrac{|x|}{x^2}$

23. The graph of a function h is given.

(a) Find $h(-2)$, $h(0)$, $h(2)$, and $h(3)$.

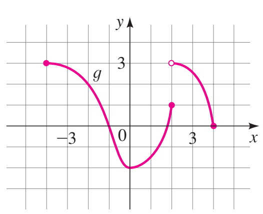
(b) Find the domain and range of h.



24. The graph of a function g is given.

(a) Find $g(-4)$, $g(-2)$, $g(0)$, $g(2)$, and $g(4)$.

(b) Find the domain and range of g.

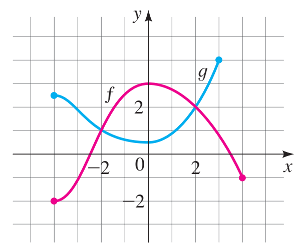


25. Graphs of the functions f and g are given.

(a) Which is larger, $f(0)$ or $g(0)$?

(b) Which is larger, $f(-3)$ or $g(-3)$?

(c) For which values of x is $f(x) = g(x)$?

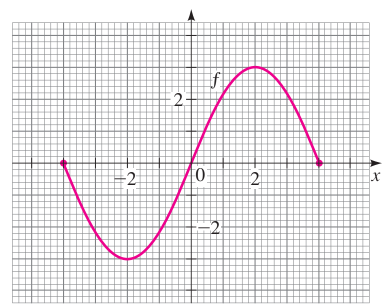


26. The graph of a function f is given.

(a) Estimate $f(0.5)$ to the nearest tenth.

(b) Estimate $f(3)$ to the nearest tenth.

(c) Find all the numbers x in the domain of f for which $f(x) = 1$.



27–36 ■ A function f is given.

(a) Use a graphing calculator to draw the graph of f.

(b) Find the domain and range of f from the graph.

27. $f(x) = x - 1$

28. $f(x) = 2(x + 1)$

29. $f(x) = 4$

30. $f(x) = -x^2$

31. $f(x) = 4 - x^2$

32. $f(x) = x^2 + 4$

33. $f(x) = \sqrt{16 - x^2}$

34. $f(x) = -\sqrt{25 - x^2}$

35. $f(x) = \sqrt{x - 1}$

36. $f(x) = \sqrt{x + 2}$

37–50 ■ Sketch the graph of the piecewise defined function.

37. $f(x) = \begin{cases} 0 & \text{if } x < 2 \\ 1 & \text{if } x \ge 2 \end{cases}$

57050_02_ch02_p146-247.qxd 07/04/2008 04:59 PM Page 168

38. $f(x) = \begin{cases} 1 & \text{if } x \le 1 \\ x + 1 & \text{if } x > 1 \end{cases}$

39. $f(x) = \begin{cases} 3 & \text{if } x < 2 \\ x - 1 & \text{if } x \ge 2 \end{cases}$

40. $f(x) = \begin{cases} 1 - x & \text{if } x < -2 \\ 5 & \text{if } x \ge -2 \end{cases}$

41. $f(x) = \begin{cases} x & \text{if } x \le 0 \\ x + 1 & \text{if } x > 0 \end{cases}$

42. $f(x) = \begin{cases} 2x + 3 & \text{if } x < -1 \\ 3 - x & \text{if } x \ge -1 \end{cases}$

43. $f(x) = \begin{cases} -1 & \text{if } x < -1 \\ 1 & \text{if } -1 \le x \le 1 \\ -1 & \text{if } x > 1 \end{cases}$

44. $f(x) = \begin{cases} -1 & \text{if } x < -1 \\ x & \text{if } -1 \le x \le 1 \\ 1 & \text{if } x > 1 \end{cases}$

45. $f(x) = \begin{cases} 2 & \text{if } x \le -1 \\ x^2 & \text{if } x > -1 \end{cases}$

46. $f(x) = \begin{cases} 1 - x^2 & \text{if } x \le 2 \\ x & \text{if } x > 2 \end{cases}$

47. $f(x) = \begin{cases} 0 & \text{if } |x| \le 2 \\ 3 & \text{if } |x| > 2 \end{cases}$

48. $f(x) = \begin{cases} x^2 & \text{if } |x| \le 1 \\ 1 & \text{if } |x| > 1 \end{cases}$

49. $f(x) = \begin{cases} 4 & \text{if } x < -2 \\ x^2 & \text{if } -2 \le x \le 2 \\ -x + 6 & \text{if } x > 2 \end{cases}$

50. $f(x) = \begin{cases} -x & \text{if } x \le 0 \\ 9 - x^2 & \text{if } 0 < x \le 3 \\ x - 3 & \text{if } x > 3 \end{cases}$

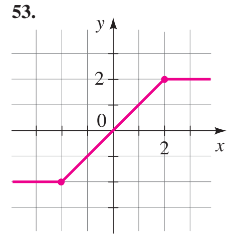
51–52 ■ Use a graphing device to draw the graph of the piecewise defined function. (See the margin note on page 162.)

51. $f(x) = \begin{cases} x + 2 & \text{if } x \le -1 \\ x^2 & \text{if } x > -1 \end{cases}$

52. $f(x) = \begin{cases} 2x - x^2 & \text{if } x > 1 \\ (x - 1)^3 & \text{if } x \le 1 \end{cases}$

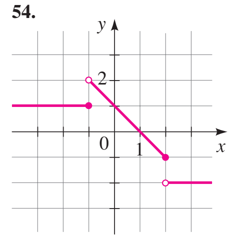
53–54 ■ The graph of a piecewise defined function is given. Find a formula for the function in the indicated form.

53.



$f(x) = \begin{cases} & \text{if } x < -2 \\ & \text{if } -2 \le x \le 2 \\ & \text{if } x > 2 \end{cases}$

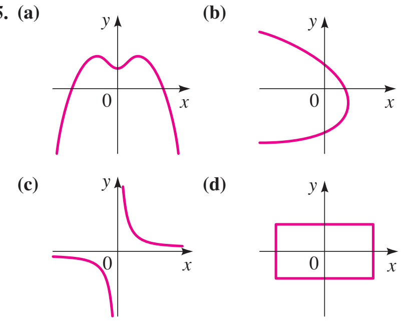
54.



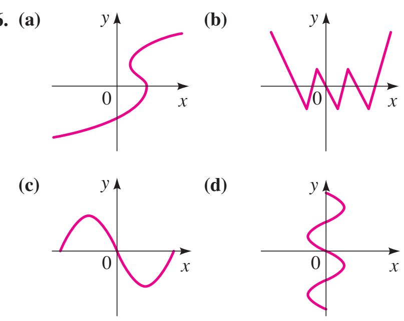
$f(x) = \begin{cases} & \text{if } x \le -1 \\ & \text{if } -1 < x \le 2 \\ & \text{if } x > 2 \end{cases}$

55–56 ■ Determine whether the curve is the graph of a function of x.

55.



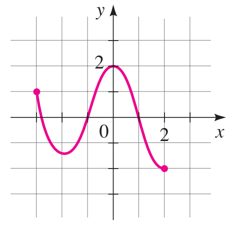
56.



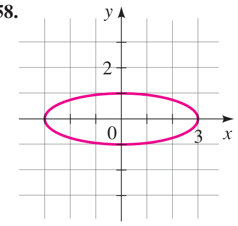
57050_02_ch02_p146-247.qxd 07/04/2008 04:59 PM Page 169

57–60 ■ Determine whether the curve is the graph of a function x. If it is, state the domain and range of the function.

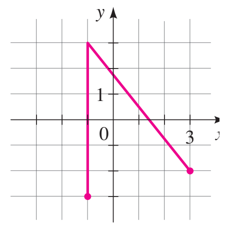
57.



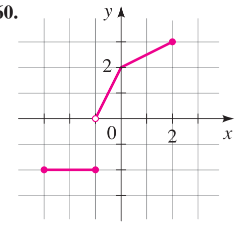
58.



59.



60.



61–72 ■ Determine whether the equation defines y as a function of x. (See Example 10.)

61. $x^2 + 2y = 4$

62. $3x + 7y = 21$

63. $x = y^2$

64. $x^2 + (y - 1)^2 = 4$

65. $x + y^2 = 9$

66. $x^2 + y = 9$

67. $x^2y + y = 1$

68. $\sqrt{x} + y = 12$

69. $2|x| + y = 0$

70. $2x + |y| = 0$

71. $x = y^3$

72. $x = y^4$

73–78 ■ A family of functions is given. In parts (a) and (b) graph all the given members of the family in the viewing rectangle indicated. In part (c) state the conclusions you can make from your graphs.

73. $f(x) = x^2 + c$

(a) $c = 0, 2, 4, 6;\quad [-5, 5]$ by $[-10, 10]$

(b) $c = 0, -2, -4, -6;\quad [-5, 5]$ by $[-10, 10]$

(c) How does the value of c affect the graph?

74. $f(x) = (x - c)^2$

(a) $c = 0, 1, 2, 3;\quad [-5, 5]$ by $[-10, 10]$

(b) $c = 0, -1, -2, -3;\quad [-5, 5]$ by $[-10, 10]$

(c) How does the value of c affect the graph?

75. $f(x) = (x - c)^3$

(a) $c = 0, 2, 4, 6;\quad [-10, 10]$ by $[-10, 10]$

(b) $c = 0, -2, -4, -6;\quad [-10, 10]$ by $[-10, 10]$

(c) How does the value of c affect the graph?

76. $f(x) = cx^2$

(a) $c = 1, \frac{1}{2}, 2, 4;\quad [-5, 5]$ by $[-10, 10]$

(b) $c = 1, -1, -\frac{1}{2}, -2;\quad [-5, 5]$ by $[-10, 10]$

(c) How does the value of c affect the graph?

77. $f(x) = x^c$

(a) $c = \frac{1}{2}, \frac{1}{4}, \frac{1}{6};\quad [-1, 4]$ by $[-1, 3]$

(b) $c = 1, \frac{1}{3}, \frac{1}{5};\quad [-3, 3]$ by $[-2, 2]$

(c) How does the value of c affect the graph?

78. $f(x) = 1/x^n$

(a) $n = 1, 3;\quad [-3, 3]$ by $[-3, 3]$

(b) $n = 2, 4;\quad [-3, 3]$ by $[-3, 3]$

(c) How does the value of n affect the graph?

79–82 ■ Find a function whose graph is the given curve.

79. The line segment joining the points $(-2, 1)$ and $(4, -6)$

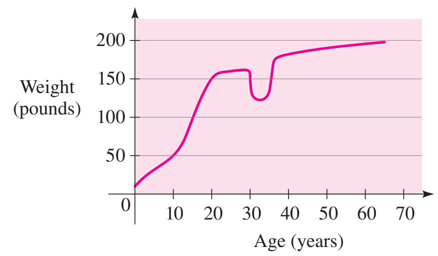
80. The line segment joining the points $(-3, -2)$ and $(6, 3)$

81. The top half of the circle $x^2 + y^2 = 9$

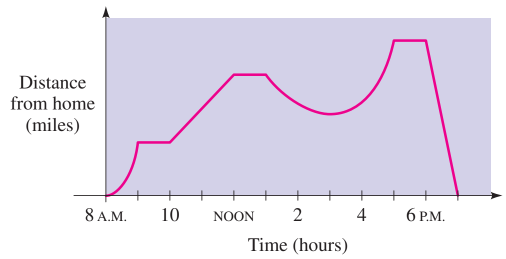
82. The bottom half of the circle $x^2 + y^2 = 9$

Applications

83. Weight Function The graph gives the weight of a certain person as a function of age. Describe in words how this person's weight has varied over time. What do you think happened when this person was 30 years old?

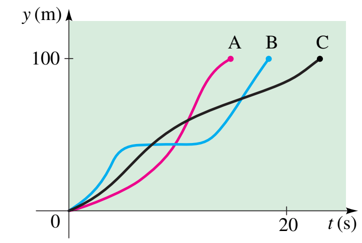


84. Distance Function The graph gives a salesman's distance from his home as a function of time on a certain day. Describe in words what the graph indicates about his travels on this day.



57050_02_ch02_p146-247.qxd 07/04/2008 05:00 PM Page 170

85. Hurdle Race Three runners compete in a 100-meter hurdle race. The graph depicts the distance run as a function of time for each runner. Describe in words what the graph tells you about this race. Who won the race? Did each runner finish the race? What do you think happened to runner B?

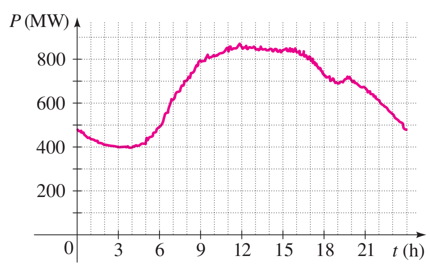


86. Power Consumption The figure shows the power consumption in San Francisco for September 19, 1996 (P is measured in megawatts; t is measured in hours starting at midnight).

(a) What was the power consumption at 6 A.M.? At 6 P.M.?

(b) When was the power consumption the lowest?

(c) When was the power consumption the highest?



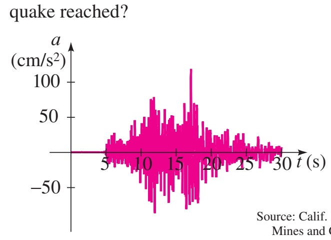
Source: Pacific Gas & Electric

87. Earthquake The graph shows the vertical acceleration of the ground from the 1994 Northridge earthquake in Los Angeles, as measured by a seismograph. (Here t represents the time in seconds.)

(a) At what time t did the earthquake first make noticeable movements of the earth?

(b) At what time t did the earthquake seem to end?

(c) At what time t was the maximum intensity of the earthquake reached?



Source: Calif. Dept. of Mines and Geology

88. Utility Rates Westside Energy charges its electric customers a base rate of \$6.00 per month, plus 10¢ per kilowatt-hour (kWh) for the first 300 kWh used and 6¢ per kWh for all usage over 300 kWh. Suppose a customer uses x kWh of electricity in one month.

(a) Express the monthly cost E as a function of x.

(b) Graph the function E for $0 \le x \le 600$.

89. Taxicab Function A taxi company charges \$2.00 for the first mile (or part of a mile) and 20 cents for each succeeding tenth of a mile (or part). Express the cost C (in dollars) of a ride as a function of the distance x traveled (in miles) for $0 < x < 2$, and sketch the graph of this function.

90. Postage Rates The domestic postage rate for first-class letters weighing 12 oz or less is 37 cents for the first ounce (or less), plus 23 cents for each additional ounce (or part of an ounce). Express the postage P as a function of the weight x of a letter, with $0 < x \le 12$, and sketch the graph of this function.

Discovery • Discussion

91. When Does a Graph Represent a Function? For every integer n, the graph of the equation $y = x^n$ is the graph of a function, namely $f(x) = x^n$. Explain why the graph of $x = y^2$ is *not* the graph of a function of x. Is the graph of $x = y^3$ the graph of a function of x? If so, of what function of x is it the graph? Determine for what integers n the graph of $x = y^n$ is the graph of a function of x.

92. Step Functions In Example 8 and Exercises 89 and 90 we are given functions whose graphs consist of horizontal line segments. Such functions are often called *step functions*, because their graphs look like stairs. Give some other examples of step functions that arise in everyday life.

93. Stretched Step Functions Sketch graphs of the functions $f(x) = [\![x]\!]$, $g(x) = [\![2x]\!]$, and $h(x) = [\![3x]\!]$ on separate graphs. How are the graphs related? If n is a positive integer, what does the graph of $k(x) = [\![nx]\!]$ look like?

94. Graph of the Absolute Value of a Function

(a) Draw the graphs of the functions $f(x) = x^2 + x - 6$ and $g(x) = |x^2 + x - 6|$. How are the graphs of f and g related?

(b) Draw the graphs of the functions $f(x) = x^4 - 6x^2$ and $g(x) = |x^4 - 6x^2|$. How are the graphs of f and g related?

(c) In general, if $g(x) = |f(x)|$, how are the graphs of f and g related? Draw graphs to illustrate your answer.

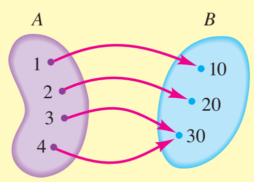
57050_02_ch02_p146-247.qxd 07/04/2008 05:00 PM Page 171

DISCOVERY PROJECT

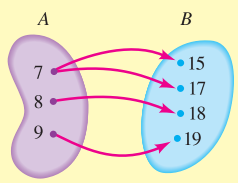
Relations and Functions

A function f can be represented as a set of ordered pairs (x, y) where x is the input and $y = f(x)$ is the output. For example, the function that squares each natural number can be represented by the ordered pairs $\{(1, 1), (2, 4), (3, 9), \ldots\}$.

A **relation** is *any* collection of ordered pairs. If we denote the ordered pairs in a relation by (x, y) then the set of x-values (or inputs) is the **domain** and the set of y-values (or outputs) is the **range**. With this terminology a **function** is a relation where for each x-value there is *exactly one* y-value (or for each input there is *exactly one* output). The correspondences in the figure below are relations—the first is a function but the second is not because the input 7 in A corresponds to two different outputs, 15 and 17, in B.

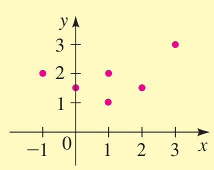


Function



Not a function

We can describe a relation by listing all the ordered pairs in the relation or giving the rule of correspondence. Also, since a relation consists of ordered pairs we can sketch its graph. Let's consider the following relations and try to decide which are functions.



(a) The relation that consists of the ordered pairs $\{(1, 1), (2, 3), (3, 3), (4, 2)\}$.
(b) The relation that consists of the ordered pairs $\{(1, 2), (1, 3), (2, 4), (3, 2)\}$.
(c) The relation whose graph is shown to the left.
(d) The relation whose input values are days in January 2005 and whose output values are the maximum temperature in Los Angeles on that day.
(e) The relation whose input values are days in January 2005 and whose output values are the persons born in Los Angeles on that day.

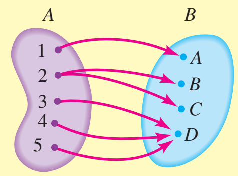
The relation in part (a) is a function because each input corresponds to exactly one output. But the relation in part (b) is not, because the input 1 corresponds to two different outputs (2 and 3). The relation in part (c) is not a function because the input 1 corresponds to two different outputs (1 and 2). The relation in (d) is a function because each day corresponds to exactly one maximum temperature. The relation in (e) is not a function because many persons (not just one) were born in Los Angeles on most days in January 2005.

1. Let $A = \{1, 2, 3, 4\}$ and $B = \{-1, 0, 1\}$. Is the given relation a function from A to B?
(a) $\{(1, 0), (2, -1), (3, 0), (4, 1)\}$
(b) $\{(1, 0), (2, -1), (3, 0), (3, -1), (4, 0)\}$

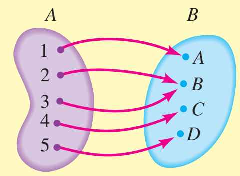
57050_02_ch02_p146-247.qxd 07/04/2008 05:00 PM Page 172

2. Determine if the correspondence is a function.

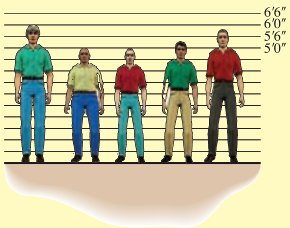
(a)



(b)



3. The following data were collected from members of a college precalculus class. Is the set of ordered pairs (x, y) a function?



(a)

x Height	y Weight
72 in.	180 lb
60 in.	204 lb
60 in.	120 lb
63 in.	145 lb
70 in.	184 lb

(b)

x Age	y ID Number
19	82-4090
21	80-4133
40	66-8295
21	64-9110
21	20-6666

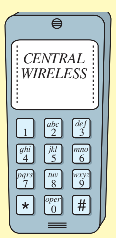
(c)

x Year of graduation	y Number of graduates
2005	2
2006	12
2007	18
2008	7
2009	1

4. An equation in x and y defines a relation, which may or may not be a function (see page 164). Decide whether the relation consisting of all ordered pairs of real numbers (x, y) satisfying the given condition is a function.

(a) $y = x^2$ (b) $x = y^2$ (c) $x \le y$ (d) $2x + 7y = 11$

5. In everyday life we encounter many relations which may or may not define functions. For example, we match up people with their telephone number(s), baseball players with their batting averages, or married men with their wives. Does this last correspondence define a function? In a society in which each married man has exactly one wife the rule is a function. But the rule is not a function. Which of the following everyday relations are functions?



(a) x is the daughter of y (x and y are women in the United States)

(b) x is taller than y (x and y are people in California)

(c) x has received dental treatment from y (x and y are millionaires in the United States)

(d) x is a digit (0 to 9) on a telephone dial and y is a corresponding letter

57050_02_ch02_p146-247.qxd 07/04/2008 05:00 PM Page 173

2.3 Increasing and Decreasing Functions; Average Rate of Change

SUGGESTED TIME AND EMPHASIS
1 class.
Essential material.

Functions are often used to model changing quantities. In this section we learn how to determine if a function is increasing or decreasing, and how to find the rate at which its values change as the variable changes.

Increasing and Decreasing Functions

It is very useful to know where the graph of a function rises and where it falls. The graph shown in Figure 1 rises, falls, then rises again as we move from left to right: It rises from A to B, falls from B to C, and rises again from C to D. The function f is said to be *increasing* when its graph rises and *decreasing* when its graph falls.

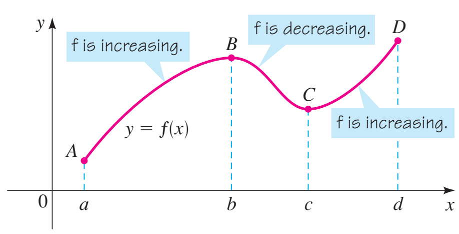


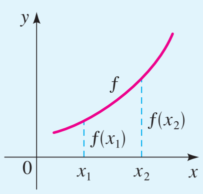
Figure 1
f is increasing on $[a, b]$ and $[c, d]$.
f is decreasing on $[b, c]$.

We have the following definition.

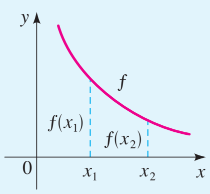
Definition of Increasing and Decreasing Functions

f is **increasing** on an interval I if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ in I.

f is **decreasing** on an interval I if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$ in I.



f is increasing



f is decreasing

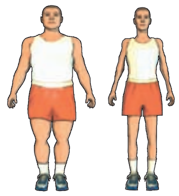
POINTS TO STRESS

1. Algebraic and geometric definitions of increasing and decreasing.
2. Average rate of change.

57050_02_ch02_p146-247.qxd 07/04/2008 05:00 PM Page 174

Example 1 Intervals on which a Function Increases and Decreases

The graph in Figure 2 gives the weight W of a person at age x. Determine the intervals on which the function W is increasing and on which it is decreasing.



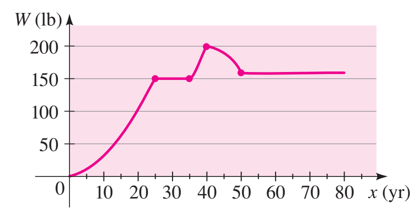


Figure 2
Weight as a function of age

Solution The function is increasing on $[0, 25]$ and $[35, 40]$. It is decreasing on $[40, 50]$. The function is constant (neither increasing nor decreasing) on $[25, 35]$ and $[50, 80]$. This means that the person gained weight until age 25, then gained weight again between ages 35 and 40. He lost weight between ages 40 and 50. ■

Example 2 Using a Graph to Find Intervals where a Function Increases and Decreases

(a) Sketch the graph of the function $f(x) = x^{2/3}$.
(b) Find the domain and range of the function.
(c) Find the intervals on which f increases and decreases.

Solution
(a) We use a graphing calculator to sketch the graph in Figure 3.
(b) From the graph we observe that the domain of f is $\mathbb{R}$ and the range is $[0, \infty)$.
(c) From the graph we see that f is decreasing on $(-\infty, 0]$ and increasing on $[0, \infty)$. ■

Some graphing calculators, such as the TI-82, do not evaluate $x^{2/3}$ [entered as x^(2/3)] for negative x. To graph a function like $f(x) = x^{2/3}$, we enter it as $y_1 = (x^{\wedge}(1/3))^{\wedge}2$ because these calculators correctly evaluate powers of the form $x^{\wedge}(1/n)$. Newer calculators, such as the TI-83 and TI-86, do not have this problem.

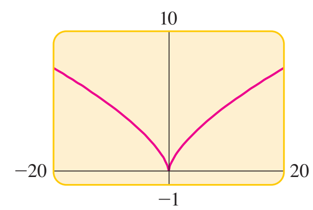


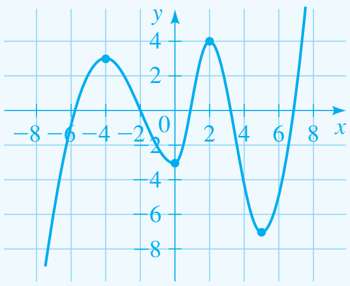
Figure 3
Graph of $f(x) = x^{2/3}$

Average Rate of Change

We are all familiar with the concept of speed: If you drive a distance of 120 miles in 2 hours, then your average speed, or rate of travel, is $\frac{120 \text{ mi}}{2 \text{ h}} = 60$ mi/h.

ALTERNATE EXAMPLE 1

State the intervals on which the function whose graph is shown in figure below is decreasing.



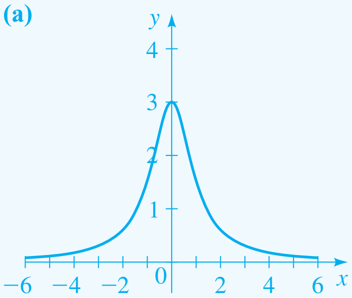
ANSWER
$[-4, 0] \cup [2, 5]$

ALTERNATE EXAMPLE 2

(a) Sketch the graph of the function $f(x) = \dfrac{3}{1 + x^2}$.
(b) Find the domain and range of the function.
(c) Find the intervals on which f increases and decreases.

ANSWERS

(a)



(b) From the graph we observe that the domain of f is $\mathbb{R}$ and the range is $(0, 3]$.
(c) f increases on $(-\infty, 0]$ and f decreases on $[0, \infty)$.

IN-CLASS MATERIALS

Draw a graph of electrical power consumption in the classroom versus time on a typical weekday, pointing out important features throughout, and using the vocabulary of this section as much as possible.

SAMPLE QUESTIONS

Text Questions

Let $f(t) = 3t + 2$.

(a) What is the average rate of change of f from $t = 1$ to $t = 3$?
(b) What is the average rate of change of f from $t = 1$ to $t = \pi$?

Answers

(a) 3
(b) 3

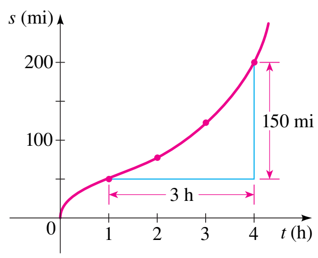


Figure 4
Average speed

Now suppose you take a car trip and record the distance that you travel every few minutes. The distance s you have traveled is a function of the time t:

$$s(t) = \text{total distance traveled at time } t$$

We graph the function s as shown in Figure 4. The graph shows that you have traveled a total of 50 miles after 1 hour, 75 miles after 2 hours, 140 miles after 3 hours, and so on. To find your *average* speed between any two points on the trip, we divide the distance traveled by the time elapsed.

Let's calculate your average speed between 1:00 P.M. and 4:00 P.M. The time elapsed is $4 - 1 = 3$ hours. To find the distance you traveled, we subtract the distance at 1:00 P.M. from the distance at 4:00 P.M., that is, $200 - 50 = 150$ mi. Thus, your average speed is

$$\text{average speed} = \frac{\text{distance traveled}}{\text{time elapsed}} = \frac{150 \text{ mi}}{3 \text{ h}} = 50 \text{ mi/h}$$

The average speed we have just calculated can be expressed using function notation:

$$\text{average speed} = \frac{s(4) - s(1)}{4 - 1} = \frac{200 - 50}{3} = 50 \text{ mi/h}$$

Note that the average speed is different over different time intervals. For example, between 2:00 P.M. and 3:00 P.M. we find that

$$\text{average speed} = \frac{s(3) - s(2)}{3 - 2} = \frac{140 - 75}{1} = 65 \text{ mi/h}$$

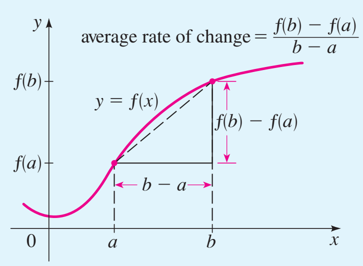
Finding average rates of change is important in many contexts. For instance, we may be interested in knowing how quickly the air temperature is dropping as a storm approaches, or how fast revenues are increasing from the sale of a new product. So we need to know how to determine the average rate of change of the functions that model these quantities. In fact, the concept of average rate of change can be defined for any function.

Average Rate of Change

The **average rate of change** of the function $y = f(x)$ between $x = a$ and $x = b$ is

$$\text{average rate of change} = \frac{\text{change in } y}{\text{change in } x} = \frac{f(b) - f(a)}{b - a}$$

The average rate of change is the slope of the **secant line** between $x = a$ and $x = b$ on the graph of f, that is, the line that passes through $(a, f(a))$ and $(b, f(b))$.



DRILL QUESTION

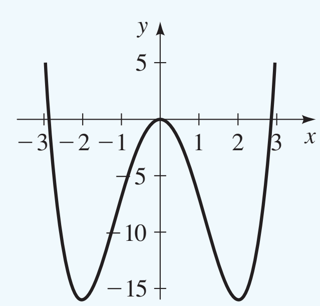
If $f(t) = t^2 - |3t|$, what is the average rate of change between $t = -3$ and $t = -1$?

Answer

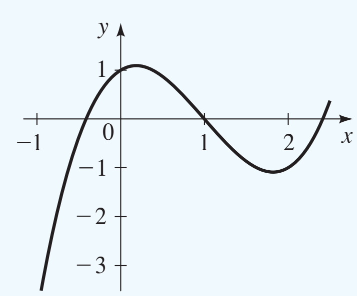
-1

IN-CLASS MATERIALS

Notice that it is fairly easy to tell where some functions are increasing and decreasing by looking at their graphs. For example, the graph of $f(x) = x^4 - 8x^2$ makes things very clear. Note that in this case, the intervals are not immediately apparent from looking at the formula. However, for many functions such as $g(x) = x^3 - 3x^2 + x + 1$, it is very difficult to find the exact intervals where the function is increasing/decreasing. In this example, the endpoints of the intervals will occur at precisely $x = 1 \pm \frac{1}{3}\sqrt{6}$.



$f(x) = x^4 - 8x^2$



$g(x) = x^3 - 3x^2 + x + 1$

ALTERNATE EXAMPLE 3a
For the function $f(x) = x^2 + 7$, find the average rate of change between $x = 2$ and $x = 7$.

ANSWER
9

ALTERNATE EXAMPLE 3b
For the function $f(x) = x^2 + 6$, find the average rate of change of the function between the points $x = y$ and $x = y + d$ $(d \neq 0)$.

ANSWER
$2y + d$

ALTERNATE EXAMPLE 4
Find the direction and the value when vertically shifting the graph of the function $f(x) = x^3 - 3x$ to obtain the graph of the function $g(x) = x^3 - 3x - 25$.

ANSWER
Downward, 25

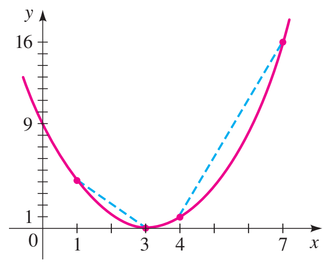


Figure 5
$f(x) = (x - 3)^2$

Example 3 Calculating the Average Rate of Change

For the function $f(x) = (x - 3)^2$, whose graph is shown in Figure 5, find the average rate of change between the following points:

(a) $x = 1$ and $x = 3$ (b) $x = 4$ and $x = 7$

Solution

(a) Average rate of change $= \dfrac{f(3) - f(1)}{3 - 1}$ Definition

$= \dfrac{(3-3)^2 - (1-3)^2}{3 - 1}$ Use $f(x) = (x-3)^2$

$= \dfrac{0 - 4}{2} = -2$

(b) Average rate of change $= \dfrac{f(7) - f(4)}{7 - 4}$ Definition

$= \dfrac{(7-3)^2 - (4-3)^2}{7 - 4}$ Use $f(x) = (x-3)^2$

$= \dfrac{16 - 1}{3} = 5$ ■

Example 4 Average Speed of a Falling Object

If an object is dropped from a tall building, then the distance it has fallen after t seconds is given by the function $d(t) = 16t^2$. Find its average speed (average rate of change) over the following intervals:

(a) Between 1 s and 5 s (b) Between $t = a$ and $t = a + h$

Solution

(a) Average rate of change $= \dfrac{d(5) - d(1)}{5 - 1}$ Definition

$= \dfrac{16(5)^2 - 16(1)^2}{5 - 1}$ Use $d(t) = 16t^2$

$= \dfrac{400 - 16}{4} = 96$ ft/s

(b) Average rate of change $= \dfrac{d(a+h) - d(a)}{(a+h) - a}$ Definition

$= \dfrac{16(a+h)^2 - 16(a)^2}{(a+h) - a}$ Use $d(t) = 16t^2$

$= \dfrac{16(a^2 + 2ah + h^2 - a^2)}{h}$ Expand and factor 16

$= \dfrac{16(2ah + h^2)}{h}$ Simplify numerator

$= \dfrac{16h(2a + h)}{h}$ Factor h

$= 16(2a + h)$ Simplify ■

IN-CLASS MATERIALS

Students should see the geometry of the average rate of change—that the average rate of change from $x = a$ to $x = b$ is the slope of the line from $(a, f(a))$ to $(b, f(b))$. Armed with this knowledge, students now have a way of estimating average rate of change: graph the function (making *sure* that the x- and y-scales are the same), plot the relevant points, and then estimate the slope of the line between them.

57050_02_ch02_p146-247.qxd 07/04/2008 05:00 PM Page 177

The average rate of change calculated in Example 4(b) is known as a *difference quotient*. In calculus we use difference quotients to calculate *instantaneous* rates of change. An example of an instantaneous rate of change is the speed shown on the speedometer of your car. This changes from one instant to the next as your car's speed changes.

Time	Temperature (°F)
8:00 A.M.	38
9:00 A.M.	40
10:00 A.M.	44
11:00 A.M.	50
12:00 NOON	56
1:00 P.M.	62
2:00 P.M.	66
3:00 P.M.	67
4:00 P.M.	64
5:00 P.M.	58
6:00 P.M.	55
7:00 P.M.	51

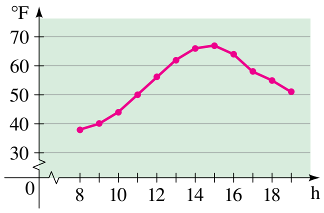


Figure 6

Example 5 Average Rate of Temperature Change

The table gives the outdoor temperatures observed by a science student on a spring day. Draw a graph of the data, and find the average rate of change of temperature between the following times:

(a) 8:00 A.M. and 9:00 A.M.

(b) 1:00 P.M. and 3:00 P.M.

(c) 4:00 P.M. and 7:00 P.M.

Solution A graph of the temperature data is shown in Figure 6. Let t represent time, measured in hours since midnight (so that 2:00 P.M., for example, corresponds to $t = 14$). Define the function F by

$$F(t) = \text{temperature at time } t$$

(a) $$\text{Average rate of change} = \frac{\text{temperature at 9 A.M.} - \text{temperature at 8 A.M.}}{9 - 8} = \frac{F(9) - F(8)}{9 - 8} = \frac{40 - 38}{9 - 8} = 2$$

The average rate of change was 2°F per hour.

(b) $$\text{Average rate of change} = \frac{\text{temperature at 3 P.M.} - \text{temperature at 1 P.M.}}{15 - 13} = \frac{F(15) - F(13)}{15 - 13} = \frac{67 - 62}{2} = 2.5$$

The average rate of change was 2.5°F per hour.

(c) $$\text{Average rate of change} = \frac{\text{temperature at 7 P.M.} - \text{temperature at 4 P.M.}}{19 - 16} = \frac{F(19) - F(16)}{19 - 16} = \frac{51 - 64}{3} \approx -4.3$$

The average rate of change was about −4.3°F per hour during this time interval. The negative sign indicates that the temperature was dropping. ■

ALTERNATE EXAMPLE 5a
The table gives the outdoor temperatures observed by a science student on a spring day. Find the average rate of change of temperature between the following times: 10:00 A.M. and 11:00 A.M.

Time	Temperature (°F)
8:00 A.M.	35
9:00 A.M.	38
10:00 A.M.	40
11:00 A.M.	45
12:00 NOON	50
1:00 P.M.	53
2:00 P.M.	55
3:00 P.M.	56
4:00 P.M.	54
5:00 P.M.	51
6:00 P.M.	47
7:00 P.M.	45

ANSWER
5

ALTERNATE EXAMPLE 5b
The table gives the outdoor temperatures observed by a science student on a spring day. Find the average rate of change of temperature between the following times: 5:00 P.M. and 7:00 P.M.

Time	Temperature (°F)
8:00 A.M.	31
9:00 A.M.	33
10:00 A.M.	36
11:00 A.M.	39
12:00 NOON	41
1:00 P.M.	47
2:00 P.M.	49
3:00 P.M.	50
4:00 P.M.	47
5:00 P.M.	45
6:00 P.M.	41
7:00 P.M.	39

ANSWER
−3

IN-CLASS MATERIALS

It is possible, at this point, to foreshadow calculus nicely. Take a simple function such as $l(t) = t^2$ and look at the average rate of change from $t = 1$ to $t = 2$. Then look at the average rate of change from $t = 1$ to $t = \frac{3}{2}$. If students work in parallel, they should be able to fill in the following table:

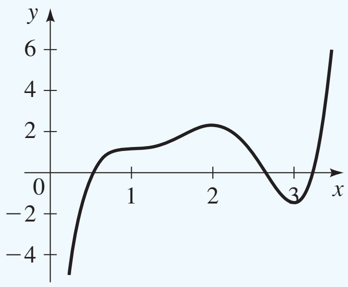
From	To	Average Rate of Change
$t = 1$	2	2
$t = 1$	1.5	2.5
$t = 1$	1.25	2.25
$t = 1$	1.1	2.1
$t = 1$	1.01	2.01
$t = 1$	1.001	2.001

Note that these numbers seem to be approaching 2.

57050_02_ch02_p146-247.qxd 07/04/2008 05:00 PM Page 178

EXAMPLE

A function with two integer turning points and a flat spot:



$\frac{1}{6}(12x^5 - 105x^4 + 340x^3 - 510x^2 + 360x - 90)$

ALTERNATE EXAMPLE 6

Let $f(x) = 5x - 10$. Find the average rate of change between $x = 4$ and $x = 8$.

ANSWER

5

ALTERNATE EXAMPLE 6c

Let $f(x) = 3x - 4$. Find the average rate of change of f between the points $x = b$ and $x = b + d$.

ANSWER

3

Mathematics in the Modern World

Computers

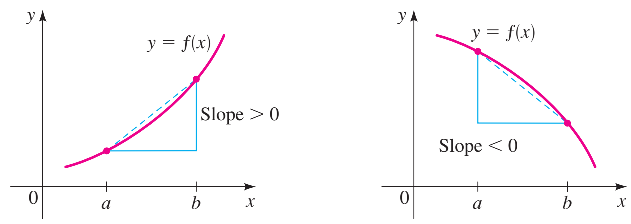
For centuries machines have been designed to perform specific tasks. For example, a washing machine washes clothes, a weaving machine weaves cloth, an adding machine adds numbers, and so on. The computer has changed all that.

The computer is a machine that does nothing—until it is given instructions on what to do. So your computer can play games, draw pictures, or calculate π to a million decimal places; it all depends on what program (or instructions) you give the computer. The computer can do all this because it is able to accept instructions and logically change those instructions based on incoming data. This versatility makes computers useful in nearly every aspect of human endeavor.

The idea of a computer was described theoretically in the 1940s by the mathematician Allan Turing (see page 103) in what he called a *universal machine*. In 1945 the mathematician John Von Neumann, extending Turing's ideas, built one of the first electronic computers.

Mathematicians continue to develop new theoretical bases for the design of computers. The heart of the computer is the "chip," which is capable of processing logical instructions. To get an idea of the chip's complexity, consider that the Pentium chip has over 3.5 million logic circuits!

The graphs in Figure 7 show that if a function is increasing on an interval, then the average rate of change between any two points is positive, whereas if a function is decreasing on an interval, then the average rate of change between any two points is negative.



f increasing
Average rate of change positive

f decreasing
Average rate of change negative

Figure 7

Example 6 Linear Functions Have Constant Rate of Change

Let $f(x) = 3x - 5$. Find the average rate of change of f between the following points.

(a) $x = 0$ and $x = 1$ (b) $x = 3$ and $x = 7$ (c) $x = a$ and $x = a + h$

What conclusion can you draw from your answers?

Solution

(a) $$\text{Average rate of change} = \frac{f(1) - f(0)}{1 - 0} = \frac{(3 \cdot 1 - 5) - (3 \cdot 0 - 5)}{1} = \frac{(-2) - (-5)}{1} = 3$$

(b) $$\text{Average rate of change} = \frac{f(7) - f(3)}{7 - 3} = \frac{(3 \cdot 7 - 5) - (3 \cdot 3 - 5)}{4} = \frac{16 - 4}{4} = 3$$

(c) $$\text{Average rate of change} = \frac{f(a + h) - f(a)}{(a + h) - a} = \frac{[3(a + h) - 5] - [3a - 5]}{h} = \frac{3a + 3h - 5 - 3a + 5}{h} = \frac{3h}{h} = 3$$

It appears that the average rate of change is always 3 for this function. In fact, part (c) proves that the rate of change between any two arbitrary points $x = a$ and $x = a + h$ is 3. ■

As Example 6 indicates, for a linear function $f(x) = mx + b$, the average rate of change between any two points is the slope m of the line. This agrees with what we learned in Section 1.10, that the slope of a line represents the rate of change of y with respect to x.

IN-CLASS MATERIALS

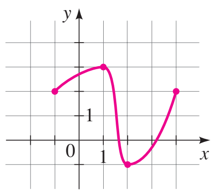
Revisit $f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$, pointing out that it is neither increasing nor decreasing near $x = 0$. Stress that when dealing with new sorts of functions, it becomes important to know the precise mathematical definitions of such terms.

57050_02_ch02_p146-247.qxd 07/04/2008 05:00 PM Page 179

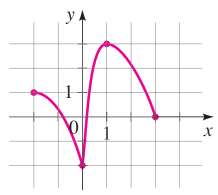
2.3 Exercises

1–4 ■ The graph of a function is given. Determine the intervals on which the function is **(a)** increasing and **(b)** decreasing.

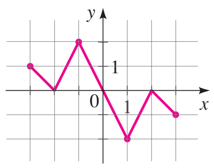
1.



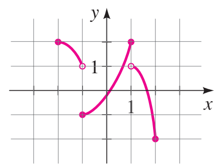
2.



3.



4.



5–12 ■ A function f is given.

(a) Use a graphing device to draw the graph of f.

(b) State approximately the intervals on which f is increasing and on which f is decreasing.

5. $f(x) = x^{2/5}$

6. $f(x) = 4 - x^{2/3}$

7. $f(x) = x^2 - 5x$

8. $f(x) = x^3 - 4x$

9. $f(x) = 2x^3 - 3x^2 - 12x$

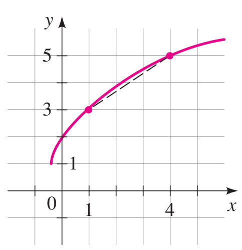
10. $f(x) = x^4 - 16x^2$

11. $f(x) = x^3 + 2x^2 - x - 2$

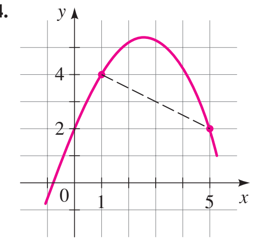
12. $f(x) = x^4 - 4x^3 + 2x^2 + 4x - 3$

13–16 ■ The graph of a function is given. Determine the average rate of change of the function between the indicated values of the variable.

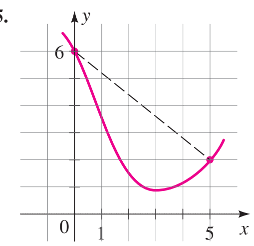
13.



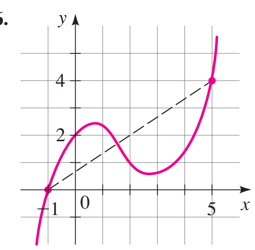
14.



15.



16.



17–28 ■ A function is given. Determine the average rate of change of the function between the given values of the variable.

17. $f(x) = 3x - 2$; $x = 2$, $x = 3$

18. $g(x) = 5 + \frac{1}{2}x$; $x = 1, x = 5$

19. $h(t) = t^2 + 2t$; $t = -1, t = 4$

20. $f(z) = 1 - 3z^2$; $z = -2, z = 0$

21. $f(x) = x^3 - 4x^2$; $x = 0, x = 10$

22. $f(x) = x + x^4$; $x = -1, x = 3$

23. $f(x) = 3x^2$; $x = 2, x = 2 + h$

24. $f(x) = 4 - x^2$; $x = 1, x = 1 + h$

25. $g(x) = \dfrac{1}{x}$; $x = 1, x = a$

26. $g(x) = \dfrac{2}{x + 1}$; $x = 0, x = h$

57050_02_ch02_p146-247.qxd 07/04/2008 05:00 PM Page 180

27. $f(t) = \frac{2}{t}; \quad t = a, t = a + h$

28. $f(t) = \sqrt{t}; \quad t = a, t = a + h$

29–30 ■ A linear function is given.

(a) Find the average rate of change of the function between $x = a$ and $x = a + h$.

(b) Show that the average rate of change is the same as the slope of the line.

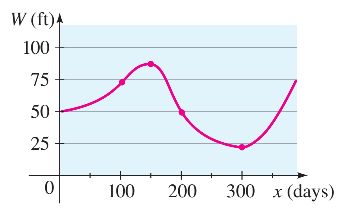
29. $f(x) = \frac{1}{2}x + 3$ **30.** $g(x) = -4x + 2$

Applications

31. Changing Water Levels The graph shows the depth of water W in a reservoir over a one-year period, as a function of the number of days x since the beginning of the year.

(a) Determine the intervals on which the function W is increasing and on which it is decreasing.

(b) What was the average rate of change of W between $x = 100$ and $x = 200$?

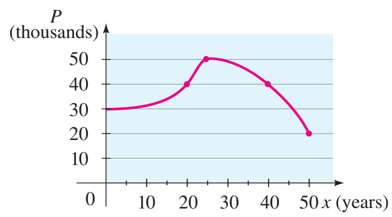


32. Population Growth and Decline The graph shows the population P in a small industrial city from 1950 to 2000. The variable x represents the number of years since 1950.

(a) Determine the intervals on which the function P is increasing and on which it is decreasing.

(b) What was the average rate of change of P between $x = 20$ and $x = 40$?

(c) Interpret the value of the average rate of change that you found in part (b).



33. Population Growth and Decline The table gives the population in a small coastal community for the period 1997–2006. Figures shown are for January 1 in each year.

(a) What was the average rate of change of population between 1998 and 2001?

(b) What was the average rate of change of population between 2002 and 2004?

(c) For what period of time was the population increasing?

(d) For what period of time was the population decreasing?

Year	Population
1997	624
1998	856
1999	1,336
2000	1,578
2001	1,591
2002	1,483
2003	994
2004	826
2005	801
2006	745

34. Running Speed A man is running around a circular track 200 m in circumference. An observer uses a stopwatch to record the runner's time at the end of each lap, obtaining the data in the following table.

(a) What was the man's average speed (rate) between 68 s and 152 s?

(b) What was the man's average speed between 263 s and 412 s?

(c) Calculate the man's speed for each lap. Is he slowing down, speeding up, or neither?

Time (s)	Distance (m)
32	200
68	400
108	600
152	800
203	1000
263	1200
335	1400
412	1600

35. CD Player Sales The table shows the number of CD players sold in a small electronics store in the years 1993–2003.

(a) What was the average rate of change of sales between 1993 and 2003?

57050_02_ch02_p146-247.qxd 07/04/2008 05:00 PM Page 181

(b) What was the average rate of change of sales between 1993 and 1994?

(c) What was the average rate of change of sales between 1994 and 1996?

(d) Between which two successive years did CD player sales *increase* most quickly? *Decrease* most quickly?

Year	CD players sold
1993	512
1994	520
1995	413
1996	410
1997	468
1998	510
1999	590
2000	607
2001	732
2002	612
2003	584

36. Book Collection Between 1980 and 2000, a rare book collector purchased books for his collection at the rate of 40 books per year. Use this information to complete the following table. (Note that not every year is given in the table.)

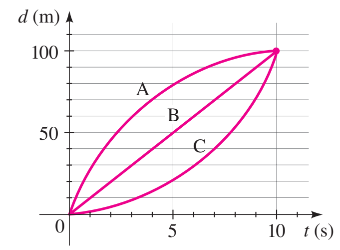
Year	Number of books
1980	420
1981	460
1982	
1985	
1990	
1992	
1995	
1997	
1998	
1999	
2000	1220

Discovery • Discussion

37. 100-meter Race A 100-m race ends in a three-way tie for first place. The graph shows distance as a function of time for each of the three winners.

(a) Find the average speed for each winner.

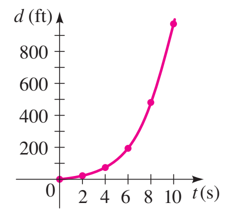
(b) Describe the differences between the way the three runners ran the race.



38. Changing Rates of Change: Concavity The two tables and graphs give the distances traveled by a racing car during two different 10-s portions of a race. In each case, calculate the average speed at which the car is traveling between the observed data points. Is the speed increasing or decreasing? In other words, is the car *accelerating* or *decelerating* on each of these intervals? How does the shape of the graph tell you whether the car is accelerating or decelerating? (The first graph is said to be *concave up* and the second graph *concave down*.)

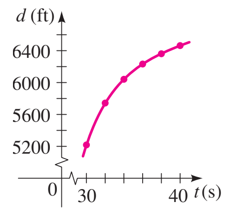
(a)

Time (s)	Distance (ft)
0	0
2	34
4	70
6	196
8	490
10	964



(b)

Time (s)	Distance (ft)
30	5208
32	5734
34	6022
36	6204
38	6352
40	6448



39. Functions That Are Always Increasing or Decreasing Sketch rough graphs of functions that are defined for all real numbers and that exhibit the indicated behavior (or explain why the behavior is impossible).

(a) f is always increasing, and $f(x) > 0$ for all x

(b) f is always decreasing, and $f(x) > 0$ for all x

(c) f is always increasing, and $f(x) < 0$ for all x

(d) f is always decreasing, and $f(x) < 0$ for all x

SUGGESTED TIME AND EMPHASIS
1 class.
Essential material.

ALTERNATE EXAMPLE 1
Find the direction and the value of vertically shifting the graph of the function $y = x^2$ to obtain the graph of the function $f(x) = x^2 - 3$.

ANSWER
Downward, 3

2.4 Transformations of Functions

In this section we study how certain transformations of a function affect its graph. This will give us a better understanding of how to graph functions. The transformations we study are shifting, reflecting, and stretching.

Vertical Shifting

Adding a constant to a function shifts its graph vertically: upward if the constant is positive and downward if it is negative.

Example 1 Vertical Shifts of Graphs

Use the graph of $f(x) = x^2$ to sketch the graph of each function.

(a) $g(x) = x^2 + 3$ (b) $h(x) = x^2 - 2$

Solution The function $f(x) = x^2$ was graphed in Example 1(a), Section 2.2. It is sketched again in Figure 1.

(a) Observe that

$$g(x) = x^2 + 3 = f(x) + 3$$

So the y-coordinate of each point on the graph of g is 3 units above the corresponding point on the graph of f. This means that to graph g we shift the graph of f upward 3 units, as in Figure 1.

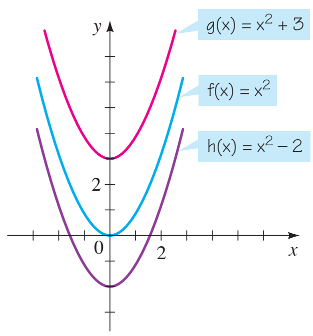


Figure 1

(b) Similarly, to graph h we shift the graph of f downward 2 units, as shown. ■

Recall that the graph of the function f is the same as the graph of the equation $y = f(x)$.

In general, suppose we know the graph of $y = f(x)$. How do we obtain from it the graphs of

$$y = f(x) + c \quad \text{and} \quad y = f(x) - c \quad (c > 0)$$

The y-coordinate of each point on the graph of $y = f(x) + c$ is c units above the y-coordinate of the corresponding point on the graph of $y = f(x)$. So, we obtain the graph of $y = f(x) + c$ simply by shifting the graph of $y = f(x)$ upward c units. Similarly, we obtain the graph of $y = f(x) - c$ by shifting the graph of $y = f(x)$ downward c units.

POINTS TO STRESS

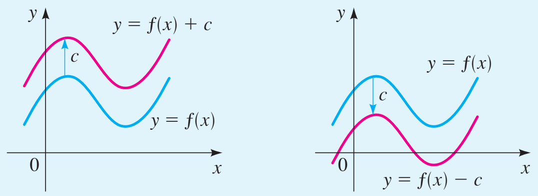
1. Transforming a given function to a different one by shifting, stretching, and reflection.
2. Using the technique of reflection to better understand the concepts of even and odd functions.

Vertical Shifts of Graphs

Suppose $c > 0$.

To graph $y = f(x) + c$, shift the graph of $y = f(x)$ upward c units.

To graph $y = f(x) - c$, shift the graph of $y = f(x)$ downward c units.



Example 2 Vertical Shifts of Graphs

Use the graph of $f(x) = x^3 - 9x$, which was sketched in Example 12, Section 1.8, to sketch the graph of each function.

(a) $g(x) = x^3 - 9x + 10$ (b) $h(x) = x^3 - 9x - 20$

Solution The graph of f is sketched again in Figure 2.

(a) To graph g we shift the graph of f upward 10 units, as shown.

(b) To graph h we shift the graph of f downward 20 units, as shown.

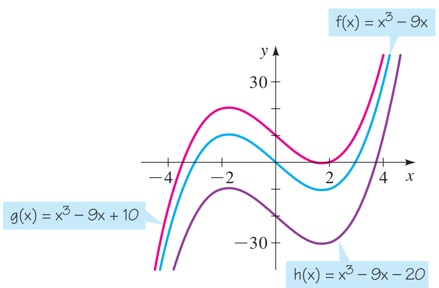


Figure 2

■

Horizontal Shifting

Suppose that we know the graph of $y = f(x)$. How do we use it to obtain the graphs of

$$y = f(x + c) \quad \text{and} \quad y = f(x - c) \quad (c > 0)$$

The value of $f(x - c)$ at x is the same as the value of $f(x)$ at $x - c$. Since $x - c$ is c units to the left of x, it follows that the graph of $y = f(x - c)$ is just the graph of

ALTERNATE EXAMPLE 2
Find the direction and the value when vertically shifting the graph of the function $f(x) = x^3 - 3x$ to obtain the graph of the function $g(x) = x^3 - 3x - 25$.

ANSWER
Downward, 25

SAMPLE QUESTION
Text Question
What is the difference between a vertical stretch and a vertical shift?

Answer
A vertical stretch extends the graph in the vertical direction, changing its shape. A vertical shift simply moves the graph in the vertical direction, preserving its shape.

57050_02_ch02_p146-247.qxd 07/04/2008 05:00 PM Page 184

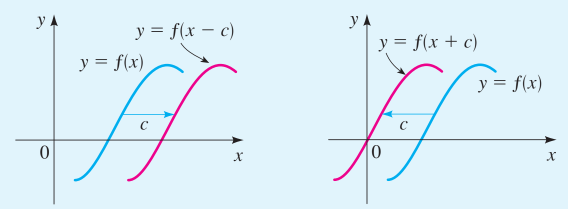
$y = f(x)$ shifted to the right c units. Similar reasoning shows that the graph of $y = f(x + c)$ is the graph of $y = f(x)$ shifted to the left c units. The following box summarizes these facts.

Horizontal Shifts of Graphs

Suppose $c > 0$.

To graph $y = f(x - c)$, shift the graph of $y = f(x)$ to the right c units.

To graph $y = f(x + c)$, shift the graph of $y = f(x)$ to the left c units.



Example 3 Horizontal Shifts of Graphs

Use the graph of $f(x) = x^2$ to sketch the graph of each function.

(a) $g(x) = (x + 4)^2$ (b) $h(x) = (x - 2)^2$

Solution

(a) To graph g, we shift the graph of f to the left 4 units.

(b) To graph h, we shift the graph of f to the right 2 units.

The graphs of g and h are sketched in Figure 3.

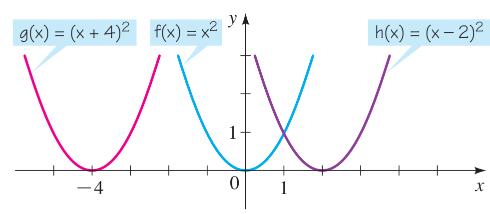


Figure 3 ■

Example 4 Combining Horizontal and Vertical Shifts

Sketch the graph of $f(x) = \sqrt{x - 3} + 4$.

Solution We start with the graph of $y = \sqrt{x}$ (Example 1(c), Section 2.2) and shift it to the right 3 units to obtain the graph of $y = \sqrt{x - 3}$. Then we shift

ALTERNATE EXAMPLE 3
Find the direction and the value of horizontally shifting the graph of the function $y = x^2$ to obtain the graph of the function $f(x) = (x + 2)^2$.

ANSWER
Left, 2

ALTERNATE EXAMPLE 4
Find the direction and the value of horizontal and vertical shifts of the graph of the function $y = \sqrt{x}$ to obtain the graph of the function $f(x) = \sqrt{x + 5} - 4$.

ANSWER
(Downward, 4), (left, 5)

IN-CLASS MATERIALS

Students will often view this section as a process of memorizing eight similar formulas. While it doesn't hurt to memorize how to shift, reflect, or stretch a graph, emphasize to students the importance of understanding what they are doing when they transform a graph.

57050_02_ch02_p146-247.qxd 07/04/2008 05:00 PM Page 185

the resulting graph upward 4 units to obtain the graph of $f(x) = \sqrt{x - 3} + 4$ shown in Figure 4.

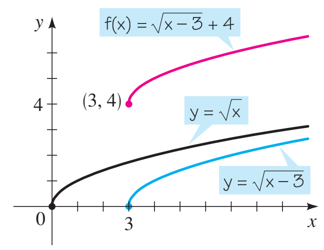


Figure 4

■

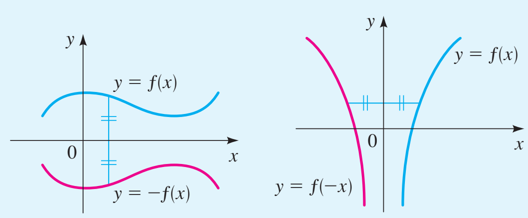
Reflecting Graphs

Suppose we know the graph of $y = f(x)$. How do we use it to obtain the graphs of $y = -f(x)$ and $y = f(-x)$? The y-coordinate of each point on the graph of $y = -f(x)$ is simply the negative of the y-coordinate of the corresponding point on the graph of $y = f(x)$. So the desired graph is the reflection of the graph of $y = f(x)$ in the x-axis. On the other hand, the value of $y = f(-x)$ at x is the same as the value of $y = f(x)$ at $-x$ and so the desired graph here is the reflection of the graph of $y = f(x)$ in the y-axis. The following box summarizes these observations.

Reflecting Graphs

To graph $y = -f(x)$, reflect the graph of $y = f(x)$ in the x-axis.

To graph $y = f(-x)$, reflect the graph of $y = f(x)$ in the y-axis.



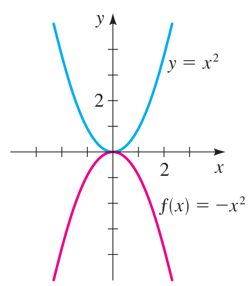


Figure 5

Example 5 **Reflecting Graphs**

Sketch the graph of each function.

(a) $f(x) = -x^2$ (b) $g(x) = \sqrt{-x}$

Solution

(a) We start with the graph of $y = x^2$. The graph of $f(x) = -x^2$ is the graph of $y = x^2$ reflected in the x-axis (see Figure 5).

ALTERNATE EXAMPLE 5

Find the axis of reflection needed to obtain the graph of the function $f(x) = -3x^2$ from the graph of the function $g(x) = 3x^2$.

ANSWER

x

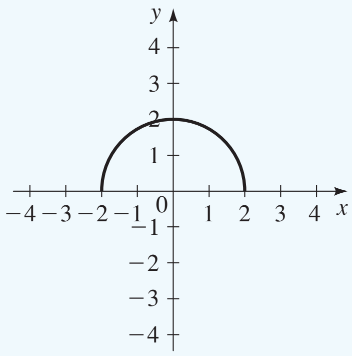
IN-CLASS MATERIALS

Show the class a function they have not learned about yet, such as $f(x) = \sin x$. (If students know about sin, then show them arctan or e^{-x^2}—any function with which they are unfamiliar.) Point out that even though they don't know a lot about $\sin x$, once they've seen the graph, they can graph $\sin x + 3$, $\sin(x - 1)$, $2 \sin x$, $-\sin x$, etc.

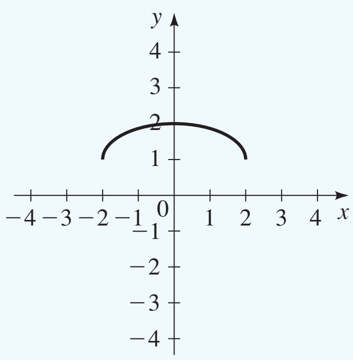
57050_02_ch02_p146-247.qxd 07/04/2008 05:00 PM Page 186

DRILL QUESTION

Given the graph of $f(x)$ below, sketch the graph of $\frac{1}{2}f(x) + 1$.



Answer



ALTERNATE EXAMPLE 6

Find what is needed to perform (stretching or shrinking) with the graph of the function $y = x^2$ to obtain the graph of the function $f(x) = 2x^2$. Find the corresponding factor of that stretching or shrinking.

ANSWER

Stretching, 2

(b) We start with the graph of $y = \sqrt{x}$ (Example 1(c) in Section 2.2). The graph of $g(x) = \sqrt{-x}$ is the graph of $y = \sqrt{x}$ reflected in the y-axis (see Figure 6). Note that the domain of the function $g(x) = \sqrt{-x}$ is $\{x \mid x \le 0\}$.

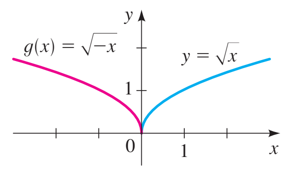


Figure 6 ■

Vertical Stretching and Shrinking

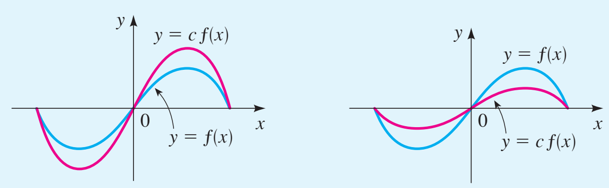
Suppose we know the graph of $y = f(x)$. How do we use it to obtain the graph of $y = cf(x)$? The y-coordinate of $y = cf(x)$ at x is the same as the corresponding y-coordinate of $y = f(x)$ multiplied by c. Multiplying the y-coordinates by c has the effect of vertically stretching or shrinking the graph by a factor of c.

Vertical Stretching and Shrinking of Graphs

To graph $y = cf(x)$:

If $c > 1$, stretch the graph of $y = f(x)$ vertically by a factor of c.

If $0 < c < 1$, shrink the graph of $y = f(x)$ vertically by a factor of c.



$c > 1$ $0 < c < 1$

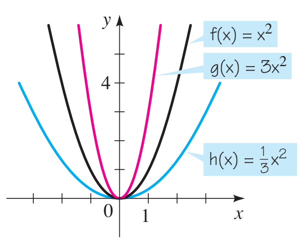


Figure 7

Example 6 Vertical Stretching and Shrinking of Graphs

Use the graph of $f(x) = x^2$ to sketch the graph of each function.

(a) $g(x) = 3x^2$ (b) $h(x) = \frac{1}{3}x^2$

Solution

(a) The graph of g is obtained by multiplying the y-coordinate of each point on the graph of f by 3. That is, to obtain the graph of g we stretch the graph of f vertically by a factor of 3. The result is the narrower parabola in Figure 7.

(b) The graph of h is obtained by multiplying the y-coordinate of each point on the graph of f by $\frac{1}{3}$. That is, to obtain the graph of h we shrink the graph of f vertically by a factor of $\frac{1}{3}$. The result is the wider parabola in Figure 7. ■

We illustrate the effect of combining shifts, reflections, and stretching in the following example.

IN-CLASS MATERIALS

Graph $f(x) = x^2$ with the class. Then anticipate Section 2.6 by having students graph $(x - 2)^2 - 3$ and $(x + 1)^2 + 2$, finally working up to $g(x) = (x - h)^2 + k$. If you are able to then point out that any equation of the form $g(x) = ax^2 + bx + c$ can be so written, students will have a very good start on the next section in addition to learning this one.

57050_02_ch02_p146-247.qxd 07/04/2008 05:00 PM Page 187

Example 7 Combining Shifting, Stretching, and Reflecting

Sketch the graph of the function $f(x) = 1 - 2(x - 3)^2$.

Solution Starting with the graph of $y = x^2$, we first shift to the right 3 units to get the graph of $y = (x - 3)^2$. Then we reflect in the x-axis and stretch by a factor of 2 to get the graph of $y = -2(x - 3)^2$. Finally, we shift upward 1 unit to get the graph of $f(x) = 1 - 2(x - 3)^2$ shown in Figure 8.

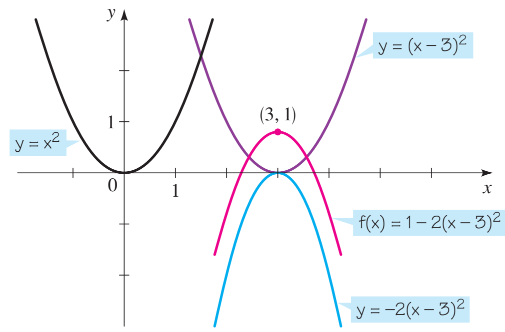


Figure 8

Horizontal Stretching and Shrinking

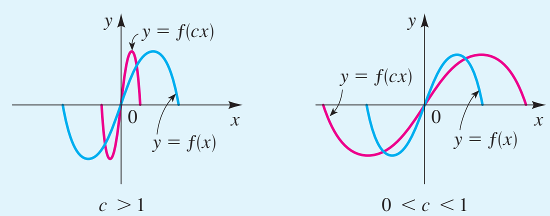
Now we consider horizontal shrinking and stretching of graphs. If we know the graph of $y = f(x)$, then how is the graph of $y = f(cx)$ related to it? The y-coordinate of $y = f(cx)$ at x is the same as the y-coordinate of $y = f(x)$ at cx. Thus, the x-coordinates in the graph of $y = f(x)$ correspond to the x-coordinates in the graph of $y = f(cx)$ multiplied by c. Looking at this the other way around, we see that the x-coordinates in the graph of $y = f(cx)$ are the x-coordinates in the graph of $y = f(x)$ multiplied by $1/c$. In other words, to change the graph of $y = f(x)$ to the graph of $y = f(cx)$, we must shrink (or stretch) the graph horizontally by a factor of $1/c$, as summarized in the following box.

Horizontal Shrinking and Stretching of Graphs

To graph $y = f(cx)$:

If $c > 1$, shrink the graph of $y = f(x)$ horizontally by a factor of $1/c$.

If $0 < c < 1$, stretch the graph of $y = f(x)$ horizontally by a factor of $1/c$.

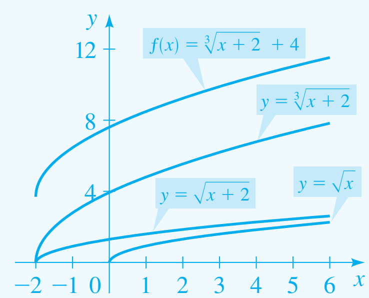


ALTERNATE EXAMPLE 7

Sketch the graph of the function $f(x) = 3\sqrt{x + 2} + 4$.

ANSWER

Starting with the graph of $y = \sqrt{x}$, we first shift to the left 2 units to get the graph of $y = \sqrt{x + 2}$. Then we stretch by a factor of 3 to get the graph of $y = \sqrt[3]{x + 2}$. Finally, we shift upward 4 units to get the graph of $f(x) = \sqrt[3]{x + 2} + 4$ as shown below:



IN-CLASS MATERIALS

This is a good time in the course to start seriously discussing parameters. Ask your students to imagine a scientist who knows that a given function will be shaped like a stretched parabola, but has to do some more measurements to find out exactly what the stretching factor is. In other words, she can write $f(x) = -ax^2$, noting that she will have to figure out the a experimentally. The a is not a variable, it is a parameter. Similarly, if we are going to do a bunch of calculations with the function $f(x) = \sqrt[3]{x + 2}$, and then do the same calculations with $\sqrt[3]{x + 3}$, $\sqrt[3]{x - \pi}$, and $\sqrt[3]{x - \frac{2}{3}}$, it is faster and easier to do the set of calculations just once, with the function $g(x) = \sqrt[3]{x + h}$, and then fill in the different values for h at the end. Again, this letter h is called a parameter. Ask the class how, in the expression $f(t) = t + 3s$, they can tell which is the variable, and which is the parameter—the answer may encourage them to use careful notation.

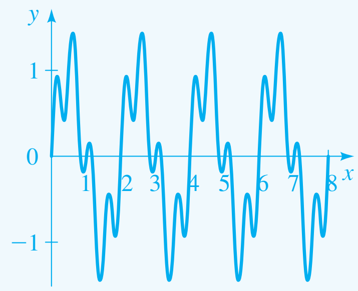
57050_02_ch02_p146-247.qxd 07/04/2008 05:00 PM Page 188

ALTERNATE EXAMPLE 8

The graph of $y = f(x)$ is shown below. Sketch the graph of each function:

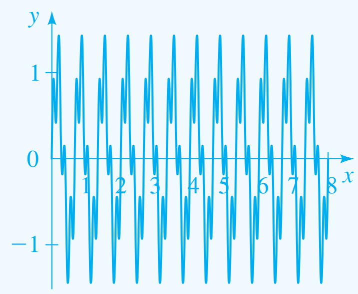
(a) $y = f(3x)$

(b) $y = f\left(\frac{1}{3}x\right)$

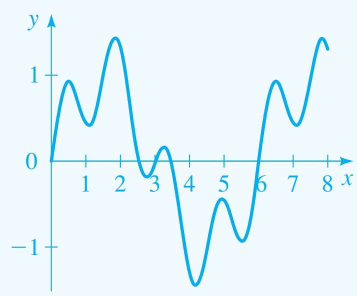


ANSWERS

(a)



(b)



Sonya Kovalevsky (1850–1891) is considered the most important woman mathematician of the 19th century. She was born in Moscow to an aristocratic family. While a child, she was exposed to the principles of calculus in a very unusual fashion—her bedroom was temporarily wallpapered with the pages of a calculus book. She later wrote that she "spent many hours in front of that wall, trying to understand it." Since Russian law forbade women from studying in universities, she entered a marriage of convenience, which allowed her to travel to Germany and obtain a doctorate in mathematics from the University of Göttingen. She eventually was awarded a full professorship at the University of Stockholm, where she taught for eight years before dying in an influenza epidemic at the age of 41. Her research was instrumental in helping put the ideas and applications of functions and calculus on a sound and logical foundation. She received many accolades and prizes for her research work.

Example 8 Horizontal Stretching and Shrinking of Graphs

The graph of $y = f(x)$ is shown in Figure 9. Sketch the graph of each function.

(a) $y = f(2x)$ (b) $y = f(\frac{1}{2}x)$

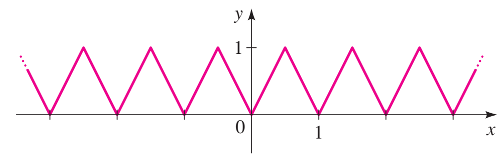


Figure 9
$y = f(x)$

Solution Using the principles described in the preceding box, we obtain the graphs shown in Figures 10 and 11.

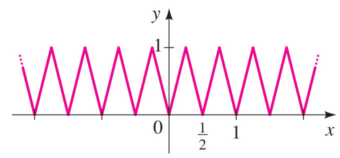


Figure 10
$y = f(2x)$

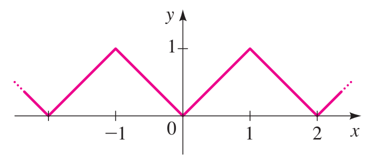


Figure 11
$y = f(\frac{1}{2}x)$ ■

Even and Odd Functions

If a function f satisfies $f(-x) = f(x)$ for every number x in its domain, then f is called an **even function**. For instance, the function $f(x) = x^2$ is even because

$$f(-x) = (-x)^2 = (-1)^2x^2 = x^2 = f(x)$$

The graph of an even function is symmetric with respect to the y-axis (see Figure 12). This means that if we have plotted the graph of f for $x \geq 0$, then we can obtain the entire graph simply by reflecting this portion in the y-axis.

If f satisfies $f(-x) = -f(x)$ for every number x in its domain, then f is called an **odd function**. For example, the function $f(x) = x^3$ is odd because

$$f(-x) = (-x)^3 = (-1)^3x^3 = -x^3 = -f(x)$$

The graph of an odd function is symmetric about the origin (see Figure 13). If we have plotted the graph of f for $x \geq 0$, then we can obtain the entire graph by rotating

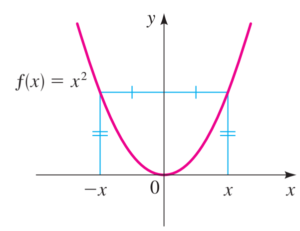


Figure 12
$f(x) = x^2$ is an even function.

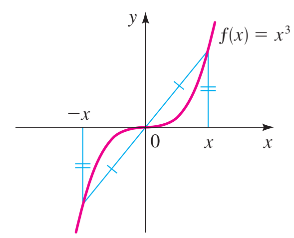
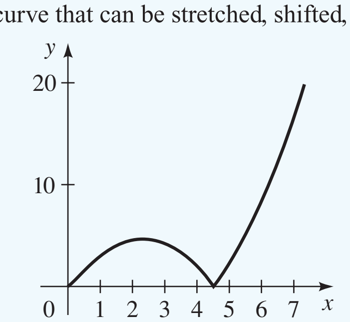


Figure 13
$f(x) = x^3$ is an odd function.

EXAMPLE

A distinctive-looking, asymmetric curve that can be stretched, shifted, and reflected:



$f(x) = |x^2 - 5x + \sqrt{x}|$

57050_02_ch02_p146-247.qxd 07/04/2008 05:01 PM Page 189

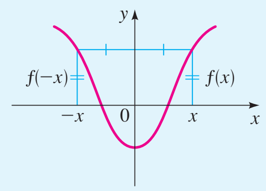
this portion through 180° about the origin. (This is equivalent to reflecting first in the x-axis and then in the y-axis.)

Even and Odd Functions

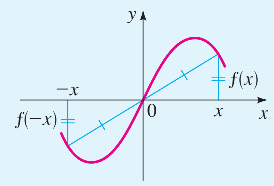
Let f be a function.

f is **even** if $f(-x) = f(x)$ for all x in the domain of f.

f is **odd** if $f(-x) = -f(x)$ for all x in the domain of f



The graph of an even function is symmetric with respect to the y-axis.



The graph of an odd function is symmetric with respect to the origin.

Example 9 Even and Odd Functions

Determine whether the functions are even, odd, or neither even nor odd.

(a) $f(x) = x^5 + x$ (b) $g(x) = 1 - x^4$ (c) $h(x) = 2x - x^2$

Solution

(a) $f(-x) = (-x)^5 + (-x)$

$= -x^5 - x = -(x^5 + x)$

$= -f(x)$

Therefore, f is an odd function.

(b) $g(-x) = 1 - (-x)^4 = 1 - x^4 = g(x)$

So g is even.

(c) $h(-x) = 2(-x) - (-x)^2 = -2x - x^2$

Since $h(-x) \neq h(x)$ and $h(-x) \neq -h(x)$, we conclude that h is neither even nor odd. ■

The graphs of the functions in Example 9 are shown in Figure 14. The graph of f is symmetric about the origin, and the graph of g is symmetric about the y-axis. The graph of h is not symmetric either about the y-axis or the origin.

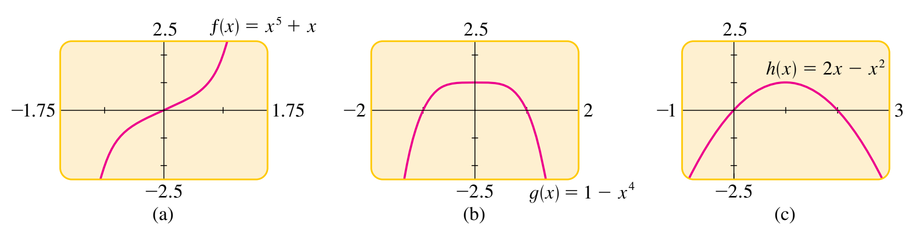


Figure 14

ALTERNATE EXAMPLE 9

Determine whether the following functions are even, odd, or neither even nor odd:

(a) $f(x) = x^3 + x$
(b) $f(x) = 7 - x^6$
(c) $f(x) = 3x - x^3$

ANSWER

(a) Odd
(b) Even
(c) Odd

57050_02_ch02_p146-247.qxd 07/04/2008 05:01 PM Page 190

2.4 Exercises

1–10 ■ Suppose the graph of f is given. Describe how the graph of each function can be obtained from the graph of f.

1. **(a)** $y = f(x) - 5$ **(b)** $y = f(x - 5)$

2. **(a)** $y = f(x + 7)$ **(b)** $y = f(x) + 7$

3. **(a)** $y = f(x + \frac{1}{2})$ **(b)** $y = f(x) + \frac{1}{2}$

4. **(a)** $y = -f(x)$ **(b)** $y = f(-x)$

5. **(a)** $y = -2f(x)$ **(b)** $y = -\frac{1}{2}f(x)$

6. **(a)** $y = -f(x) + 5$ **(b)** $y = 3f(x) - 5$

7. **(a)** $y = f(x - 4) + \frac{3}{4}$ **(b)** $y = f(x + 4) - \frac{3}{4}$

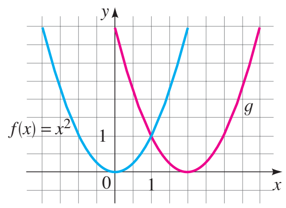
8. **(a)** $y = 2f(x + 2) - 2$ **(b)** $y = 2f(x - 2) + 2$

9. **(a)** $y = f(4x)$ **(b)** $y = f(\frac{1}{4}x)$

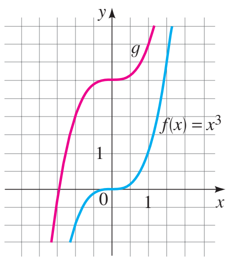
10. **(a)** $y = -f(2x)$ **(b)** $y = f(2x) - 1$

11–16 ■ The graphs of f and g are given. Find a formula for the function g.

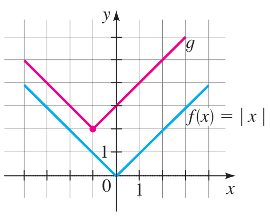
11.



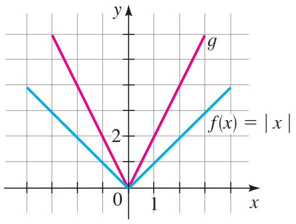
12.



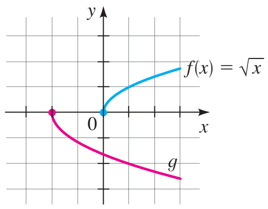
13.



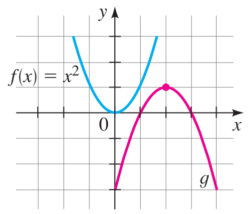
14.



15.



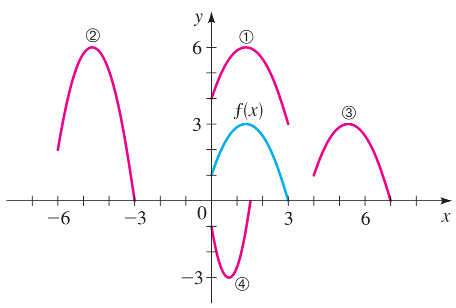
16.



17–18 ■ The graph of $y = f(x)$ is given. Match each equation with its graph.

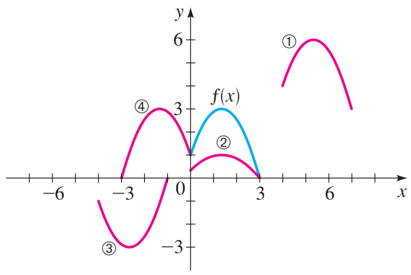
17. **(a)** $y = f(x - 4)$ **(b)** $y = f(x) + 3$

(c) $y = 2f(x + 6)$ **(d)** $y = -f(2x)$



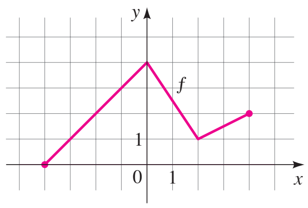
57050_02_ch02_p146-247.qxd 07/04/2008 05:01 PM Page 191

18. **(a)** $y = \frac{1}{3}f(x)$ **(b)** $y = -f(x + 4)$
(c) $y = f(x - 4) + 3$ **(d)** $y = f(-x)$



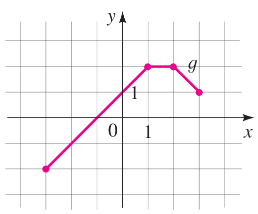
19. The graph of f is given. Sketch the graphs of the following functions.

(a) $y = f(x - 2)$ **(b)** $y = f(x) - 2$
(c) $y = 2f(x)$ **(d)** $y = -f(x) + 3$
(e) $y = f(-x)$ **(f)** $y = \frac{1}{2}f(x - 1)$



20. The graph of g is given. Sketch the graphs of the following functions.

(a) $y = g(x + 1)$ **(b)** $y = -g(x + 1)$
(c) $y = g(x - 2)$ **(d)** $y = g(x) - 2$
(e) $y = -g(x) + 2$ **(f)** $y = 2g(x)$



21. **(a)** Sketch the graph of $f(x) = \dfrac{1}{x}$ by plotting points.

(b) Use the graph of f to sketch the graphs of the following functions.

(i) $y = -\dfrac{1}{x}$ (ii) $y = \dfrac{1}{x - 1}$

(iii) $y = \dfrac{2}{x + 2}$ (iv) $y = 1 + \dfrac{1}{x - 3}$

22. **(a)** Sketch the graph of $g(x) = \sqrt[3]{x}$ by plotting points.

(b) Use the graph of g to sketch the graphs of the following functions.

(i) $y = \sqrt[3]{x - 2}$ (ii) $y = \sqrt[3]{x + 2} + 2$
(iii) $y = 1 - \sqrt[3]{x}$ (iv) $y = 2\sqrt[3]{x}$

23–26 ■ Explain how the graph of g is obtained from the graph of f.

23. **(a)** $f(x) = x^2$, $g(x) = (x + 2)^2$
(b) $f(x) = x^2$, $g(x) = x^2 + 2$

24. **(a)** $f(x) = x^3$, $g(x) = (x - 4)^3$
(b) $f(x) = x^3$, $g(x) = x^3 - 4$

25. **(a)** $f(x) = \sqrt{x}$, $g(x) = 2\sqrt{x}$
(b) $f(x) = \sqrt{x}$, $g(x) = \frac{1}{2}\sqrt{x - 2}$

26. **(a)** $f(x) = |x|$, $g(x) = 3|x| + 1$
(b) $f(x) = |x|$, $g(x) = -|x + 1|$

27–32 ■ A function f is given, and the indicated transformations are applied to its graph (in the given order). Write the equation for the final transformed graph.

27. $f(x) = x^2$; shift upward 3 units and shift 2 units to the right

28. $f(x) = x^3$; shift downward 1 unit and shift 4 units to the left

29. $f(x) = \sqrt{x}$; shift 3 units to the left, stretch vertically by a factor of 5, and reflect in the x-axis

30. $f(x) = \sqrt[3]{x}$; reflect in the y-axis, shrink vertically by a factor of $\frac{1}{2}$, and shift upward $\frac{3}{5}$ unit

31. $f(x) = |x|$; shift to the right $\frac{1}{2}$ unit, shrink vertically by a factor of 0.1, and shift downward 2 units

32. $f(x) = |x|$; shift to the left 1 unit, stretch vertically by a factor of 3, and shift upward 10 units

33–48 ■ Sketch the graph of the function, not by plotting points, but by starting with the graph of a standard function and applying transformations.

33. $f(x) = (x - 2)^2$ **34.** $f(x) = (x + 7)^2$
35. $f(x) = -(x + 1)^2$ **36.** $f(x) = 1 - x^2$
37. $f(x) = x^3 + 2$ **38.** $f(x) = -x^3$
39. $y = 1 + \sqrt{x}$ **40.** $y = 2 - \sqrt{x + 1}$
41. $y = \frac{1}{2}\sqrt{x + 4} - 3$ **42.** $y = 3 - 2(x - 1)^2$
43. $y = 5 + (x + 3)^2$ **44.** $y = \frac{1}{3}x^3 - 1$
45. $y = |x| - 1$ **46.** $y = |x - 1|$
47. $y = |x + 2| + 2$ **48.** $y = 2 - |x|$

57050_02_ch02_p146-247.qxd 07/04/2008 05:01 PM Page 192

49–52 ■ Graph the functions on the same screen using the given viewing rectangle. How is each graph related to the graph in part (a)?

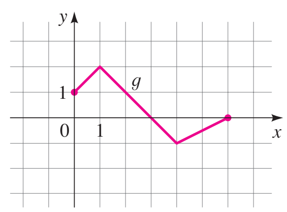
49. Viewing rectangle $[-8, 8]$ by $[-2, 8]$
(a) $y = \sqrt[4]{x}$ **(b)** $y = \sqrt[4]{x+5}$
(c) $y = 2\sqrt[4]{x+5}$ **(d)** $y = 4 + 2\sqrt[4]{x+5}$

50. Viewing rectangle $[-8, 8]$ by $[-6, 6]$
(a) $y = |x|$ **(b)** $y = -|x|$
(c) $y = -3|x|$ **(d)** $y = -3|x - 5|$

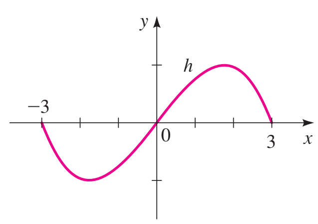
51. Viewing rectangle $[-4, 6]$ by $[-4, 4]$
(a) $y = x^6$ **(b)** $y = \frac{1}{3}x^6$
(c) $y = -\frac{1}{3}x^6$ **(d)** $y = -\frac{1}{3}(x-4)^6$

52. Viewing rectangle $[-6, 6]$ by $[-4, 4]$
(a) $y = \dfrac{1}{\sqrt{x}}$ **(b)** $y = \dfrac{1}{\sqrt{x+3}}$
(c) $y = \dfrac{1}{2\sqrt{x+3}}$ **(d)** $y = \dfrac{1}{2\sqrt{x+3}} - 3$

53. The graph of g is given. Use it to graph each of the following functions.
(a) $y = g(2x)$ **(b)** $y = g(\frac{1}{2}x)$

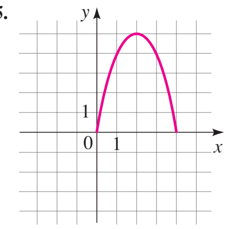


54. The graph of h is given. Use it to graph each of the following functions.
(a) $y = h(3x)$ **(b)** $y = h(\frac{1}{3}x)$

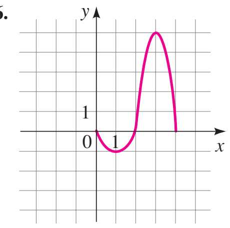


55–56 ■ The graph of a function defined for $x \geq 0$ is given. Complete the graph for $x < 0$ to make
(a) an even function
(b) an odd function

55.



56.



57–58 ■ Use the graph of $f(x) = [\![x]\!]$ described on pages 162–163 to graph the indicated function.

57. $y = [\![2x]\!]$ **58.** $y = [\![\frac{1}{4}x]\!]$

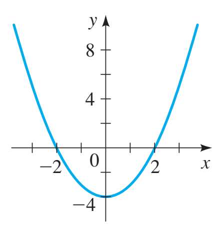
59. If $f(x) = \sqrt{2x - x^2}$, graph the following functions in the viewing rectangle $[-5, 5]$ by $[-4, 4]$. How is each graph related to the graph in part (a)?
(a) $y = f(x)$ **(b)** $y = f(2x)$ **(c)** $y = f(\frac{1}{2}x)$

60. If $f(x) = \sqrt{2x - x^2}$, graph the following functions in the viewing rectangle $[-5, 5]$ by $[-4, 4]$. How is each graph related to the graph in part (a)?
(a) $y = f(x)$ **(b)** $y = f(-x)$ **(c)** $y = -f(-x)$
(d) $y = f(-2x)$ **(e)** $y = f(-\frac{1}{2}x)$

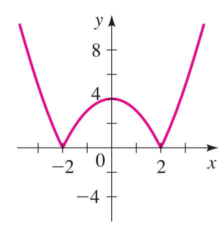
61–68 ■ Determine whether the function f is even, odd, or neither. If f is even or odd, use symmetry to sketch its graph.

61. $f(x) = x^{-2}$ **62.** $f(x) = x^{-3}$
63. $f(x) = x^2 + x$ **64.** $f(x) = x^4 - 4x^2$
65. $f(x) = x^3 - x$ **66.** $f(x) = 3x^3 + 2x^2 + 1$
67. $f(x) = 1 - \sqrt[3]{x}$ **68.** $f(x) = x + \dfrac{1}{x}$

69. The graphs of $f(x) = x^2 - 4$ and $g(x) = |x^2 - 4|$ are shown. Explain how the graph of g is obtained from the graph of f.



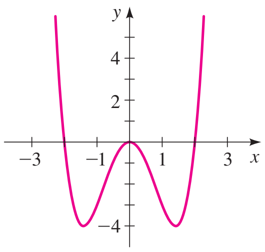
$f(x) = x^2 - 4$



$g(x) = |x^2 - 4|$

57050_02_ch02_p146-247.qxd 07/04/2008 05:01 PM Page 193

70. The graph of $f(x) = x^4 - 4x^2$ is shown. Use this graph to sketch the graph of $g(x) = |x^4 - 4x^2|$.



71–72 ■ Sketch the graph of each function.

71. **(a)** $f(x) = 4x - x^2$ **(b)** $g(x) = |4x - x^2|$

72. **(a)** $f(x) = x^3$ **(b)** $g(x) = |x^3|$

Applications

73. Sales Growth The annual sales of a certain company can be modeled by the function $f(t) = 4 + 0.01t^2$, where t represents years since 1990 and $f(t)$ is measured in millions of dollars.

(a) What shifting and shrinking operations must be performed on the function $y = t^2$ to obtain the function $y = f(t)$?

(b) Suppose you want t to represent years since 2000 instead of 1990. What transformation would you have to apply to the function $y = f(t)$ to accomplish this? Write the new function $y = g(t)$ that results from this transformation.

74. Changing Temperature Scales The temperature on a certain afternoon is modeled by the function

$$C(t) = \tfrac{1}{2}t^2 + 2$$

where t represents hours after 12 noon $(0 \le t \le 6)$, and C is measured in °C.

(a) What shifting and shrinking operations must be performed on the function $y = t^2$ to obtain the function $y = C(t)$?

(b) Suppose you want to measure the temperature in °F instead. What transformation would you have to apply to the function $y = C(t)$ to accomplish this? (Use the fact that the relationship between Celsius and Fahrenheit degrees is given by $F = \tfrac{9}{5}C + 32$.) Write the new function $y = F(t)$ that results from this transformation.

Discovery • Discussion

75. Sums of Even and Odd Functions If f and g are both even functions, is $f + g$ necessarily even? If both are odd, is their sum necessarily odd? What can you say about the sum if one is odd and one is even? In each case, prove your answer.

76. Products of Even and Odd Functions Answer the same questions as in Exercise 75, except this time consider the *product* of f and g instead of the sum.

77. Even and Odd Power Functions What must be true about the integer n if the function

$$f(x) = x^n$$

is an even function? If it is an odd function? Why do you think the names "even" and "odd" were chosen for these function properties?

2.5 Quadratic Functions; Maxima and Minima

A maximum or minimum value of a function is the largest or smallest value of the function on an interval. For a function that represents the profit in a business, we would be interested in the maximum value; for a function that represents the amount of material to be used in a manufacturing process, we would be interested in the minimum value. In this section we learn how to find the maximum and minimum values of quadratic and other functions.

SUGGESTED TIME AND EMPHASIS

1 class.
Essential material.

POINTS TO STRESS

1. Graphing quadratic functions, including obtaining the exact coordinates of the vertex by completing the square.
2. Finding the local extrema of a function by looking at its graph.

57050_02_ch02_p146-247.qxd 07/04/2008 05:01 PM Page 194

Graphing Quadratic Functions Using the Standard Form

A **quadratic function** is a function f of the form

$$f(x) = ax^2 + bx + c$$

where a, b, and c are real numbers and $a \neq 0$.

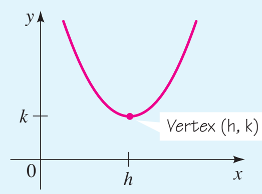
In particular, if we take $a = 1$ and $b = c = 0$, we get the simple quadratic function $f(x) = x^2$ whose graph is the parabola that we drew in Example 1 of Section 2.2. In fact, the graph of any quadratic function is a **parabola**; it can be obtained from the graph of $f(x) = x^2$ by the transformations given in Section 2.4.

Standard Form of a Quadratic Function

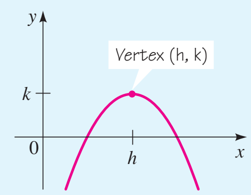
A quadratic function $f(x) = ax^2 + bx + c$ can be expressed in the **standard form**

$$f(x) = a(x - h)^2 + k$$

by completing the square. The graph of f is a parabola with **vertex** (h, k); the parabola opens upward if $a > 0$ or downward if $a < 0$.



$f(x) = a(x - h)^2 + k, \; a > 0$



$f(x) = a(x - h)^2 + k, \; a < 0$

ALTERNATE EXAMPLE 1a
For the graph of the quadratic function $f(x) = 3x^2 - 6x + 10$, find the coordinates of the vertex and the y-intercept.

ANSWER
(1, 7), 10

Example 1 Standard Form of a Quadratic Function

Let $f(x) = 2x^2 - 12x + 23$.

(a) Express f in standard form.

(b) Sketch the graph of f.

Solution

Completing the square is discussed in Section 1.5.

(a) Since the coefficient of x^2 is not 1, we must factor this coefficient from the terms involving x before we complete the square.

$$f(x) = 2x^2 - 12x + 23$$
$$= 2(x^2 - 6x) + 23 \quad \text{Factor 2 from the x-terms}$$
$$= 2(x^2 - 6x + 9) + 23 - 2 \cdot 9 \quad \text{Complete the square: Add 9 inside parentheses, subtract } 2 \cdot 9 \text{ outside}$$
$$= 2(x - 3)^2 + 5 \quad \text{Factor and simplify}$$

$f(x) = 2(x - 3)^2 + 5$
Vertex is (3, 5)

The standard form is $f(x) = 2(x - 3)^2 + 5$.

SAMPLE QUESTION

Text Question

If $f(x) = 4x^2 + 16x + 5$, why would it be useful to complete the square?

Answer

Completing the square would reveal the zeros of the function.

57050_02_ch02_p146-247.qxd 07/04/2008 05:01 PM Page 195

(b) The standard form tells us that we get the graph of f by taking the parabola $y = x^2$, shifting it to the right 3 units, stretching it by a factor of 2, and moving it upward 5 units. The vertex of the parabola is at $(3, 5)$ and the parabola opens upward. We sketch the graph in Figure 1 after noting that the y-intercept is $f(0) = 23$.

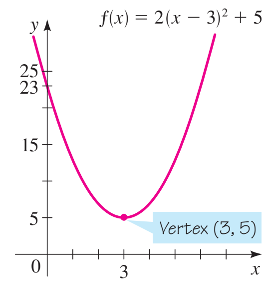


Figure 1

Maximum and Minimum Values of Quadratic Functions

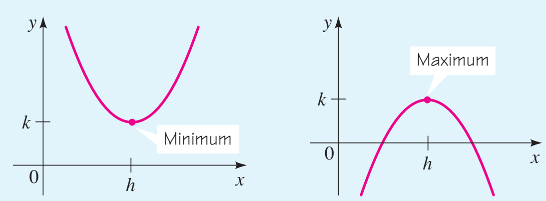
If a quadratic function has vertex (h, k), then the function has a minimum value at the vertex if it opens upward and a maximum value at the vertex if it opens downward. For example, the function graphed in Figure 1 has minimum value 5 when $x = 3$, since the vertex $(3, 5)$ is the lowest point on the graph.

Maximum or Minimum Value of a Quadratic Function

Let f be a quadratic function with standard form $f(x) = a(x - h)^2 + k$. The maximum or minimum value of f occurs at $x = h$.

If $a > 0$, then the **minimum value** of f is $f(h) = k$.

If $a < 0$, then the **maximum value** of f is $f(h) = k$.



$f(x) = a(x - h)^2 + k,\ a > 0$ $\qquad$ $f(x) = a(x - h)^2 + k,\ a < 0$

IN-CLASS MATERIALS

A straightforward way to demonstrate the utility of quadratic functions is to demonstrate how thrown objects follow parabolic paths. Physically throw an actual ball (perhaps trying to get it into the wastebasket) and have the class observe the shape of the path. Note that not only can thrown objects' paths be modeled by parabolas, their height as a function of time can (on Earth) be modeled by quadratic functions of the form $f(x) = -16t^2 + v_0t + s_0$, where t is in seconds, f is in feet, v_0 is initial velocity, and s_0 is initial height.

57050_02_ch02_p146-247.qxd 07/04/2008 05:01 PM Page 196

ALTERNATE EXAMPLE 2a
Express the quadratic function f in standard form: $f(x) = 2x^2 - 20x + 53$

ANSWER
$f(x) = 2(x - 5)^2 + 3$

ALTERNATE EXAMPLE 2c
Find the minimum value of $f(x) = 3x^2 - 30x + 77$.

ANSWER
2

ALTERNATE EXAMPLE 3a
Express the quadratic function f in standard form: $f(x) = -x^2 + x + 5$

ANSWER
$f(x) = -\left(x - \dfrac{1}{2}\right)^2 + \dfrac{21}{4}$

ALTERNATE EXAMPLE 3c
Find the maximum value of the function: $f(x) = -x^2 + x + 5$

ANSWER
$\dfrac{21}{4}$

Example 2 Minimum Value of a Quadratic Function

Consider the quadratic function $f(x) = 5x^2 - 30x + 49$.

(a) Express f in standard form.

(b) Sketch the graph of f.

(c) Find the minimum value of f.

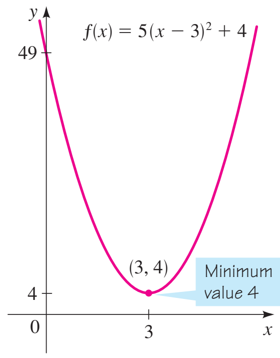


Figure 2

Solution

(a) To express this quadratic function in standard form, we complete the square.

$$\begin{aligned} f(x) &= 5x^2 - 30x + 49 \\ &= 5(x^2 - 6x) + 49 && \text{Factor 5 from the } x\text{-terms} \\ &= 5(x^2 - 6x + 9) + 49 - 5\cdot 9 && \text{Complete the square: Add 9 inside parentheses, subtract } 5\cdot 9 \text{ outside} \\ &= 5(x - 3)^2 + 4 && \text{Factor and simplify} \end{aligned}$$

(b) The graph is a parabola that has its vertex at $(3, 4)$ and opens upward, as sketched in Figure 2.

(c) Since the coefficient of x^2 is positive, f has a minimum value. The minimum value is $f(3) = 4$. ■

Example 3 Maximum Value of a Quadratic Function

Consider the quadratic function $f(x) = -x^2 + x + 2$.

(a) Express f in standard form.

(b) Sketch the graph of f.

(c) Find the maximum value of f.

Solution

(a) To express this quadratic function in standard form, we complete the square.

$$\begin{aligned} y &= -x^2 + x + 2 \\ &= -(x^2 - x) + 2 && \text{Factor } -1 \text{ from the } x\text{-terms} \\ &= -(x^2 - x + \tfrac{1}{4}) + 2 - (-1)\tfrac{1}{4} && \text{Complete the square: Add } \tfrac{1}{4} \text{ inside parentheses, subtract } (-1)\tfrac{1}{4} \text{ outside} \\ &= -(x - \tfrac{1}{2})^2 + \tfrac{9}{4} && \text{Factor and simplify} \end{aligned}$$

(b) From the standard form we see that the graph is a parabola that opens downward and has vertex $(\frac{1}{2}, \frac{9}{4})$. As an aid to sketching the graph, we find the intercepts. The y-intercept is $f(0) = 2$. To find the x-intercepts, we set $f(x) = 0$ and factor the resulting equation.

$$\begin{aligned} -x^2 + x + 2 &= 0 \\ -(x^2 - x - 2) &= 0 \\ -(x - 2)(x + 1) &= 0 \end{aligned}$$

IN-CLASS MATERIALS

Show students how quadratic functions can come up in an applied context. If a demand function is linear then the revenue will be quadratic. For example, if the number of shoes you can sell is given by $10000 - 3c$, where c is the cost per shoe, then the revenue is $R = c(10000 - 3c)$. If we are thinking of costs in a narrow possible range, we can usually approximate revenue in such a way.

57050_02_ch02_p146-247.qxd 07/04/2008 05:01 PM Page 197

Thus, the x-intercepts are $x = 2$ and $x = -1$. The graph of f is sketched in Figure 3.

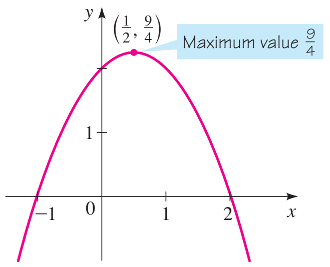


Figure 3
Graph of $f(x) = -x^2 + x + 2$

(c) Since the coefficient of x^2 is negative, f has a maximum value, which is $f\left(\frac{1}{2}\right) = \frac{9}{4}$. ■

Expressing a quadratic function in standard form helps us sketch its graph as well as find its maximum or minimum value. If we are interested only in finding the maximum or minimum value, then a formula is available for doing so. This formula is obtained by completing the square for the general quadratic function as follows:

$$f(x) = ax^2 + bx + c$$

$$= a\left(x^2 + \frac{b}{a}x\right) + c \qquad \text{Factor } a \text{ from the } x\text{-terms}$$

$$= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + c - a\left(\frac{b^2}{4a^2}\right) \qquad \text{Complete the square: Add } \frac{b^2}{4a^2} \text{ inside parentheses, subtract } a\left(\frac{b^2}{4a^2}\right) \text{ outside}$$

$$= a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} \qquad \text{Factor}$$

This equation is in standard form with $h = -b/(2a)$ and $k = c - b^2/(4a)$. Since the maximum or minimum value occurs at $x = h$, we have the following result.

Maximum or Minimum Value of a Quadratic Function

The maximum or minimum value of a quadratic function $f(x) = ax^2 + bx + c$ occurs at

$$x = -\frac{b}{2a}$$

If $a > 0$, then the **minimum value** is $f\left(-\frac{b}{2a}\right)$.

If $a < 0$, then the **maximum value** is $f\left(-\frac{b}{2a}\right)$.

IN-CLASS MATERIALS

Having covered quadratic functions, it is not a big leap to talk about quadratic inequalities. After graphing $f(x) = x^2 + 4x - 5$, find the intervals described by $x^2 + 4x + 5 > 0$, $x^2 + 4x + 5 \geq 0$, $x^2 + 4x + 5 < 0$, and $x^2 + 4x + 5 \leq 0$.

57050_02_ch02_p146-247.qxd 07/04/2008 05:01 PM Page 198

ALTERNATE EXAMPLE 4a
Find the maximum or minimum value of the quadratic function: $f(x) = x^2 + 8x$

ANSWER
-16

ALTERNATE EXAMPLE 4b
Find the maximum or minimum value of the quadratic function: $g(x) = -4x^2 + 8x - 2$

ANSWER
2

ALTERNATE EXAMPLE 5
Most cars get their best gas mileage when traveling at a relatively modest speed. The gas mileage M for a certain new car is modeled by the function

$$M(s) = -\frac{1}{32}s^2 + 3s - 41,$$

$15 \le s \le 70$ where s is the speed in miles per gallon. What is the car's best gas mileage, and at what speed is it attained?

ANSWER
31, 48

DRILL QUESTION
If $f(x) = -x^2 + 9x + 2$, find the extreme value of f. Is it a maximum or a minimum?

Answer
$f\left(\frac{9}{2}\right) = \frac{89}{4}$ is a maximum.

Example 4 Finding Maximum and Minimum Values of Quadratic Functions

Find the maximum or minimum value of each quadratic function.

(a) $f(x) = x^2 + 4x$ (b) $g(x) = -2x^2 + 4x - 5$

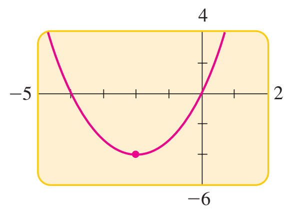
Solution

(a) This is a quadratic function with $a = 1$ and $b = 4$. Thus, the maximum or minimum value occurs at

$$x = -\frac{b}{2a} = -\frac{4}{2 \cdot 1} = -2$$

Since $a > 0$, the function has the *minimum* value

$$f(-2) = (-2)^2 + 4(-2) = -4$$



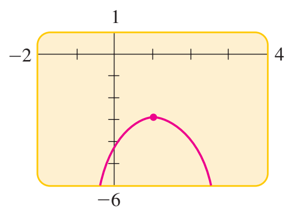
The minimum value occurs at $x = -2$.

(b) This is a quadratic function with $a = -2$ and $b = 4$. Thus, the maximum or minimum value occurs at

$$x = -\frac{b}{2a} = -\frac{4}{2 \cdot (-2)} = 1$$

Since $a < 0$, the function has the *maximum* value

$$f(1) = -2(1)^2 + 4(1) - 5 = -3$$ ■



The maximum value occurs at $x = 1$.

Many real-world problems involve finding a maximum or minimum value for a function that models a given situation. In the next example we find the maximum value of a quadratic function that models the gas mileage for a car.

Example 5 Maximum Gas Mileage for a Car

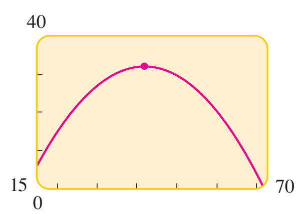
Most cars get their best gas mileage when traveling at a relatively modest speed. The gas mileage M for a certain new car is modeled by the function

$$M(s) = -\frac{1}{28}s^2 + 3s - 31, \qquad 15 \le s \le 70$$

where s is the speed in mi/h and M is measured in mi/gal. What is the car's best gas mileage, and at what speed is it attained?

Solution The function M is a quadratic function with $a = -\frac{1}{28}$ and $b = 3$. Thus, its maximum value occurs when

$$s = -\frac{b}{2a} = -\frac{3}{2(-\frac{1}{28})} = 42$$



The maximum gas mileage occurs at 42 mi/h.

The maximum is $M(42) = -\frac{1}{28}(42)^2 + 3(42) - 31 = 32$. So the car's best gas mileage is 32 mi/gal, when it is traveling at 42 mi/h. ■

Using Graphing Devices to Find Extreme Values

The methods we have discussed apply to finding extreme values of quadratic functions only. We now show how to locate extreme values of any function that can be graphed with a calculator or computer.

If there is a viewing rectangle such that the point $(a, f(a))$ is the highest point on the graph of f *within* the viewing rectangle (not on the edge), then the number $f(a)$ is called a **local maximum value** of f (see Figure 4). Notice that $f(a) \geq f(x)$ for all numbers x that are close to a.

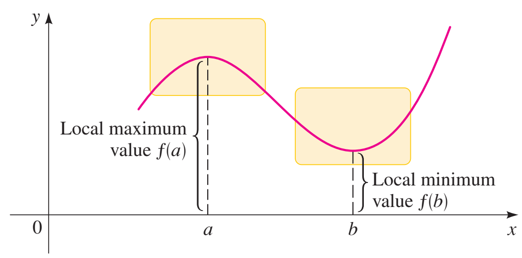


Figure 4

Similarly, if there is a viewing rectangle such that the point $(b, f(b))$ is the lowest point on the graph of f within the viewing rectangle, then the number $f(b)$ is called a **local minimum value** of f. In this case, $f(b) \leq f(x)$ for all numbers x that are close to b.

Example 6 Finding Local Maxima and Minima from a Graph

Find the local maximum and minimum values of the function $f(x) = x^3 - 8x + 1$, correct to three decimals.

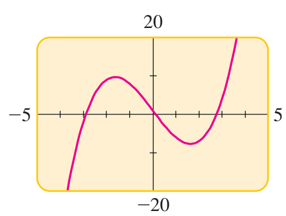


Figure 5
Graph of $f(x) = x^3 - 8x + 1$

Solution The graph of f is shown in Figure 5. There appears to be one local maximum between $x = -2$ and $x = -1$, and one local minimum between $x = 1$ and $x = 2$.

Let's find the coordinates of the local maximum point first. We zoom in to enlarge the area near this point, as shown in Figure 6. Using the TRACE feature on the graphing device, we move the cursor along the curve and observe how the y-coordinates change. The local maximum value of y is 9.709, and this value occurs when x is -1.633, correct to three decimals.

We locate the minimum value in a similar fashion. By zooming in to the viewing rectangle shown in Figure 7, we find that the local minimum value is about -7.709, and this value occurs when $x \approx 1.633$.

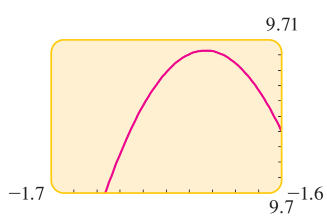


Figure 6

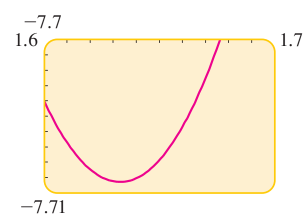
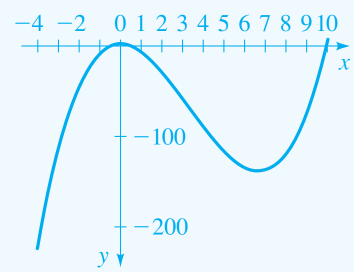


Figure 7 ■

ALTERNATE EXAMPLE 6
Find the local maximum and minimum values of the function $f(x) = x^3 - 10x^2 + x + 4$ correct to three decimals.

ANSWER
The graph is shown below. There appears to be a local maximum between $x = -1$ and $x = 1$, and a local minimum between $x = 6$ and $x = 8$.



Using the calculator as done in the text, we obtain a local maximum of 4.025 (at $x = 0.0503$) and a local minimum of -137.507 (at $x = 6.616$).

IN-CLASS MATERIALS

Point out that while quadratic functions have exactly one local extremum, cubics can have up to two, and in general an nth-degree polynomial function can have up to $n - 1$ local extrema.

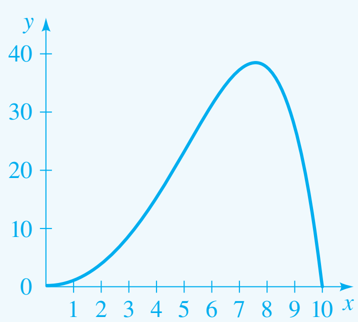
57050_02_ch02_p146-247.qxd 07/04/2008 05:01 PM Page 200

ALTERNATE EXAMPLE 7
The profit, in millions of dollars, a factory makes by producing x thousand items is approximated by the function

$$f(x) = -\frac{x^6}{10000} + x^2$$

when x is between 0 and 10 thousand. Estimate the amount of items that the factory should make to maximize its profit.

ANSWER
The graph of f as a function of x is shown below. There appears to be a maximum between $x = 7$ and $x = 9$. Using the **maximum** command, we see that the maximum value of f is 38.49, or \$38,490,000. It occurs when the factory produces 7,598 items.



The `maximum` and `minimum` commands on a TI-82 or TI-83 calculator provide another method for finding extreme values of functions. We use this method in the next example.

Example 7 A Model for the Food Price Index

A model for the food price index (the price of a representative "basket" of foods) between 1990 and 2000 is given by the function

$$I(t) = -0.0113t^3 + 0.0681t^2 + 0.198t + 99.1$$

where t is measured in years since midyear 1990, so $0 \le t \le 10$, and $I(t)$ is scaled so that $I(3) = 100$. Estimate the time when food was most expensive during the period 1990–2000.

Solution The graph of I as a function of t is shown in Figure 8(a). There appears to be a maximum between $t = 4$ and $t = 7$. Using the `maximum` command, as shown in Figure 8(b), we see that the maximum value of I is about 100.38, and it occurs when $t \approx 5.15$, which corresponds to August 1995.

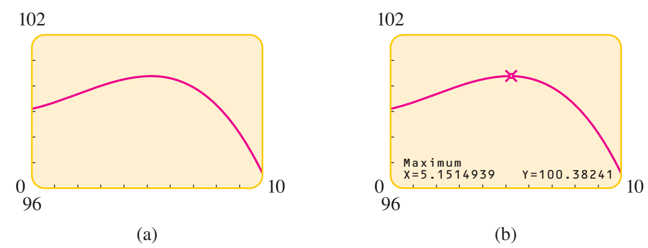
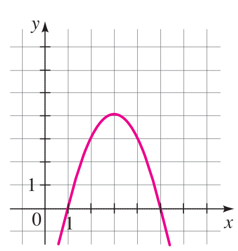


Figure 8 ■

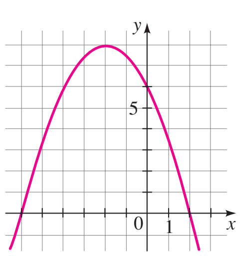
2.5 Exercises

1–4 ■ The graph of a quadratic function f is given.
(a) Find the coordinates of the vertex.
(b) Find the maximum or minimum value of f.

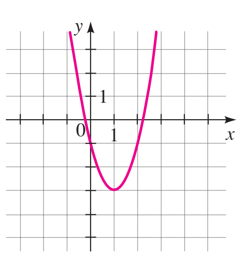
1. $f(x) = -x^2 + 6x - 5$



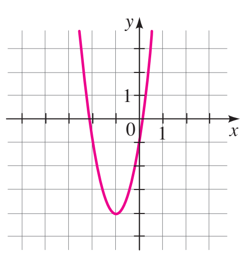
2. $f(x) = -\frac{1}{2}x^2 - 2x + 6$



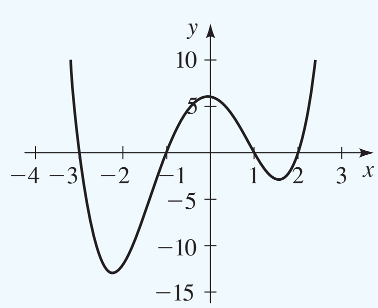
3. $f(x) = 2x^2 - 4x - 1$



4. $f(x) = 3x^2 + 6x - 1$



EXAMPLES
1. A quadratic function that can be graphed by hand:
$f(x) = -2x^2 + 12x - 13 = -2(x - 3)^2 + 5$
2. A function with several local extrema:
$f(x) = x^4 + x^3 - 7x^2 - x + 6 = (x + 3)(x + 1)(x - 1)(x - 2)$



The extrema occur at $x \approx -2.254$, $x \approx -0.0705$, and $x \approx 1.5742$.

57050_02_ch02_p146-247.qxd 07/04/2008 05:01 PM Page 201

5–18 ■ A quadratic function is given.

(a) Express the quadratic function in standard form.

(b) Find its vertex and its x- and y-intercept(s).

(c) Sketch its graph.

5. $f(x) = x^2 - 6x$ **6.** $f(x) = x^2 + 8x$

7. $f(x) = 2x^2 + 6x$ **8.** $f(x) = -x^2 + 10x$

9. $f(x) = x^2 + 4x + 3$ **10.** $f(x) = x^2 - 2x + 2$

11. $f(x) = -x^2 + 6x + 4$ **12.** $f(x) = -x^2 - 4x + 4$

13. $f(x) = 2x^2 + 4x + 3$ **14.** $f(x) = -3x^2 + 6x - 2$

15. $f(x) = 2x^2 - 20x + 57$ **16.** $f(x) = 2x^2 + x - 6$

17. $f(x) = -4x^2 - 16x + 3$ **18.** $f(x) = 6x^2 + 12x - 5$

19–28 ■ A quadratic function is given.

(a) Express the quadratic function in standard form.

(b) Sketch its graph.

(c) Find its maximum or minimum value.

19. $f(x) = 2x - x^2$ **20.** $f(x) = x + x^2$

21. $f(x) = x^2 + 2x - 1$ **22.** $f(x) = x^2 - 8x + 8$

23. $f(x) = -x^2 - 3x + 3$ **24.** $f(x) = 1 - 6x - x^2$

25. $g(x) = 3x^2 - 12x + 13$ **26.** $g(x) = 2x^2 + 8x + 11$

27. $h(x) = 1 - x - x^2$ **28.** $h(x) = 3 - 4x - 4x^2$

29–38 ■ Find the maximum or minimum value of the function.

29. $f(x) = x^2 + x + 1$ **30.** $f(x) = 1 + 3x - x^2$

31. $f(t) = 100 - 49t - 7t^2$ **32.** $f(t) = 10t^2 + 40t + 113$

33. $f(s) = s^2 - 1.2s + 16$ **34.** $g(x) = 100x^2 - 1500x$

35. $h(x) = \frac{1}{2}x^2 + 2x - 6$ **36.** $f(x) = -\frac{x^2}{3} + 2x + 7$

37. $f(x) = 3 - x - \frac{1}{2}x^2$ **38.** $g(x) = 2x(x - 4) + 7$

39. Find a function whose graph is a parabola with vertex $(1, -2)$ and that passes through the point $(4, 16)$.

40. Find a function whose graph is a parabola with vertex $(3, 4)$ and that passes through the point $(1, -8)$.

41–44 ■ Find the domain and range of the function.

41. $f(x) = -x^2 + 4x - 3$ **42.** $f(x) = x^2 - 2x - 3$

43. $f(x) = 2x^2 + 6x - 7$ **44.** $f(x) = -3x^2 + 6x + 4$

45–46 ■ A quadratic function is given.

(a) Use a graphing device to find the maximum or minimum value of the quadratic function f, correct to two decimal places.

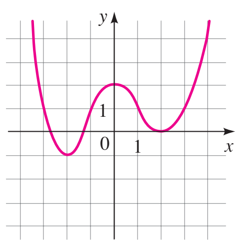
(b) Find the exact maximum or minimum value of f, and compare with your answer to part (a).

45. $f(x) = x^2 + 1.79x - 3.21$

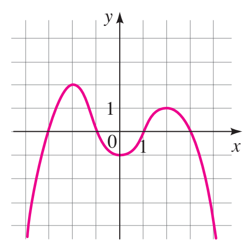
46. $f(x) = 1 + x - \sqrt{2}x^2$

47–50 ■ Find all local maximum and minimum values of the function whose graph is shown.

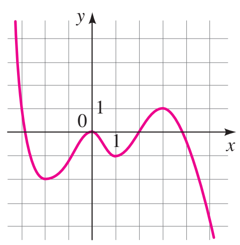
47.



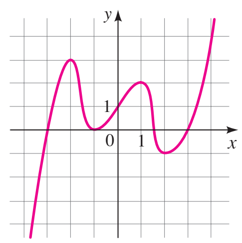
48.



49.



50.



51–58 ■ Find the local maximum and minimum values of the function and the value of x at which each occurs. State each answer correct to two decimal places.

51. $f(x) = x^3 - x$ **52.** $f(x) = 3 + x + x^2 - x^3$

53. $g(x) = x^4 - 2x^3 - 11x^2$ **54.** $g(x) = x^5 - 8x^3 + 20x$

55. $U(x) = x\sqrt{6 - x}$ **56.** $U(x) = x\sqrt{x - x^2}$

57. $V(x) = \frac{1 - x^2}{x^3}$ **58.** $V(x) = \frac{1}{x^2 + x + 1}$

Applications

59. Height of a Ball If a ball is thrown directly upward with a velocity of 40 ft/s, its height (in feet) after t seconds is given by $y = 40t - 16t^2$. What is the maximum height attained by the ball?

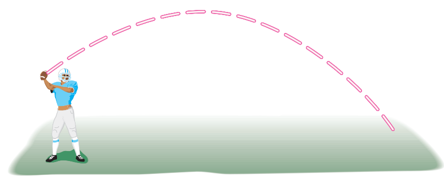
60. Path of a Ball A ball is thrown across a playing field. Its path is given by the equation $y = -0.005x^2 + x + 5$,

57050_02_ch02_p146-247.qxd 07/04/2008 05:01 PM Page 202

where x is the distance the ball has traveled horizontally, and y is its height above ground level, both measured in feet.

(a) What is the maximum height attained by the ball?

(b) How far has it traveled horizontally when it hits the ground?



61. Revenue A manufacturer finds that the revenue generated by selling x units of a certain commodity is given by the function $R(x) = 80x - 0.4x^2$, where the revenue $R(x)$ is measured in dollars. What is the maximum revenue, and how many units should be manufactured to obtain this maximum?

62. Sales A soft-drink vendor at a popular beach analyzes his sales records, and finds that if he sells x cans of soda pop in one day, his profit (in dollars) is given by

$$P(x) = -0.001x^2 + 3x - 1800$$

What is his maximum profit per day, and how many cans must he sell for maximum profit?

63. Advertising The effectiveness of a television commercial depends on how many times a viewer watches it. After some experiments an advertising agency found that if the effectiveness E is measured on a scale of 0 to 10, then

$$E(n) = \tfrac{2}{3}n - \tfrac{1}{90}n^2$$

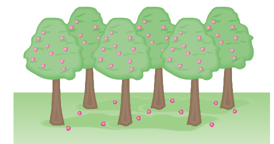
where n is the number of times a viewer watches a given commercial. For a commercial to have maximum effectiveness, how many times should a viewer watch it?

64. Pharmaceuticals When a certain drug is taken orally, the concentration of the drug in the patient's bloodstream after t minutes is given by $C(t) = 0.06t - 0.0002t^2$, where $0 \le t \le 240$ and the concentration is measured in mg/L. When is the maximum serum concentration reached, and what is that maximum concentration?

65. Agriculture The number of apples produced by each tree in an apple orchard depends on how densely the trees are planted. If n trees are planted on an acre of land, then each tree produces $900 - 9n$ apples. So the number of apples produced per acre is

$$A(n) = n(900 - 9n)$$

How many trees should be planted per acre in order to obtain the maximum yield of apples?



66. Migrating Fish A fish swims at a speed v relative to the water, against a current of 5 mi/h. Using a mathematical model of energy expenditure, it can be shown that the total energy E required to swim a distance of 10 mi is given by

$$E(v) = 2.73v^3\frac{10}{v - 5}$$

Biologists believe that migrating fish try to minimize the total energy required to swim a fixed distance. Find the value of v that minimizes energy required.

NOTE This result has been verified; migrating fish swim against a current at a speed 50% greater than the speed of the current.

67. Highway Engineering A highway engineer wants to estimate the maximum number of cars that can safely travel a particular highway at a given speed. She assumes that each car is 17 ft long, travels at a speed s, and follows the car in front of it at the "safe following distance" for that speed. She finds that the number N of cars that can pass a given point per minute is modeled by the function

$$N(s) = \frac{88s}{17 + 17\left(\dfrac{s}{20}\right)^2}$$

At what speed can the greatest number of cars travel the highway safely?

68. Volume of Water Between 0°C and 30°C, the volume V (in cubic centimeters) of 1 kg of water at a temperature T is given by the formula

$$V = 999.87 - 0.06426T + 0.0085043T^2 - 0.0000679T^3$$

Find the temperature at which the volume of 1 kg of water is a minimum.

57050_02_ch02_p146-247.qxd 07/04/2008 05:01 PM Page 203

69. Coughing When a foreign object lodged in the trachea (windpipe) forces a person to cough, the diaphragm thrusts upward causing an increase in pressure in the lungs. At the same time, the trachea contracts, causing the expelled air to move faster and increasing the pressure on the foreign object. According to a mathematical model of coughing, the velocity v of the airstream through an average-sized person's trachea is related to the radius r of the trachea (in centimeters) by the function

$$v(r) = 3.2(1 - r)r^2, \qquad \tfrac{1}{2} \le r \le 1$$

Determine the value of r for which v is a maximum.

Discovery • Discussion

70. Maxima and Minima In Example 5 we saw a real-world situation in which the maximum value of a function is important. Name several other everyday situations in which a maximum or minimum value is important.

71. Minimizing a Distance When we seek a minimum or maximum value of a function, it is sometimes easier to work with a simpler function instead.

(a) Suppose $g(x) = \sqrt{f(x)}$, where $f(x) \ge 0$ for all x. Explain why the local minima and maxima of f and g occur at the same values of x.

(b) Let $g(x)$ be the distance between the point $(3, 0)$ and the point (x, x^2) on the graph of the parabola $y = x^2$. Express g as a function of x.

(c) Find the minimum value of the function g that you found in part (b). Use the principle described in part (a) to simplify your work.

72. Maximum of a Fourth-Degree Polynomial Find the maximum value of the function

$$f(x) = 3 + 4x^2 - x^4$$

[*Hint:* Let $t = x^2$.]

2.6 Modeling with Functions

Many of the processes studied in the physical and social sciences involve understanding how one quantity varies with respect to another. Finding a function that describes the dependence of one quantity on another is called *modeling*. For example, a biologist observes that the number of bacteria in a certain culture increases with time. He tries to model this phenomenon by finding the precise function (or rule) that relates the bacteria population to the elapsed time.

In this section we will learn how to find models that can be constructed using geometric or algebraic properties of the object under study. (Finding models from *data* is studied in the *Focus on Modeling* at the end of this chapter.) Once the model is found, we use it to analyze and predict properties of the object or process being studied.

Modeling with Functions

We begin with a simple real-life situation that illustrates the modeling process.

Example 1 Modeling the Volume of a Box

A breakfast cereal company manufactures boxes to package their product. For aesthetic reasons, the box must have the following proportions: Its width is 3 times its depth and its height is 5 times its depth.

(a) Find a function that models the volume of the box in terms of its depth.

(b) Find the volume of the box if the depth is 1.5 in.

(c) For what depth is the volume 90 in^3?

(d) For what depth is the volume greater than 60 in^3?

SUGGESTED TIME AND EMPHASIS

1–2 classes.

Essential material (assuming students will be going on to take calculus).

ALTERNATE EXAMPLE 1a

A breakfast cereal company manufactures boxes to package their product. For aesthetic reasons, the box must have the following proportions: Its width is 3 times its depth, and its height is 4 times its depth. Find a function that models the volume V of the box in terms of its depth x.

ANSWER

$V(x) = 12x^3$

POINTS TO STRESS

1. Transforming an applied optimization problem, stated in words, into a one-variable function to be optimized.
2. Answering applied questions carefully, being sure to provide the information asked for, including correct units.

57050_02_ch02_p146-247.qxd 07/04/2008 05:01 PM Page 204

ALTERNATE EXAMPLE 1b
A breakfast cereal company manufactures boxes to package their product. For aesthetic reasons, the box must have the following proportions: Its width is 4 times its depth, and its height is 5 times its depth. Find the volume of the box if the depth is 1.1 in.

ANSWER
$V(1.1) = 26.62$

ALTERNATE EXAMPLE 1c
A breakfast cereal company manufactures boxes to package their product. For aesthetic reasons, the box must have the following proportions: Its width is 4 times its depth, and its height is 7 times its depth. For what depth is the volume 112 in^3? Give the answer to two decimal places.

ANSWER
1.59

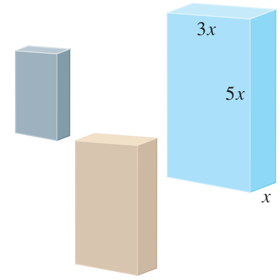
ALTERNATE EXAMPLE 1d
A breakfast cereal company manufactures boxes to package their product. For aesthetic reasons, the box must have the following proportions: Its width is 4 times its depth, and its height is 5 times its depth. For what depth x is the volume greater than or equal to 60 in^3? Round the answer to two decimal places.

ANSWER
$x \geq 1.44$

■ Thinking About the Problem

Let's experiment with the problem. If the depth is 1 in, then the width is 3 in. and the height is 5 in. So in this case, the volume is $V = 1 \times 3 \times 5 = 15 \text{ in}^3$. The table gives other values. Notice that all the boxes have the same shape, and the greater the depth the greater the volume.

Depth	Volume
1	$1 \times 3 \times 5 = 15$
2	$2 \times 6 \times 10 = 120$
3	$3 \times 9 \times 15 = 405$
4	$4 \times 12 \times 20 = 960$



Solution

(a) To find the function that models the volume of the box, we use the following steps.

■ Express the Model in Words

We know that the volume of a rectangular box is

$$\text{volume} = \text{depth} \times \text{width} \times \text{height}$$

■ Choose the Variable

There are three varying quantities—width, depth, and height. Since the function we want depends on the depth, we let

$$x = \text{depth of the box}$$

Then we express the other dimensions of the box in terms of x.

In Words	In Algebra
Depth	x
Width	$3x$
Height	$5x$

■ Set up the Model

The model is the function V that gives the volume of the box in terms of the depth x.

$$\text{volume} = \text{depth} \times \text{width} \times \text{height}$$
$$V(x) = x \cdot 3x \cdot 5x$$
$$V(x) = 15x^3$$

The volume of the box is modeled by the function $V(x) = 15x^3$. The function V is graphed in Figure 1.

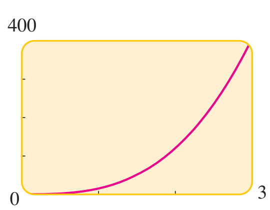
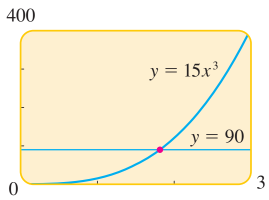


Figure 1

SAMPLE QUESTION
Text Question
Why is it important that we express our models in terms of one variable?

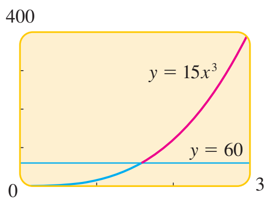
Answer
It helps us to use the model—we can find a maximum or minimum using graphical methods, and we can plug single numbers into our model to get results.

57050_02_ch02_p146-247.qxd 07/04/2008 05:01 PM Page 205



$15x^3 = 90$

Figure 2



$15x^3 > 60$

Figure 3

■ **Use the Model**

We use the model to answer the questions in parts (b), (c), and (d).

(b) If the depth is 1.5 in., the volume is $V(1.5) = 15(1.5)^3 = 50.625\ \text{in}^3$.

(c) We need to solve the equation $V(x) = 90$ or

$$15x^3 = 90$$
$$x^3 = 6$$
$$x = \sqrt[3]{6} \approx 1.82 \text{ in.}$$

The volume is 90 in^3 when the depth is about 1.82 in. (We can also solve this equation graphically, as shown in Figure 2.)

(d) We need to solve the inequality $V(x) > 60$ or

$$15x^3 > 60$$
$$x^3 > 4$$
$$x > \sqrt[3]{4} \approx 1.59$$

The volume will be greater than 60 in^3 if the depth is greater than 1.59 in. (We can also solve this inequality graphically, as shown in Figure 3.) ■

The steps in Example 1 are typical of how we model with functions. They are summarized in the following box.

Guidelines for Modeling with Functions

1. **Express the Model in Words.** Identify the quantity you want to model and express it, in words, as a function of the other quantities in the problem.
2. **Choose the Variable.** Identify all the variables used to express the function in Step 1. Assign a symbol, such as x, to one variable and express the other variables in terms of this symbol.
3. **Set up the Model.** Express the function in the language of algebra by writing it as a function of the single variable chosen in Step 2.
4. **Use the Model.** Use the function to answer the questions posed in the problem. (To find a maximum or a minimum, use the algebraic or graphical methods described in Section 2.5.)

Example 2 Fencing a Garden

A gardener has 140 feet of fencing to fence in a rectangular vegetable garden.

(a) Find a function that models the area of the garden she can fence.

(b) For what range of widths is the area greater than or equal to 825 ft^2?

(c) Can she fence a garden with area 1250 ft^2?

(d) Find the dimensions of the largest area she can fence.

ALTERNATE EXAMPLE 2a

A gardener has 120 feet of fencing to fence in a rectangular vegetable garden. Find a function that models the area A of the garden she can fence.

ANSWER

$A(x) = 60x - x^2$

IN-CLASS MATERIALS

Go over some examples, making a point of using the four steps in the text. It is very easy for the students to confuse the equations that they use in Step 2 with the function they are trying to optimize. Perhaps write the steps of the textbook method off to one side, and check off each step as it is completed.

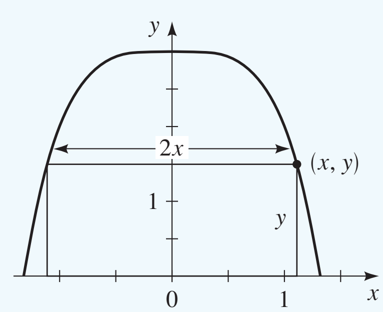
EXAMPLE

Find the maximum area of a rectangle inscribed between the x-axis and $f(x) = -x^4 + 3$.

ANSWER

Thinking about the problem.

We are inscribing a rectangle under a curve. We can make the rectangle tall and narrow, short and fat, or in between, and we want it to have a large area. The rectangle will touch the curve at two points; let's call the one on the right (x, y). (*Continued*)



57050_02_ch02_p146-247.qxd 07/04/2008 05:01 PM Page 206

ALTERNATE EXAMPLE 2b
A gardener has 180 feet of fencing to fence in a rectangular vegetable garden. For what range of widths is the area greater than or equal to 1400 ft^2?

ANSWER
$20 \le x \le 70$

ALTERNATE EXAMPLE 2d
A gardener has 240 feet of fencing to fence in a rectangular vegetable garden. Find the dimensions of the largest area she can fence.

ANSWER
(60, 60)

■ **Thinking About the Problem**

If the gardener fences a plot with width 10 ft, then the length must be 60 ft, because $10 + 10 + 60 + 60 = 140$. So the area is

$$A = \text{width} \times \text{length} = 10 \cdot 60 = 600 \text{ ft}^2$$

The table shows various choices for fencing the garden. We see that as the width increases, the fenced area increases, then decreases.

Width	Length	Area
10	60	600
20	50	1000
30	40	1200
40	30	1200
50	20	1000
60	10	600

Solution

(a) The model we want is a function that gives the area she can fence.

■ **Express the Model in Words**

We know that the area of a rectangular garden is

$$\text{area} = \text{width} \times \text{length}$$

■ **Choose the Variable**

There are two varying quantities—width and length. Since the function we want depends on only one variable, we let

$$x = \text{width of the garden}$$

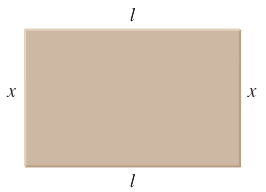


Figure 4

Then we must express the length in terms of x. The perimeter is fixed at 140 ft, so the length is determined once we choose the width. If we let the length be l as in Figure 4, then $2x + 2l = 140$, so $l = 70 - x$. We summarize these facts.

In Words	In Algebra
Width	x
Length	$70 - x$

■ **Set up the Model**

The model is the function A that gives the area of the garden for any width x.

$$\text{area} = \text{width} \times \text{length}$$

$$A(x) = x(70 - x)$$

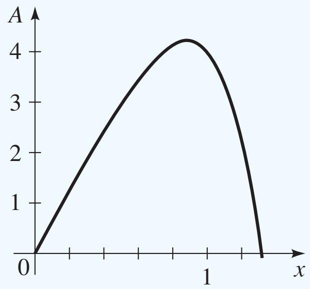
$$A(x) = 70x - x^2$$

The area she can fence is modeled by the function $A(x) = 70x - x^2$.

ANSWER (Continued from p. 205)
The table below shows various choices for the x-coordinate of the point, and the resultant area. We see that as x increases, the area increases, then decreases.

x	y	Area
0.50	2.9378	2.938
0.75	2.684	4.025
1	2	4
1.25	0.559	1.396

1. **Express the model in words.** We know that the area of a rectangle is area = width × length.
2. **Choose the variable.** There are two varying quantities, width and length. Length = $2x$ and width $= -x^4 + 3$.
3. **Set up the model.** $A(x) = 2x(-x^4 + 3)$
4. **Use the model.** We graph $A(x)$, and find the maximum value of A.



The maximum area is approximately 4.225, and it occurs when $x \approx 0.880$.

57050_02_ch02_p146-247.qxd 07/04/2008 05:02 PM Page 207

■ **Use the Model**

We use the model to answer the questions in parts (b)–(d).

Maximum values of quadratic functions are discussed on page 195.

(b) We need to solve the inequality $A(x) \geq 825$. To solve graphically, we graph $y = 70x - x^2$ and $y = 825$ in the same viewing rectangle (see Figure 5). We see that $15 \leq x \leq 55$.

(c) From Figure 6 we see that the graph of $A(x)$ always lies below the line $y = 1250$, so an area of 1250 ft^2 is never attained.

(d) We need to find the maximum value of the function $A(x) = 70x - x^2$. Since this is a quadratic function with $a = -1$ and $b = 70$, the maximum occurs at

$$x = -\frac{b}{2a} = -\frac{70}{2(-1)} = 35$$

So the maximum area that she can fence has width 35 ft and length $70 - 35 = 35$ ft.

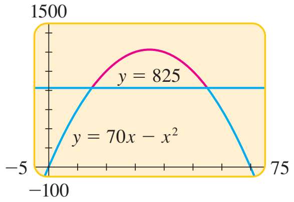


Figure 5

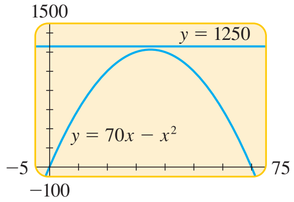


Figure 6 ■

Example 3 Maximizing Revenue from Ticket Sales

A hockey team plays in an arena with a seating capacity of 15,000 spectators. With the ticket price set at \$14, average attendance at recent games has been 9500. A market survey indicates that for each dollar the ticket price is lowered, the average attendance increases by 1000.

(a) Find a function that models the revenue in terms of ticket price.
(b) What ticket price is so high that no one attends, and hence no revenue is generated?
(c) Find the price that maximizes revenue from ticket sales.

■ **Thinking About the Problem**

With a ticket price of \$14, the revenue is $9500 \times \$14 = \$133{,}000$. If the ticket price is lowered to \$13, attendance increases to $9500 + 1000 = 10{,}500$, so the revenue becomes $10{,}500 \times \$13 = \$136{,}500$. The table shows the revenue for several ticket prices. Note that if the ticket price is lowered, revenue increases, but if the ticket price is lowered too much, revenue decreases.

Price	Attendance	Revenue
\$15	8,500	\$127,500
\$14	9,500	\$133,500
\$13	10,500	\$136,500
\$12	11,500	\$138,500
\$11	12,500	\$137,500
\$10	13,500	\$135,500
\$9	14,500	\$130,500

ALTERNATE EXAMPLE 3b

A hockey team plays in an arena with a seating capacity of 10,500 spectators. With the ticket price set at \$10, average attendance at recent games has been 9000. A market survey indicates that for each dollar the ticket price is lowered, the average attendance increases by 1000. What ticket price is so high that no one attends, and hence no revenue is generated?

ANSWER

\$19

ALTERNATE EXAMPLE 3c

A hockey team plays in an arena with a seating capacity of 15,250 spectators. With the ticket price set at \$16, average attendance at recent games has been 9500. A market survey indicates that for each dollar the ticket price is lowered the average attendance increases by 1000. Find the price that maximizes revenue from ticket sales.

Round your answer to the nearest cent, if necessary.

ANSWER

\$12.75

57050_02_ch02_p146-247.qxd 07/04/2008 05:02 PM Page 208

DRILL QUESTION

We cut 10 cm of wire into two pieces, and make a square out of each piece. What is the total combined area, if we make the cut x cm from the end?

Answer

$$\left(\frac{x}{4}\right)^2 + \left(\frac{10-x}{4}\right)^2 = \frac{1}{8}x^2 - \frac{5}{4}x + \frac{25}{4}$$

Solution

(a) The model we want is a function that gives the revenue for any ticket price.

■ **Express the Model in Words**

We know that

$$\text{revenue} = \text{ticket price} \times \text{attendance}$$

■ **Choose the Variable**

There are two varying quantities—ticket price and attendance. Since the function we want depends on price, we let

$$x = \text{ticket price}$$

Next, we must express the attendance in terms of x.

In Words	In Algebra
Ticket price	x
Amount ticket price is lowered	$14 - x$
Increase in attendance	$1000(14 - x)$
Attendance	$9500 + 1000(14 - x) = 23{,}500 - 1000x$

■ **Set up the Model**

The model is the function R that gives the revenue for a given ticket price x.

$$\text{revenue} = \text{ticket price} \times \text{attendance}$$

$$R(x) = x(23{,}500 - 1000x)$$

$$R(x) = 23{,}500x - 1000x^2$$

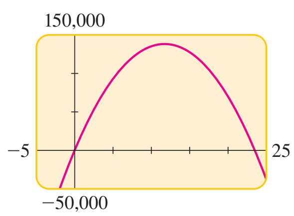


Figure 7

Maximum values of quadratic functions are discussed on page 195.

■ **Use the Model**

We use the model to answer the questions in parts (b) and (c).

(b) We want to find the ticket price x for which $R(x) = 23{,}500x - 1000x^2 = 0$. We can solve this quadratic equation algebraically or graphically. From the graph in Figure 7 we see that $R(x) = 0$ when $x = 0$ or $x = 23.5$. So, according to our model, the revenue would drop to zero if the ticket price is \$23.50 or higher. (Of course, revenue is also zero if the ticket price is zero!)

(c) Since $R(x) = 23{,}500x - 1000x^2$ is a quadratic function with $a = -1000$ and $b = 23{,}500$, the maximum occurs at

$$x = -\frac{b}{2a} = -\frac{23{,}500}{2(-1000)} = 11.75$$

So a ticket price of \$11.75 yields the maximum revenue. At this price the revenue is

$$R(11.75) = 23{,}500(11.75) - 1000(11.75)^2 = \$138{,}062.50$$ ■

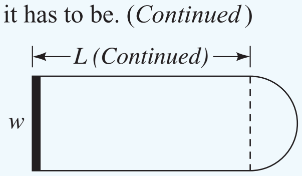
EXAMPLE

Suppose we have 500 feet of fencing and we want to enclose a field against the wall of a barn. This field is to have two parallel lengths of fencing, perpendicular to the wall, joined by a convex semicircle opposite the wall. How do we maximize the area of this field?

ANSWER

Thinking about the problem.

We have a finite amount of fencing, so we have to figure out how wide to make the field to make the area as big as possible, given that the wider we make it, the shorter it has to be. (*Continued*)



57050_02_ch02_p146-247.qxd 07/04/2008 05:02 PM Page 209

Example 4 Minimizing the Metal in a Can

A manufacturer makes a metal can that holds 1 L (liter) of oil. What radius minimizes the amount of metal in the can?

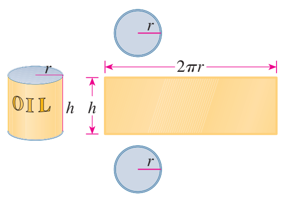


Figure 8

■ **Thinking About the Problem**

To use the least amount of metal, we must minimize the surface area of the can, that is, the area of the top, bottom, and the sides. The area of the top and bottom is $2\pi r^2$ and the area of the sides is $2\pi rh$ (see Figure 8), so the surface area of the can is

$$S = 2\pi r^2 + 2\pi rh$$

The radius and height of the can must be chosen so that the volume is exactly 1 L, or 1000 cm^3. If we want a small radius, say $r = 3$, then the height must be just tall enough to make the total volume 1000 cm^3. In other words, we must have

$$\pi(3)^2 h = 1000 \qquad \text{Volume of the can is } \pi r^2 h$$

$$h = \frac{1000}{9\pi} \approx 35.4 \text{ cm} \qquad \text{Solve for } h$$

Now that we know the radius and height, we can find the surface area of the can:

$$\text{surface area} = 2\pi(3)^2 + 2\pi(3)(35.4) \approx 729.1 \text{ cm}^3$$

If we want a different radius, we can find the corresponding height and surface area in a similar fashion.

Solution The model we want is a function that gives the surface area of the can.

■ **Express the Model in Words**

We know that for a cylindrical can

surface area = area of top and bottom + area of sides

■ **Choose the Variable**

There are two varying quantities—radius and height. Since the function we want depends on the radius, we let

$$r = \text{radius of can}$$

Next, we must express the height in terms of the radius r. Since the volume of a cylindrical can is $V = \pi r^2 h$ and the volume must be 1000 cm^3, we have

$$\pi r^2 h = 1000 \qquad \text{Volume of can is 1000 cm}^3$$

$$h = \frac{1000}{\pi r^2} \qquad \text{Solve for } h$$

ALTERNATE EXAMPLE 4
A manufacturer makes a metal can that holds 3 L (liters) of oil. What radius minimizes the amount of metal in the can? Give the answer to the nearest tenth.

ANSWER
7.8

ANSWER (Continued from p. 208)

1. Express the model in words. We note that the radius of the semicircle is half the width of the pen. The combined area is the area of the rectangular piece plus the area of the semicircle:

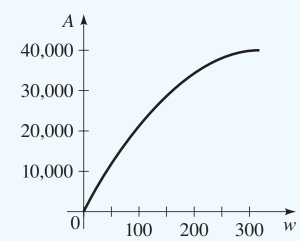
$$\text{Area} = \text{length} \times \text{width} + \tfrac{1}{2}\pi\left(\tfrac{\text{width}}{2}\right)^2$$

2. Choose the variable. Because both pieces of the area depend on width, we will let w = the width of the pen. Then, since $500 = 2l + \pi\left(\frac{w}{2}\right)$, we have $l = 250 - \pi\left(\frac{w}{4}\right)$.

3. Set up the model.

$$A(w) = \left[250 - \pi\left(\tfrac{w}{4}\right)\right]w + \tfrac{1}{2}\pi\left(\tfrac{w}{2}\right)^2$$
$$= 250w - \tfrac{1}{8}\pi w^2$$

4. Use the model.



When we graph A, we see it is always increasing. The maximum occurs when w is as large as possible. This occurs when $l = 0$, and we maximize the area of the field by letting $l = 0$ ft, so $w = \frac{1000}{\pi}$ ft.

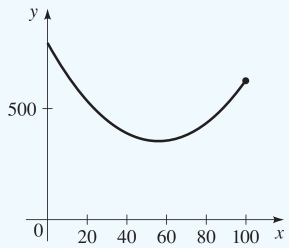
Point out that the one part of this process in which we use approximation techniques is in looking at the graph of a function to find its maximum or minimum. In calculus, we will learn how to find the exact location of a function's maximum or minimum.

57050_02_ch02_p146-247.qxd 07/04/2008 05:02 PM Page 210

EXAMPLE
We cut a 100 cm wire into two pieces, x cm from the left. The first piece is used to make a square. The second piece is used to make a circle. Find an expression for the total combined area in terms of x.

ANSWER

$$A(x) = \left(\frac{x}{4}\right)^2 + \pi\left(\frac{100 - x}{2\pi}\right)^2$$



Note the maximum combined area occurs when all of the wire is used to make a circle, and the minimum occurs when $x = \dfrac{400}{\pi + 4} \approx 56.0$. Make sure to justify ending the graph at $x = 100$.

We can now express the areas of the top, bottom, and sides in terms of r only.

In Words	In Algebra
Radius of can	r
Height of can	$\dfrac{1000}{\pi r^2}$
Area of top and bottom	$2\pi r^2$
Area of sides ($2\pi rh$)	$2\pi r\left(\dfrac{1000}{\pi r^2}\right)$

■ **Set up the Model**

The model is the function S that gives the surface area of the can as a function of the radius r.

surface area = area of top and bottom + area of sides

$$S(r) = 2\pi r^2 + 2\pi r\left(\frac{1000}{\pi r^2}\right)$$

$$S(r) = 2\pi r^2 + \frac{2000}{r}$$

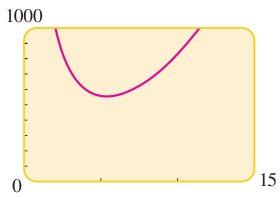


Figure 9
$S = 2\pi r^2 + \dfrac{2000}{r}$

■ **Use the Model**

We use the model to find the minimum surface area of the can. We graph S in Figure 9 and zoom in on the minimum point to find that the minimum value of S is about 554 cm^2 and occurs when the radius is about 5.4 cm. ■

2.6 Exercises

1–18 ■ In these exercises you are asked to find a function that models a real-life situation. Use the guidelines for modeling described in the text to help you.

1. **Area** A rectangular building lot is three times as long as it is wide. Find a function that models its area A in terms of its width w.

2. **Area** A poster is 10 inches longer than it is wide. Find a function that models its area A in terms of its width w.

3. **Volume** A rectangular box has a square base. Its height is half the width of the base. Find a function that models its volume V in terms of its width w.

4. **Volume** The height of a cylinder is four times its radius. Find a function that models the volume V of the cylinder in terms of its radius r.

5. **Area** A rectangle has a perimeter of 20 ft. Find a function that models its area A in terms of the length x of one of its sides.

6. **Perimeter** A rectangle has an area of 16 m^2. Find a function that models its perimeter P in terms of the length x of one of its sides.

7. **Area** Find a function that models the area A of an equilateral triangle in terms of the length x of one of its sides.

8. **Area** Find a function that models the surface area S of a cube in terms of its volume V.

9. **Radius** Find a function that models the radius r of a circle in terms of its area A.

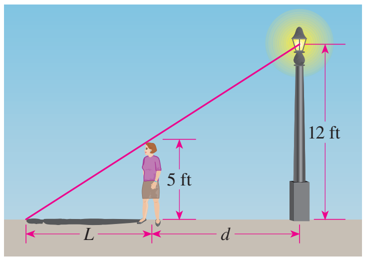
10. **Area** Find a function that models the area A of a circle in terms of its circumference C.

11. **Area** A rectangular box with a volume of 60 ft^3 has a square base. Find a function that models its surface area S in terms of the length x of one side of its base.

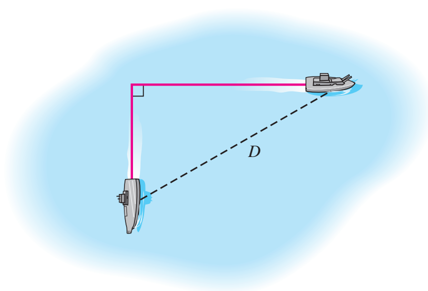
12. **Length** A woman 5 ft tall is standing near a street lamp that is 12 ft tall, as shown in the figure. Find a function that

57050_02_ch02_p146-247.qxd 07/04/2008 05:02 PM Page 211

models the length L of her shadow in terms of her distance d from the base of the lamp.



13. Distance Two ships leave port at the same time. One sails south at 15 mi/h and the other sails east at 20 mi/h. Find a function that models the distance D between the ships in terms of the time t (in hours) elapsed since their departure.

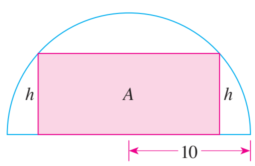


14. Product The sum of two positive numbers is 60. Find a function that models their product P in terms of x, one of the numbers.

15. Area An isosceles triangle has a perimeter of 8 cm. Find a function that models its area A in terms of the length of its base b.

16. Perimeter A right triangle has one leg twice as long as the other. Find a function that models its perimeter P in terms of the length x of the shorter leg.

17. Area A rectangle is inscribed in a semicircle of radius 10, as shown in the figure. Find a function that models the area A of the rectangle in terms of its height h.



18. Height The volume of a cone is 100 in^3. Find a function that models the height h of the cone in terms of its radius r.

19–36 ■ In these problems you are asked to find a function that models a real-life situation, and then use the model to answer questions about the situation. Use the guidelines on page 205 to help you.

19. Maximizing a Product Consider the following problem: Find two numbers whose sum is 19 and whose product is as large as possible.

(a) Experiment with the problem by making a table like the one below, showing the product of different pairs of numbers that add up to 19. Based on the evidence in your table, estimate the answer to the problem.

First number	Second number	Product
1	18	18
2	17	34
3	16	48
⋮	⋮	⋮

(b) Find a function that models the product in terms of one of the two numbers.

(c) Use your model to solve the problem, and compare with your answer to part (a).

20. Minimizing a Sum Find two positive numbers whose sum is 100 and the sum of whose squares is a minimum.

21. Maximizing a Product Find two numbers whose sum is -24 and whose product is a maximum.

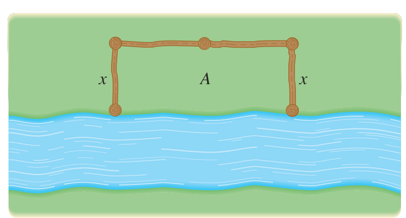
22. Maximizing Area Among all rectangles that have a perimeter of 20 ft, find the dimensions of the one with the largest area.

23. Fencing a Field Consider the following problem: A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He does not need a fence along the river (see the figure). What are the dimensions of the field of largest area that he can fence?

(a) Experiment with the problem by drawing several diagrams illustrating the situation. Calculate the area of each configuration, and use your results to estimate the dimensions of the largest possible field.

(b) Find a function that models the area of the field in terms of one of its sides.

(c) Use your model to solve the problem, and compare with your answer to part (a).

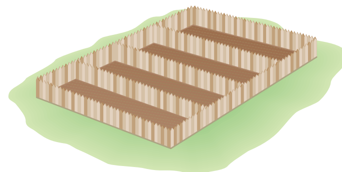


57050_02_ch02_p146-247.qxd 07/04/2008 05:02 PM Page 212

24. Dividing a Pen A rancher with 750 ft of fencing wants to enclose a rectangular area and then divide it into four pens with fencing parallel to one side of the rectangle (see the figure).

(a) Find a function that models the total area of the four pens.

(b) Find the largest possible total area of the four pens.

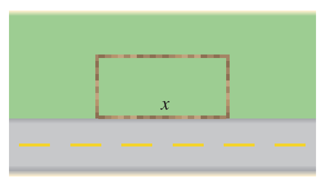


25. Fencing a Garden Plot A property owner wants to fence a garden plot adjacent to a road, as shown in the figure. The fencing next to the road must be sturdier and costs \$5 per foot, but the other fencing costs just \$3 per foot. The garden is to have an area of 1200 ft^2.

(a) Find a function that models the cost of fencing the garden.

(b) Find the garden dimensions that minimize the cost of fencing.

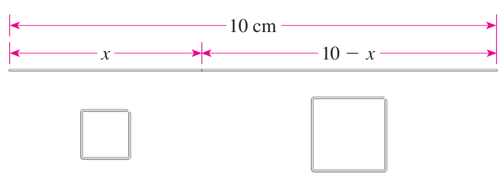
(c) If the owner has at most \$600 to spend on fencing, find the range of lengths he can fence along the road.



26. Maximizing Area A wire 10 cm long is cut into two pieces, one of length x and the other of length $10 - x$, as shown in the figure. Each piece is bent into the shape of a square.

(a) Find a function that models the total area enclosed by the two squares.

(b) Find the value of x that minimizes the total area of the two squares.



27. Stadium Revenue A baseball team plays in a stadium that holds 55,000 spectators. With the ticket price at \$10, the average attendance at recent games has been 27,000. A market survey indicates that for every dollar the ticket price is lowered, attendance increases by 3000.

(a) Find a function that models the revenue in terms of ticket price.

(b) What ticket price is so high that no revenue is generated?

(c) Find the price that maximizes revenue from ticket sales.

28. Maximizing Profit A community bird-watching society makes and sells simple bird feeders to raise money for its conservation activities. The materials for each feeder cost \$6, and they sell an average of 20 per week at a price of \$10 each. They have been considering raising the price, so they conduct a survey and find that for every dollar increase they lose 2 sales per week.

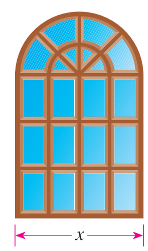
(a) Find a function that models weekly profit in terms of price per feeder.

(b) What price should the society charge for each feeder to maximize profits? What is the maximum profit?

29. Light from a Window A Norman window has the shape of a rectangle surmounted by a semicircle, as shown in the figure. A Norman window with perimeter 30 ft is to be constructed.

(a) Find a function that models the area of the window.

(b) Find the dimensions of the window that admits the greatest amount of light.



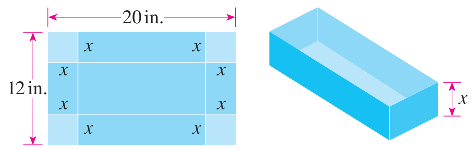
30. Volume of a Box A box with an open top is to be constructed from a rectangular piece of cardboard with dimensions 12 in. by 20 in. by cutting out equal squares of side x at each corner and then folding up the sides (see the figure).

(a) Find a function that models the volume of the box.

57050_02_ch02_p146-247.qxd 07/04/2008 05:02 PM Page 213

(b) Find the values of x for which the volume is greater than 200 in^3.

(c) Find the largest volume that such a box can have.

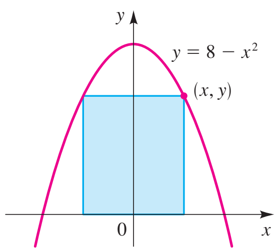


31. Area of a Box An open box with a square base is to have a volume of 12 ft^3.

(a) Find a function that models the surface area of the box.

(b) Find the box dimensions that minimize the amount of material used.

32. Inscribed Rectangle Find the dimensions that give the largest area for the rectangle shown in the figure. Its base is on the x-axis and its other two vertices are above the x-axis, lying on the parabola $y = 8 - x^2$.



33. Minimizing Costs A rancher wants to build a rectangular pen with an area of 100 m^2.

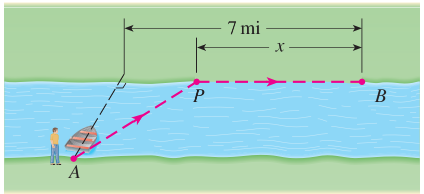
(a) Find a function that models the length of fencing required.

(b) Find the pen dimensions that require the minimum amount of fencing.

34. Minimizing Time A man stands at a point A on the bank of a straight river, 2 mi wide. To reach point B, 7 mi downstream on the opposite bank, he first rows his boat to point P on the opposite bank and then walks the remaining distance x to B, as shown in the figure. He can row at a speed of 2 mi/h and walk at a speed of 5 mi/h.

(a) Find a function that models the time needed for the trip.

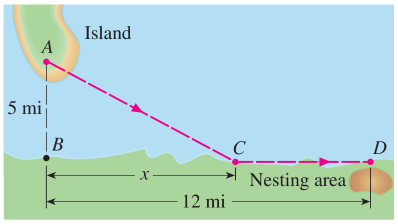
(b) Where should he land so that he reaches B as soon as possible?



35. Bird Flight A bird is released from point A on an island, 5 mi from the nearest point B on a straight shoreline. The bird flies to a point C on the shoreline, and then flies along the shoreline to its nesting area D (see the figure). Suppose the bird requires 10 kcal/mi of energy to fly over land and 14 kcal/mi to fly over water (see Example 9 in Section 1.6).

(a) Find a function that models the energy expenditure of the bird.

(b) If the bird instinctively chooses a path that minimizes its energy expenditure, to what point does it fly?

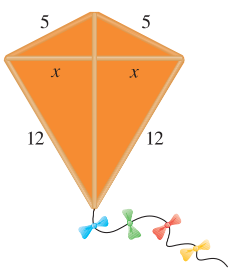


36. Area of a Kite A kite frame is to be made from six pieces of wood. The four pieces that form its border have been cut to the lengths indicated in the figure. Let x be as shown in the figure.

(a) Show that the area of the kite is given by the function

$$A(x) = x\left(\sqrt{25 - x^2} + \sqrt{144 - x^2}\right)$$

(b) How long should each of the two crosspieces be to maximize the area of the kite?



SUGGESTED TIME AND EMPHASIS

$\frac{1}{2}$–1 class.

Essential material.

2.7 Combining Functions

In this section we study different ways to combine functions to make new functions.

Sums, Differences, Products, and Quotients

Two functions f and g can be combined to form new functions $f + g$, $f - g$, fg, and f/g in a manner similar to the way we add, subtract, multiply, and divide real numbers. For example, we define the function $f + g$ by

$$(f + g)(x) = f(x) + g(x)$$

The new function $f + g$ is called the **sum** of the functions f and g; its value at x is $f(x) + g(x)$. Of course, the sum on the right-hand side makes sense only if both $f(x)$ and $g(x)$ are defined, that is, if x belongs to the domain of f and also to the domain of g. So, if the domain of f is A and the domain of g is B, then the domain of $f + g$ is the intersection of these domains, that is, $A \cap B$. Similarly, we can define the **difference** $f - g$, the **product** fg, and the **quotient** f/g of the functions f and g. Their domains are $A \cap B$, but in the case of the quotient we must remember not to divide by 0.

The sum of f and g is defined by

$$(f + g)(x) = f(x) + g(x)$$

The name of the new function is "$f + g$." So this $+$ sign stands for the operation of addition of *functions*. The $+$ sign on the right side, however, stands for addition of the *numbers* $f(x)$ and $g(x)$.

Algebra of Functions

Let f and g be functions with domains A and B. Then the functions $f + g$, $f - g$, fg, and f/g are defined as follows.

$(f + g)(x) = f(x) + g(x)$	Domain $A \cap B$
$(f - g)(x) = f(x) - g(x)$	Domain $A \cap B$
$(fg)(x) = f(x)g(x)$	Domain $A \cap B$
$\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$	Domain $\{x \in A \cap B \mid g(x) \neq 0\}$

Example 1 Combinations of Functions and Their Domains

Let $f(x) = \dfrac{1}{x - 2}$ and $g(x) = \sqrt{x}$.

(a) Find the functions $f + g$, $f - g$, fg, and f/g and their domains.

(b) Find $(f + g)(4)$, $(f - g)(4)$, $(fg)(4)$, and $(f/g)(4)$.

Solution

(a) The domain of f is $\{x \mid x \neq 2\}$ and the domain of g is $\{x \mid x \geq 0\}$. The intersection of the domains of f and g is

$$\{x \mid x \geq 0 \text{ and } x \neq 2\} = [0, 2) \cup (2, \infty)$$

ALTERNATE EXAMPLE 1a

Let $f(x) = x^2$ and $g(x) = \sqrt{2x}$. Find the function $f + g$ and its domain.

ANSWER

$x^2 + \sqrt{2x}$

ALTERNATE EXAMPLE 1b

Let $f(x) = x^2$ and $g(x) = \sqrt{3x}$. Find $(f + g)(3)$.

ANSWER

12

POINTS TO STRESS

1. Addition, subtraction, multiplication, and division of functions.
2. Composition of functions.
3. Finding the domain of a function based on analysis of the domain of its components.

57050_02_ch02_p146-247.qxd 4/10/08 10:05 PM Page 215

Thus, we have

$$(f+g)(x) = f(x) + g(x) = \frac{1}{x-2} + \sqrt{x} \qquad \text{Domain } \{x \mid x \geq 0 \text{ and } x \neq 2\}$$

$$(f-g)(x) = f(x) - g(x) = \frac{1}{x-2} - \sqrt{x} \qquad \text{Domain } \{x \mid x \geq 0 \text{ and } x \neq 2\}$$

$$(fg)(x) = f(x)g(x) = \frac{\sqrt{x}}{x-2} \qquad \text{Domain } \{x \mid x \geq 0 \text{ and } x \neq 2\}$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{1}{(x-2)\sqrt{x}} \qquad \text{Domain } \{x \mid x > 0 \text{ and } x \neq 2\}$$

To divide fractions, invert the denominator and multiply:

$$\frac{1/(x-2)}{\sqrt{x}} = \frac{1/(x-2)}{\sqrt{x}/1} = \frac{1}{x-2} \cdot \frac{1}{\sqrt{x}} = \frac{1}{(x-2)\sqrt{x}}$$

Note that in the domain of f/g we exclude 0 because $g(0) = 0$.

(b) Each of these values exist because $x = 4$ is in the domain of each function.

$$(f+g)(4) = f(4) + g(4) = \frac{1}{4-2} + \sqrt{4} = \frac{5}{2}$$

$$(f-g)(4) = f(4) - g(4) = \frac{1}{4-2} - \sqrt{4} = -\frac{3}{2}$$

$$(fg)(4) = f(4)g(4) = \left(\frac{1}{4-2}\right)\sqrt{4} = 1$$

$$\left(\frac{f}{g}\right)(4) = \frac{f(4)}{g(4)} = \frac{1}{(4-2)\sqrt{4}} = \frac{1}{4}$$

■

The graph of the function $f + g$ can be obtained from the graphs of f and g by **graphical addition**. This means that we add corresponding y-coordinates, as illustrated in the next example.

Example 2 Using Graphical Addition

The graphs of f and g are shown in Figure 1. Use graphical addition to graph the function $f + g$.

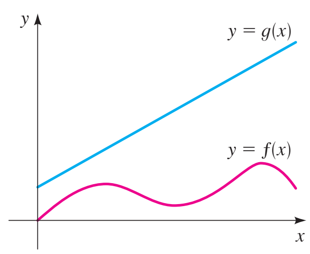


Figure 1

Solution We obtain the graph of $f + g$ by "graphically adding" the value of $f(x)$ to $g(x)$ as shown in Figure 2. This is implemented by copying the line segment PQ on top of PR to obtain the point S on the graph of $f + g$.

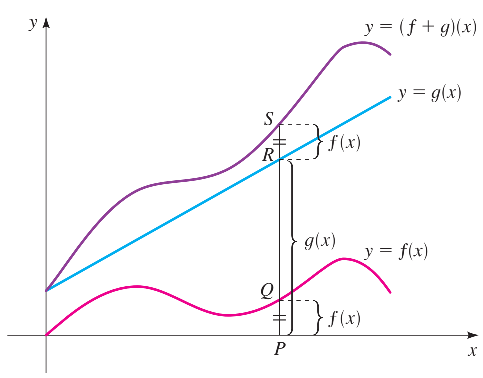


Figure 2
Graphical addition

■

ALTERNATE EXAMPLE 2

If $f(x) = \dfrac{x-4}{x-1}$ and $g(x) = \dfrac{x-3}{x-2}$, find the functions $f+g$, $f-g$, fg, and f/g and their domains.

ANSWER

The domain of $f(x) = \dfrac{x-4}{x-1}$ is the real numbers that are not equal to one. The domain of $g(x) = \dfrac{x-3}{x-2}$ is the real numbers that are not equal to two. The intersection of the domains of f and g is $(-\infty, 1) \cup (1, 2) \cup (2, \infty)$

$$(f+g)(x) = \frac{x-4}{x-1} + \frac{x-3}{x-2}$$

Domain $\{x \mid x \neq 1, x \neq 2\}$

$$(f-g)(x) = \frac{x-4}{x-1} - \frac{x-3}{x-2}$$

Domain $\{x \mid x \neq 1, x \neq 2\}$

$$(fg)(x) = \left(\frac{x-4}{x-1}\right)\left(\frac{x-3}{x-2}\right)$$

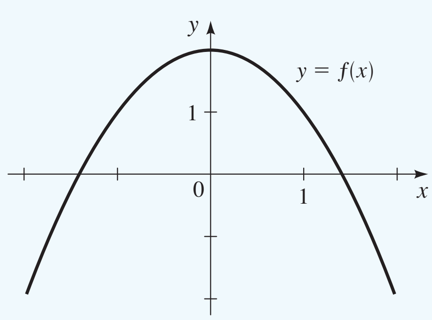
Domain $\{x \mid x \neq 1, x \neq 2\}$

$$\left(\frac{f}{g}\right)(x) = \left(\frac{x-4}{x-1}\right)\left(\frac{x-2}{x-3}\right)$$

Domain $\{x \mid x \neq 1, x \neq 2, x \neq 3\}$

IN-CLASS MATERIALS

Do the following problem with the class:



From the graph of $y = f(x) = -x^2 + 2$ shown above, compute $f \circ f$ at $x = -1$, 0, and 1. First do it graphically, then algebraically.

57050_02_ch02_p146-247.qxd 07/04/2008 05:02 PM Page 216

Composition of Functions

Now let's consider a very important way of combining two functions to get a new function. Suppose $f(x) = \sqrt{x}$ and $g(x) = x^2 + 1$. We may define a function h as

$$h(x) = f(g(x)) = f(x^2 + 1) = \sqrt{x^2 + 1}$$

The function h is made up of the functions f and g in an interesting way: Given a number x, we first apply to it the function g, then apply f to the result. In this case, f is the rule "take the square root," g is the rule "square, then add 1," and h is the rule "square, then add 1, then take the square root." In other words, we get the rule h by applying the rule g and then the rule f. Figure 3 shows a machine diagram for h.

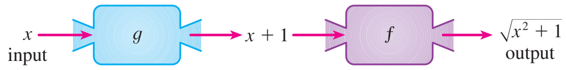


Figure 3
The h machine is composed of the g machine (first) and then the f machine.

In general, given any two functions f and g, we start with a number x in the domain of g and find its image $g(x)$. If this number $g(x)$ is in the domain of f, we can then calculate the value of $f(g(x))$. The result is a new function $h(x) = f(g(x))$ obtained by substituting g into f. It is called the *composition* (or *composite*) of f and g and is denoted by $f \circ g$ ("f composed with g").

Composition of Functions

Given two functions f and g, the **composite function** $f \circ g$ (also called the **composition** of f and g) is defined by

$$(f \circ g)(x) = f(g(x))$$

The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f. In other words, $(f \circ g)(x)$ is defined whenever both $g(x)$ and $f(g(x))$ are defined. We can picture $f \circ g$ using an arrow diagram (Figure 4).

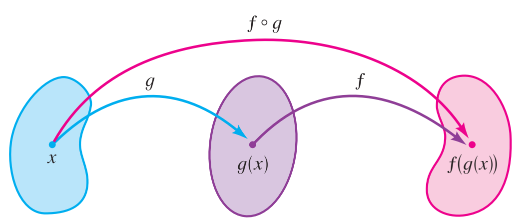
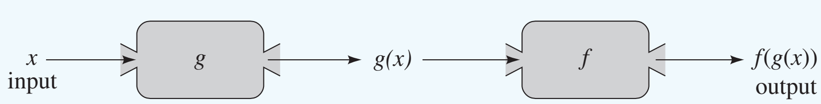


Figure 4
Arrow diagram for $f \circ g$

SAMPLE QUESTION
Text Question
The text describes addition, multiplication, division, and composition of functions. Which of these operations is represented by the following diagram?



Answer
Composition

57050_02_ch02_p146-247.qxd 07/04/2008 05:02 PM Page 217

Example 3 **Finding the Composition of Functions**

Let $f(x) = x^2$ and $g(x) = x - 3$.

(a) Find the functions $f \circ g$ and $g \circ f$ and their domains.

(b) Find $(f \circ g)(5)$ and $(g \circ f)(7)$.

In Example 3, f is the rule "square" and g is the rule "subtract 3." The function $f \circ g$ *first* subtracts 3 and *then* squares; the function $g \circ f$ *first* squares and *then* subtracts 3.

Solution

(a) We have

$$\begin{aligned} (f \circ g)(x) &= f(g(x)) && \text{Definition of } f \circ g \\ &= f(x - 3) && \text{Definition of } g \\ &= (x - 3)^2 && \text{Definition of } f \end{aligned}$$

and

$$\begin{aligned} (g \circ f)(x) &= g(f(x)) && \text{Definition of } g \circ f \\ &= g(x^2) && \text{Definition of } f \\ &= x^2 - 3 && \text{Definition of } g \end{aligned}$$

The domains of both $f \circ g$ and $g \circ f$ are $\mathbb{R}$.

(b) We have

$$(f \circ g)(5) = f(g(5)) = f(2) = 2^2 = 4$$

$$(g \circ f)(7) = g(f(7)) = g(49) = 49 - 3 = 46$$

■

You can see from Example 3 that, in general, $f \circ g \neq g \circ f$. Remember that the notation $f \circ g$ means that the function g is applied first and then f is applied second.

Example 4 **Finding the Composition of Functions**

If $f(x) = \sqrt{x}$ and $g(x) = \sqrt{2 - x}$, find the following functions and their domains.

(a) $f \circ g$ (b) $g \circ f$ (c) $f \circ f$ (d) $g \circ g$

Solution

(a)

$$\begin{aligned} (f \circ g)(x) &= f(g(x)) && \text{Definition of } f \circ g \\ &= f(\sqrt{2 - x}) && \text{Definition of } g \\ &= \sqrt{\sqrt{2 - x}} && \text{Definition of } f \\ &= \sqrt[4]{2 - x} \end{aligned}$$

The domain of $f \circ g$ is $\{x \mid 2 - x \geq 0\} = \{x \mid x \leq 2\} = (-\infty, 2]$.

(b)

$$\begin{aligned} (g \circ f)(x) &= g(f(x)) && \text{Definition of } g \circ f \\ &= g(\sqrt{x}) && \text{Definition of } f \\ &= \sqrt{2 - \sqrt{x}} && \text{Definition of } g \end{aligned}$$

For $\sqrt{x}$ to be defined, we must have $x \geq 0$. For $\sqrt{2 - \sqrt{x}}$ to be defined, we

ALTERNATE EXAMPLE 3a

Let $f(x) = x^2$ and $g(x) = x - 5$. Find the function $f \circ g$ and its domain.

ANSWER

$(x - 5)^2, (-\infty, \infty)$

ALTERNATE EXAMPLE 3b

Let $f(x) = x^2$ and $g(x) = x - 4$. Find $(g \circ f)(3)$.

ANSWER

5

ALTERNATE EXAMPLE 4a

If $f(x) = \sqrt{x}$ and $g(x) = \sqrt{4 - x}$, find the function $f \circ g$ and its domain.

ANSWER

$\sqrt[4]{4 - x}, (-\infty, 4]$

ALTERNATE EXAMPLE 4d

If $f(x) = \sqrt{x}$ and $g(x) = \sqrt{4 - x}$, find the function $g \circ g$ and its domain.

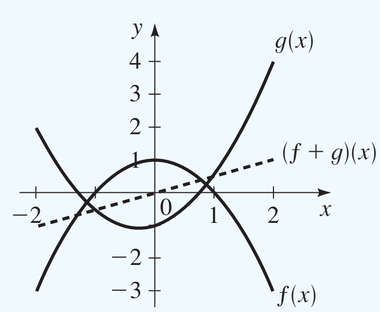
ANSWER

$\sqrt{4 - \sqrt{4 - x}}, [-12, 4]$

IN-CLASS MATERIALS

Show the tie between algebraic addition of functions and graphical addition. For example let $f(x) = 1 - x^2$ and $g(x) = x^2 + \frac{1}{2}x - 1$. First add the functions graphically, as shown right, and then show how this result can be obtained algebraically:

$$(1 - x^2) + \left(x^2 + \frac{1}{2}x - 1\right) = \frac{1}{2}x$$



57050_02_ch02_p146-247.qxd 07/04/2008 05:02 PM Page 218

must have $2 - \sqrt{x} \geq 0$, that is, $\sqrt{x} \leq 2$, or $x \leq 4$. Thus, we have $0 \leq x \leq 4$, so the domain of $g \circ f$ is the closed interval $[0, 4]$.

(c)

$$
\begin{aligned}
(f \circ f)(x) &= f(f(x)) && \text{Definition of } f \circ f \\
&= f(\sqrt{x}) && \text{Definition of } f \\
&= \sqrt{\sqrt{x}} && \text{Definition of } f \\
&= \sqrt[4]{x}
\end{aligned}
$$

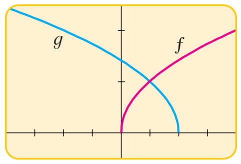
The domain of $f \circ f$ is $[0, \infty)$.

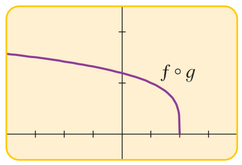
(d)

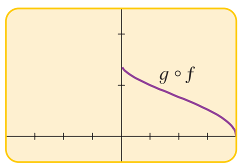
$$
\begin{aligned}
(g \circ g)(x) &= g(g(x)) && \text{Definition of } g \circ g \\
&= g(\sqrt{2 - x}) && \text{Definition of } g \\
&= \sqrt{2 - \sqrt{2 - x}} && \text{Definition of } g
\end{aligned}
$$

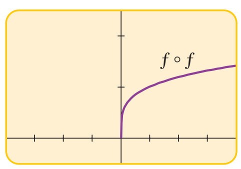
This expression is defined when both $2 - x \geq 0$ and $2 - \sqrt{2 - x} \geq 0$. The first inequality means $x \leq 2$, and the second is equivalent to $\sqrt{2 - x} \leq 2$, or $2 - x \leq 4$, or $x \geq -2$. Thus, $-2 \leq x \leq 2$, so the domain of $g \circ g$ is $[-2, 2]$. ■

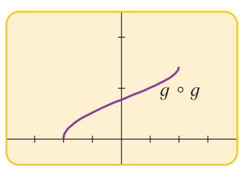
The graphs of f and g of Example 4, as well as $f \circ g$, $g \circ f$, $f \circ f$, and $g \circ g$, are shown below. These graphs indicate that the operation of composition can produce functions quite different from the original functions.











It is possible to take the composition of three or more functions. For instance, the composite function $f \circ g \circ h$ is found by first applying h, then g, and then f as follows:

$$(f \circ g \circ h)(x) = f(g(h(x)))$$

Example 5 A Composition of Three Functions

Find $f \circ g \circ h$ if $f(x) = x/(x + 1)$, $g(x) = x^{10}$ and $h(x) = x + 3$.

Solution

$$
\begin{aligned}
(f \circ g \circ h)(x) &= f(g(h(x))) && \text{Definition of } f \circ g \circ h \\
&= f(g(x + 3)) && \text{Definition of } h \\
&= f((x + 3)^{10}) && \text{Definition of } g \\
&= \frac{(x + 3)^{10}}{(x + 3)^{10} + 1} && \text{Definition of } f
\end{aligned}
$$

■

So far we have used composition to build complicated functions from simpler ones. But in calculus it is useful to be able to "decompose" a complicated function into simpler ones, as shown in the following example.

Example 6 Recognizing a Composition of Functions

Given $F(x) = \sqrt[4]{x + 9}$, find functions f and g such that $F = f \circ g$.

Solution Since the formula for F says to first add 9 and then take the fourth root, we let

$$g(x) = x + 9 \quad \text{and} \quad f(x) = \sqrt[4]{x}$$

ALTERNATE EXAMPLE 5

If $f(x) = 1/x$ and $g(x) = \sqrt[4]{x - 1}$, find the following functions and their domains.

(a) $f \circ g$ **(b)** $g \circ f$
(c) $f \circ f$ **(d)** $g \circ g$

ANSWERS

(a) $f(g(x)) = f\left(\sqrt[4]{x - 1}\right) = \dfrac{1}{\sqrt[4]{x - 1}}$. The domain of $f \circ g$ is $\{x \mid x > 1\}$.

(b) $g(f(x)) = g(1/x) = \sqrt[4]{\dfrac{1}{x} - 1}$. The domain of $g \circ f$ is $\{x \mid 0 < x \leq 1\}$.

(c) $f(f(x)) = x$. The domain of $f \circ f$ is $\{x \mid x \neq 0\}$.

(d) $g(g(x)) = \sqrt[4]{\sqrt[4]{x - 1} - 1}$. The domain of $g \circ g$ is $\{x \mid x \geq 2\}$.

ALTERNATE EXAMPLE 6

Find $f \circ g \circ h$ if $f(x) = \dfrac{x + 1}{x - 1}$, $g(x) = 2x + 1$, and $h(x) = x^2$

ANSWER

$$
\begin{aligned}
f(g(h(x))) &= f(g(x^2)) \\
&= f(2x^2 + 1) \\
&= \frac{2x^2 + 2}{2x^2}
\end{aligned}
$$

IN-CLASS MATERIALS

After doing a few basic examples of composition, it is possible to foreshadow the idea of inverses, which will be covered in the next section. Let $f(x) = 2x^3 + 3$ and $g(x) = x^2 - x$. Compute $f \circ g$ and $g \circ f$ for your students. Then ask them to come up with a function $h(x)$ with the property that $(f \circ h)(x) = x$.

They may not be used to the idea of coming up with examples for themselves, and so the main hints they will need might be "don't give up," "when in doubt, just try something and see what happens," and "I'm not expecting you to get it in fifteen seconds." If the class is really stuck, have them try $f(x) = 2x^3$ to get a feel for how the game is played. Once they have determined that $h(x) = \sqrt[3]{\dfrac{x - 3}{2}}$, have them compute $(h \circ f)(x)$ and have them conjecture whether, in general, if $(f \circ g)(x) = x$ then $(g \circ f)(x)$ must also equal x.

57050_02_ch02_p146-247.qxd 07/04/2008 05:02 PM Page 219

Then

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) && \text{Definition of } f \circ g\\ &= f(x+9) && \text{Definition of } g\\ &= \sqrt[4]{x+9} && \text{Definition of } f\\ &= F(x)\end{aligned}$$

■

Example 7 An Application of Composition of Functions

A ship is traveling at 20 mi/h parallel to a straight shoreline. The ship is 5 mi from shore. It passes a lighthouse at noon.

(a) Express the distance s between the lighthouse and the ship as a function of d, the distance the ship has traveled since noon; that is, find f so that $s = f(d)$.

(b) Express d as a function of t, the time elapsed since noon; that is, find g so that $d = g(t)$.

(c) Find $f \circ g$. What does this function represent?

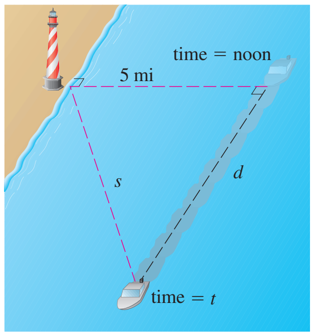


Figure 5

Solution We first draw a diagram as in Figure 5.

(a) We can relate the distances s and d by the Pythagorean Theorem. Thus, s can be expressed as a function of d by

$$s = f(d) = \sqrt{25 + d^2}$$

(b) Since the ship is traveling at 20 mi/h, the distance d it has traveled is a function of t as follows:

distance = rate × time

$$d = g(t) = 20t$$

(c) We have

$$\begin{aligned}(f \circ g)(t) &= f(g(t)) && \text{Definition of } f \circ g\\ &= f(20t) && \text{Definition of } g\\ &= \sqrt{25 + (20t)^2} && \text{Definition of } f\end{aligned}$$

The function $f \circ g$ gives the distance of the ship from the lighthouse as a function of time. ■

2.7 Exercises

1–6 ■ Find $f + g$, $f - g$, fg, and f/g and their domains.

1. $f(x) = x - 3, \quad g(x) = x^2$

2. $f(x) = x^2 + 2x, \quad g(x) = 3x^2 - 1$

3. $f(x) = \sqrt{4 - x^2}, \quad g(x) = \sqrt{1 + x}$

4. $f(x) = \sqrt{9 - x^2}, \quad g(x) = \sqrt{x^2 - 4}$

5. $f(x) = \frac{2}{x}, \quad g(x) = \frac{4}{x + 4}$

6. $f(x) = \frac{2}{x + 1}, \quad g(x) = \frac{x}{x + 1}$

7–10 ■ Find the domain of the function.

7. $f(x) = \sqrt{x} + \sqrt{1 - x}$

8. $g(x) = \sqrt{x + 1} - \frac{1}{x}$

9. $h(x) = (x - 3)^{-1/4}$

10. $k(x) = \frac{\sqrt{x + 3}}{x - 1}$

DRILL QUESTION

Let $f(x) = 4x$ and $g(x) = x^3 + x$.

(a) Compute $(f \circ g)(x)$.
(b) Compute $(g \circ f)(x)$.

Answers

(a) $4(x^3 + x) = 4x^3 + 4x$
(b) $(4x)^3 + 4x = 64x^3 + 4x$

ALTERNATE EXAMPLE 7

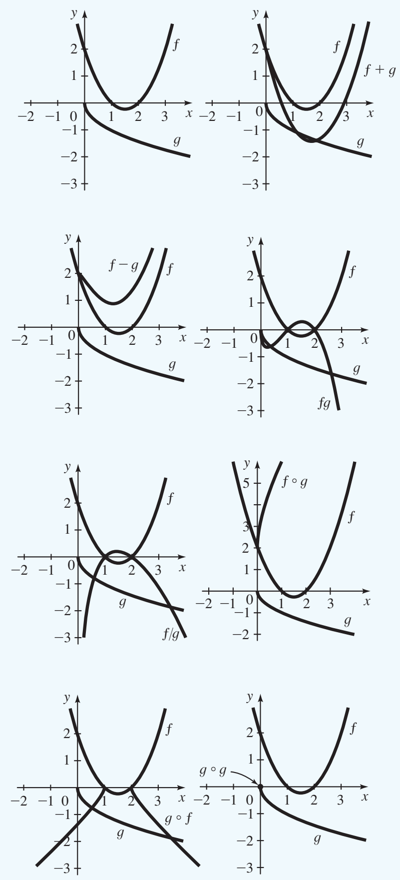
Given $F(x) = (3x + 1)^{10}$, find functions f and g such that $F = f \circ g$.

ANSWER

We let $g(x) = 3x + 1$ and $f(x) = x^{10}$.

EXAMPLE

Combined functions with graphs: Let $f(x) = x^2 - 3x + 2$ and $g(x) = -\sqrt{x}$.



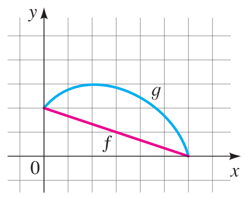
IN-CLASS MATERIALS

Point out that it is important to keep track of domains, especially when doing algebraic simplification. For example, if $f(x) = x + \sqrt{x}$ and $g(x) = 3x^2 + \sqrt{x}$, even though $(f - g)(x) = x - 3x^2$, its domain is not the real numbers, but $\{x \mid x \geq 0\}$.

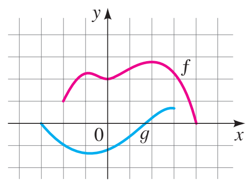
57050_02_ch02_p146-247.qxd 07/04/2008 05:02 PM Page 220

11–12 ■ Use graphical addition to sketch the graph of $f + g$.

11.



12.



13–16 ■ Draw the graphs of f, g, and $f + g$ on a common screen to illustrate graphical addition.

13. $f(x) = \sqrt{1 + x}, \quad g(x) = \sqrt{1 - x}$

14. $f(x) = x^2, \quad g(x) = \sqrt{x}$

15. $f(x) = x^2, \quad g(x) = \frac{1}{3}x^3$

16. $f(x) = \sqrt[4]{1 - x}, \quad g(x) = \sqrt{1 - \dfrac{x^2}{9}}$

17–22 ■ Use $f(x) = 3x - 5$ and $g(x) = 2 - x^2$ to evaluate the expression.

17. **(a)** $f(g(0))$ **(b)** $g(f(0))$

18. **(a)** $f(f(4))$ **(b)** $g(g(3))$

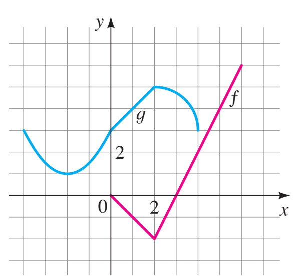
19. **(a)** $(f \circ g)(-2)$ **(b)** $(g \circ f)(-2)$

20. **(a)** $(f \circ f)(-1)$ **(b)** $(g \circ g)(2)$

21. **(a)** $(f \circ g)(x)$ **(b)** $(g \circ f)(x)$

22. **(a)** $(f \circ f)(x)$ **(b)** $(g \circ g)(x)$

23–28 ■ Use the given graphs of f and g to evaluate the expression.



23. $f(g(2))$

24. $g(f(0))$

25. $(g \circ f)(4)$

26. $(f \circ g)(0)$

27. $(g \circ g)(-2)$

28. $(f \circ f)(4)$

29–40 ■ Find the functions $f \circ g$, $g \circ f$, $f \circ f$, and $g \circ g$ and their domains.

29. $f(x) = 2x + 3, \quad g(x) = 4x - 1$

30. $f(x) = 6x - 5, \quad g(x) = \dfrac{x}{2}$

31. $f(x) = x^2, \quad g(x) = x + 1$

32. $f(x) = x^3 + 2, \quad g(x) = \sqrt[3]{x}$

33. $f(x) = \dfrac{1}{x}, \quad g(x) = 2x + 4$

34. $f(x) = x^2, \quad g(x) = \sqrt{x - 3}$

35. $f(x) = |x|, \quad g(x) = 2x + 3$

36. $f(x) = x - 4, \quad g(x) = |x + 4|$

37. $f(x) = \dfrac{x}{x + 1}, \quad g(x) = 2x - 1$

38. $f(x) = \dfrac{1}{\sqrt{x}}, \quad g(x) = x^2 - 4x$

39. $f(x) = \sqrt[3]{x}, \quad g(x) = \sqrt[4]{x}$

40. $f(x) = \dfrac{2}{x}, \quad g(x) = \dfrac{x}{x + 2}$

41–44 ■ Find $f \circ g \circ h$.

41. $f(x) = x - 1, \quad g(x) = \sqrt{x}, \quad h(x) = x - 1$

42. $f(x) = \dfrac{1}{x}, \quad g(x) = x^3, \quad h(x) = x^2 + 2$

43. $f(x) = x^4 + 1, \quad g(x) = x - 5, \quad h(x) = \sqrt{x}$

44. $f(x) = \sqrt{x}, \quad g(x) = \dfrac{x}{x - 1}, \quad h(x) = \sqrt[3]{x}$

45–50 ■ Express the function in the form $f \circ g$.

45. $F(x) = (x - 9)^5$

46. $F(x) = \sqrt{x} + 1$

57050_02_ch02_p146-247.qxd 07/04/2008 05:02 PM Page 221

47. $G(x) = \dfrac{x^2}{x^2 + 4}$

48. $G(x) = \dfrac{1}{x + 3}$

49. $H(x) = |1 - x^3|$

50. $H(x) = \sqrt{1 + \sqrt{x}}$

51–54 ■ Express the function in the form $f \circ g \circ h$.

51. $F(x) = \dfrac{1}{x^2 + 1}$

52. $F(x) = \sqrt[3]{\sqrt{x} - 1}$

53. $G(x) = (4 + \sqrt[3]{x})^9$

54. $G(x) = \dfrac{2}{(3 + \sqrt{x})^2}$

Applications

55–56 ■ **Revenue, Cost, and Profit** A print shop makes bumper stickers for election campaigns. If x stickers are ordered (where $x < 10{,}000$), then the price per sticker is $0.15 - 0.000002x$ dollars, and the total cost of producing the order is $0.095x - 0.0000005x^2$ dollars.

55. Use the fact that

revenue = price per item × number of items sold

to express $R(x)$, the revenue from an order of x stickers, as a product of two functions of x.

56. Use the fact that

profit = revenue − cost

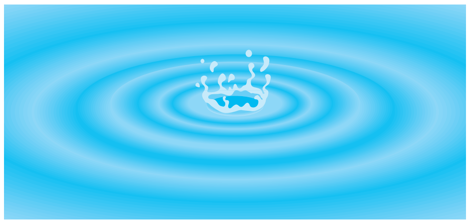
to express $P(x)$, the profit on an order of x stickers, as a difference of two functions of x.

57. **Area of a Ripple** A stone is dropped in a lake, creating a circular ripple that travels outward at a speed of 60 cm/s.

(a) Find a function g that models the radius as a function of time.

(b) Find a function f that models the area of the circle as a function of the radius.

(c) Find $f \circ g$. What does this function represent?



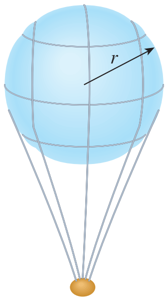
58. **Inflating a Balloon** A spherical balloon is being inflated. The radius of the balloon is increasing at the rate of 1 cm/s.

(a) Find a function f that models the radius as a function of time.

(b) Find a function g that models the volume as a function of the radius.

(c) Find $g \circ f$. What does this function represent?

59. **Area of a Balloon** A spherical weather balloon is being inflated. The radius of the balloon is increasing at the rate of 2 cm/s. Express the surface area of the balloon as a function of time t (in seconds).



60. **Multiple Discounts** You have a $50 coupon from the manufacturer good for the purchase of a cell phone. The store where you are purchasing your cell phone is offering a 20% discount on all cell phones. Let x represent the regular price of the cell phone.

(a) Suppose only the 20% discount applies. Find a function f that models the purchase price of the cell phone as a function of the regular price x.

(b) Suppose only the $50 coupon applies. Find a function g that models the purchase price of the cell phone as a function of the sticker price x.

57050_02_ch02_p146-247.qxd 07/04/2008 05:02 PM Page 222

(c) If you can use the coupon and the discount, then the purchase price is either $f \circ g(x)$ or $g \circ f(x)$, depending on the order in which they are applied to the price. Find both $f \circ g(x)$ and $g \circ f(x)$. Which composition gives the lower price?

61. Multiple Discounts An appliance dealer advertises a 10% discount on all his washing machines. In addition, the manufacturer offers a \$100 rebate on the purchase of a washing machine. Let x represent the sticker price of the washing machine.

(a) Suppose only the 10% discount applies. Find a function f that models the purchase price of the washer as a function of the sticker price x.

(b) Suppose only the \$100 rebate applies. Find a function g that models the purchase price of the washer as a function of the sticker price x.

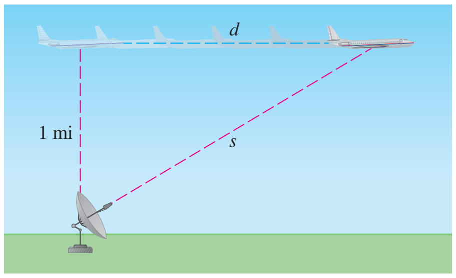
(c) Find $f \circ g$ and $g \circ f$. What do these functions represent? Which is the better deal?

62. Airplane Trajectory An airplane is flying at a speed of 350 mi/h at an altitude of one mile. The plane passes directly above a radar station at time $t = 0$.

(a) Express the distance s (in miles) between the plane and the radar station as a function of the horizontal distance d (in miles) that the plane has flown.

(b) Express d as a function of the time t (in hours) that the plane has flown.

(c) Use composition to express s as a function of t.



Discovery • Discussion

63. Compound Interest A savings account earns 5% interest compounded annually. If you invest x dollars in such an account, then the amount $A(x)$ of the investment after one year is the initial investment plus 5%; that is, $A(x) = x + 0.05x = 1.05x$. Find

$$A \circ A$$

$$A \circ A \circ A$$

$$A \circ A \circ A \circ A$$

What do these compositions represent? Find a formula for what you get when you compose n copies of A.

64. Composing Linear Functions The graphs of the functions

$$f(x) = m_1 x + b_1$$

$$g(x) = m_2 x + b_2$$

are lines with slopes m_1 and m_2, respectively. Is the graph of $f \circ g$ a line? If so, what is its slope?

65. Solving an Equation for an Unknown Function Suppose that

$$g(x) = 2x + 1$$

$$h(x) = 4x^2 + 4x + 7$$

Find a function f such that $f \circ g = h$. (Think about what operations you would have to perform on the formula for g to end up with the formula for h.) Now suppose that

$$f(x) = 3x + 5$$

$$h(x) = 3x^2 + 3x + 2$$

Use the same sort of reasoning to find a function g such that $f \circ g = h$.

66. Compositions of Odd and Even Functions Suppose that

$$h = f \circ g$$

If g is an even function, is h necessarily even? If g is odd, is h odd? What if g is odd and f is odd? What if g is odd and f is even?

57050_02_ch02_p146-247.qxd 07/04/2008 05:02 PM Page 223

DISCOVERY PROJECT

Iteration and Chaos

The **iterates** of a function f at a point x_0 are $f(x_0)$, $f(f(x_0))$, $f(f(f(x_0)))$, and so on. We write

$$x_1 = f(x_0) \qquad \text{The first iterate}$$

$$x_2 = f(f(x_0)) \qquad \text{The second iterate}$$

$$x_3 = f(f(f(x_0))) \qquad \text{The third iterate}$$

For example, if $f(x) = x^2$, then the iterates of f at 2 are $x_1 = 4$, $x_2 = 16$, $x_3 = 256$, and so on. (Check this.) Iterates can be described graphically as in Figure 1. Start with x_0 on the x-axis, move vertically to the graph of f, then horizontally to the line $y = x$, then vertically to the graph of f, and so on. The x-coordinates of the points on the graph of f are the iterates of f at x_0.

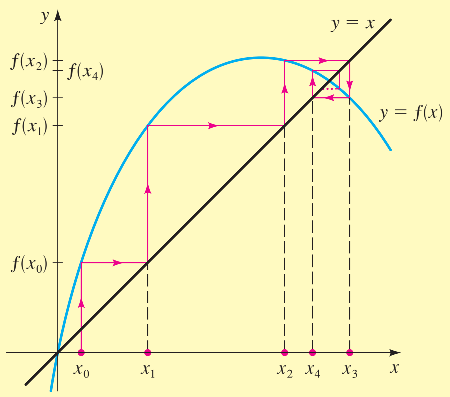


Figure 1

Iterates are important in studying the **logistic function**

$$f(x) = kx(1 - x)$$

n	x_n
0	0.1
1	0.234
2	0.46603
3	0.64700
4	0.59382
5	0.62712
6	0.60799
7	0.61968
8	0.61276
9	0.61694
10	0.61444
11	0.61595
12	0.61505

which models the population of a species with limited potential for growth (such as rabbits on an island or fish in a pond). In this model the maximum population that the environment can support is 1 (that is, 100%). If we start with a fraction of that population, say 0.1 (10%), then the iterates of f at 0.1 give the population after each time interval (days, months, or years, depending on the species). The constant k depends on the rate of growth of the species being modeled; it is called the **growth constant**. For example, for $k = 2.6$ and $x_0 = 0.1$ the iterates shown in the table to the left give the population of the species for the first 12 time intervals. The population seems to be stabilizing around 0.615 (that is, 61.5% of maximum).

In the three graphs in Figure 2, we plot the iterates of f at 0.1 for different values of the growth constant k. For $k = 2.6$ the population appears to stabilize at a value 0.615 of maximum, for $k = 3.1$ the population appears to oscillate

57050_02_ch02_p146-247.qxd 07/04/2008 05:02 PM Page 224

between two values, and for $k = 3.8$ no obvious pattern emerges. This latter situation is described mathematically by the word **chaos**.

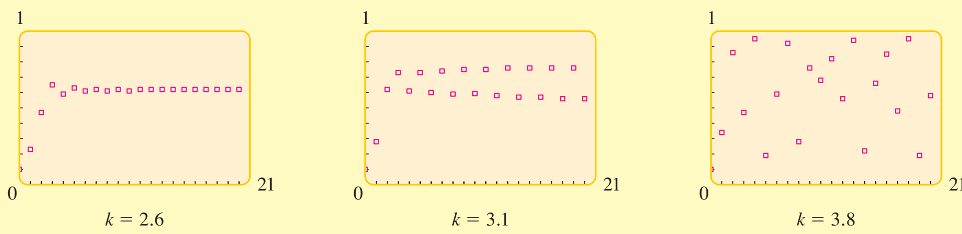


Figure 2

The following TI-83 program draws the first graph in Figure 2. The other graphs are obtained by choosing the appropriate value for K in the program.

```
PROGRAM:ITERATE
:ClrDraw
:2.6→K
:0.1→X
:For(N,1,20)
:K*X*(1-X)→Z
:Pt-On(N,Z,2)
:Z→X
:End
```

1. Use the graphical procedure illustrated in Figure 1 to find the first five iterates of $f(x) = 2x(1 - x)$ at $x = 0.1$.

2. Find the iterates of $f(x) = x^2$ at $x = 1$.

3. Find the iterates of $f(x) = 1/x$ at $x = 2$.

4. Find the first six iterates of $f(x) = 1/(1 - x)$ at $x = 2$. What is the 1000th iterate of f at 2?

5. Find the first 10 iterates of the logistic function at $x = 0.1$ for the given value of k. Does the population appear to stabilize, oscillate, or is it chaotic?

(a) $k = 2.1$ (b) $k = 3.2$ (c) $k = 3.9$

6. It's easy to find iterates using a graphing calculator. The following steps show how to find the iterates of $f(x) = kx(1 - x)$ at 0.1 for $k = 3$ on a TI-83 calculator. (The procedure can be adapted for any graphing calculator.)

Y₁ = K * X * (1 − X)	Enter f as Y_1 on the graph list
3 → K	Store 3 in the variable K
0.1 → X	Store 0.1 in the variable X
Y_1 → X	Evaluate f at X and store result back in X
0.27	Press ENTER and obtain first iterate
0.5913	Keep pressing ENTER to re-execute the command and obtain successive iterates
0.72499293	
0.59813454435	

You can also use the program in the margin to graph the iterates and study them visually.

Use a graphing calculator to experiment with how the value of k affects the iterates of $f(x) = kx(1 - x)$ at 0.1. Find several different values of k that make the iterates stabilize at one value, oscillate between two values, and exhibit chaos. (Use values of k between 1 and 4.) Can you find a value of k that makes the iterates oscillate between *four* values?

57050_02_ch02_p146-247.qxd 07/04/2008 05:03 PM Page 225

2.8 One-to-One Functions and Their Inverses

SUGGESTED TIME AND EMPHASIS
$1\text{–}1\frac{1}{2}$ classes.
Essential material.

The *inverse* of a function is a rule that acts on the output of the function and produces the corresponding input. So, the inverse "undoes" or reverses what the function has done. Not all functions have inverses; those that do are called *one-to-one*.

One-to-One Functions

Let's compare the functions f and g whose arrow diagrams are shown in Figure 1. Note that f never takes on the same value twice (any two numbers in A have different images), whereas g does take on the same value twice (both 2 and 3 have the same image, 4). In symbols, $g(2) = g(3)$ but $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$. Functions that have this latter property are called *one-to-one*.

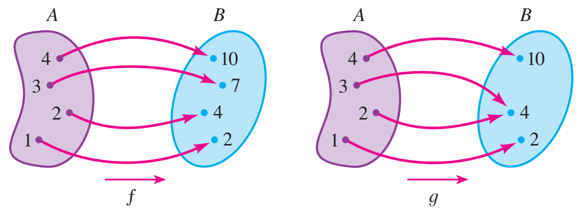


Figure 1 f is one-to-one $\quad$ g is not one-to-one

Definition of a One-to-one Function

A function with domain A is called a **one-to-one function** if no two elements of A have the same image, that is,

$$f(x_1) \neq f(x_2) \qquad \text{whenever } x_1 \neq x_2$$

An equivalent way of writing the condition for a one-to-one function is this:

$$\text{If } f(x_1) = f(x_2), \text{ then } x_1 = x_2.$$

If a horizontal line intersects the graph of f at more than one point, then we see from Figure 2 that there are numbers $x_1 \neq x_2$ such that $f(x_1) = f(x_2)$. This means that f is not one-to-one. Therefore, we have the following geometric method for determining whether a function is one-to-one.

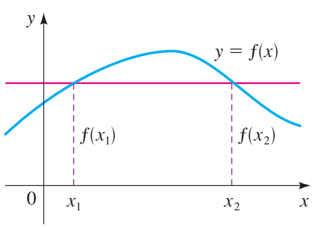


Figure 2
This function is not one-to-one because $f(x_1) = f(x_2)$.

Horizontal Line Test

A function is one-to-one if and only if no horizontal line intersects its graph more than once.

POINTS TO STRESS

1. One-to-one functions: their definition and the Horizontal Line Test.
2. Algebraic and geometric properties of inverse functions.
3. Finding inverse functions.

57050_02_ch02_p146-247.qxd 07/04/2008 05:03 PM Page 226

ALTERNATE EXAMPLE 1
Is the function $f(x) = 2x^3$ one-to-one?

ANSWER
Yes

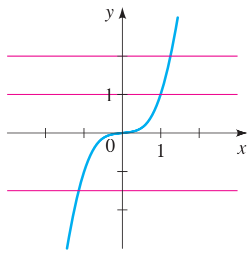


Figure 3
$f(x) = x^3$ is one-to-one.

Example 1 Deciding whether a Function Is One-to-One

Is the function $f(x) = x^3$ one-to-one?

Solution 1 If $x_1 \neq x_2$, then $x_1^3 \neq x_2^3$ (two different numbers cannot have the same cube). Therefore, $f(x) = x^3$ is one-to-one.

Solution 2 From Figure 3 we see that no horizontal line intersects the graph of $f(x) = x^3$ more than once. Therefore, by the Horizontal Line Test, f is one-to-one. ■

Notice that the function f of Example 1 is increasing and is also one-to-one. In fact, it can be proved that *every increasing function and every decreasing function is one-to-one*.

ALTERNATE EXAMPLE 2
Is the function $f(x) = x^2 - 3x + 2$ one-to-one?

ANSWER
Solution 1: The function is not one-to-one because, for instance, $f(1) = 0 = f(2)$ and so 1 and 2 have the same image.
Solution 2: When we graph $f(x)$, we see that it fails the horizontal line test.

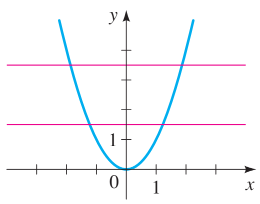


Figure 4
$f(x) = x^2$ is not one-to-one.

Example 2 Deciding whether a Function Is One-to-One

Is the function $g(x) = x^2$ one-to-one?

Solution 1 This function is not one-to-one because, for instance,

$$g(1) = 1 \quad \text{and} \quad g(-1) = 1$$

and so 1 and -1 have the same image.

Solution 2 From Figure 4 we see that there are horizontal lines that intersect the graph of g more than once. Therefore, by the Horizontal Line Test, g is not one-to-one. ■

Although the function g in Example 2 is not one-to-one, it is possible to restrict its domain so that the resulting function is one-to-one. In fact, if we define

$$h(x) = x^2, \qquad x \geq 0$$

then h is one-to-one, as you can see from Figure 5 and the Horizontal Line Test.

ALTERNATE EXAMPLE 3
If $f(3) = 1$, $f(4) = 5$, and $f(9) = -8$, find $f^{-1}(1)$, $f^{-1}(5)$, and $f^{-1}(-8)$.

ANSWER
3, 4, 9

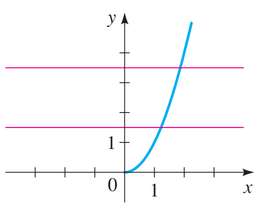


Figure 5
$f(x) = x^2 \quad (x \geq 0)$ is one-to-one.

Example 3 Showing That a Function Is One-to-One

Show that the function $f(x) = 3x + 4$ is one-to-one.

Solution
Suppose there are numbers x_1 and x_2 such that $f(x_1) = f(x_2)$. Then

$$
\begin{aligned}
3x_1 + 4 &= 3x_2 + 4 && \text{Suppose } f(x_1) = f(x_2) \\
3x_1 &= 3x_2 && \text{Subtract 4} \\
x_1 &= x_2 && \text{Divide by 3}
\end{aligned}
$$

Therefore, f is one-to-one. ■

The Inverse of a Function

One-to-one functions are important because they are precisely the functions that possess inverse functions according to the following definition.

IN-CLASS MATERIALS

Make sure students understand the notation: f^{-1} is not the same thing as $\frac{1}{f}$.

57050_02_ch02_p146-247.qxd 07/04/2008 05:03 PM Page 227

Don't mistake the -1 in f^{-1} for an exponent.

$$f^{-1} \quad \textit{does not mean} \quad \frac{1}{f(x)}$$

The reciprocal $1/f(x)$ is written as $(f(x))^{-1}$.

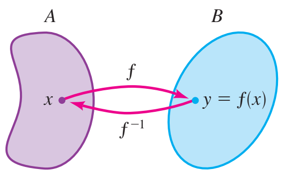


Figure 6

Definition of the Inverse of a Function

Let f be a one-to-one function with domain A and range B. Then its **inverse function** f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \quad \Leftrightarrow \quad f(x) = y$$

for any y in B.

This definition says that if f takes x into y, then f^{-1} takes y back into x. (If f were not one-to-one, then f^{-1} would not be defined uniquely.) The arrow diagram in Figure 6 indicates that f^{-1} reverses the effect of f. From the definition we have

$$\text{domain of } f^{-1} = \text{range of } f$$

$$\text{range of } f^{-1} = \text{domain of } f$$

Example 4 Finding f^{-1} for Specific Values

If $f(1) = 5$, $f(3) = 7$, and $f(8) = -10$, find $f^{-1}(5)$, $f^{-1}(7)$, and $f^{-1}(-10)$.

Solution From the definition of f^{-1} we have

$$f^{-1}(5) = 1 \quad \text{because} \quad f(1) = 5$$

$$f^{-1}(7) = 3 \quad \text{because} \quad f(3) = 7$$

$$f^{-1}(-10) = 8 \quad \text{because} \quad f(8) = -10$$

Figure 7 shows how f^{-1} reverses the effect of f in this case.

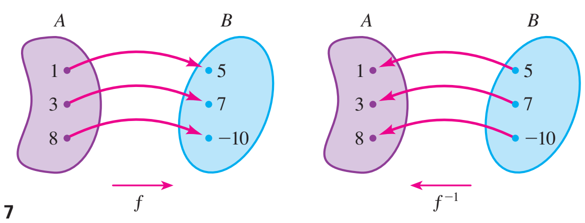


Figure 7 ■

By definition the inverse function f^{-1} undoes what f does: If we start with x, apply f, and then apply f^{-1}, we arrive back at x, where we started. Similarly, f undoes what f^{-1} does. In general, any function that reverses the effect of f in this way must be the inverse of f. These observations are expressed precisely as follows.

Inverse Function Property

Let f be a one-to-one function with domain A and range B. The inverse function f^{-1} satisfies the following cancellation properties.

$$f^{-1}(f(x)) = x \qquad \text{for every } x \text{ in } A$$

$$f(f^{-1}(x)) = x \qquad \text{for every } x \text{ in } B$$

Conversely, any function f^{-1} satisfying these equations is the inverse of f.

DRILL QUESTION

If $f(-2) = 4$, $f(-1) = 3$, $f(0) = 2$, $f(1) = 1$, and $f(2) = 3$, what is $f^{-1}(2)$?

Answer

0

ALTERNATE EXAMPLE 4

Are the functions $f(x) = x^5$ and $g(x) = x^{1/5}$ inverses of each other?

ANSWER

Yes

IN-CLASS MATERIALS

Starting with $f(x) = \sqrt[3]{x - 4}$ compute $f^{-1}(-2)$ and $f^{-1}(0)$. Then use algebra to find a formula for $f^{-1}(x)$. Have the class try to repeat the process with $g(x) = x^3 + x - 2$. Note that facts such as $g^{-1}(-2) = 0$, $g^{-1}(0) = 1$, and $g^{-1}(8) = 2$ can be found by looking at a table of values for $g(x)$ but that the algebraic approach fails to give us a general formula for $g^{-1}(x)$. Finally, draw graphs of f, f^{-1}, g, and g^{-1}.

57050_02_ch02_p146-247.qxd 07/04/2008 05:03 PM Page 228

ALTERNATE EXAMPLE 5
Find the inverse of the function $f(x) = 3x - 7$.

ANSWER
$y = \dfrac{x + 7}{3}$

These properties indicate that f is the inverse function of f^{-1}, so we say that f and f^{-1} are *inverses of each other*.

Example 5 Verifying That Two Functions Are Inverses

Show that $f(x) = x^3$ and $g(x) = x^{1/3}$ are inverses of each other.

Solution Note that the domain and range of both f and g is $\mathbb{R}$. We have

$$g(f(x)) = g(x^3) = (x^3)^{1/3} = x$$
$$f(g(x)) = f(x^{1/3}) = (x^{1/3})^3 = x$$

So, by the Property of Inverse Functions, f and g are inverses of each other. These equations simply say that the cube function and the cube root function, when composed, cancel each other. ■

Now let's examine how we compute inverse functions. We first observe from the definition of f^{-1} that

$$y = f(x) \quad \Leftrightarrow \quad f^{-1}(y) = x$$

So, if $y = f(x)$ and if we are able to solve this equation for x in terms of y, then we must have $x = f^{-1}(y)$. If we then interchange x and y, we have $y = f^{-1}(x)$, which is the desired equation.

How to Find the Inverse of a One-to-One Function

1. Write $y = f(x)$.
2. Solve this equation for x in terms of y (if possible).
3. Interchange x and y. The resulting equation is $y = f^{-1}(x)$.

Note that Steps 2 and 3 can be reversed. In other words, we can interchange x and y first and then solve for y in terms of x.

In Example 6 note how f^{-1} reverses the effect of f. The function f is the rule "multiply by 3, then subtract 2," whereas f^{-1} is the rule "add 2, then divide by 3."

ALTERNATE EXAMPLE 6
Find the inverse of the function $f(x) = 6x - 8$.

ANSWER
$f(x) = (x + 8)/6$

Example 6 Finding the Inverse of a Function

Find the inverse of the function $f(x) = 3x - 2$.

Solution First we write $y = f(x)$.

$$y = 3x - 2$$

Then we solve this equation for x:

$$3x = y + 2 \qquad \text{Add 2}$$
$$x = \frac{y + 2}{3} \qquad \text{Divide by 3}$$

Finally, we interchange x and y:

$$y = \frac{x + 2}{3}$$

Therefore, the inverse function is $f^{-1}(x) = \dfrac{x + 2}{3}$. ■

Check Your Answer

We use the Inverse Function Property.

$$f^{-1}(f(x)) = f^{-1}(3x - 2) = \frac{(3x - 2) + 2}{3} = \frac{3x}{3} = x$$

$$f(f^{-1}(x)) = f\left(\frac{x + 2}{3}\right) = 3\left(\frac{x + 2}{3}\right) - 2 = x + 2 - 2 = x \quad ✓$$

IN-CLASS MATERIALS

Pose the question: If f is always increasing, is f^{-1} always increasing? Give students time to try to prove their answer.

Answer

This is true. Proofs may involve diagrams and reflections about $y = x$, or you may try to get them to be more rigorous. This is an excellent opportunity to discuss concavity, noting that if f is concave up and increasing, then f^{-1} is concave down and increasing.

57050_02_ch02_p146-247.qxd 07/04/2008 05:03 PM Page 229

In Example 7 note how f^{-1} reverses the effect of f. The function f is the rule "take the fifth power, subtract 3, then divide by 2," whereas f^{-1} is the rule "multiply by 2, add 3, then take the fifth root."

Check Your Answer

We use the Inverse Function Property.

$$f^{-1}(f(x)) = f^{-1}\left(\frac{x^5 - 3}{2}\right)$$
$$= \left[2\left(\frac{x^5 - 3}{2}\right) + 3\right]^{1/5}$$
$$= (x^5 - 3 + 3)^{1/5}$$
$$= (x^5)^{1/5} = x$$

$$f(f^{-1}(x)) = f((2x + 3)^{1/5})$$
$$= \frac{[(2x + 3)^{1/5}]^5 - 3}{2}$$
$$= \frac{2x + 3 - 3}{2}$$
$$= \frac{2x}{2} = x \quad ✓$$

Example 7 Finding the Inverse of a Function

Find the inverse of the function $f(x) = \dfrac{x^5 - 3}{2}$.

Solution We first write $y = (x^5 - 3)/2$ and solve for x.

$$y = \frac{x^5 - 3}{2} \qquad \text{Equation defining function}$$
$$2y = x^5 - 3 \qquad \text{Multiply by 2}$$
$$x^5 = 2y + 3 \qquad \text{Add 3}$$
$$x = (2y + 3)^{1/5} \qquad \text{Take fifth roots}$$

Then we interchange x and y to get $y = (2x + 3)^{1/5}$. Therefore, the inverse function is $f^{-1}(x) = (2x + 3)^{1/5}$. ■

The principle of interchanging x and y to find the inverse function also gives us a method for obtaining the graph of f^{-1} from the graph of f. If $f(a) = b$, then $f^{-1}(b) = a$. Thus, the point (a, b) is on the graph of f if and only if the point (b, a) is on the graph of f^{-1}. But we get the point (b, a) from the point (a, b) by reflecting in the line $y = x$ (see Figure 8). Therefore, as Figure 9 illustrates, the following is true.

> The graph of f^{-1} is obtained by reflecting the graph of f in the line $y = x$.

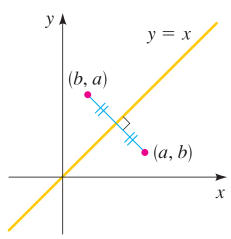


Figure 8

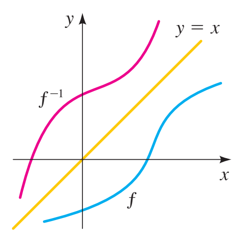


Figure 9

Example 8 Finding the Inverse of a Function

(a) Sketch the graph of $f(x) = \sqrt{x - 2}$.

(b) Use the graph of f to sketch the graph of f^{-1}.

(c) Find an equation for f^{-1}.

Solution

(a) Using the transformations from Section 2.4, we sketch the graph of $y = \sqrt{x - 2}$ by plotting the graph of the function $y = \sqrt{x}$ (Example 1(c) in Section 2.2) and moving it to the right 2 units.

(b) The graph of f^{-1} is obtained from the graph of f in part (a) by reflecting it in the line $y = x$, as shown in Figure 10.

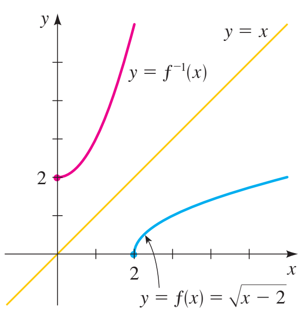


Figure 10

ALTERNATE EXAMPLE 7

Find the inverse of the function $f(x) = \dfrac{x^7 - 7}{6}$.

ANSWER

$y = (6x + 7)^{1/7}$

ALTERNATE EXAMPLE 8

Find the inverse function and the domain of the inverse function for the expression $f(x) = \sqrt{x - 6}$.

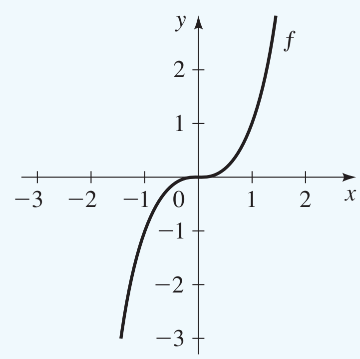
ANSWER

$[0, \infty), y = x^2 + 6$

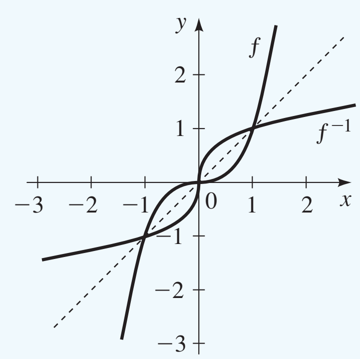
SAMPLE QUESTION

Text Question

Sketch the inverse function of the function graphed below.



Answer

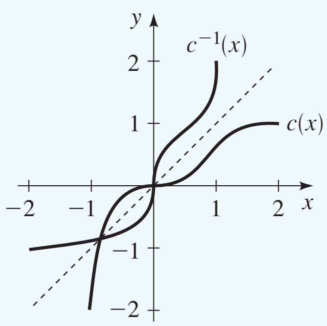


IN-CLASS MATERIALS

Make sure to discuss units carefully: when comparing $y = f(x)$ to $y = f^{-1}(x)$, the units of y and x trade places.

57050_02_ch02_p146-247.qxd 07/04/2008 05:03 PM Page 230

EXAMPLE
The graph of a complicated function and its inverse:



In Example 8 note how f^{-1} reverses the effect of f. The function f is the rule "subtract 2, then take the square root," whereas f^{-1} is the rule "square, then add 2."

(c) Solve $y = \sqrt{x-2}$ for x, noting that $y \geq 0$.

$$\sqrt{x-2} = y$$

$$x - 2 = y^2 \qquad \text{Square each side}$$

$$x = y^2 + 2, \quad y \geq 0 \qquad \text{Add 2}$$

Interchange x and y:

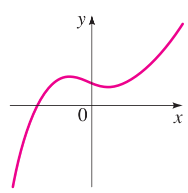
$$y = x^2 + 2, \qquad x \geq 0$$

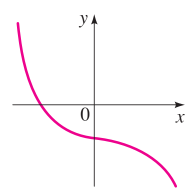
Thus $$f^{-1}(x) = x^2 + 2, \qquad x \geq 0$$

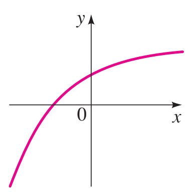
This expression shows that the graph of f^{-1} is the right half of the parabola $y = x^2 + 2$ and, from the graph shown in Figure 10, this seems reasonable. ■

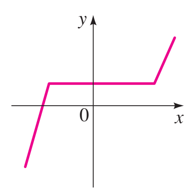
2.8 Exercises

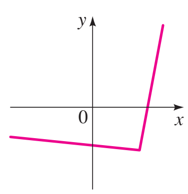
1–6 ■ The graph of a function f is given. Determine whether f is one-to-one.

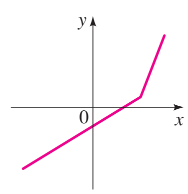
1.


2.


3.


4.


5.


6.


7–16 ■ Determine whether the function is one-to-one.

7. $f(x) = -2x + 4$ **8.** $f(x) = 3x - 2$

9. $g(x) = \sqrt{x}$ **10.** $g(x) = |x|$

11. $h(x) = x^2 - 2x$ **12.** $h(x) = x^3 + 8$

13. $f(x) = x^4 + 5$

14. $f(x) = x^4 + 5, \quad 0 \leq x \leq 2$

15. $f(x) = \dfrac{1}{x^2}$ **16.** $f(x) = \dfrac{1}{x}$

17–18 ■ Assume f is a one-to-one function.

17. **(a)** If $f(2) = 7$, find $f^{-1}(7)$.
(b) If $f^{-1}(3) = -1$, find $f(-1)$.

18. **(a)** If $f(5) = 18$, find $f^{-1}(18)$.
(b) If $f^{-1}(4) = 2$, find $f(2)$.

19. If $f(x) = 5 - 2x$, find $f^{-1}(3)$.

20. If $g(x) = x^2 + 4x$ with $x \geq -2$, find $g^{-1}(5)$.

21–30 ■ Use the Inverse Function Property to show that f and g are inverses of each other.

21. $f(x) = x - 6, \quad g(x) = x + 6$

22. $f(x) = 3x, \quad g(x) = \dfrac{x}{3}$

23. $f(x) = 2x - 5; \quad g(x) = \dfrac{x+5}{2}$

24. $f(x) = \dfrac{3-x}{4}; \quad g(x) = 3 - 4x$

25. $f(x) = \dfrac{1}{x}, \quad g(x) = \dfrac{1}{x}$

26. $f(x) = x^5, \quad g(x) = \sqrt[5]{x}$

27. $f(x) = x^2 - 4, \quad x \geq 0;$
$g(x) = \sqrt{x+4}, \quad x \geq -4$

IN-CLASS MATERIALS

Point out that the idea of "reversing input and output" permeates the idea of inverse functions, in all four representations of "function." When finding inverse functions algebraically, we explicitly reverse x and y. When drawing the inverse function of a graph, by reflecting across the line $y = x$ we are reversing the y- and x-axes. If $c(x)$ is the cost (in dollars) to make x fruit roll-ups, then $c^{-1}(x)$ is the number of fruit roll-ups that could be made for x dollars—again reversing the input and the output. Finally, show the class how to find the inverse of a function given a numeric data table, and note that again the inputs and outputs are reversed.

x	$f(x)$
1	3
2	4.2
3	5.7
4	8

x	$f^{-1}(x)$
3	1
4.2	2
5.7	3
8	4

57050_02_ch02_p146-247.qxd 07/04/2008 05:03 PM Page 231

28. $f(x) = x^3 + 1; \quad g(x) = (x - 1)^{1/3}$

29. $f(x) = \dfrac{1}{x - 1}, \quad x \neq 1;$

$g(x) = \dfrac{1}{x} + 1, \quad x \neq 0$

30. $f(x) = \sqrt{4 - x^2}, \quad 0 \le x \le 2;$

$g(x) = \sqrt{4 - x^2}, \quad 0 \le x \le 2$

31–50 ■ Find the inverse function of f.

31. $f(x) = 2x + 1$

32. $f(x) = 6 - x$

33. $f(x) = 4x + 7$

34. $f(x) = 3 - 5x$

35. $f(x) = \dfrac{x}{2}$

36. $f(x) = \dfrac{1}{x^2}, \quad x > 0$

37. $f(x) = \dfrac{1}{x + 2}$

38. $f(x) = \dfrac{x - 2}{x + 2}$

39. $f(x) = \dfrac{1 + 3x}{5 - 2x}$

40. $f(x) = 5 - 4x^3$

41. $f(x) = \sqrt{2 + 5x}$

42. $f(x) = x^2 + x, \quad x \ge -\frac{1}{2}$

43. $f(x) = 4 - x^2, \quad x \ge 0$

44. $f(x) = \sqrt{2x - 1}$

45. $f(x) = 4 + \sqrt[3]{x}$

46. $f(x) = (2 - x^3)^5$

47. $f(x) = 1 + \sqrt{1 + x}$

48. $f(x) = \sqrt{9 - x^2}, \quad 0 \le x \le 3$

49. $f(x) = x^4, \quad x \ge 0$

50. $f(x) = 1 - x^3$

51–54 ■ A function f is given.

(a) Sketch the graph of f.

(b) Use the graph of f to sketch the graph of f^{-1}.

(c) Find f^{-1}.

51. $f(x) = 3x - 6$

52. $f(x) = 16 - x^2, \quad x \ge 0$

53. $f(x) = \sqrt{x + 1}$

54. $f(x) = x^3 - 1$

55–60 ■ Draw the graph of f and use it to determine whether the function is one-to-one.

55. $f(x) = x^3 - x$

56. $f(x) = x^3 + x$

57. $f(x) = \dfrac{x + 12}{x - 6}$

58. $f(x) = \sqrt{x^3 - 4x + 1}$

59. $f(x) = |x| - |x - 6|$

60. $f(x) = x \cdot |x|$

61–64 ■ A one-to-one function is given.

(a) Find the inverse of the function.

(b) Graph both the function and its inverse on the same screen to verify that the graphs are reflections of each other in the line $y = x$.

61. $f(x) = 2 + x$

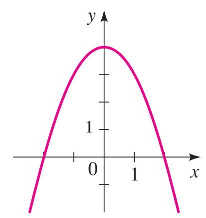
62. $f(x) = 2 - \frac{1}{2}x$

63. $g(x) = \sqrt{x + 3}$

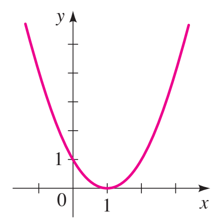
64. $g(x) = x^2 + 1, \quad x \ge 0$

65–68 ■ The given function is not one-to-one. Restrict its domain so that the resulting function *is* one-to-one. Find the inverse of the function with the restricted domain. (There is more than one correct answer.)

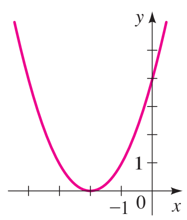
65. $f(x) = 4 - x^2$



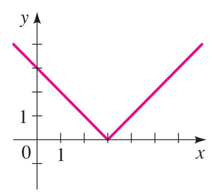
66. $g(x) = (x - 1)^2$



67. $h(x) = (x + 2)^2$

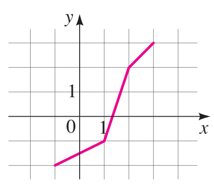


68. $k(x) = |x - 3|$

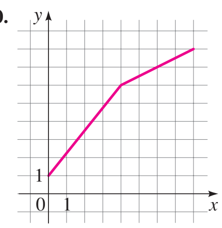


69–70 ■ Use the graph of f to sketch the graph of f^{-1}.

69.



70.



Applications

71. Fee for Service For his services, a private investigator requires a \$500 retention fee plus \$80 per hour. Let x represent the number of hours the investigator spends working on a case.

(a) Find a function f that models the investigator's fee as a function of x.

(b) Find f^{-1}. What does f^{-1} represent?

(c) Find $f^{-1}(1220)$. What does your answer represent?

57050_02_ch02_p146-247.qxd 07/04/2008 05:03 PM Page 232

72. Toricelli's Law A tank holds 100 gallons of water, which drains from a leak at the bottom, causing the tank to empty in 40 minutes. Toricelli's Law gives the volume of water remaining in the tank after t minutes as

$$V(t) = 100\left(1 - \frac{t}{40}\right)^2$$

(a) Find V^{-1}. What does V^{-1} represent?

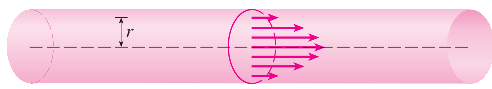
(b) Find $V^{-1}(15)$. What does your answer represent?

73. Blood Flow As blood moves through a vein or artery, its velocity v is greatest along the central axis and decreases as the distance r from the central axis increases (see the figure below). For an artery with radius 0.5 cm, v is given as a function of r by

$$v(r) = 18{,}500(0.25 - r^2)$$

(a) Find v^{-1}. What does v^{-1} represent?

(b) Find $v^{-1}(30)$. What does your answer represent?



74. Demand Function The amount of a commodity sold is called the *demand* for the commodity. The demand D for a certain commodity is a function of the price given by

$$D(p) = -3p + 150$$

(a) Find D^{-1}. What does D^{-1} represent?

(b) Find $D^{-1}(30)$. What does your answer represent?

75. Temperature Scales The relationship between the Fahrenheit (F) and Celsius (C) scales is given by

$$F(C) = \tfrac{9}{5}C + 32$$

(a) Find F^{-1}. What does F^{-1} represent?

(b) Find $F^{-1}(86)$. What does your answer represent?

76. Exchange Rates The relative value of currencies fluctuates every day. When this problem was written, one Canadian dollar was worth 0.8159 U.S. dollar.

(a) Find a function f that gives the U.S. dollar value $f(x)$ of x Canadian dollars.

(b) Find f^{-1}. What does f^{-1} represent?

(c) How much Canadian money would \$12,250 in U.S. currency be worth?

77. Income Tax In a certain country, the tax on incomes less than or equal to €20,000 is 10%. For incomes more than €20,000, the tax is €2000 plus 20% of the amount over €20,000.

(a) Find a function f that gives the income tax on an income x. Express f as a piecewise defined function.

(b) Find f^{-1}. What does f^{-1} represent?

(c) How much income would require paying a tax of €10,000?

78. Multiple Discounts A car dealership advertises a 15% discount on all its new cars. In addition, the manufacturer offers a \$1000 rebate on the purchase of a new car. Let x represent the sticker price of the car.

(a) Suppose only the 15% discount applies. Find a function f that models the purchase price of the car as a function of the sticker price x.

(b) Suppose only the \$1000 rebate applies. Find a function g that models the purchase price of the car as a function of the sticker price x.

(c) Find a formula for $H = f \circ g$.

(d) Find H^{-1}. What does H^{-1} represent?

(e) Find $H^{-1}(13{,}000)$. What does your answer represent?

79. Pizza Cost Marcello's Pizza charges a base price of \$7 for a large pizza, plus \$2 for each topping. Thus, if you order a large pizza with x toppings, the price of your pizza is given by the function $f(x) = 7 + 2x$. Find f^{-1}. What does the function f^{-1} represent?

Discovery • Discussion

80. Determining when a Linear Function Has an Inverse For the linear function $f(x) = mx + b$ to be one-to-one, what must be true about its slope? If it is one-to-one, find its inverse. Is the inverse linear? If so, what is its slope?

81. Finding an Inverse "In Your Head" In the margin notes in this section we pointed out that the inverse of a function can be found by simply reversing the operations that make up the function. For instance, in Example 6 we saw that the inverse of

$$f(x) = 3x - 2 \quad \text{is} \quad f^{-1}(x) = \frac{x+2}{3}$$

because the "reverse" of "multiply by 3 and subtract 2" is "add 2 and divide by 3." Use the same procedure to find the inverse of the following functions.

(a) $f(x) = \dfrac{2x+1}{5}$ **(b)** $f(x) = 3 - \dfrac{1}{x}$

(c) $f(x) = \sqrt{x^3 + 2}$ **(d)** $f(x) = (2x - 5)^3$

Now consider another function:

$$f(x) = x^3 + 2x + 6$$

Is it possible to use the same sort of simple reversal of operations to find the inverse of this function? If so, do it. If not, explain what is different about this function that makes this task difficult.

82. The Identity Function The function $I(x) = x$ is called the **identity function**. Show that for any function f we have $f \circ I = f$, $I \circ f = f$, and $f \circ f^{-1} = f^{-1} \circ f = I$. (This means that the identity function I behaves for functions and composition just like the number 1 behaves for real numbers and multiplication.)

83. Solving an Equation for an Unknown Function In Exercise 65 of Section 2.7 you were asked to solve equations in which the unknowns were functions. Now that we know about inverses and the identity function (see Exercise 82), we can use algebra to solve such equations. For instance, to solve $f \circ g = h$ for the unknown function f, we perform the following steps:

$f \circ g = h$	Problem: Solve for f
$f \circ g \circ g^{-1} = h \circ g^{-1}$	Compose with g^{-1} on the right
$f \circ I = h \circ g^{-1}$	$g \circ g^{-1} = I$
$f = h \circ g^{-1}$	$f \circ I = f$

So the solution is $f = h \circ g^{-1}$. Use this technique to solve the equation $f \circ g = h$ for the indicated unknown function.

(a) Solve for f, where $g(x) = 2x + 1$ and $h(x) = 4x^2 + 4x + 7$

(b) Solve for g, where $f(x) = 3x + 5$ and $h(x) = 3x^2 + 3x + 2$

2 Review

Concept Check

1. Define each concept in your own words. (Check by referring to the definition in the text.)
- **(a)** Function
- **(b)** Domain and range of a function
- **(c)** Graph of a function
- **(d)** Independent and dependent variables

2. Give an example of each type of function.
- **(a)** Constant function
- **(b)** Linear function
- **(c)** Quadratic function

3. Sketch by hand, on the same axes, the graphs of the following functions.
- **(a)** $f(x) = x$ **(b)** $g(x) = x^2$
- **(c)** $h(x) = x^3$ **(d)** $j(x) = x^4$

4. **(a)** State the Vertical Line Test.
- **(b)** State the Horizontal Line Test.

5. How is the average rate of change of the function f between two points defined?

6. Define each concept in your own words.
- **(a)** Increasing function
- **(b)** Decreasing function
- **(c)** Constant function

7. Suppose the graph of f is given. Write an equation for each graph that is obtained from the graph of f as follows.
- **(a)** Shift 3 units upward
- **(b)** Shift 3 units downward
- **(c)** Shift 3 units to the right
- **(d)** Shift 3 units to the left
- **(e)** Reflect in the x-axis
- **(f)** Reflect in the y-axis
- **(g)** Stretch vertically by a factor of 3
- **(h)** Shrink vertically by a factor of $\frac{1}{3}$
- **(i)** Stretch horizontally by a factor of 2
- **(j)** Shrink horizontally by a factor of $\frac{1}{2}$

8. **(a)** What is an even function? What symmetry does its graph possess? Give an example of an even function.
- **(b)** What is an odd function? What symmetry does its graph possess? Give an example of an odd function.

9. Write the standard form of a quadratic function.

10. What does it mean to say that $f(3)$ is a local maximum value of f?

11. Suppose that f has domain A and g has domain B.
- **(a)** What is the domain of $f + g$?
- **(b)** What is the domain of fg?
- **(c)** What is the domain of f/g?

57050_02_ch02_p146-247.qxd 07/04/2008 05:03 PM Page 234

12. How is the composite function $f \circ g$ defined?

13. (a) What is a one-to-one function?

(b) How can you tell from the graph of a function whether it is one-to-one?

(c) Suppose f is a one-to-one function with domain A and range B. How is the inverse function f^{-1} defined? What is the domain of f^{-1}? What is the range of f^{-1}?

(d) If you are given a formula for f, how do you find a formula for f^{-1}?

(e) If you are given the graph of f, how do you find the graph of f^{-1}?

Exercises

1. If $f(x) = x^2 - 4x + 6$, find $f(0)$, $f(2)$, $f(-2)$, $f(a)$, $f(-a)$, $f(x+1)$, $f(2x)$, and $2f(x) - 2$.

2. If $f(x) = 4 - \sqrt{3x - 6}$, find $f(5)$, $f(9)$, $f(a+2)$, $f(-x)$, $f(x^2)$, and $[f(x)]^2$.

3. The graph of a function f is given.

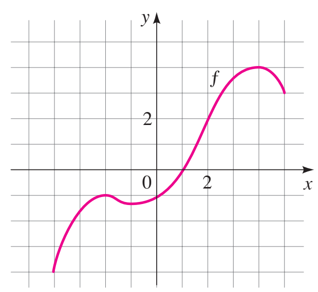
(a) Find $f(-2)$ and $f(2)$.

(b) Find the domain of f.

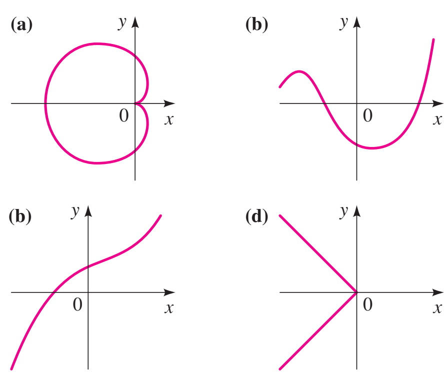
(c) Find the range of f.

(d) On what intervals is f increasing? On what intervals is f decreasing?

(e) Is f one-to-one?



4. Which of the following figures are graphs of functions? Which of the functions are one-to-one?



5–6 ■ Find the domain and range of the function.

5. $f(x) = \sqrt{x+3}$

6. $F(t) = t^2 + 2t + 5$

7–14 ■ Find the domain of the function.

7. $f(x) = 7x + 15$

8. $f(x) = \dfrac{2x+1}{2x-1}$

9. $f(x) = \sqrt{x+4}$

10. $f(x) = 3x - \dfrac{2}{\sqrt{x+1}}$

11. $f(x) = \dfrac{1}{x} + \dfrac{1}{x+1} + \dfrac{1}{x+2}$

12. $g(x) = \dfrac{2x^2 + 5x + 3}{2x^2 - 5x - 3}$

13. $h(x) = \sqrt{4-x} + \sqrt{x^2 - 1}$

14. $f(x) = \dfrac{\sqrt[3]{2x+1}}{\sqrt[3]{2x} + 2}$

15–32 ■ Sketch the graph of the function.

15. $f(x) = 1 - 2x$

16. $f(x) = \frac{1}{3}(x - 5), \; 2 \le x \le 8$

17. $f(t) = 1 - \frac{1}{2}t^2$

18. $g(t) = t^2 - 2t$

19. $f(x) = x^2 - 6x + 6$

20. $f(x) = 3 - 8x - 2x^2$

21. $g(x) = 1 - \sqrt{x}$

22. $g(x) = -|x|$

23. $h(x) = \frac{1}{2}x^3$

24. $h(x) = \sqrt{x+3}$

25. $h(x) = \sqrt[3]{x}$

26. $H(x) = x^3 - 3x^2$

27. $g(x) = \dfrac{1}{x^2}$

28. $G(x) = \dfrac{1}{(x-3)^2}$

29. $f(x) = \begin{cases} 1 - x & \text{if } x < 0 \\ 1 & \text{if } x \ge 0 \end{cases}$

30. $f(x) = \begin{cases} 1 - 2x & \text{if } x \le 0 \\ 2x - 1 & \text{if } x > 0 \end{cases}$

31. $f(x) = \begin{cases} x + 6 & \text{if } x < -2 \\ x^2 & \text{if } x \ge -2 \end{cases}$

32. $f(x) = \begin{cases} -x & \text{if } x < 0 \\ x^2 & \text{if } 0 \le x < 2 \\ 1 & \text{if } x \ge 2 \end{cases}$

33. Determine which viewing rectangle produces the most appropriate graph of the function $f(x) = 6x^3 - 15x^2 + 4x - 1$.

(i) $[-2, 2]$ by $[-2, 2]$

(ii) $[-8, 8]$ by $[-8, 8]$

(iii) $[-4, 4]$ by $[-12, 12]$

(iv) $[-100, 100]$ by $[-100, 100]$

57050_02_ch02_p146-247.qxd 07/04/2008 05:03 PM Page 235

34. Determine which viewing rectangle produces the most appropriate graph of the function $f(x) = \sqrt{100 - x^3}$.
(i) $[-4, 4]$ by $[-4, 4]$
(ii) $[-10, 10]$ by $[-10, 10]$
(iii) $[-10, 10]$ by $[-10, 40]$
(iv) $[-100, 100]$ by $[-100, 100]$

35–38 ■ Draw the graph of the function in an appropriate viewing rectangle.

35. $f(x) = x^2 + 25x + 173$

36. $f(x) = 1.1x^3 - 9.6x^2 - 1.4x + 3.2$

37. $f(x) = \dfrac{x}{\sqrt{x^2 + 16}}$

38. $f(x) = |x(x + 2)(x + 4)|$

39. Find, approximately, the domain of the function $f(x) = \sqrt{x^3 - 4x + 1}$.

40. Find, approximately, the range of the function $f(x) = x^4 - x^3 + x^2 + 3x - 6$.

41–44 ■ Find the average rate of change of the function between the given points.

41. $f(x) = x^2 + 3x;\quad x = 0, x = 2$

42. $f(x) = \dfrac{1}{x - 2};\quad x = 4, x = 8$

43. $f(x) = \dfrac{1}{x};\quad x = 3, x = 3 + h$

44. $f(x) = (x + 1)^2;\quad x = a, x = a + h$

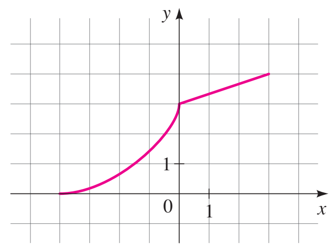
45–46 ■ Draw a graph of the function f, and determine the intervals on which f is increasing and on which f is decreasing.

45. $f(x) = x^3 - 4x^2$

46. $f(x) = |x^4 - 16|$

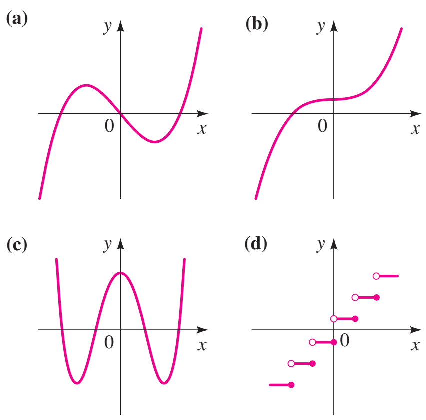
47. Suppose the graph of f is given. Describe how the graphs of the following functions can be obtained from the graph of f.
(a) $y = f(x) + 8$ **(b)** $y = f(x + 8)$
(c) $y = 1 + 2f(x)$ **(d)** $y = f(x - 2) - 2$
(e) $y = f(-x)$ **(f)** $y = -f(-x)$
(g) $y = -f(x)$ **(h)** $y = f^{-1}(x)$

48. The graph of f is given. Draw the graphs of the following functions.
(a) $y = f(x - 2)$ **(b)** $y = -f(x)$
(c) $y = 3 - f(x)$ **(d)** $y = \frac{1}{2}f(x) - 1$
(e) $y = f^{-1}(x)$ **(f)** $y = f(-x)$



49. Determine whether f is even, odd, or neither.
(a) $f(x) = 2x^5 - 3x^2 + 2$ **(b)** $f(x) = x^3 - x^7$
(c) $f(x) = \dfrac{1 - x^2}{1 + x^2}$ **(d)** $f(x) = \dfrac{1}{x + 2}$

50. Determine whether the function in the figure is even, odd, or neither.



51. Express the quadratic function $f(x) = x^2 + 4x + 1$ in standard form.

52. Express the quadratic function $f(x) = -2x^2 + 12x + 12$ in standard form.

53. Find the minimum value of the function $g(x) = 2x^2 + 4x - 5$.

54. Find the maximum value of the function $f(x) = 1 - x - x^2$.

57050_02_ch02_p146-247.qxd 07/04/2008 05:03 PM Page 236

55. A stone is thrown upward from the top of a building. Its height (in feet) above the ground after t seconds is given by $h(t) = -16t^2 + 48t + 32$. What maximum height does it reach?

56. The profit P (in dollars) generated by selling x units of a certain commodity is given by

$$P(x) = -1500 + 12x - 0.0004x^2$$

What is the maximum profit, and how many units must be sold to generate it?

57–58 ■ Find the local maximum and minimum values of the function and the values of x at which they occur. State each answer correct to two decimal places.

57. $f(x) = 3.3 + 1.6x - 2.5x^3$

58. $f(x) = x^{2/3}(6 - x)^{1/3}$

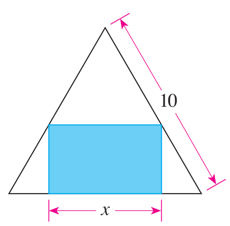
59. The number of air conditioners sold by an appliance store depends on the time of year. Sketch a rough graph of the number of A/C units sold as a function of the time of year.

60. An isosceles triangle has a perimeter of 8 cm. Express the area A of the triangle as a function of the length b of the base of the triangle.

61. A rectangle is inscribed in an equilateral triangle with a perimeter of 30 cm as in the figure.

(a) Express the area A of the rectangle as a function of the length x shown in the figure.

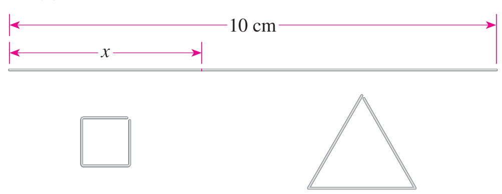
(b) Find the dimensions of the rectangle with the largest area.



62. A piece of wire 10 m long is cut into two pieces. One piece, of length x, is bent into the shape of a square. The other piece is bent into the shape of an equilateral triangle.

(a) Express the total area enclosed as a function of x.

(b) For what value of x is this total area a minimum?



63. If $f(x) = x^2 - 3x + 2$ and $g(x) = 4 - 3x$, find the following functions.

(a) $f + g$ **(b)** $f - g$ **(c)** fg
(d) f/g **(e)** $f \circ g$ **(f)** $g \circ f$

64. If $f(x) = 1 + x^2$ and $g(x) = \sqrt{x - 1}$, find the following.

(a) $f \circ g$ **(b)** $g \circ f$ **(c)** $(f \circ g)(2)$
(d) $(f \circ f)(2)$ **(e)** $f \circ g \circ f$ **(f)** $g \circ f \circ g$

65–66 ■ Find the functions $f \circ g$, $g \circ f$, $f \circ f$, and $g \circ g$ and their domains.

65. $f(x) = 3x - 1$, $g(x) = 2x - x^2$

66. $f(x) = \sqrt{x}$, $g(x) = \dfrac{2}{x - 4}$

67. Find $f \circ g \circ h$, where $f(x) = \sqrt{1 - x}$, $g(x) = 1 - x^2$, and $h(x) = 1 + \sqrt{x}$.

68. If $T(x) = \dfrac{1}{\sqrt{1 + \sqrt{x}}}$, find functions f, g, and h such that $f \circ g \circ h = T$.

69–74 ■ Determine whether the function is one-to-one.

69. $f(x) = 3 + x^3$

70. $g(x) = 2 - 2x + x^2$

71. $h(x) = \dfrac{1}{x^4}$

72. $r(x) = 2 + \sqrt{x + 3}$

73. $p(x) = 3.3 + 1.6x - 2.5x^3$

74. $q(x) = 3.3 + 1.6x + 2.5x^3$

75–78 ■ Find the inverse of the function.

75. $f(x) = 3x - 2$

76. $f(x) = \dfrac{2x + 1}{3}$

77. $f(x) = (x + 1)^3$

78. $f(x) = 1 + \sqrt[5]{x - 2}$

79. **(a)** Sketch the graph of the function

$$f(x) = x^2 - 4, \quad x \geq 0$$

(b) Use part (a) to sketch the graph of f^{-1}.

(c) Find an equation for f^{-1}.

80. **(a)** Show that the function $f(x) = 1 + \sqrt[4]{x}$ is one-to-one.

(b) Sketch the graph of f.

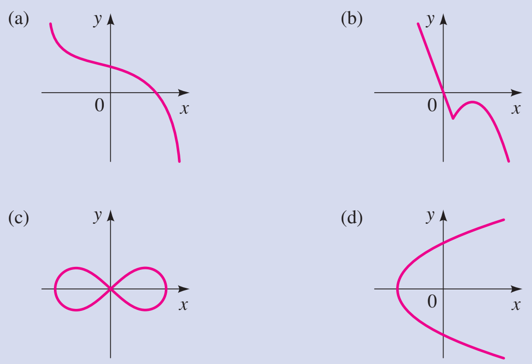
(c) Use part (b) to sketch the graph of f^{-1}.

(d) Find an equation for f^{-1}.

57050_02_ch02_p146-247.qxd 07/04/2008 05:03 PM Page 237

2 Test

1. Which of the following are graphs of functions? If the graph is that of a function, is it one-to-one?



2. Let $f(x) = \dfrac{\sqrt{x+1}}{x}$.

(a) Evaluate $f(3)$, $f(5)$, and $f(a-1)$.
(b) Find the domain of f.

3. Determine the average rate of change for the function $f(t) = t^2 - 2t$ between $t = 2$ and $t = 5$.

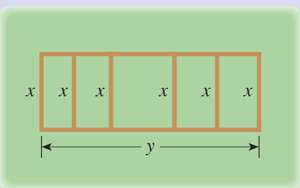
4. (a) Sketch the graph of the function $f(x) = x^3$.
(b) Use part (a) to graph the function $g(x) = (x-1)^3 - 2$.

5. (a) How is the graph of $y = f(x-3) + 2$ obtained from the graph of f?
(b) How is the graph of $y = f(-x)$ obtained from the graph of f?

6. (a) Write the quadratic function $f(x) = 2x^2 - 8x + 13$ in standard form.
(b) Sketch a graph of f.
(c) What is the minimum value of f?

7. Let $f(x) = \begin{cases} 1 - x^2 & \text{if } x \le 0 \\ 2x + 1 & \text{if } x > 0 \end{cases}$

(a) Evaluate $f(-2)$ and $f(1)$.
(b) Sketch the graph of f.



8. (a) If 1800 ft of fencing is available to build five adjacent pens, as shown in the diagram to the left, express the total area of the pens as a function of x.
(b) What value of x will maximize the total area?

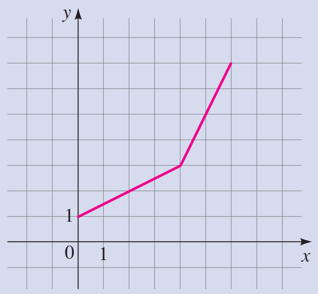
9. If $f(x) = x^2 + 1$ and $g(x) = x - 3$, find the following.

(a) $f \circ g$ (b) $g \circ f$
(c) $f(g(2))$ (d) $g(f(2))$
(e) $g \circ g \circ g$

57050_02_ch02_p146-247.qxd 07/04/2008 05:03 PM Page 238

10. (a) If $f(x) = \sqrt{3 - x}$, find the inverse function f^{-1}.
(b) Sketch the graphs of f and f^{-1} on the same coordinate axes.

11. The graph of a function f is given.
(a) Find the domain and range of f.
(b) Sketch the graph of f^{-1}.
(c) Find the average rate of change of f between $x = 2$ and $x = 6$.



12. Let $f(x) = 3x^4 - 14x^2 + 5x - 3$.
(a) Draw the graph of f in an appropriate viewing rectangle.
(b) Is f one-to-one?
(c) Find the local maximum and minimum values of f and the values of x at which they occur. State each answer correct to two decimal places.
(d) Use the graph to determine the range of f.
(e) Find the intervals on which f is increasing and on which f is decreasing.

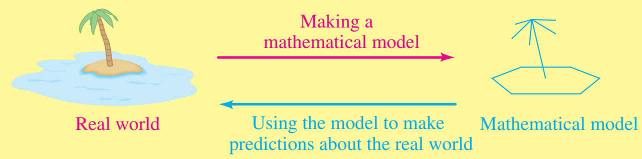
57050_02_ch02_p146-247.qxd 07/04/2008 05:03 PM Page 239

Focus on Modeling

Fitting Lines to Data

A model is a representation of an object or process. For example, a toy Ferrari is a *model* of the actual car; a road map is a model of the streets and highways in a city. A model usually represents just one aspect of the original thing. The toy Ferrari is not an actual car, but it does represent what a real Ferrari looks like; a road map does not contain the actual streets in a city, but it does represent the relationship of the streets to each other.

A **mathematical model** is a mathematical representation of an object or process. Often a mathematical model is a function that describes a certain phenomenon. In Example 12 of Section 1.10 we found that the function $T = -10h + 20$ models the atmospheric temperature T at elevation h. We then used this function to predict the temperature at a certain height. The figure below illustrates the process of mathematical modeling.



Mathematical models are useful because they enable us to isolate critical aspects of the thing we are studying and then to predict how it will behave. Models are used extensively in engineering, industry, and manufacturing. For example, engineers use computer models of skyscrapers to predict their strength and how they would behave in an earthquake. Aircraft manufacturers use elaborate mathematical models to predict the aerodynamic properties of a new design *before* the aircraft is actually built.

How are mathematical models developed? How are they used to predict the behavior of a process? In the next few pages and in subsequent *Focus on Modeling* sections, we explain how mathematical models can be constructed from real-world data, and we describe some of their applications.

Linear Equations as Models

The data in Table 1 were obtained by measuring pressure at various ocean depths. From the table it appears that pressure increases with depth. To see this trend better, we make a **scatter plot** as in Figure 1. It appears that the data lie more or less along a line. We can try to fit a line visually to approximate the points in the scatter plot (see Figure 2),

Table 1

Depth (ft)	Pressure (lb/in^2)
5	15.5
8	20.3
12	20.7
15	20.8
18	23.2
22	23.8
25	24.9
30	29.3

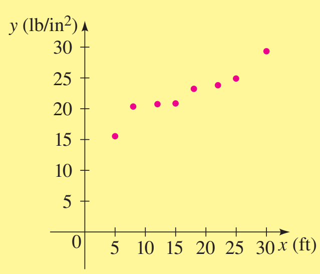


Figure 1
Scatter plot

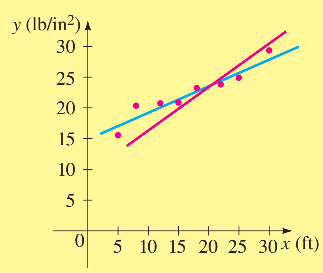


Figure 2
Attempts to fit line to data visually

57050_02_ch02_p146-247.qxd 07/04/2008 05:03 PM Page 240

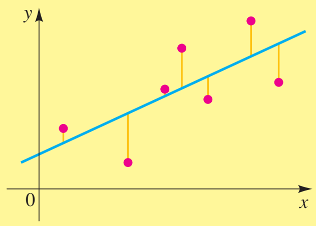


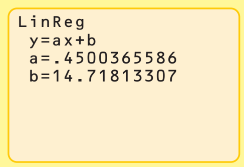
Figure 3
Distances from the points to the line

but this method is not accurate. So how do we find the line that fits the data as best as possible?

It seems reasonable to choose the line that is as close as possible to all the data points. This is the line for which the sum of the distances from the data points to the line is as small as possible (see Figure 3). For technical reasons it is better to find the line where the sum of the squares of these distances is smallest. The resulting line is called the **regression line**. The formula for the regression line is found using calculus. Fortunately, this formula is programmed into most graphing calculators. Using a calculator (see Figure 4(a)), we find that the regression line for the depth-pressure data in Table 1 is

$$P = 0.45d + 14.7 \qquad \text{Model}$$

The regression line and the scatter plot are graphed in Figure 4(b).



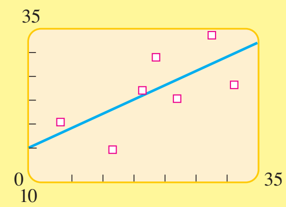


Figure 4
Linear regression on a graphing calculator

(a) Output of the `LinReg` command on a TI-83 calculator

(b) Scatter plot and regression line for depth-pressure data

Example 1 Olympic Pole Vaults

Table 2 gives the men's Olympic pole vault records up to 2004.

(a) Find the regression line for the data.

(b) Make a scatter plot of the data and graph the regression line. Does the regression line appear to be a suitable model for the data?

(c) Use the model to predict the winning pole vault height for the 2008 Olympics.

Table 2

Year	Gold medalist	Height (m)	Year	Gold medalist	Height (m)
1896	William Hoyt, USA	3.30	1956	Robert Richards USA	4.56
1900	Irving Baxter, USA	3.30	1960	Don Bragg, USA	4.70
1904	Charles Dvorak, USA	3.50	1964	Fred Hansen, USA	5.10
1906	Fernand Gonder, France	3.50	1968	Bob Seagren, USA	5.40
1908	A. Gilbert, E. Cook, USA	3.71	1972	W. Nordwig, E. Germany	5.64
1912	Harry Babcock, USA	3.95	1976	Tadeusz Slusarski, Poland	5.64
1920	Frank Foss, USA	4.09	1980	W. Kozakiewicz, Poland	5.78
1924	Lee Barnes, USA	3.95	1984	Pierre Quinon, France	5.75
1928	Sabin Carr, USA	4.20	1988	Sergei Bubka, USSR	5.90
1932	William Miller, USA	4.31	1992	M. Tarassob, Unified Team	5.87
1936	Earle Meadows, USA	4.35	1996	Jean Jalfione, France	5.92
1948	Guinn Smith, USA	4.30	2000	Nick Hysong, USA	5.90
1952	Robert Richards, USA	4.55	2004	Timothy Mack, USA	5.95

57050_02_ch02_p146-247.qxd 07/04/2008 05:03 PM Page 241


```
LinReg
 y=ax+b
 a=.0265652857
 b=3.400989881
```

Output of the `LinReg` function on the TI-83 Plus

Alexandr Satinsky/AFP/Getty Images

Solution

(a) Let x = year – 1900, so that 1896 corresponds to $x = -4$, 1900 to $x = 0$, and so on. Using a calculator, we find the regression line:

$$y = 0.0266x + 3.40$$

(b) The scatter plot and the regression line are shown in Figure 5. The regression line appears to be a good model for the data.

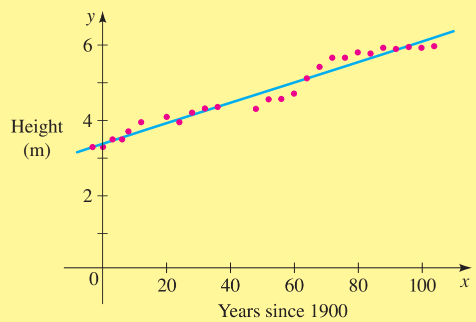


Figure 5
Scatter plot and regression line for pole-vault data

(c) The year 2008 corresponds to $x = 108$ in our model. The model gives

$$y = 0.0266(108) + 3.40 \approx 6.27 \text{ m}$$ ■

If you are reading this after the 2008 Olympics, look up the actual record for 2008 and compare with this prediction. Such predictions are reasonable for points close to our measured data, but we can't predict too far away from the measured data. Is it reasonable to use this model to predict the record 100 years from now?

Example 2 Asbestos Fibers and Cancer

When laboratory rats are exposed to asbestos fibers, some of them develop lung tumors. Table 3 lists the results of several experiments by different scientists.

(a) Find the regression line for the data.

(b) Make a scatter plot of the data and graph the regression line. Does the regression line appear to be a suitable model for the data?

Eric & David Hosking/Corbis

Table 3

Asbestos exposure (fibers/mL)	Percent that develop lung tumors
50	2
400	6
500	5
900	10
1100	26
1600	42
1800	37
2000	28
3000	50

57050_02_ch02_p146-247.qxd 07/04/2008 05:03 PM Page 242

Solution

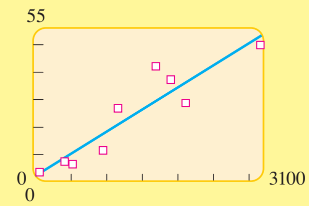
(a) Using a calculator, we find the regression line (see Figure 6(a)):

$$y = 0.0177x + 0.5405$$

(b) The scatter plot and the regression line are shown in Figure 6(b). The regression line appears to be a reasonable model for the data.

```
LinReg
 y=ax+b
 a=.0177212141
 b=.5404689256
```

(a) Output of the `LinReg` command on a TI-83 calculator



(b) Scatter plot and regression line

Figure 6
Linear regression for the asbestos-tumor data

■

How Good Is the Fit?

For any given set of data it is always possible to find the regression line, even if the data do not tend to lie along a line. Consider the three scatter plots in Figure 7.

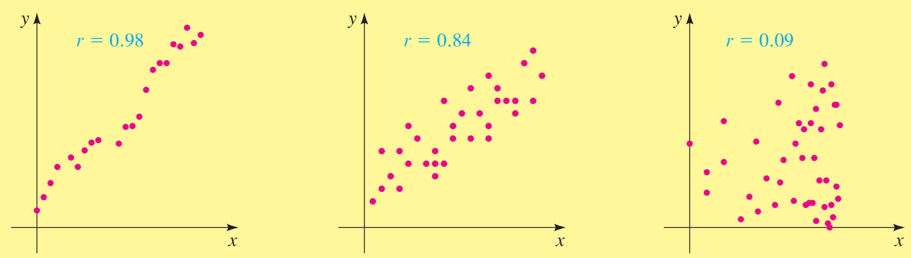


Figure 7

The data in the first scatter plot appear to lie along a line. In the second plot they also appear to display a linear trend, but it seems more scattered. The third does not have a discernible trend. We can easily find the regression lines for each scatter plot using a graphing calculator. But how well do these lines represent the data? The calculator gives a **correlation coefficient** r, which is a statistical measure of how well the data lie along the regression line, or how well the two variables are **correlated**. The correlation coefficient is a number between -1 and 1. A correlation coefficient r close to 1 or -1 indicates strong correlation and a coefficient close to 0 indicates very little correlation; the slope of the line determines whether the correlation coefficient is positive or negative. Also, the more data points we have, the more meaningful the correlation coefficient will be. Using a calculator we find that the correlation coefficient between asbestos fibers and lung tumors in the rats of Example 2 is $r = 0.92$. We can reasonably conclude that the presence of asbestos and the risk of lung tumors in rats are related. Can we conclude that asbestos *causes* lung tumors in rats?

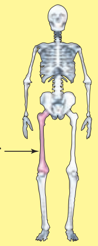
If two variables are correlated, it does not necessarily mean that a change in one variable *causes* a change in the other. For example, the mathematician John Allen Paulos points out that shoe size is strongly correlated to mathematics scores among school children. Does this mean that big feet cause high math scores? Certainly

57050_02_ch02_p146-247.qxd 09/04/2008 10:07 AM Page 243

not—both shoe size and math skills increase independently as children get older. So it is important not to jump to conclusions: Correlation and causation are not the same thing. Correlation is a useful tool in bringing important cause-and-effect relationships to light, but to prove causation, we must explain the mechanism by which one variable affects the other. For example, the link between smoking and lung cancer was observed as a correlation long before science found the mechanism through which smoking causes lung cancer.

Problems

1. **Femur Length and Height** Anthropologists use a linear model that relates femur length to height. The model allows an anthropologist to determine the height of an individual when only a partial skeleton (including the femur) is found. In this problem we find the model by analyzing the data on femur length and height for the eight males given in the table.
   (a) Make a scatter plot of the data.
   (b) Find and graph a linear function that models the data.
   (c) An anthropologist finds a femur of length 58 cm. How tall was the person?



Femur length (cm)	Height (cm)
50.1	178.5
48.3	173.6
45.2	164.8
44.7	163.7
44.5	168.3
42.7	165.0
39.5	155.4
38.0	155.8

2. **Demand for Soft Drinks** A convenience store manager notices that sales of soft drinks are higher on hotter days, so he assembles the data in the table.
   (a) Make a scatter plot of the data.
   (b) Find and graph a linear function that models the data.
   (c) Use the model to predict soft-drink sales if the temperature is 95 °F.

High temperature (°F)	Number of cans sold
55	340
58	335
64	410
68	460
70	450
75	610
80	735
84	780

3. **Tree Diameter and Age** To estimate ages of trees, forest rangers use a linear model that relates tree diameter to age. The model is useful because tree diameter is much easier to measure than tree age (which requires special tools for extracting a representative

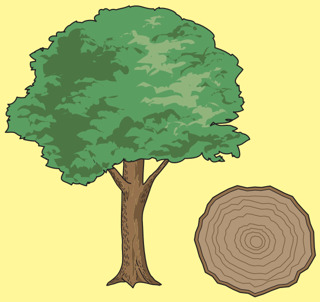
57050_02_ch02_p146-247.qxd 07/04/2008 05:03 PM Page 244

cross section of the tree and counting the rings). To find the model, use the data in the table collected for a certain variety of oaks.

(a) Make a scatter plot of the data.

(b) Find and graph a linear function that models the data.

(c) Use the model to estimate the age of an oak whose diameter is 18 in.



Diameter (in.)	Age (years)
2.5	15
4.0	24
6.0	32
8.0	56
9.0	49
9.5	76
12.5	90
15.5	89

Year	CO_2 level (ppm)
1984	344.3
1986	347.0
1988	351.3
1990	354.0
1992	356.3
1994	358.9
1996	362.7
1998	366.5
2000	369.4

4. Carbon Dioxide Levels The table lists average carbon dioxide (CO_2) levels in the atmosphere, measured in parts per million (ppm) at Mauna Loa Observatory from 1984 to 2000.

(a) Make a scatter plot of the data.

(b) Find and graph the regression line.

(c) Use the linear model in part (b) to estimate the CO_2 level in the atmosphere in 2001. Compare your answer with the actual CO_2 level of 371.1 measured in 2001.

5. Temperature and Chirping Crickets Biologists have observed that the chirping rate of crickets of a certain species appears to be related to temperature. The table shows the chirping rates for various temperatures.

(a) Make a scatter plot of the data.

(b) Find and graph the regression line.

(c) Use the linear model in part (b) to estimate the chirping rate at 100°F.

Temperature (°F)	Chirping rate (chirps/min)
50	20
55	46
60	79
65	91
70	113
75	140
80	173
85	198
90	211

Income	Ulcer rate
$4,000	14.1
$6,000	13.0
$8,000	13.4
$12,000	12.4
$16,000	12.0
$20,000	12.5
$30,000	10.5
$45,000	9.4
$60,000	8.2

6. Ulcer Rates The table in the margin shows (lifetime) peptic ulcer rates (per 100 population) for various family incomes as reported by the 1989 National Health Interview Survey.

(a) Make a scatter plot of the data.

(b) Find and graph the regression line.

57050_02_ch02_p146-247.qxd 09/04/2008 10:07 AM Page 245

Mathematics in the Modern World

Ed Kashi/Corbis

Model Airplanes

When we think of the word "model," we often think of a model car or a model airplane. In fact, this everyday use of the word *model* corresponds to its use in mathematics. A model usually represents a certain aspect of the original thing. So a model airplane represents what the real airplane looks like. Before the 1980s airplane manufacturers built full scale mock-ups of new airplane designs to test their aerodynamic properties. Today, manufacturers "build" mathematical models of airplanes, which are stored in the memory of computers. The aerodynamic properties of "mathematical airplanes" correspond to those of real planes, but the mathematical planes can be flown and tested without leaving the computer memory!

(c) Estimate the peptic ulcer rate for an income level of \$25,000 according to the linear model in part (b).

(d) Estimate the peptic ulcer rate for an income level of \$80,000 according to the linear model in part (b).

7. Mosquito Prevalence The table lists the relative abundance of mosquitoes (as measured by the mosquito positive rate) versus the flow rate (measured as a percentage of maximum flow) of canal networks in Saga City, Japan.

(a) Make a scatter plot of the data.

(b) Find and graph the regression line.

(c) Use the linear model in part (b) to estimate the mosquito positive rate if the canal flow is 70% of maximum.

Flow rate (%)	Mosquito positive rate (%)
0	22
10	16
40	12
60	11
90	6
100	2

8. Noise and Intelligibility Audiologists study the intelligibility of spoken sentences under different noise levels. Intelligibility, the MRT score, is measured as the percent of a spoken sentence that the listener can decipher at a certain noise level in decibels (dB). The table shows the results of one such test.

(a) Make a scatter plot of the data.

(b) Find and graph the regression line.

(c) Find the correlation coefficient. Is a linear model appropriate?

(d) Use the linear model in part (b) to estimate the intelligibility of a sentence at a 94-dB noise level.

Noise level (dB)	MRT score (%)
80	99
84	91
88	84
92	70
96	47
100	23
104	11

9. Life Expectancy The average life expectancy in the United States has been rising steadily over the past few decades, as shown in the table.

(a) Make a scatter plot of the data.

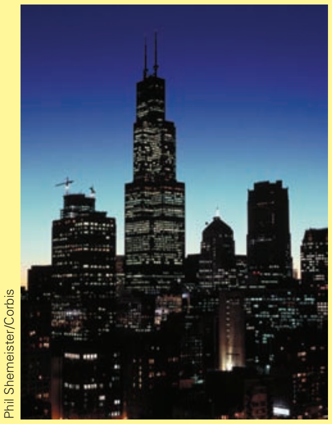
(b) Find and graph the regression line.

(c) Use the linear model you found in part (b) to predict the life expectancy in the year 2004.

(d) Search the Internet or your campus library to find the actual 2004 average life expectancy. Compare to your answer in part (c).

Year	Life expectancy
1920	54.1
1930	59.7
1940	62.9
1950	68.2
1960	69.7
1970	70.8
1980	73.7
1990	75.4
2000	76.9

57050_02_ch02_p146-247.qxd 07/04/2008 05:04 PM Page 246



Phil Shemeister/Corbis

10. Heights of Tall Buildings The table gives the heights and number of stories for 11 tall buildings.

(a) Make a scatter plot of the data.

(b) Find and graph the regression line.

(c) What is the slope of your regression line? What does its value indicate?

Building	Height (ft)	Stories
Empire State Building, New York	1250	102
One Liberty Place, Philadelphia	945	61
Canada Trust Tower, Toronto	863	51
Bank of America Tower, Seattle	943	76
Sears Tower, Chicago	1450	110
Petronas Tower I, Malaysia	1483	88
Commerzbank Tower, Germany	850	60
Palace of Culture and Science, Poland	758	42
Republic Plaza, Singapore	919	66
Transamerica Pyramid, San Francisco	853	48
Taipei 101 Building, Taiwan	1679	101

11. Olympic Swimming Records The tables give the gold medal times in the men's and women's 100-m freestyle Olympic swimming event.

(a) Find the regression lines for the men's data and the women's data.

(b) Sketch both regression lines on the same graph. When do these lines predict that the women will overtake the men in the event? Does this conclusion seem reasonable?

MEN

Year	Gold medalist	Time (s)
1908	C. Daniels, USA	65.6
1912	D. Kahanamoku, USA	63.4
1920	D. Kahanamoku, USA	61.4
1924	J. Weissmuller, USA	59.0
1928	J. Weissmuller, USA	58.6
1932	Y. Miyazaki, Japan	58.2
1936	F. Csik, Hungary	57.6
1948	W. Ris, USA	57.3
1952	C. Scholes, USA	57.4
1956	J. Henricks, Australia	55.4
1960	J. Devitt, Australia	55.2
1964	D. Schollander, USA	53.4
1968	M. Wenden, Australia	52.2
1972	M. Spitz, USA	51.22
1976	J. Montgomery, USA	49.99
1980	J. Woithe, E. Germany	50.40
1984	R. Gaines, USA	49.80
1988	M. Biondi, USA	48.63
1992	A. Popov, Russia	49.02
1996	A. Popov, Russia	48.74
2000	P. van den Hoogenband, Netherlands	48.30
2004	P. van den Hoogenband, Netherlands	48.17

WOMEN

Year	Gold medalist	Time (s)
1912	F. Durack, Australia	82.2
1920	E. Bleibtrey, USA	73.6
1924	E. Lackie, USA	72.4
1928	A. Osipowich, USA	71.0
1932	H. Madison, USA	66.8
1936	H. Mastenbroek, Holland	65.9
1948	G. Andersen, Denmark	66.3
1952	K. Szoke, Hungary	66.8
1956	D. Fraser, Australia	62.0
1960	D. Fraser, Australia	61.2
1964	D. Fraser, Australia	59.5
1968	J. Henne, USA	60.0
1972	S. Nielson, USA	58.59
1976	K. Ender, E. Germany	55.65
1980	B. Krause, E. Germany	54.79
1984	(Tie) C. Steinseifer, USA	55.92
	N. Hogshead, USA	55.92
1988	K. Otto, E. Germany	54.93
1992	Z. Yong, China	54.64
1996	L. Jingyi, China	54.50
2000	I. DeBruijn, Netherlands	53.83
2004	J. Henry, Australia	53.84

57050_02_ch02_p146-247.qxd 09/04/2008 10:07 AM Page 247

12. Parent Height and Offspring Height In 1885 Sir Francis Galton compared the height of children to the height of their parents. His study is considered one of the first uses of regression. The table gives some of Galton's original data. The term "midparent height" means the average of the heights of the father and mother.

(a) Find a linear equation that models the data.

(b) How well does the model predict your own height (based on your parents' heights)?

Midparent height (in.)	Offspring height (in.)
64.5	66.2
65.5	66.2
66.5	67.2
67.5	69.2
68.5	67.2
68.5	69.2
69.5	71.2
69.5	70.2
70.5	69.2
70.5	70.2
72.5	72.2
73.5	73.2

13. Shoe Size and Height Do you think that shoe size and height are correlated? Find out by surveying the shoe sizes and heights of people in your class. (Of course, the data for men and women should be separate.) Find the correlation coefficient.

Would you buy a candy bar from the vending machine in the hallway if the price is as indicated?

Price	Yes or No
30¢	
40¢	
50¢	
60¢	
70¢	
80¢	
90¢	
$1.00	
$1.10	
$1.20	

14. Demand for Candy Bars In this problem you will determine a linear demand equation that describes the demand for candy bars in your class. Survey your classmates to determine what price they would be willing to pay for a candy bar. Your survey form might look like the sample to the left.

(a) Make a table of the number of respondents who answered "yes" at each price level.

(b) Make a scatter plot of your data.

(c) Find and graph the regression line $y = mp + b$, which gives the number of responents y who would buy a candy bar if the price were p cents. This is the *demand equation*. Why is the slope m negative?

(d) What is the p-intercept of the demand equation? What does this intercept tell you about pricing candy bars?