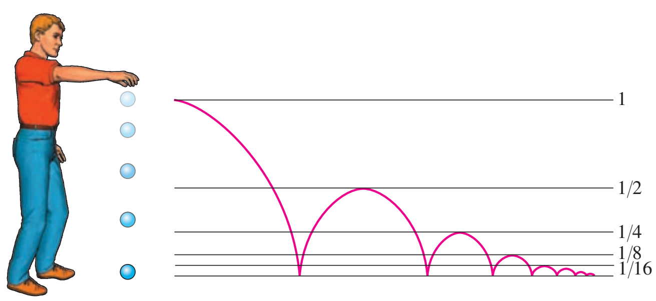
11 Sequences and Series

Chapter Overview

In this chapter we study sequences and series of numbers. Roughly speaking, a sequence is a list of numbers written in a specific order. The numbers in the sequence are often written as $a_1, a_2, a_3, \ldots$. The dots mean that the list continues forever. A simple example is the sequence

$$\begin{array}{ccccc} 5, & 10, & 15, & 20, & 25, \ldots \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ a_1 & a_2 & a_3 & a_4 & a_5 \ldots \end{array}$$

Sequences arise in many real-world situations. For example, if you deposit a sum of money into an interest-bearing account, the interest earned each month forms a sequence. If you drop a ball and let it bounce, the height the ball reaches at each successive bounce is a sequence. An interesting sequence is hidden in the internal structure of a nautilus shell.



Susan Van Etten/PhotoEdit

We can describe the pattern of the sequence displayed above by the *formula*:

$$a_n = 5n$$

You may have already thought of a different way to describe the pattern—namely, "you go from one number to the next by adding 5." This natural way of describing the sequence is expressed by the *recursive formula*:

$$a_n = a_{n-1} + 5$$

starting with $a_1 = 5$. Try substituting $n = 1, 2, 3, \ldots$ in each of these formulas to see

SUGGESTED TIME AND EMPHASIS

1 class.
Optional material.

POINTS TO STRESS

1. Definition and notation of sequences.
2. Recursively defined sequences, including the Fibonacci sequence.
3. Partial sums, including summation notation.

how they produce the numbers in the sequence.

We often use sequences to model real-world phenomena—for example, the monthly payments on a mortgage form a sequence. We will explore many other applications of sequences in this chapter and in *Focus on Modeling* on page 874.

11.1 Sequences and Summation Notation

Many real-world processes generate lists of numbers. For instance, the balance in a bank account at the end of each month forms a list of numbers when tracked over time. Mathematicians call such lists *sequences*. In this section we study sequences and their applications.

Sequences

A *sequence* is a set of numbers written in a specific order:

$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$$

The number a_1 is called the *first term*, a_2 is the *second term*, and in general a_n is the *nth term*. Since for every natural number n there is a corresponding number a_n, we can define a sequence as a function.

Definition of a Sequence

A **sequence** is a function f whose domain is the set of natural numbers. The values $f(1), f(2), f(3), \ldots$ are called the **terms** of the sequence.

We usually write a_n instead of the function notation $f(n)$ for the value of the function at the number n.

Here is a simple example of a sequence:

$$2, 4, 6, 8, 10, \ldots$$

The dots indicate that the sequence continues indefinitely. We can write a sequence in this way when it's clear what the subsequent terms of the sequence are. This sequence consists of even numbers. To be more accurate, however, we need to specify a procedure for finding *all* the terms of the sequence. This can be done by giving a formula for the nth term a_n of the sequence. In this case,

$$a_n = 2n$$

and the sequence can be written as

2,	4,	6,	8,	...,	$2n$,	...
1st term	2nd term	3rd term	4th term		nth term	

Another way to write this sequence is to use function notation:

$$a(n) = 2n$$

so $a(1) = 2, a(2) = 4, a(3) = 6, \ldots$

IN-CLASS MATERIALS

Students often confuse the idea of a function with a sequence. Have the students graph the function $f(n) = \sin(2\pi n)$, and then the sequence $a_n = \sin(2\pi n)$ to make sure they understand the difference.

Notice how the formula $a_n = 2n$ gives all the terms of the sequence. For instance, substituting 1, 2, 3, and 4 for n gives the first four terms:

$$a_1 = 2 \cdot 1 = 2 \qquad a_2 = 2 \cdot 2 = 4$$
$$a_3 = 2 \cdot 3 = 6 \qquad a_4 = 2 \cdot 4 = 8$$

To find the 103rd term of this sequence, we use $n = 103$ to get

$$a_{103} = 2 \cdot 103 = 206$$

Example 1 Finding the Terms of a Sequence

Find the first five terms and the 100th term of the sequence defined by each formula.

(a) $a_n = 2n - 1$ (b) $c_n = n^2 - 1$

(c) $t_n = \dfrac{n}{n+1}$ (d) $r_n = \dfrac{(-1)^n}{2^n}$

Solution To find the first five terms, we substitute $n = 1, 2, 3, 4$, and 5 in the formula for the nth term. To find the 100th term, we substitute $n = 100$. This gives the following.

nth term	First five terms	100th term
(a) $2n - 1$	1, 3, 5, 7, 9	199
(b) $n^2 - 1$	0, 3, 8, 15, 24	9999
(c) $\dfrac{n}{n+1}$	$\dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}, \dfrac{4}{5}, \dfrac{5}{6}$	$\dfrac{100}{101}$
(d) $\dfrac{(-1)^n}{2^n}$	$-\dfrac{1}{2}, \dfrac{1}{4}, -\dfrac{1}{8}, \dfrac{1}{16}, -\dfrac{1}{32}$	$\dfrac{1}{2^{100}}$

■

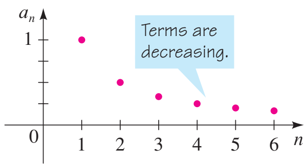


Figure 1

In Example 1(d) the presence of $(-1)^n$ in the sequence has the effect of making successive terms alternately negative and positive.

It is often useful to picture a sequence by sketching its graph. Since a sequence is a function whose domain is the natural numbers, we can draw its graph in the Cartesian plane. For instance, the graph of the sequence

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots, \frac{1}{n}, \ldots$$

is shown in Figure 1. Compare this to the graph of

$$1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{6}, \ldots, \frac{(-1)^{n+1}}{n}, \ldots$$

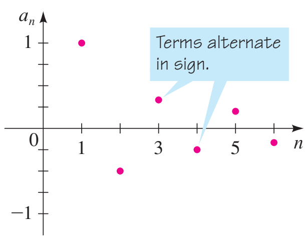


Figure 2

shown in Figure 2. The graph of every sequence consists of isolated points that are *not* connected.

Graphing calculators are useful in analyzing sequences. To work with sequences on a TI-83, we put the calculator in `Seq` mode ("sequence" mode) as in

ALTERNATE EXAMPLE 1a
Find the first five terms and the 150th term of the sequence defined by the following formula: $a_n = 8n - 2$.

ANSWER
$a_1 = 6, a_2 = 14, a_3 = 22, a_4 = 30, a_5 = 38, a_{150} = 1198$

ALTERNATE EXAMPLE 1b
Find the first five terms and the 100th term of the sequence defined by the following formula: $c_n = n^2 - 4$.

ANSWER
$c_1 = 3, c_2 = 0, c_3 = 5, c_4 = 12, c_5 = 21, c_{100} = 9996$

IN-CLASS MATERIALS

Have the students try to figure out the pattern of this sequence: 3, 3, 5, 4, 4, 3, 5, 5, 4, 3, ...

Answer

It counts the number of letters in the words 'one,' 'two,' 'three,' . . . .

IN-CLASS MATERIALS

Craig High School teacher Melissa Pfohl's favorite sequence is $a_n = n^2 + (-1)^n n$. Starting with $n = 0$, the sequence goes as follows: 0, 0, 6, 6, 20, 20, 42, 42, ... Note that the formula is far from obvious, and the tantalizing "doubling property" can be proved using elementary methods.

EXAMPLE
A non-obvious telescoping series:
Consider the sequence $\frac{1}{2}, \frac{1}{6}, \frac{1}{12}, \frac{1}{20}, \ldots, a_n = \frac{1}{n(n+1)}$. This can be rewritten as $a_n = \frac{1}{n} - \frac{1}{n+1}$. Thus $\sum_{n=1}^{k} a_n = 1 - \frac{1}{k+1}$ (as shown in the text).

Figure 3(a). If we enter the sequence $u(n) = n/(n + 1)$ of Example 1(c), we can display the terms using the TABLE command as shown in Figure 3(b). We can also graph the sequence as shown in Figure 3(c).

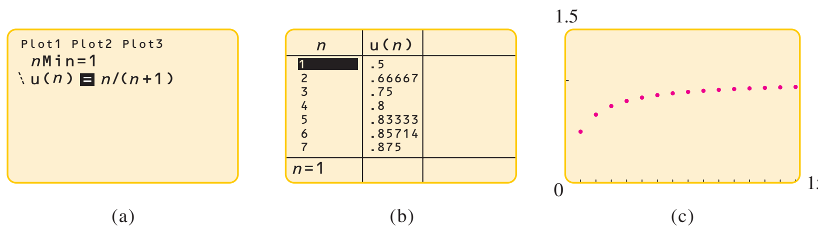


Figure 3
$u(n) = n/(n + 1)$

Finding patterns is an important part of mathematics. Consider a sequence that begins

$$1, 4, 9, 16, \ldots$$

Can you detect a pattern in these numbers? In other words, can you define a sequence whose first four terms are these numbers? The answer to this question seems easy; these numbers are the squares of the numbers 1, 2, 3, 4. Thus, the sequence we are looking for is defined by $a_n = n^2$. However, this is not the *only* sequence whose first four terms are 1, 4, 9, 16. In other words, the answer to our problem is not unique (see Exercise 78). In the next example we are interested in finding an *obvious* sequence whose first few terms agree with the given ones.

Not all sequences can be defined by a formula. For example, there is no known formula for the sequence of prime numbers:

$$2, 3, 5, 7, 11, 13, 17, 19, 23, \ldots$$

Large Prime Numbers

The search for large primes fascinates many people. As of this writing, the largest known prime number is

$$2^{25,964,951} - 1$$

It was discovered in 2005 by Dr. Martin Nowak, an eye surgeon and math hobbyist in Michelfeld, Germany, using a 2.4-GHz Pentium 4 computer. In decimal notation this number contains 7,816,230 digits. If it were written in full, it would occupy almost twice as many pages as this book contains. Nowak was working with a large Internet group known as GIMPS (the Great Internet Mersenne Prime Search). Numbers of the form $2^p - 1$, where p is prime, are called Mersenne numbers and are more easily checked for primality than others. That is why the largest known primes are of this form.

Example 2 Finding the *n*th Term of a Sequence

Find the nth term of a sequence whose first several terms are given.

(a) $\frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8}, \ldots$ (b) $-2, 4, -8, 16, -32, \ldots$

Solution

(a) We notice that the numerators of these fractions are the odd numbers and the denominators are the even numbers. Even numbers are of the form $2n$, and odd numbers are of the form $2n - 1$ (an odd number differs from an even number by 1). So, a sequence that has these numbers for its first four terms is given by

$$a_n = \frac{2n - 1}{2n}$$

(b) These numbers are powers of 2 and they alternate in sign, so a sequence that agrees with these terms is given by

$$a_n = (-1)^n 2^n$$

You should check that these formulas do indeed generate the given terms. ■

Recursively Defined Sequences

Some sequences do not have simple defining formulas like those of the preceding example. The nth term of a sequence may depend on some or all of the terms preceding it. A sequence defined in this way is called **recursive**. Here are two examples.

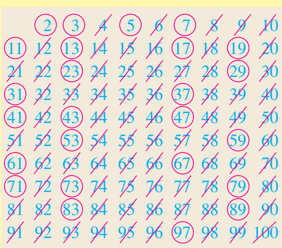
ALTERNATE EXAMPLE 2
Find the first five terms of the sequence defined recursively by $a_1 = 4$ and $a_n = 2(a_{n-1} + 5)$.

ANSWER
$a_1 = 4, a_2 = 18, a_3 = 46, a_4 = 102, a_5 = 214$

IN-CLASS MATERIALS

There are several ways of making the terms of a sequence alternate. The text gives the term $(-1)^n$. Another alternating sequence is $\cos \pi n$. Ask the students to come up with a formula for the following sequence: 0, 1, 0, −1, 0, 1, 0, −1, ... There are several ways to do this, but the cleanest is $\sin\left(\frac{\pi}{2}n\right)$.

Eratosthenes (circa 276–195 B.C.) was a renowned Greek geographer, mathematician, and astronomer. He accurately calculated the circumference of the earth by an ingenious method (see Exercise 72, page 476). He is most famous, however, for his method for finding primes, now called the *sieve of Eratosthenes*. The method consists of listing the integers, beginning with 2 (the first prime), and then crossing out all the multiples of 2, which are not prime. The next number remaining on the list is 3 (the second prime), so we again cross out all multiples of it. The next remaining number is 5 (the third prime number), and we cross out all multiples of it, and so on. In this way all numbers that are not prime are crossed out, and the remaining numbers are the primes.



Example 3 Finding the Terms of a Recursively Defined Sequence

Find the first five terms of the sequence defined recursively by $a_1 = 1$ and

$$a_n = 3(a_{n-1} + 2)$$

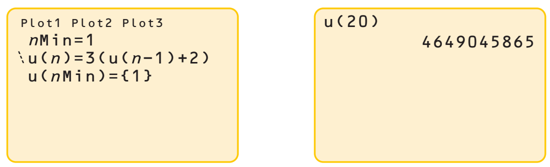
Solution The defining formula for this sequence is recursive. It allows us to find the nth term a_n if we know the preceding term a_{n-1}. Thus, we can find the second term from the first term, the third term from the second term, the fourth term from the third term, and so on. Since we are given the first term $a_1 = 1$, we can proceed as follows.

$$a_2 = 3(a_1 + 2) = 3(1 + 2) = 9$$
$$a_3 = 3(a_2 + 2) = 3(9 + 2) = 33$$
$$a_4 = 3(a_3 + 2) = 3(33 + 2) = 105$$
$$a_5 = 3(a_4 + 2) = 3(105 + 2) = 321$$

Thus, the first five terms of this sequence are

$$1, 9, 33, 105, 321, \ldots$$

Note that in order to find the 20th term of the sequence in Example 3, we must first find all 19 preceding terms. This is most easily done using a graphing calculator. Figure 4(a) shows how to enter this sequence on the TI-83 calculator. From Figure 4(b) we see that the 20th term of the sequence is $a_{20} = 4{,}649{,}045{,}865$.



(a) (b)

Figure 4
$u(n) = 3(u(n-1) + 2),\ u(1) = 1$

Example 4 The Fibonacci Sequence

Find the first 11 terms of the sequence defined recursively by $F_1 = 1$, $F_2 = 1$ and

$$F_n = F_{n-1} + F_{n-2}$$

Solution To find F_n, we need to find the two preceding terms F_{n-1} and F_{n-2}. Since we are given F_1 and F_2, we proceed as follows.

$$F_3 = F_2 + F_1 = 1 + 1 = 2$$
$$F_4 = F_3 + F_2 = 2 + 1 = 3$$
$$F_5 = F_4 + F_3 = 3 + 2 = 5$$

ALTERNATE EXAMPLE 3
Find the first 11 terms of the sequence defined recursively by $F_1 = 3$, $F_2 = 8$, and $F_n = F_{n-1} + F_{n-2}$.

ANSWER
$F_1 = 3,\ F_2 = 8,\ F_3 = 11,\ F_4 = 19,\ F_5 = 30,\ F_6 = 49,\ F_7 = 79,\ F_8 = 128,\ F_9 = 207,\ F_{10} = 335,\ F_{11} = 542$

EXAMPLE
A sequence whose values look random: $a_n = \sin(n^2)$

ALTERNATE EXAMPLE 4
Find the nth term of a sequence whose first five terms are given below.
$-2, 4, -8, 16, -32, \ldots$

ANSWER
$(-1)^n \cdot 2^n$

IN-CLASS MATERIALS

Note that there are many sequences that have no pattern: $1, \pi, 3, e, -27, \ldots$. Also point out that not every sequence with a pattern has a rule that is easily written as a formula. For example,

$3, 1, 4, 1, 5, 9, 2, 6, 5, 3, \ldots$
$3, 3.1, 3.14, 3.141, 3.1415, \ldots$
$0, 0.1, 0.12, 0.123, \ldots, 0.123456789, 0.12345678910, 0.1234567891011, \ldots$

(The limit of this last sequence is called the Champernowe constant.)

Fibonacci (1175–1250) was born in Pisa, Italy, and educated in North Africa. He traveled widely in the Mediterranean area and learned the various methods then in use for writing numbers. On returning to Pisa in 1202, Fibonacci advocated the use of the Hindu-Arabic decimal system, the one we use today, over the Roman numeral system used in Europe in his time. His most famous book, *Liber Abaci*, expounds on the advantages of the Hindu-Arabic numerals. In fact, multiplication and division were so complicated using Roman numerals that a college degree was necessary to master these skills. Interestingly, in 1299 the city of Florence outlawed the use of the decimal system for merchants and businesses, requiring numbers to be written in Roman numerals or words. One can only speculate about the reasons for this law.

It's clear what is happening here. Each term is simply the sum of the two terms that precede it, so we can easily write down as many terms as we please. Here are the first 11 terms:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$$ ■

The sequence in Example 4 is called the **Fibonacci sequence**, named after the 13th-century Italian mathematician who used it to solve a problem about the breeding of rabbits (see Exercise 77). The sequence also occurs in numerous other applications in nature. (See Figures 5 and 6.) In fact, so many phenomena behave like the Fibonacci sequence that one mathematical journal, the *Fibonacci Quarterly*, is devoted entirely to its properties.

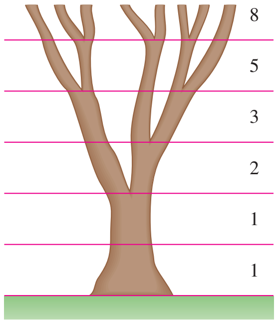


Figure 5
The Fibonacci sequence in the branching of a tree

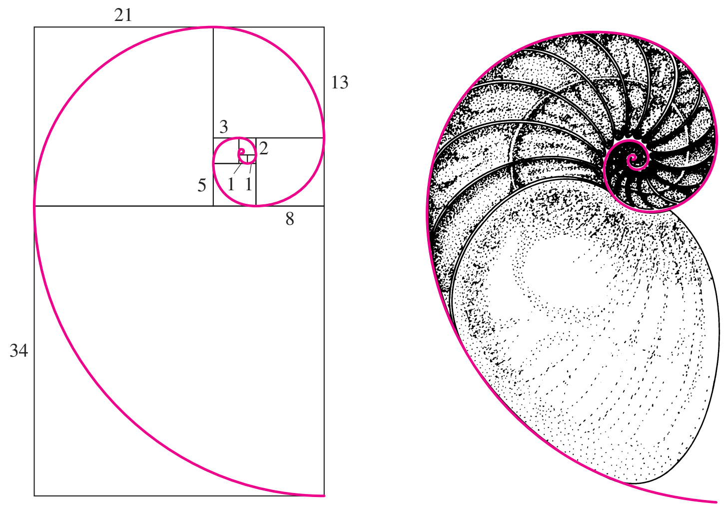


Figure 6 Fibonacci spiral Nautilus shell

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The Partial Sums of a Sequence

In calculus we are often interested in adding the terms of a sequence. This leads to the following definition.

The Partial Sums of a Sequence

For the sequence

$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$$

the **partial sums** are

$$\begin{aligned} S_1 &= a_1 \\ S_2 &= a_1 + a_2 \\ S_3 &= a_1 + a_2 + a_3 \\ S_4 &= a_1 + a_2 + a_3 + a_4 \\ &\vdots \\ S_n &= a_1 + a_2 + a_3 + \cdots + a_n \\ &\vdots \end{aligned}$$

S_1 is called the **first partial sum**, S_2 is the **second partial sum**, and so on. S_n is called the ***n*th partial sum**. The sequence $S_1, S_2, S_3, \ldots, S_n, \ldots$ is called the **sequence of partial sums**.

Example 5 Finding the Partial Sums of a Sequence

Find the first four partial sums and the nth partial sum of the sequence given by $a_n = 1/2^n$.

Solution The terms of the sequence are

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$$

The first four partial sums are

$$\begin{aligned} S_1 &= \frac{1}{2} && = \frac{1}{2} \\ S_2 &= \frac{1}{2} + \frac{1}{4} && = \frac{3}{4} \\ S_3 &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} && = \frac{7}{8} \\ S_4 &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} && = \frac{15}{16} \end{aligned}$$

DRILL QUESTION

If we have a sequence defined by $a_1 = 4$, $a_2 = -3$, and $a_n = a_{n-2} + a_{n-1}$, what is a_4?

Answer

-2

ALTERNATE EXAMPLE 5

Find the first four partial sums and the nth partial sum of the sequence given by $a_n = 2n - 1$.

ANSWER

$S_1 = 1, S_2 = 4, S_3 = 9, S_4 = 16, S_n = n^2$

IN-CLASS MATERIALS

Have students compute some partial sums that converge quickly to a recognizable number, such as the one associated with $a_n = \dfrac{1}{(n-1)!}$ (with $a_1 = 1$) which converges to e, and $a_n = \left(\dfrac{3}{4}\right)^n$ which converges to 3. Then have them look at some partial sums that go off to infinity, such as the ones associated with $a_n = 10^n$ and $a_n = n$. Then have them conjecture about the fate of $a_n = \dfrac{1}{n}$.

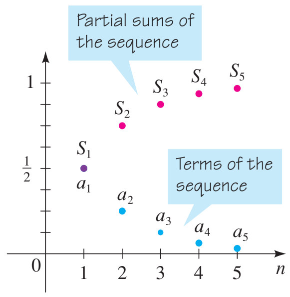


Figure 7
Graph of the sequence a_n and the sequence of partial sums S_n

Notice that in the value of each partial sum the denominator is a power of 2 and the numerator is one less than the denominator. In general, the nth partial sum is

$$S_n = \frac{2^n - 1}{2^n} = 1 - \frac{1}{2^n}$$

The first five terms of a_n and S_n are graphed in Figure 7. ■

Example 6 Finding the Partial Sums of a Sequence

Find the first four partial sums and the nth partial sum of the sequence given by

$$a_n = \frac{1}{n} - \frac{1}{n+1}$$

Solution The first four partial sums are

$$S_1 = \left(1 - \frac{1}{2}\right) = 1 - \frac{1}{2}$$

$$S_2 = \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \frac{1}{3}\right) = 1 - \frac{1}{3}$$

$$S_3 = \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \frac{1}{4}\right) = 1 - \frac{1}{4}$$

$$S_4 = \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{4}} - \frac{1}{5}\right) = 1 - \frac{1}{5}$$

Do you detect a pattern here? Of course. The nth partial sum is

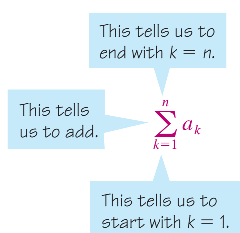
$$S_n = 1 - \frac{1}{n+1}$$ ■

Sigma Notation

Given a sequence

$$a_1, a_2, a_3, a_4, \ldots$$

we can write the sum of the first n terms using **summation notation**, or **sigma notation**. This notation derives its name from the Greek letter Σ (capital sigma, corresponding to our S for "sum"). Sigma notation is used as follows:



$$\boxed{\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + a_4 + \cdots + a_n}$$

The left side of this expression is read "The sum of a_k from $k = 1$ to $k = n$." The letter k is called the **index of summation**, or the **summation variable**, and the idea is to replace k in the expression after the sigma by the integers $1, 2, 3, \ldots, n$, and add the resulting expressions, arriving at the right side of the equation.

ALTERNATE EXAMPLE 6
Find the first four partial sums and the nth partial sum of the sequence given by

$$a_n = \frac{1}{n+1} - \frac{1}{n+2}.$$

ANSWER

$S_1 = \frac{1}{2} - \frac{1}{3}$, $S_2 = \frac{1}{2} - \frac{1}{4}$,
$S_3 = \frac{1}{2} - \frac{1}{5}$, $S_4 = \frac{1}{2} - \frac{1}{6}$,
$S_n = \frac{1}{2} - \frac{1}{n+2}$

The ancient Greeks considered a line segment to be divided into the **golden ratio** if the ratio of the shorter part to the longer part is the same as the ratio of the longer part to the whole segment.

Thus, the segment shown is divided into the golden ratio if

$$\frac{1}{x} = \frac{x}{1 + x}$$

This leads to a quadratic equation whose positive solution is

$$x = \frac{1 + \sqrt{5}}{2} \approx 1.618$$

This ratio occurs naturally in many places. For instance, psychological experiments show that the most pleasing shape of rectangle is one whose sides are in golden ratio. The ancient Greeks agreed with this and built their temples in this ratio.

The golden ratio is related to the Fibonacci sequence. In fact, it can be shown using calculus* that the ratio of two successive Fibonacci numbers

$$\frac{F_{n+1}}{F_n}$$

gets closer to the golden ratio the larger the value of n. Try finding this ratio for $n = 10$.

*James Stewart, *Calculus*, 5th ed. (Pacific Grove, CA: Brooks/Cole, 2003) p. 748.

Example 7 Sigma Notation

Find each sum.

(a) $\displaystyle\sum_{k=1}^{5} k^2$ (b) $\displaystyle\sum_{j=3}^{5} \frac{1}{j}$ (c) $\displaystyle\sum_{i=5}^{10} i$ (d) $\displaystyle\sum_{i=1}^{6} 2$

Solution

(a) $\displaystyle\sum_{k=1}^{5} k^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55$

(b) $\displaystyle\sum_{j=3}^{5} \frac{1}{j} = \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{47}{60}$

(c) $\displaystyle\sum_{i=5}^{10} i = 5 + 6 + 7 + 8 + 9 + 10 = 45$

(d) $\displaystyle\sum_{i=1}^{6} 2 = 2 + 2 + 2 + 2 + 2 + 2 = 12$ ■

We can use a graphing calculator to evaluate sums. For instance, Figure 8 shows how the TI-83 can be used to evaluate the sums in parts (a) and (b) of Example 7.

```
sum(seq(K²,K,1,5,1))
                  55
sum(seq(1/J,J,3,5,
1))▶Frac
               47/60
```

Figure 8

Example 8 Writing Sums in Sigma Notation

Write each sum using sigma notation.

(a) $1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3$

(b) $\sqrt{3} + \sqrt{4} + \sqrt{5} + \cdots + \sqrt{77}$

Solution

(a) We can write

$$1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3 = \sum_{k=1}^{7} k^3$$

(b) A natural way to write this sum is

$$\sqrt{3} + \sqrt{4} + \sqrt{5} + \cdots + \sqrt{77} = \sum_{k=3}^{77} \sqrt{k}$$

However, there is no unique way of writing a sum in sigma notation. We could also write this sum as

$$\sqrt{3} + \sqrt{4} + \sqrt{5} + \cdots + \sqrt{77} = \sum_{k=0}^{74} \sqrt{k + 3}$$

or $$\sqrt{3} + \sqrt{4} + \sqrt{5} + \cdots + \sqrt{77} = \sum_{k=1}^{75} \sqrt{k + 2}$$ ■

ALTERNATE EXAMPLE 7a

Find the sum.

$\displaystyle\sum_{k+5}^{9} k$

ANSWER

1925

ALTERNATE EXAMPLE 7c

Find the sum.

$\displaystyle\sum_{i=2}^{9} i$

ANSWER

44

ALTERNATE EXAMPLE 7d

Find the sum.

$\displaystyle\sum_{i=2}^{7} 2$

ANSWER

12

ALTERNATE EXAMPLE 8

Write the sum using sigma notation, where the index of summation k runs from 1 to 7.

$3^4 + 4^4 + 5^4 + 6^4 + 7^4 + 8^4 + 9^4$

ANSWER

$\displaystyle\sum_{i=2}^{7} (k + 2)^4$

SAMPLE QUESTION
Text Question

Compute $\displaystyle\sum_{k+1}^{5} k$.

Answer

15

The following properties of sums are natural consequences of properties of the real numbers.

Properties of Sums

Let $a_1, a_2, a_3, a_4, \ldots$ and $b_1, b_2, b_3, b_4, \ldots$ be sequences. Then for every positive integer n and any real number c, the following properties hold.

1. $\displaystyle\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$
2. $\displaystyle\sum_{k=1}^{n}(a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$
3. $\displaystyle\sum_{k=1}^{n} ca_k = c\left(\sum_{k=1}^{n} a_k\right)$

Proof To prove Property 1, we write out the left side of the equation to get

$$\sum_{k=1}^{n}(a_k + b_k) = (a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) + \cdots + (a_n + b_n)$$

Because addition is commutative and associative, we can rearrange the terms on the right side to read

$$\sum_{k=1}^{n}(a_k + b_k) = (a_1 + a_2 + a_3 + \cdots + a_n) + (b_1 + b_2 + b_3 + \cdots + b_n)$$

Rewriting the right side using sigma notation gives Property 1. Property 2 is proved in a similar manner. To prove Property 3, we use the Distributive Property:

$$\begin{aligned}\sum_{k=1}^{n} ca_k &= ca_1 + ca_2 + ca_3 + \cdots + ca_n \\ &= c(a_1 + a_2 + a_3 + \cdots + a_n) = c\left(\sum_{k=1}^{n} a_k\right)\end{aligned}$$

■

11.1 Exercises

1–10 ■ Find the first four terms and the 100th term of the sequence.

1. $a_n = n + 1$

2. $a_n = 2n + 3$

3. $a_n = \dfrac{1}{n + 1}$

4. $a_n = n^2 + 1$

5. $a_n = \dfrac{(-1)^n}{n^2}$

6. $a_n = \dfrac{1}{n^2}$

7. $a_n = 1 + (-1)^n$

8. $a_n = (-1)^{n+1}\dfrac{n}{n + 1}$

9. $a_n = n^n$

10. $a_n = 3$

11–16 ■ Find the first five terms of the given recursively defined sequence.

11. $a_n = 2(a_{n-1} - 2)$ and $a_1 = 3$

12. $a_n = \dfrac{a_{n-1}}{2}$ and $a_1 = -8$

13. $a_n = 2a_{n-1} + 1$ and $a_1 = 1$

14. $a_n = \dfrac{1}{1 + a_{n-1}}$ and $a_1 = 1$

15. $a_n = a_{n-1} + a_{n-2}$ and $a_1 = 1, a_2 = 2$

16. $a_n = a_{n-1} + a_{n-2} + a_{n-3}$ and $a_1 = a_2 = a_3 = 1$

17–22 ■ Use a graphing calculator to do the following.

(a) Find the first 10 terms of the sequence.

(b) Graph the first 10 terms of the sequence.

17. $a_n = 4n + 3$

18. $a_n = n^2 + n$

19. $a_n = \dfrac{12}{n}$

20. $a_n = 4 - 2(-1)^n$

21. $a_n = \dfrac{1}{a_{n-1}}$ and $a_1 = 2$

22. $a_n = a_{n-1} - a_{n-2}$ and $a_1 = 1, a_2 = 3$

23–30 ■ Find the nth term of a sequence whose first several terms are given.

23. $2, 4, 8, 16, \ldots$

24. $-\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}, \frac{1}{81}, \ldots$

25. $1, 4, 7, 10, \ldots$

26. $5, -25, 125, -625, \ldots$

27. $1, \frac{3}{4}, \frac{5}{9}, \frac{7}{16}, \frac{9}{25}, \ldots$

28. $\frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \ldots$

29. $0, 2, 0, 2, 0, 2, \ldots$

30. $1, \frac{1}{2}, 3, \frac{1}{4}, 5, \frac{1}{6}, \ldots$

31–34 ■ Find the first six partial sums $S_1, S_2, S_3, S_4, S_5, S_6$ of the sequence.

31. $1, 3, 5, 7, \ldots$

32. $1^2, 2^2, 3^2, 4^2, \ldots$

33. $\dfrac{1}{3}, \dfrac{1}{3^2}, \dfrac{1}{3^3}, \dfrac{1}{3^4}, \ldots$

34. $-1, 1, -1, 1, \ldots$

35–38 ■ Find the first four partial sums and the nth partial sum of the sequence a_n.

35. $a_n = \dfrac{2}{3^n}$

36. $a_n = \dfrac{1}{n+1} - \dfrac{1}{n+2}$

37. $a_n = \sqrt{n} - \sqrt{n+1}$

38. $a_n = \log\left(\dfrac{n}{n+1}\right)$ [*Hint:* Use a property of logarithms to write the nth term as a difference.]

39–46 ■ Find the sum.

39. $\sum_{k=1}^{4} k$

40. $\sum_{k=1}^{4} k^2$

41. $\sum_{k=1}^{3} \dfrac{1}{k}$

42. $\sum_{j=1}^{100} (-1)^j$

43. $\sum_{i=1}^{8} [1 + (-1)^i]$

44. $\sum_{i=4}^{12} 10$

45. $\sum_{k=1}^{5} 2^{k-1}$

46. $\sum_{i=1}^{3} i2^i$

47–52 ■ Use a graphing calculator to evaluate the sum.

47. $\sum_{k=1}^{10} k^2$

48. $\sum_{k=1}^{100} (3k + 4)$

49. $\sum_{j=7}^{20} j^2(1 + j)$

50. $\sum_{j=5}^{15} \dfrac{1}{j^2 + 1}$

51. $\sum_{n=0}^{22} (-1)^n 2n$

52. $\sum_{n=1}^{100} \dfrac{(-1)^n}{n}$

53–58 ■ Write the sum without using sigma notation.

53. $\sum_{k=1}^{5} \sqrt{k}$

54. $\sum_{i=0}^{4} \dfrac{2i-1}{2i+1}$

55. $\sum_{k=0}^{6} \sqrt{k+4}$

56. $\sum_{k=6}^{9} k(k+3)$

57. $\sum_{k=3}^{100} x^k$

58. $\sum_{j=1}^{n} (-1)^{j+1} x^j$

59–66 ■ Write the sum using sigma notation.

59. $1 + 2 + 3 + 4 + \cdots + 100$

60. $2 + 4 + 6 + \cdots + 20$

61. $1^2 + 2^2 + 3^2 + \cdots + 10^2$

62. $\dfrac{1}{2\ln 2} - \dfrac{1}{3\ln 3} + \dfrac{1}{4\ln 4} - \dfrac{1}{5\ln 5} + \cdots + \dfrac{1}{100\ln 100}$

63. $\dfrac{1}{1\cdot 2} + \dfrac{1}{2\cdot 3} + \dfrac{1}{3\cdot 4} + \cdots + \dfrac{1}{999\cdot 1000}$

64. $\dfrac{\sqrt{1}}{1^2} + \dfrac{\sqrt{2}}{2^2} + \dfrac{\sqrt{3}}{3^2} + \cdots + \dfrac{\sqrt{n}}{n^2}$

65. $1 + x + x^2 + x^3 + \cdots + x^{100}$

66. $1 - 2x + 3x^2 - 4x^3 + 5x^4 + \cdots - 100x^{99}$

67. Find a formula for the nth term of the sequence

$$\sqrt{2},\ \sqrt{2\sqrt{2}},\ \sqrt{2\sqrt{2\sqrt{2}}},\ \sqrt{2\sqrt{2\sqrt{2\sqrt{2}}}}, \ldots$$

[*Hint:* Write each term as a power of 2.]

68. Define the sequence

$$G_n = \frac{1}{\sqrt{5}}\left(\frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n}\right)$$

Use the $\boxed{\text{TABLE}}$ command on a graphing calculator to find the first 10 terms of this sequence. Compare to the Fibonacci sequence F_n.

Applications

69. Compound Interest Julio deposits \$2000 in a savings account that pays 2.4% interest per year compounded monthly. The amount in the account after n months is given by the sequence

$$A_n = 2000\left(1 + \frac{0.024}{12}\right)^n$$

(a) Find the first six terms of the sequence.

(b) Find the amount in the account after 3 years.

70. Compound Interest Helen deposits \$100 at the end of each month into an account that pays 6% interest per year compounded monthly. The amount of interest she has accumulated after n months is given by the sequence

$$I_n = 100\left(\frac{1.005^n - 1}{0.005} - n\right)$$

(a) Find the first six terms of the sequence.

(b) Find the interest she has accumulated after 5 years.

71. Population of a City A city was incorporated in 2004 with a population of 35,000. It is expected that the population will increase at a rate of 2% per year. The population n years after 2004 is given by the sequence

$$P_n = 35{,}000(1.02)^n$$

(a) Find the first five terms of the sequence.

(b) Find the population in 2014.

72. Paying off a Debt Margarita borrows \$10,000 from her uncle and agrees to repay it in monthly installments of \$200. Her uncle charges 0.5% interest per month on the balance.

(a) Show that her balance A_n in the nth month is given recursively by $A_0 = 10{,}000$ and

$$A_n = 1.005A_{n-1} - 200$$

(b) Find her balance after six months.

73. Fish Farming A fish farmer has 5000 catfish in his pond. The number of catfish increases by 8% per month, and the farmer harvests 300 catfish per month.

(a) Show that the catfish population P_n after n months is given recursively by $P_0 = 5000$ and

$$P_n = 1.08P_{n-1} - 300$$

(b) How many fish are in the pond after 12 months?

74. Price of a House The median price of a house in Orange County increases by about 6% per year. In 2002 the median price was \$240,000. Let P_n be the median price n years after 2002.

(a) Find a formula for the sequence P_n.

(b) Find the expected median price in 2010.

75. Salary Increases A newly hired salesman is promised a beginning salary of \$30,000 a year with a \$2000 raise every year. Let S_n be his salary in his nth year of employment.

(a) Find a recursive definition of S_n.

(b) Find his salary in his fifth year of employment.

76. Concentration of a Solution A biologist is trying to find the optimal salt concentration for the growth of a certain species of mollusk. She begins with a brine solution that has 4 g/L of salt and increases the concentration by 10% every day. Let C_0 denote the initial concentration and C_n the concentration after n days.

(a) Find a recursive definition of C_n.

(b) Find the salt concentration after 8 days.

77. Fibonacci's Rabbits Fibonacci posed the following problem: Suppose that rabbits live forever and that every month each pair produces a new pair that becomes productive at age 2 months. If we start with one newborn pair, how many pairs of rabbits will we have in the nth month? Show that the answer is F_n, where F_n is the nth term of the Fibonacci sequence.

Discovery • Discussion

78. Different Sequences That Start the Same

(a) Show that the first four terms of the sequence $a_n = n^2$ are

$$1, 4, 9, 16, \ldots$$

(b) Show that the first four terms of the sequence $a_n = n^2 + (n-1)(n-2)(n-3)(n-4)$ are also

$$1, 4, 9, 16, \ldots$$

(c) Find a sequence whose first six terms are the same as those of $a_n = n^2$ but whose succeeding terms differ from this sequence.

(d) Find two different sequences that begin

$$2, 4, 8, 16, \ldots$$

79. A Recursively Defined Sequence Find the first 40 terms of the sequence defined by

$$a_{n+1} = \begin{cases} \dfrac{a_n}{2} & \text{if } a_n \text{ is an even number} \\ 3a_n + 1 & \text{if } a_n \text{ is an odd number} \end{cases}$$

and $a_1 = 11$. Do the same if $a_1 = 25$. Make a conjecture about this type of sequence. Try several other values for a_1, to test your conjecture.

80. A Different Type of Recursion Find the first 10 terms of the sequence defined by

$$a_n = a_{n-a_{n-1}} + a_{n-a_{n-2}}$$

with

$$a_1 = 1 \qquad \text{and} \qquad a_2 = 1$$

How is this recursive sequence different from the others in this section?

11.2 Arithmetic Sequences

In this section we study a special type of sequence, called an arithmetic sequence.

Arithmetic Sequences

Perhaps the simplest way to generate a sequence is to start with a number a and add to it a fixed constant d, over and over again.

Definition of an Arithmetic Sequence

An **arithmetic sequence** is a sequence of the form

$$a, a + d, a + 2d, a + 3d, a + 4d, \ldots$$

The number a is the **first term**, and d is the **common difference** of the sequence. The ***n*th term** of an arithmetic sequence is given by

$$a_n = a + (n - 1)d$$

The number d is called the common difference because any two consecutive terms of an arithmetic sequence differ by d.

Example 1 Arithmetic Sequences

(a) If $a = 2$ and $d = 3$, then we have the arithmetic sequence

$$2, 2 + 3, 2 + 6, 2 + 9, \ldots$$

or

$$2, 5, 8, 11, \ldots$$

Any two consecutive terms of this sequence differ by $d = 3$. The nth term is $a_n = 2 + 3(n - 1)$.

(b) Consider the arithmetic sequence

$$9, 4, -1, -6, -11, \ldots$$

Here the common difference is $d = -5$. The terms of an arithmetic sequence decrease if the common difference is negative. The nth term is $a_n = 9 - 5(n - 1)$.

(c) The graph of the arithmetic sequence $a_n = 1 + 2(n - 1)$ is shown in Figure 1. Notice that the points in the graph lie on a straight line with slope $d = 2$.

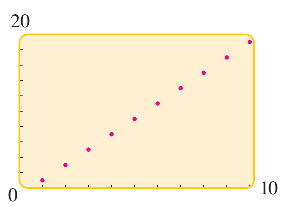


Figure 1

■

SUGGESTED TIME AND EMPHASIS

$\frac{1}{2}$–1 class.
Optional material.

POINTS TO STRESS

1. Recognizing arithmetic sequences by their formulas and by their graphs.
2. Finding the first term a and the common difference d of a given arithmetic sequence, and using this information to compute partial sums.

ALTERNATE EXAMPLE 1

Find the nth term of the arithmetic sequence $9, 4, -1, -6, -11, \ldots$.

ANSWER

$a_n = 9 - 5(n - 1)$

IN-CLASS MATERIALS

Have the students try to define an arithmetic sequence recursively. It can be done relatively simply as $a_n = a_{n-1} + d$, but obtaining this formula requires an understanding of arithmetic sequences and recursively defined sequences.

ALTERNATE EXAMPLE 2
Find the 200th term of the arithmetic sequence 14, 8, ...

ANSWER
$a_{200} = -1180$

DRILL QUESTION

Consider the sequence 3, 8, 13, 18, 23, 28, 33, 38, ... Find the 100th partial sum.

Answer
25,050

ALTERNATE EXAMPLE 3
The 11th term of an arithmetic sequence is 63, and the 15th term is 87. Find the 800th term.

ANSWER
$a_{800} = 4797$

SAMPLE QUESTION
Text Question

What distinguishes an "arithmetic sequence" from an arbitrary "sequence"?

Answer
Successive terms in an arithmetic sequence have a common difference.

IN-CLASS MATERIALS

Show the students that a constant sequence such as 5, 5, 5, 5, . . . is trivially an arithmetic sequence. Notice that the formula for the partial sum is consistent with what we know about basic arithmetic.

Mathematics in the Modern World

Fair Division of Assets

Dividing an asset fairly among a number of people is of great interest to mathematicians. Problems of this nature include dividing the national budget, disputed land, or assets in divorce cases. In 1994 Brams and Taylor found a mathematical way of dividing things fairly. Their solution has been applied to division problems in political science, legal proceedings, and other areas. To understand the problem, consider the following example. Suppose persons A and B want to divide a property fairly between them. To divide it *fairly* means that both A and B must be satisfied with the outcome of the division. Solution: A gets to divide the property into two pieces, then B gets to choose the piece he wants. Since both A and B had a part in the division process, each should be satisfied. The situation becomes much more complicated if three or more people are involved (and that's where mathematics comes in). Dividing things fairly involves much more than simply cutting things in half; it must take into account the *relative worth* each person attaches to the thing being divided. A story from the Bible illustrates this clearly. Two women appear before King Solomon, each claiming to be the mother of the same newborn baby. King Solomon's solution is to divide the baby in half! The real mother, who attaches far more worth to the baby than anyone, immediately gives up her claim to the baby in order to save its life.

Mathematical solutions to fair-division problems have recently been applied in an international treaty, the Convention on the Law of the Sea. If a country wants to develop a portion of the sea floor, it is

(continued)

An arithmetic sequence is determined completely by the first term a and the common difference d. Thus, if we know the first two terms of an arithmetic sequence, then we can find a formula for the nth term, as the next example shows.

Example 2 Finding Terms of an Arithmetic Sequence

Find the first six terms and the 300th term of the arithmetic sequence

$$13, 7, \ldots$$

Solution Since the first term is 13, we have $a = 13$. The common difference is $d = 7 - 13 = -6$. Thus, the nth term of this sequence is

$$a_n = 13 - 6(n - 1)$$

From this we find the first six terms:

$$13, 7, 1, -5, -11, -17, \ldots$$

The 300th term is $a_{300} = 13 - 6(299) = -1781$. ■

The next example shows that an arithmetic sequence is determined completely by *any* two of its terms.

Example 3 Finding Terms of an Arithmetic Sequence

The 11th term of an arithmetic sequence is 52, and the 19th term is 92. Find the 1000th term.

Solution To find the nth term of this sequence, we need to find a and d in the formula

$$a_n = a + (n - 1)d$$

From this formula we get

$$a_{11} = a + (11 - 1)d = a + 10d$$

$$a_{19} = a + (19 - 1)d = a + 18d$$

Since $a_{11} = 52$ and $a_{19} = 92$, we get the two equations:

$$\begin{cases} 52 = a + 10d \\ 92 = a + 18d \end{cases}$$

Solving this system for a and d, we get $a = 2$ and $d = 5$. (Verify this.) Thus, the nth term of this sequence is

$$a_n = 2 + 5(n - 1)$$

The 1000th term is $a_{1000} = 2 + 5(999) = 4997$. ■

Partial Sums of Arithmetic Sequences

Suppose we want to find the sum of the numbers 1, 2, 3, 4, . . . , 100, that is,

$$\sum_{k=1}^{100} k$$

When the famous mathematician C. F. Gauss was a schoolboy, his teacher posed this problem to the class and expected that it would keep the students busy for a long time. But Gauss answered the question almost immediately. His idea was this: Since we are

EXAMPLES

Finding details of an arithmetic sequence given two terms: If an arithmetic sequence has $a_3 = 39$ and $a_{10} = 25$, then we can calculate $a = 43$, $d = -2$. $a_n = 43 - 2(n - 1)$, so $a_{100} = -155$ and $\sum_{n=1}^{100} 43 - 2(n - 1) = -5600$.

Make sure to point out that this trick only works if we know ahead of time that this is an arithmetic sequence.

required to divide the portion into two parts, one part to be used by itself, the other by a consortium that will preserve it for later use by a less developed country. The consortium gets first pick.

adding numbers produced according to a fixed pattern, there must also be a pattern (or formula) for finding the sum. He started by writing the numbers from 1 to 100 and below them the same numbers in reverse order. Writing S for the sum and adding corresponding terms gives

$$\begin{array}{rcl} S &=& 1 + 2 + 3 + \cdots + 98 + 99 + 100 \\ S &=& 100 + 99 + 98 + \cdots + 3 + 2 + 1 \\ \hline 2S &=& 101 + 101 + 101 + \cdots + 101 + 101 + 101 \end{array}$$

It follows that $2S = 100(101) = 10{,}100$ and so $S = 5050$.

Of course, the sequence of natural numbers 1, 2, 3, . . . is an arithmetic sequence (with $a = 1$ and $d = 1$), and the method for summing the first 100 terms of this sequence can be used to find a formula for the nth partial sum of any arithmetic sequence. We want to find the sum of the first n terms of the arithmetic sequence whose terms are $a_k = a + (k - 1)d$; that is, we want to find

$$\begin{aligned} S_n &= \sum_{k=1}^{n} [a + (k-1)d] \\ &= a + (a + d) + (a + 2d) + (a + 3d) + \cdots + [a + (n-1)d] \end{aligned}$$

Using Gauss's method, we write

$$\begin{array}{rcl} S_n &=& a + (a + d) + \cdots + [a + (n-2)d] + [a + (n-1)d] \\ S_n &=& [a + (n-1)d] + [a + (n-2)d] + \cdots + (a + d) + a \\ \hline 2S_n &=& [2a + (n-1)d] + [2a + (n-1)d] + \cdots + [2a + (n-1)d] + [2a + (n-1)d] \end{array}$$

There are n identical terms on the right side of this equation, so

$$2S_n = n[2a + (n-1)d]$$

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

Notice that $a_n = a + (n-1)d$ is the nth term of this sequence. So, we can write

$$S_n = \frac{n}{2}[a + a + (n-1)d] = n\left(\frac{a + a_n}{2}\right)$$

This last formula says that the sum of the first n terms of an arithmetic sequence is the average of the first and nth terms multiplied by n, the number of terms in the sum. We now summarize this result.

Partial Sums of an Arithmetic Sequence

For the arithmetic sequence $a_n = a + (n-1)d$, the **nth partial sum**

$$S_n = a + (a + d) + (a + 2d) + (a + 3d) + \cdots + [a + (n-1)d]$$

is given by either of the following formulas.

1. $S_n = \dfrac{n}{2}[2a + (n-1)d]$

2. $S_n = n\left(\dfrac{a + a_n}{2}\right)$

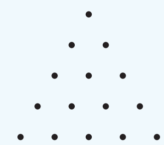
IN-CLASS MATERIALS

The perfect squares 1, 4, 9, 16, . . . can be represented as a square array of dots.

16

IN-CLASS MATERIALS

The triangular numbers are those that can be represented by a triangular array of dots: 1, 3, 6, 10, 15, ...



15

Note that the triangular numbers are precisely the partial sums of the sequence 1, 2, 3, 4, ...

IN-CLASS MATERIALS

Examine this broad triangular array:

16

Notice we run into perfect squares! This can be demonstrated by looking at the partial sums of the arithmetic sequence $1 + 2(n - 1)$, or by rearranging the dots of the broader triangle to make a square.

IN-CLASS MATERIALS

Many phenomena are either linear or locally linear (they look linear when viewing them over a narrow range). For example, assume that a company earns about $120,000 in 2011, $140,000 in 2012, $160,000 in 2013, etc. If it is linear growth over ten years, the total ten-year earning can be found from adding up the terms of an arithmetic sequence.

ALTERNATE EXAMPLE 4
Find the sum of the first 40 terms of the arithmetic sequence 3, 10, 17, 24, ...

ANSWER
5580

ALTERNATE EXAMPLE 5
Find the sum of the first 40 odd numbers.

ANSWER
1600

ALTERNATE EXAMPLE 6
An amphitheater has 40 rows of seats with 30 seats in the first row, 32 in the second, 34 in the third, and so on. Find the total number of seats.

ANSWER
2760

ALTERNATE EXAMPLE 7
How many terms of the arithmetic sequences 3, 5, 7, . . . must be added to get 624?

ANSWER
24

Example 4 Finding a Partial Sum of an Arithmetic Sequence

Find the sum of the first 40 terms of the arithmetic sequence

$$3, 7, 11, 15, \dots$$

Solution For this arithmetic sequence, $a = 3$ and $d = 4$. Using Formula 1 for the partial sum of an arithmetic sequence, we get

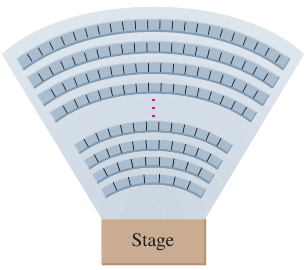
$$S_{40} = \tfrac{40}{2}[2(3) + (40 - 1)4] = 20(6 + 156) = 3240$$ ■

Example 5 Finding a Partial Sum of an Arithmetic Sequence

Find the sum of the first 50 odd numbers.

Solution The odd numbers form an arithmetic sequence with $a = 1$ and $d = 2$. The nth term is $a_n = 1 + 2(n - 1) = 2n - 1$, so the 50th odd number is $a_{50} = 2(50) - 1 = 99$. Substituting in Formula 2 for the partial sum of an arithmetic sequence, we get

$$S_{50} = 50\left(\frac{a + a_{50}}{2}\right) = 50\left(\frac{1 + 99}{2}\right) = 50 \cdot 50 = 2500$$ ■



Example 6 Finding the Seating Capacity of an Amphitheater

An amphitheater has 50 rows of seats with 30 seats in the first row, 32 in the second, 34 in the third, and so on. Find the total number of seats.

Solution The numbers of seats in the rows form an arithmetic sequence with $a = 30$ and $d = 2$. Since there are 50 rows, the total number of seats is the sum

$$\begin{aligned} S_{50} &= \tfrac{50}{2}[2(30) + 49(2)] \qquad S_n = \frac{n}{2}[2a + (n-1)d] \\ &= 3950 \end{aligned}$$

Thus, the amphitheater has 3950 seats. ■

Example 7 Finding the Number of Terms in a Partial Sum

How many terms of the arithmetic sequences 5, 7, 9, . . . must be added to get 572?

Solution We are asked to find n when $S_n = 572$. Substituting $a = 5$, $d = 2$, and $S_n = 572$ in Formula 1 for the partial sum of an arithmetic sequence, we get

$$\begin{aligned} 572 &= \frac{n}{2}[2 \cdot 5 + (n - 1)2] \qquad S_n = \frac{n}{2}[2a + (n-1)d] \\ 572 &= 5n + n(n - 1) \\ 0 &= n^2 + 4n - 572 \\ 0 &= (n - 22)(n + 26) \end{aligned}$$

This gives $n = 22$ or $n = -26$. But since n is the *number* of terms in this partial sum, we must have $n = 22$. ■

IN-CLASS MATERIALS

Consider the handshaking problem. If n people are in a room, and shake hands, we can ask the question: How many handshakes took place? If, for example, four people (Alfred, Brendel, Claude, and Debussy) all shake hands, a total of 6 shakes take place:

A & B	B & C	C & D
A & C	B & D	A & D

Allow the students to try to figure out how many handshakes take place if 12, or if n people shake hands. It turns out that the answer is $1 + 2 + \cdots + (n - 1)$ (as illustrated above), the partial sum of an arithmetic sequence.

11.2 Exercises

1–4 ■ A sequence is given.

(a) Find the first five terms of the sequence.

(b) What is the common difference d?

(c) Graph the terms you found in (a).

1. $a_n = 5 + 2(n - 1)$ **2.** $a_n = 3 - 4(n - 1)$

3. $a_n = \frac{5}{2} - (n - 1)$ **4.** $a_n = \frac{1}{2}(n - 1)$

5–8 ■ Find the nth term of the arithmetic sequence with given first term a and common difference d. What is the 10th term?

5. $a = 3, d = 5$ **6.** $a = -6, d = 3$

7. $a = \frac{5}{2}, d = -\frac{1}{2}$ **8.** $a = \sqrt{3}, d = \sqrt{3}$

9–16 ■ Determine whether the sequence is arithmetic. If it is arithmetic, find the common difference.

9. 5, 8, 11, 14, . . . **10.** 3, 6, 9, 13, . . .

11. 2, 4, 8, 16, . . . **12.** 2, 4, 6, 8, . . .

13. $3, \frac{3}{2}, 0, -\frac{3}{2}, \ldots$ **14.** ln 2, ln 4, ln 8, ln 16, . . .

15. 2.6, 4.3, 6.0, 7.7, . . . **16.** $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$

17–22 ■ Find the first five terms of the sequence and determine if it is arithmetic. If it is arithmetic, find the common difference and express the nth term of the sequence in the standard form $a_n = a + (n - 1)d$.

17. $a_n = 4 + 7n$ **18.** $a_n = 4 + 2^n$

19. $a_n = \dfrac{1}{1 + 2n}$ **20.** $a_n = 1 + \dfrac{n}{2}$

21. $a_n = 6n - 10$ **22.** $a_n = 3 + (-1)^n n$

23–32 ■ Determine the common difference, the fifth term, the nth term, and the 100th term of the arithmetic sequence.

23. 2, 5, 8, 11, . . . **24.** 1, 5, 9, 13, . . .

25. 4, 9, 14, 19, . . . **26.** 11, 8, 5, 2, . . .

27. $-12, -8, -4, 0, \ldots$ **28.** $\frac{7}{6}, \frac{5}{3}, \frac{13}{6}, \frac{8}{3}, \ldots$

29. 25, 26.5, 28, 29.5, . . . **30.** 15, 12.3, 9.6, 6.9, . . .

31. $2, 2 + s, 2 + 2s, 2 + 3s, \ldots$

32. $-t, -t + 3, -t + 6, -t + 9, \ldots$

33. The tenth term of an arithmetic sequence is $\frac{55}{2}$, and the second term is $\frac{7}{2}$. Find the first term.

34. The 12th term of an arithmetic sequence is 32, and the fifth term is 18. Find the 20th term.

35. The 100th term of an arithmetic sequence is 98, and the common difference is 2. Find the first three terms.

36. The 20th term of an arithmetic sequence is 101, and the common difference is 3. Find a formula for the nth term.

37. Which term of the arithmetic sequence 1, 4, 7, . . . is 88?

38. The first term of an arithmetic sequence is 1, and the common difference is 4. Is 11,937 a term of this sequence? If so, which term is it?

39–44 ■ Find the partial sum S_n of the arithmetic sequence that satisfies the given conditions.

39. $a = 1, d = 2, n = 10$ **40.** $a = 3, d = 2, n = 12$

41. $a = 4, d = 2, n = 20$ **42.** $a = 100, d = -5, n = 8$

43. $a_1 = 55, d = 12, n = 10$ **44.** $a_2 = 8, a_5 = 9.5, n = 15$

45–50 ■ A partial sum of an arithmetic sequence is given. Find the sum.

45. $1 + 5 + 9 + \cdots + 401$

46. $-3 + \left(-\frac{3}{2}\right) + 0 + \frac{3}{2} + 3 + \cdots + 30$

47. $0.7 + 2.7 + 4.7 + \cdots + 56.7$

48. $-10 - 9.9 - 9.8 - \cdots - 0.1$

49. $\displaystyle\sum_{k=0}^{10} (3 + 0.25k)$ **50.** $\displaystyle\sum_{n=0}^{20} (1 - 2n)$

51. Show that a right triangle whose sides are in arithmetic progression is similar to a 3–4–5 triangle.

52. Find the product of the numbers

$$10^{1/10}, 10^{2/10}, 10^{3/10}, 10^{4/10}, \ldots, 10^{19/10}$$

53. A sequence is **harmonic** if the reciprocals of the terms of the sequence form an arithmetic sequence. Determine whether the following sequence is harmonic:

$$1, \tfrac{3}{5}, \tfrac{3}{7}, \tfrac{1}{3}, \ldots$$

54. The **harmonic mean** of two numbers is the reciprocal of the average of the reciprocals of the two numbers. Find the harmonic mean of 3 and 5.

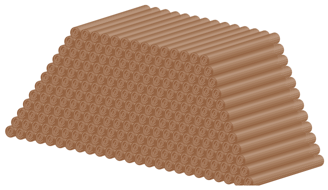
55. An arithmetic sequence has first term $a = 5$ and common difference $d = 2$. How many terms of this sequence must be added to get 2700?

56. An arithmetic sequence has first term $a_1 = 1$ and fourth term $a_4 = 16$. How many terms of this sequence must be added to get 2356?

Applications

57. Depreciation The purchase value of an office computer is \$12,500. Its annual depreciation is \$1875. Find the value of the computer after 6 years.

58. Poles in a Pile Telephone poles are stored in a pile with 25 poles in the first layer, 24 in the second, and so on. If there are 12 layers, how many telephone poles does the pile contain?



59. Salary Increases A man gets a job with a salary of \$30,000 a year. He is promised a \$2300 raise each subsequent year. Find his total earnings for a 10-year period.

60. Drive-In Theater A drive-in theater has spaces for 20 cars in the first parking row, 22 in the second, 24 in the third, and so on. If there are 21 rows in the theater, find the number of cars that can be parked.

61. Theater Seating An architect designs a theater with 15 seats in the first row, 18 in the second, 21 in the third, and so on. If the theater is to have a seating capacity of 870, how many rows must the architect use in his design?

62. Falling Ball When an object is allowed to fall freely near the surface of the earth, the gravitational pull is such that the object falls 16 ft in the first second, 48 ft in the next second, 80 ft in the next second, and so on.

(a) Find the total distance a ball falls in 6 s.

(b) Find a formula for the total distance a ball falls in n seconds.

63. The Twelve Days of Christmas In the well-known song "The Twelve Days of Christmas," a person gives his sweetheart k gifts on the kth day for each of the 12 days of Christmas. The person also repeats each gift identically on each subsequent day. Thus, on the 12th day the sweetheart receives a gift for the first day, 2 gifts for the second, 3 gifts for the third, and so on. Show that the number of gifts received on the 12th day is a partial sum of an arithmetic sequence. Find this sum.

Discovery • Discussion

64. Arithmetic Means The **arithmetic mean** (or average) of two numbers a and b is

$$m = \frac{a + b}{2}$$

Note that m is the same distance from a as from b, so a, m, b is an arithmetic sequence. In general, if $m_1, m_2, \ldots, m_k$ are equally spaced between a and b so that

$$a, m_1, m_2, \ldots, m_k, b$$

is an arithmetic sequence, then $m_1, m_2, \ldots, m_k$ are called k arithmetic means between a and b.

(a) Insert two arithmetic means between 10 and 18.

(b) Insert three arithmetic means between 10 and 18.

(c) Suppose a doctor needs to increase a patient's dosage of a certain medicine from 100 mg to 300 mg per day in five equal steps. How many arithmetic means must be inserted between 100 and 300 to give the progression of daily doses, and what are these means?

SUGGESTED TIME AND EMPHASIS

$\frac{1}{2}$–1 class.
Optional material.

11.3 Geometric Sequences

In this section we study geometric sequences. This type of sequence occurs frequently in applications to finance, population growth, and other fields.

Geometric Sequences

Recall that an arithmetic sequence is generated when we repeatedly add a number d to an initial term a. A *geometric* sequence is generated when we start with a number a and repeatedly *multiply* by a fixed nonzero constant r.

POINTS TO STRESS

1. Definition of geometric series.
2. Definition of infinite series.
3. Formulas for sums of finite and infinite geometric series.

Definition of a Geometric Sequence

A **geometric sequence** is a sequence of the form

$$a, ar, ar^2, ar^3, ar^4, \dots$$

The number a is the **first term**, and r is the **common ratio** of the sequence. The ***n*th term** of a geometric sequence is given by

$$a_n = ar^{n-1}$$

The number r is called the common ratio because the ratio of any two consecutive terms of the sequence is r.

Example 1 Geometric Sequences

(a) If $a = 3$ and $r = 2$, then we have the geometric sequence

$$3, \quad 3 \cdot 2, \quad 3 \cdot 2^2, \quad 3 \cdot 2^3, \quad 3 \cdot 2^4, \quad \dots$$

or

$$3, 6, 12, 24, 48, \dots$$

Notice that the ratio of any two consecutive terms is $r = 2$. The nth term is $a_n = 3(2)^{n-1}$.

(b) The sequence

$$2, -10, 50, -250, 1250, \dots$$

is a geometric sequence with $a = 2$ and $r = -5$. When r is negative, the terms of the sequence alternate in sign. The nth term is $a_n = 2(-5)^{n-1}$.

(c) The sequence

$$1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \dots$$

is a geometric sequence with $a = 1$ and $r = \frac{1}{3}$. The nth term is $a_n = 1\left(\frac{1}{3}\right)^{n-1}$.

(d) The graph of the geometric sequence $a_n = \frac{1}{5} \cdot 2^{n-1}$ is shown in Figure 1. Notice that the points in the graph lie on the graph of the exponential function $y = \frac{1}{5} \cdot 2^{x-1}$.

If $0 < r < 1$, then the terms of the geometric sequence ar^{n-1} decrease, but if $r > 1$, then the terms increase. (What happens if $r = 1$?) ■

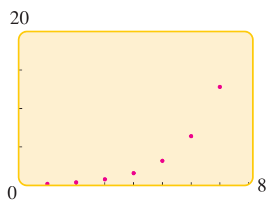


Figure 1

Geometric sequences occur naturally. Here is a simple example. Suppose a ball has elasticity such that when it is dropped it bounces up one-third of the distance it has fallen. If this ball is dropped from a height of 2 m, then it bounces up to a height of $2\left(\frac{1}{3}\right) = \frac{2}{3}$ m. On its second bounce, it returns to a height of $\left(\frac{2}{3}\right)\left(\frac{1}{3}\right) = \frac{2}{9}$ m, and so on (see Figure 2). Thus, the height h_n that the ball reaches on its nth bounce is given by the geometric sequence

$$h_n = \tfrac{2}{3}\left(\tfrac{1}{3}\right)^{n-1} = 2\left(\tfrac{1}{3}\right)^n$$

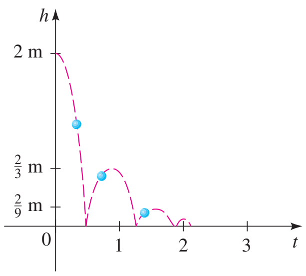


Figure 2

We can find the nth term of a geometric sequence if we know any two terms, as the following examples show.

DRILL QUESTION

Find the 100th partial sum of the sequence 2, 6, 18, 54, 162, ...

Answer

$$\frac{2(1 - 3^{100})}{1 - 3} \approx 5.154 \times 10^{47}$$

ALTERNATE EXAMPLE 1a

Find the first four terms and the nth term of the geometric sequence with $a = 2$ and $r = 3$.

ANSWER

2, 6, 18, 54, $a_n = 2(3)^{n-1}$

ALTERNATE EXAMPLE 1c

Find the nth term of the geometric sequence 2, 8, 32, 128, ...

ANSWER

$a_n = 2(4)^{n-1}$

SAMPLE QUESTION

Text Question

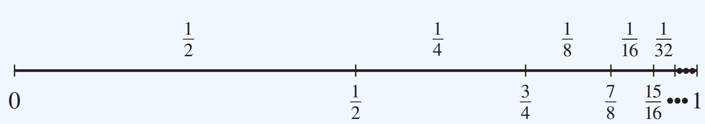
What distinguishes a "geometric sequence" from an arbitrary "sequence"?

Answer

Successive terms in a geometric sequence have a common ratio.

IN-CLASS MATERIALS

Represent a geometric series visually. For example, a geometric view of the equation $\sum_{n=1}^{\infty} 1/2^n = 1$ is given below.



ALTERNATE EXAMPLE 2
Find the eighth term of the geometric sequence 7, 42, 252, ...

ANSWER
1,959,552

ALTERNATE EXAMPLE 3
The third term of a geometric series is $\frac{75}{16}$, and the sixth term is $\frac{9375}{1024}$. Find the fifth term.

ANSWER
$\frac{1875}{256}$


Michael Ng

Srinivasa Ramanujan (1887–1920) was born into a poor family in the small town of Kumbakonam in India. Self-taught in mathematics, he worked in virtual isolation from other mathematicians. At the age of 25 he wrote a letter to G. H. Hardy, the leading British mathematician at the time, listing some of his discoveries. Hardy immediately recognized Ramanujan's genius and for the next six years the two worked together in London until Ramanujan fell ill and returned to his hometown in India, where he died a year later. Ramanujan was a genius with phenomenal ability to see hidden patterns in the properties of numbers. Most of his discoveries were written as complicated infinite series, the importance of which was not recognized until many years after his death. In the last year of his life he wrote 130 pages of mysterious formulas, many of which still defy proof. Hardy tells the story that when he visited Ramanujan in a hospital and arrived in a taxi, he remarked to Ramanujan that the cab's number, 1729, was uninteresting. Ramanujan replied "No, it is a very interesting number. It is the smallest number expressible as the sum of two cubes in two different ways." (See Problem 23 on page 144.)

Example 2 Finding Terms of a Geometric Sequence

Find the eighth term of the geometric sequence 5, 15, 45, ...

Solution To find a formula for the nth term of this sequence, we need to find a and r. Clearly, $a = 5$. To find r, we find the ratio of any two consecutive terms. For instance, $r = \frac{45}{15} = 3$. Thus

$$a_n = 5(3)^{n-1}$$

The eighth term is $a_8 = 5(3)^{8-1} = 5(3)^7 = 10{,}935$. ■

Example 3 Finding Terms of a Geometric Sequence

The third term of a geometric sequence is $\frac{63}{4}$, and the sixth term is $\frac{1701}{32}$. Find the fifth term.

Solution Since this sequence is geometric, its nth term is given by the formula $a_n = ar^{n-1}$. Thus

$$a_3 = ar^{3-1} = ar^2$$
$$a_6 = ar^{6-1} = ar^5$$

From the values we are given for these two terms, we get the following system of equations:

$$\begin{cases} \frac{63}{4} = ar^2 \\ \frac{1701}{32} = ar^5 \end{cases}$$

We solve this system by dividing.

$$\frac{ar^5}{ar^2} = \frac{\frac{1701}{32}}{\frac{63}{4}}$$

$$r^3 = \frac{27}{8} \qquad \text{Simplify}$$

$$r = \frac{3}{2} \qquad \text{Take cube root of each side}$$

Substituting for r in the first equation, $\frac{63}{4} = ar^2$, gives

$$\frac{63}{4} = a\left(\frac{3}{2}\right)^2$$

$$a = 7 \qquad \text{Solve for } a$$

It follows that the nth term of this sequence is

$$a_n = 7\left(\frac{3}{2}\right)^{n-1}$$

Thus, the fifth term is

$$a_5 = 7\left(\frac{3}{2}\right)^{5-1} = 7\left(\frac{3}{2}\right)^4 = \frac{567}{16}$$ ■

Partial Sums of Geometric Sequences

For the geometric sequence $a, ar, ar^2, ar^3, ar^4, \ldots, ar^{n-1}, \ldots$, the nth partial sum is

$$S_n = \sum_{k=1}^{n} ar^{k-1} = a + ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n-1}$$

IN-CLASS MATERIALS

One of the consequences of parenthood is that it often causes otherwise rational adults to say things like, "If I told you once, I've told you one *hundred* times!" Assume that this was true, and a child was told something on a Friday. The parental rule means that the child was previously told a fact 100 times, say on Thursday. Thus, on Wednesday, the child must have been told the information 100 times for every time on Thursday, or 10,000 times. Use the methods of this section to determine how many times the child has been told since Monday.

To find a formula for S_n, we multiply S_n by r and subtract from S_n:

$$\begin{aligned} S_n &= a + ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n-1} \\ rS_n &= \phantom{a + {}} ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n-1} + ar^n \\ \hline S_n - rS_n &= a - ar^n \end{aligned}$$

So,

$$S_n(1 - r) = a(1 - r^n)$$

$$S_n = \frac{a(1 - r^n)}{1 - r} \qquad (r \neq 1)$$

We summarize this result.

Partial Sums of a Geometric Sequence

For the geometric sequence $a_n = ar^{n-1}$, the ***n*th partial sum**

$$S_n = a + ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n-1} \qquad (r \neq 1)$$

is given by

$$S_n = a\frac{1 - r^n}{1 - r}$$

Example 4 Finding a Partial Sum of a Geometric Sequence

Find the sum of the first five terms of the geometric sequence

$$1, 0.7, 0.49, 0.343, \ldots$$

Solution The required sum is the sum of the first five terms of a geometric sequence with $a = 1$ and $r = 0.7$. Using the formula for S_n with $n = 5$, we get

$$S_5 = 1 \cdot \frac{1 - (0.7)^5}{1 - 0.7} = 2.7731$$

Thus, the sum of the first five terms of this sequence is 2.7731. ■

ALTERNATE EXAMPLE 4
Find the sum of the first 6 terms of the geometric sequence
1, 0.6, 0.36, 0.216, ...

ANSWER
2.38336

Example 5 Finding a Partial Sum of a Geometric Sequence

Find the sum $\sum_{k=1}^{5} 7\left(-\frac{2}{3}\right)^k$.

Solution The given sum is the fifth partial sum of a geometric sequence with first term $a = 7\left(-\frac{2}{3}\right) = -\frac{14}{3}$ and common ratio $r = -\frac{2}{3}$. Thus, by the formula for S_n, we have

$$S_5 = -\frac{14}{3} \cdot \frac{1 - \left(-\frac{2}{3}\right)^5}{1 - \left(-\frac{2}{3}\right)} = -\frac{14}{3} \cdot \frac{1 + \frac{32}{243}}{\frac{5}{3}} = -\frac{770}{243}$$ ■

ALTERNATE EXAMPLE 5
Find the sum

$$\sum_{k=1}^{4} 7\left(-\frac{2}{3}\right)^k.$$

ANSWER

$$-\frac{182}{81}$$

What Is an Infinite Series?

An expression of the form

$$a_1 + a_2 + a_3 + a_4 + \cdots$$

IN-CLASS MATERIALS

Introduce the idea that for any two real numbers A and B, the statement $A = B$ is the same as saying that for any integer N, $|A - B| < 1/N$. Now use this idea to show that $0.9999\ldots = 0.\overline{9} = 1$, since $|1 - 0.\overline{9}| < |1 - 0.\underbrace{99999\ldots99}_{N \text{ nines}}| = \underbrace{0.00000\ldots0001}_{N-1 \text{ zero}} = 10^{-N} = \frac{1}{10^N}$. Then use the usual approach to define $0.\overline{9}$ as $\sum_{n=1}^{\infty} 9/10^n$ and show directly that $0.\overline{9} = 1$. Generalize this result by pointing out that *any* repeating decimal ($0.\overline{3}$, $0.\overline{412}$, $0.24\overline{621}$) can be written as a geometric series, and can thus be written as a fraction using the formula for a geometric series. Demonstrate with $0.\overline{412} = \left(\frac{1}{1 - 1/1000}\right) = \frac{412}{999}$.

is called an **infinite series**. The dots mean that we are to continue the addition indefinitely. What meaning can we attach to the sum of infinitely many numbers? It seems at first that it is not possible to add infinitely many numbers and arrive at a finite number. But consider the following problem. You have a cake and you want to eat it by first eating half the cake, then eating half of what remains, then again eating half of what remains. This process can continue indefinitely because at each stage some of the cake remains. (See Figure 3.)

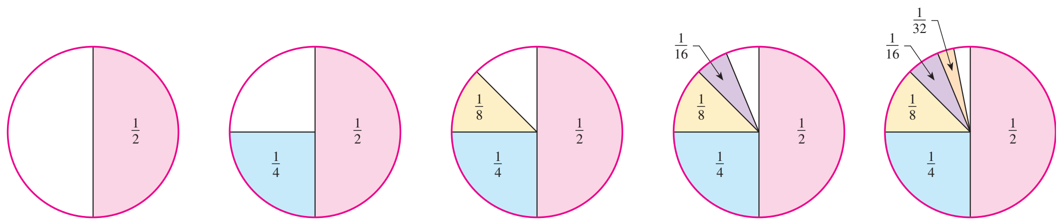


Figure 3

Does this mean that it's impossible to eat all of the cake? Of course not. Let's write down what you have eaten from this cake:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n} + \cdots$$

This is an infinite series, and we note two things about it: First, from Figure 3 it's clear that no matter how many terms of this series we add, the total will never exceed 1. Second, the more terms of this series we add, the closer the sum is to 1 (see Figure 3). This suggests that the number 1 can be written as the sum of infinitely many smaller numbers:

$$1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n} + \cdots$$

To make this more precise, let's look at the partial sums of this series:

$$\begin{aligned} S_1 &= \frac{1}{2} &&= \frac{1}{2} \\ S_2 &= \frac{1}{2} + \frac{1}{4} &&= \frac{3}{4} \\ S_3 &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} &&= \frac{7}{8} \\ S_4 &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} &&= \frac{15}{16} \end{aligned}$$

and, in general (see Example 5 of Section 11.1),

$$S_n = 1 - \frac{1}{2^n}$$

As n gets larger and larger, we are adding more and more of the terms of this series. Intuitively, as n gets larger, S_n gets closer to the sum of the series. Now notice that as n gets large, $1/2^n$ gets closer and closer to 0. Thus, S_n gets close to $1 - 0 = 1$. Using the notation of Section 3.6, we can write

$$S_n \to 1 \qquad \text{as} \qquad n \to \infty$$

In general, if S_n gets close to a finite number S as n gets large, we say that S is the **sum of the infinite series**.

Infinite Geometric Series

An **infinite geometric series** is a series of the form

$$a + ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n-1} + \cdots$$

We can apply the reasoning used earlier to find the sum of an infinite geometric series. The nth partial sum of such a series is given by the formula

$$S_n = a\frac{1 - r^n}{1 - r} \qquad (r \neq 1)$$

It can be shown that if $|r| < 1$, then r^n gets close to 0 as n gets large (you can easily convince yourself of this using a calculator). It follows that S_n gets close to $a/(1 - r)$ as n gets large, or

$$S_n \to \frac{a}{1 - r} \qquad \text{as} \qquad n \to \infty$$

Thus, the sum of this infinite geometric series is $a/(1 - r)$.

Here is another way to arrive at the formula for the sum of an infinite geometric series:

$$\begin{aligned} S &= a + ar + ar^2 + ar^3 + \cdots \\ &= a + r(a + ar + ar^2 + \cdots) \\ &= a + rS \end{aligned}$$

Solve the equation $S = a + rS$ for S to get

$$\begin{aligned} S - rS &= a \\ (1 - r)S &= a \\ S &= \frac{a}{1 - r} \end{aligned}$$

Sum of an Infinite Geometric Series

If $|r| < 1$, then the infinite geometric series

$$a + ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n-1} + \cdots$$

has the sum

$$S = \frac{a}{1 - r}$$

Example 6 Finding the Sum of an Infinite Geometric Series

Find the sum of the infinite geometric series

$$2 + \frac{2}{5} + \frac{2}{25} + \frac{2}{125} + \cdots + \frac{2}{5^n} + \cdots$$

Solution We use the formula for the sum of an infinite geometric series. In this case, $a = 2$ and $r = \frac{1}{5}$. Thus, the sum of this infinite series is

$$S = \frac{2}{1 - \frac{1}{5}} = \frac{5}{2}$$

■

Example 7 Writing a Repeated Decimal as a Fraction

Find the fraction that represents the rational number $2.3\overline{51}$.

Solution This repeating decimal can be written as a series:

$$\frac{23}{10} + \frac{51}{1000} + \frac{51}{100{,}000} + \frac{51}{10{,}000{,}000} + \frac{51}{1{,}000{,}000{,}000} + \cdots$$

EXAMPLE
Zeno's Paradox: In order to walk to a wall across the room, you have to first walk halfway to the wall, and in order to do that you have to walk halfway to the halfway point, etc. This process can be viewed as finding the sum of $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1$. The sum of this infinite series is 1.

ALTERNATE EXAMPLE 6
Find the sum, S, of the infinite geometric series

$$5 + \frac{5}{9} + \frac{5}{81} + \frac{5}{729} + \cdots + \frac{5}{9^n} \cdots + \cdots$$

ANSWER
$S = \dfrac{45}{8}$

ALTERNATE EXAMPLE 7
Find the fraction that represents the rational number $1.\overline{873}$.

ANSWER
$\dfrac{371}{198}$

IN-CLASS MATERIALS

Explore the "middle third" Cantor set with the class: This set is defined as the set of points obtained by taking the interval $[0, 1]$, throwing out the middle third to obtain $\left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$, throwing out the middle third of each remaining interval to obtain $\left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$, and repeating this process ad infinitum. Point out that there are infinitely many points left after this process. (If a point winds up as the endpoint of an interval, it never gets removed, and new intervals are created with every step.) Now calculate the total length of the sections that were thrown away: $\frac{1}{3} + 2\cdot\frac{1}{9} + 4\cdot\frac{1}{27} + \cdots = \sum_{k=0}^{\infty} \frac{2k}{3k + 1} = 1$.

Notice the apparent paradox: We've thrown away a total interval of length 1, but still infinitely many points remain. (See also Exercise 77.)

After the first term, the terms of this series form an infinite geometric series with

$$a = \frac{51}{1000} \quad \text{and} \quad r = \frac{1}{100}$$

Thus, the sum of this part of the series is

$$S = \frac{\frac{51}{1000}}{1 - \frac{1}{100}} = \frac{\frac{51}{1000}}{\frac{99}{100}} = \frac{51}{1000} \cdot \frac{100}{99} = \frac{51}{990}$$

So, $$2.3\overline{51} = \frac{23}{10} + \frac{51}{990} = \frac{2328}{990} = \frac{388}{165}$$ ■

11.3 Exercises

1–4 ■ The nth term of a sequence is given.

(a) Find the first five terms of the sequence.

(b) What is the common ratio r?

(c) Graph the terms you found in (a).

1. $a_n = 5(2)^{n-1}$ **2.** $a_n = 3(-4)^{n-1}$

3. $a_n = \frac{5}{2}\left(-\frac{1}{2}\right)^{n-1}$ **4.** $a_n = 3^{n-1}$

5–8 ■ Find the nth term of the geometric sequence with given first term a and common ratio r. What is the fourth term?

5. $a = 3, \quad r = 5$ **6.** $a = -6, \quad r = 3$

7. $a = \frac{5}{2}, \quad r = -\frac{1}{2}$ **8.** $a = \sqrt{3}, \quad r = \sqrt{3}$

9–16 ■ Determine whether the sequence is geometric. If it is geometric, find the common ratio.

9. $2, 4, 8, 16, \ldots$ **10.** $2, 6, 18, 36, \ldots$

11. $3, \frac{3}{2}, \frac{3}{4}, \frac{3}{8}, \ldots$ **12.** $27, -9, 3, -1, \ldots$

13. $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$ **14.** $e^2, e^4, e^6, e^8, \ldots$

15. $1.0, 1.1, 1.21, 1.331, \ldots$ **16.** $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \ldots$

17–22 ■ Find the first five terms of the sequence and determine if it is geometric. If it is geometric, find the common ratio and express the nth term of the sequence in the standard form $a_n = ar^{n-1}$.

17. $a_n = 2(3)^n$ **18.** $a_n = 4 + 3^n$

19. $a_n = \dfrac{1}{4^n}$ **20.** $a_n = (-1)^n 2^n$

21. $a_n = \ln(5^{n-1})$ **22.** $a_n = n^n$

23–32 ■ Determine the common ratio, the fifth term, and the nth term of the geometric sequence.

23. $2, 6, 18, 54, \ldots$ **24.** $7, \frac{14}{3}, \frac{28}{9}, \frac{56}{27}, \ldots$

25. $0.3, -0.09, 0.027, -0.0081, \ldots$

26. $1, \sqrt{2}, 2, 2\sqrt{2}, \ldots$

27. $144, -12, 1, -\frac{1}{12}, \ldots$ **28.** $-8, -2, -\frac{1}{2}, -\frac{1}{8}, \ldots$

29. $3, 3^{5/3}, 3^{7/3}, 27, \ldots$ **30.** $t, \dfrac{t^2}{2}, \dfrac{t^3}{4}, \dfrac{t^4}{8}, \ldots$

31. $1, s^{2/7}, s^{4/7}, s^{6/7}, \ldots$ **32.** $5, 5^{c+1}, 5^{2c+1}, 5^{3c+1}, \ldots$

33. The first term of a geometric sequence is 8, and the second term is 4. Find the fifth term.

34. The first term of a geometric sequence is 3, and the third term is $\frac{4}{3}$. Find the fifth term.

35. The common ratio in a geometric sequence is $\frac{2}{5}$, and the fourth term is $\frac{5}{2}$. Find the third term.

36. The common ratio in a geometric sequence is $\frac{3}{2}$, and the fifth term is 1. Find the first three terms.

37. Which term of the geometric sequence $2, 6, 18, \ldots$ is 118,098?

38. The second and the fifth terms of a geometric sequence are 10 and 1250, respectively. Is 31,250 a term of this sequence? If so, which term is it?

39–42 ■ Find the partial sum S_n of the geometric sequence that satisfies the given conditions.

39. $a = 5, \quad r = 2, \quad n = 6$ **40.** $a = \frac{2}{3}, \quad r = \frac{1}{3}, \quad n = 4$

41. $a_3 = 28, \quad a_6 = 224, \quad n = 6$

42. $a_2 = 0.12, \quad a_5 = 0.00096, \quad n = 4$

43–46 ■ Find the sum.

43. $1 + 3 + 9 + \cdots + 2187$

44. $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots - \frac{1}{512}$

45. $\displaystyle\sum_{k=0}^{10} 3\left(\tfrac{1}{2}\right)^k$ **46.** $\displaystyle\sum_{j=0}^{5} 7\left(\tfrac{3}{2}\right)^j$

47–54 ■ Find the sum of the infinite geometric series.

47. $1 + \dfrac{1}{3} + \dfrac{1}{9} + \dfrac{1}{27} + \cdots$ **48.** $1 - \dfrac{1}{2} + \dfrac{1}{4} - \dfrac{1}{8} + \cdots$

IN-CLASS MATERIALS

Do Exercise 71 (the “St. Ives” problem) with the students. After obtaining the partial sum solution, point out that traditionally people give the answer 1. The problem says “As I was going to St. Ives. . . .” So presumably all the other people were going the other way, away from St. Ives!

49. $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \cdots$

50. $\frac{2}{5} + \frac{4}{25} + \frac{8}{125} + \cdots$

51. $\frac{1}{3^6} + \frac{1}{3^8} + \frac{1}{3^{10}} + \frac{1}{3^{12}} + \cdots$

52. $3 - \frac{3}{2} + \frac{3}{4} - \frac{3}{8} + \cdots$

53. $-\frac{100}{9} + \frac{10}{3} - 1 + \frac{3}{10} - \cdots$

54. $\frac{1}{\sqrt{2}} + \frac{1}{2} + \frac{1}{2\sqrt{2}} + \frac{1}{4} + \cdots$

55–60 ■ Express the repeating decimal as a fraction.

55. $0.777\ldots$

56. $0.2\overline{53}$

57. $0.030303\ldots$

58. $2.1\overline{125}$

59. $0.\overline{112}$

60. $0.123123123\ldots$

61. If the numbers $a_1, a_2, \ldots, a_n$ form a geometric sequence, then $a_2, a_3, \ldots, a_{n-1}$ are **geometric means** between a_1 and a_n. Insert three geometric means between 5 and 80.

62. Find the sum of the first ten terms of the sequence

$$a + b,\ a^2 + 2b,\ a^3 + 3b,\ a^4 + 4b, \ldots$$

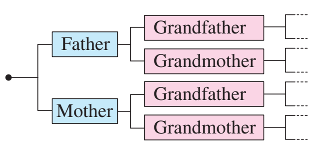
Applications

63. Depreciation A construction company purchases a bulldozer for \$160,000. Each year the value of the bulldozer depreciates by 20% of its value in the preceding year. Let V_n be the value of the bulldozer in the nth year. (Let $n = 1$ be the year the bulldozer is purchased.)

(a) Find a formula for V_n.

(b) In what year will the value of the bulldozer be less than \$100,000?

64. Family Tree A person has two parents, four grandparents, eight great-grandparents, and so on. How many ancestors does a person have 15 generations back?

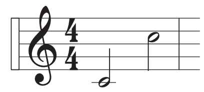


65. Bouncing Ball A ball is dropped from a height of 80 ft. The elasticity of this ball is such that it rebounds three-fourths of the distance it has fallen. How high does the ball rebound on the fifth bounce? Find a formula for how high the ball rebounds on the nth bounce.

66. Bacteria Culture A culture initially has 5000 bacteria, and its size increases by 8% every hour. How many bacteria are present at the end of 5 hours? Find a formula for the number of bacteria present after n hours.

67. Mixing Coolant A truck radiator holds 5 gal and is filled with water. A gallon of water is removed from the radiator and replaced with a gallon of antifreeze; then, a gallon of the mixture is removed from the radiator and again replaced by a gallon of antifreeze. This process is repeated indefinitely. How much water remains in the tank after this process is repeated 3 times? 5 times? n times?

68. Musical Frequencies The frequencies of musical notes (measured in cycles per second) form a geometric sequence. Middle C has a frequency of 256, and the C that is an octave higher has a frequency of 512. Find the frequency of C two octaves below middle C.



69. Bouncing Ball A ball is dropped from a height of 9 ft. The elasticity of the ball is such that it always bounces up one-third the distance it has fallen.

(a) Find the total distance the ball has traveled at the instant it hits the ground the fifth time.

(b) Find a formula for the total distance the ball has traveled at the instant it hits the ground the nth time.

70. Geometric Savings Plan A very patient woman wishes to become a billionaire. She decides to follow a simple scheme: She puts aside 1 cent the first day, 2 cents the second day, 4 cents the third day, and so on, doubling the number of cents each day. How much money will she have at the end of 30 days? How many days will it take this woman to realize her wish?

71. St. Ives The following is a well-known children's rhyme:

As I was going to St. Ives
I met a man with seven wives;
Every wife had seven sacks;
Every sack had seven cats;
Every cat had seven kits;
Kits, cats, sacks, and wives,
How many were going to St. Ives?

Assuming that the entire group is actually going to St. Ives, show that the answer to the question in the rhyme is a partial sum of a geometric sequence, and find the sum.

72. Drug Concentration A certain drug is administered once a day. The concentration of the drug in the patient's bloodstream increases rapidly at first, but each successive dose has less effect than the preceding one. The total amount of the drug (in mg) in the bloodstream after the nth dose is given by

$$\sum_{k=1}^{n} 50\left(\tfrac{1}{2}\right)^{k-1}$$

(a) Find the amount of the drug in the bloodstream after $n = 10$ days.

(b) If the drug is taken on a long-term basis, the amount in the bloodstream is approximated by the infinite series $\sum_{k=1}^{\infty} 50\left(\frac{1}{2}\right)^{k-1}$. Find the sum of this series.

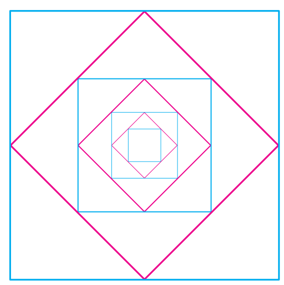
73. Bouncing Ball A certain ball rebounds to half the height from which it is dropped. Use an infinite geometric series to approximate the total distance the ball travels, after being dropped from 1 m above the ground, until it comes to rest.

74. Bouncing Ball If the ball in Exercise 73 is dropped from a height of 8 ft, then 1 s is required for its first complete bounce—from the instant it first touches the ground until it next touches the ground. Each subsequent complete bounce requires $1/\sqrt{2}$ as long as the preceding complete bounce. Use an infinite geometric series to estimate the time interval from the instant the ball first touches the ground until it stops bouncing.

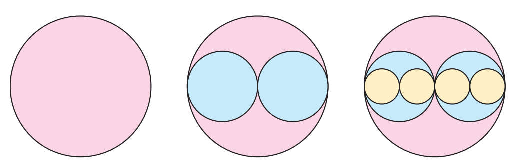
75. Geometry The midpoints of the sides of a square of side 1 are joined to form a new square. This procedure is repeated for each new square. (See the figure.)

(a) Find the sum of the areas of all the squares.

(b) Find the sum of the perimeters of all the squares.

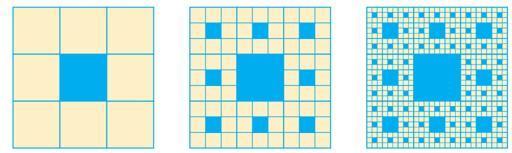


76. Geometry A circular disk of radius R is cut out of paper, as shown in figure (a). Two disks of radius $\frac{1}{2}R$ are cut out of paper and placed on top of the first disk, as in figure (b), and then four disks of radius $\frac{1}{4}R$ are placed on these two disks (figure (c)). Assuming that this process can be repeated indefinitely, find the total area of all the disks.



(a) (b) (c)

77. Geometry A yellow square of side 1 is divided into nine smaller squares, and the middle square is colored blue as shown in the figure. Each of the smaller yellow squares is in turn divided into nine squares, and each middle square is colored blue. If this process is continued indefinitely, what is the total area colored blue?



Discovery • Discussion

78. Arithmetic or Geometric? The first four terms of a sequence are given. Determine whether these terms can be the terms of an arithmetic sequence, a geometric sequence, or neither. Find the next term if the sequence is arithmetic or geometric.

(a) $5, -3, 5, -3, \ldots$ **(b)** $\frac{1}{3}, 1, \frac{5}{3}, \frac{7}{3}, \ldots$

(c) $\sqrt{3}, 3, 3\sqrt{3}, 9, \ldots$ **(d)** $1, -1, 1, -1, \ldots$

(e) $2, -1, \frac{1}{2}, 2, \ldots$ **(f)** $x - 1, x, x + 1, x + 2, \ldots$

(g) $-3, -\frac{3}{2}, 0, \frac{3}{2}, \ldots$ **(h)** $\sqrt{5}, \sqrt[3]{5}, \sqrt[6]{5}, 1, \ldots$

79. Reciprocals of a Geometric Sequence If $a_1, a_2, a_3, \ldots$ is a geometric sequence with common ratio r, show that the sequence

$$\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \ldots$$

is also a geometric sequence, and find the common ratio.

80. Logarithms of a Geometric Sequence If $a_1, a_2, a_3, \ldots$ is a geometric sequence with a common ratio $r > 0$ and $a_1 > 0$, show that the sequence

$$\log a_1, \log a_2, \log a_3, \ldots$$

is an arithmetic sequence, and find the common difference.

81. Exponentials of an Arithmetic Sequence If $a_1, a_2, a_3, \ldots$ is an arithmetic sequence with common difference d, show that the sequence

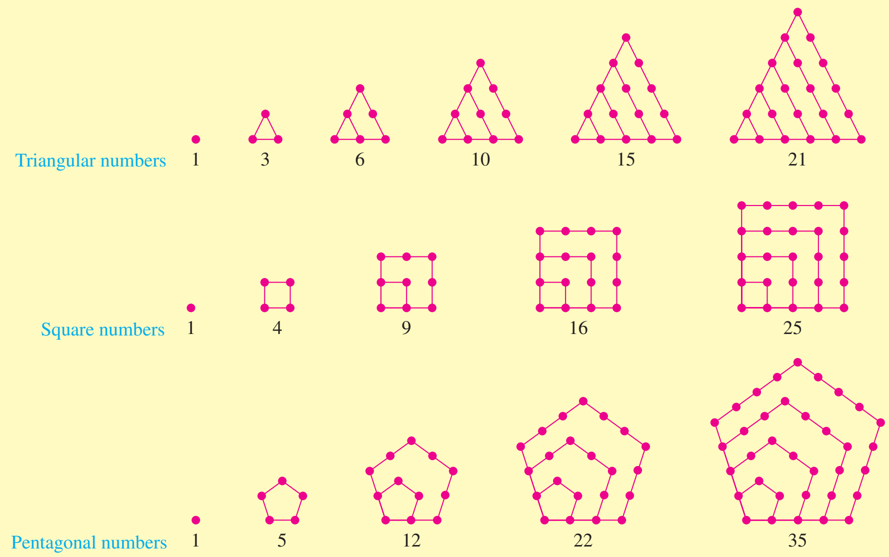
$$10^{a_1}, 10^{a_2}, 10^{a_3}, \ldots$$

is a geometric sequence, and find the common ratio.

DISCOVERY PROJECT

Finding Patterns

The ancient Greeks studied triangular numbers, square numbers, pentagonal numbers, and other **polygonal numbers**, like those shown in the figure.



To find a pattern for such numbers, we construct a **first difference sequence** by taking differences of successive terms; we repeat the process to get a **second difference sequence, third difference sequence**, and so on. For the sequence of triangular numbers T_n we get the following **difference table**:

Triangular numbers	1		3		6		10		15		21
First differences		2		3		4		5		6	
Second differences			1		1		1		1		

We stop at the second difference sequence because it's a constant sequence. Assuming that this sequence will continue to have constant value 1, we can work backward from the bottom row to find more terms of the first difference sequence, and from these, more triangular numbers.

If a sequence is given by a polynomial function and if we calculate the first differences, the second differences, the third differences, and so on, then eventually we get a constant sequence. For example, the triangular numbers are given by the polynomial $T_n = \frac{1}{2}n^2 + \frac{1}{2}n$ (see the margin note on the next page); the second difference sequence is the constant sequence 1, 1, 1, ...

The formula for the nth triangular number can be found using the formula for the sum of the first n whole numbers (Example 2, Section 11.5). From the definition of T_n we have

$$T_n = 1 + 2 + \cdots + n$$
$$= \frac{n(n+1)}{2}$$
$$= \tfrac{1}{2}n^2 + \tfrac{1}{2}n$$

1. Construct a difference table for the square numbers and the pentagonal numbers. Use your table to find the tenth pentagonal number.

2. From the patterns you've observed so far, what do you think the second difference would be for the *hexagonal numbers*? Use this, together with the fact that the first two hexagonal numbers are 1 and 6, to find the first eight hexagonal numbers.

3. Construct difference tables for $C_n = n^3$. Which difference sequence is constant? Do the same for $F_n = n^4$.

4. Make up a polynomial of degree 5 and construct a difference table. Which difference sequence is constant?

5. The first few terms of a polynomial sequence are 1, 2, 4, 8, 16, 31, 57, ... Construct a difference table and use it to find four more terms of this sequence.

11.4 Mathematics of Finance

SUGGESTED TIME AND EMPHASIS

$\frac{1}{2}$–1 class.
Optional material.

POINTS TO STRESS

1. The future and present values of an annuity.
2. Calculating the interest rate of an annuity from the size of monthly payments.

Many financial transactions involve payments that are made at regular intervals. For example, if you deposit \$100 each month in an interest-bearing account, what will the value of your account be at the end of 5 years? If you borrow \$100,000 to buy a house, how much must your monthly payments be in order to pay off the loan in 30 years? Each of these questions involves the sum of a sequence of numbers; we use the results of the preceding section to answer them here.

The Amount of an Annuity

An **annuity** is a sum of money that is paid in regular equal payments. Although the word *annuity* suggests annual (or yearly) payments, they can be made semiannually, quarterly, monthly, or at some other regular interval. Payments are usually made at the end of the payment interval. The **amount of an annuity** is the sum of all the individual payments from the time of the first payment until the last payment is made, together with all the interest. We denote this sum by A_f (the subscript f here is used to denote *final* amount).

ALTERNATE EXAMPLE 1
An investor deposits \$300 every December 15 and June 15 for 11 years in an account that earns interest at the rate of 10% per year, compounded semiannually. How much will be in the account immediately after the last payment? Please round the answer (expressed in dollars) to the nearest cent.

ANSWER
\$11,551.56

Example 1 Calculating the Amount of an Annuity

An investor deposits \$400 every December 15 and June 15 for 10 years in an account that earns interest at the rate of 8% per year, compounded semiannually. How much will be in the account immediately after the last payment?

When using interest rates in calculators, remember to convert percentages to decimals. For example, 8% is 0.08.

Solution We need to find the amount of an annuity consisting of 20 semiannual payments of \$400 each. Since the interest rate is 8% per year, compounded semiannually, the interest rate per time period is $i = 0.08/2 = 0.04$. The first payment is in the account for 19 time periods, the second for 18 time periods, and so on.

IN-CLASS MATERIALS
The key idea here is that there are many practical examples available to use as illustrations.

SAMPLE QUESTION
Text Question
What is an annuity?

Answer
An annuity is a sum of money paid in regular equal payments.

The last payment receives no interest. The situation can be illustrated by the time line in Figure 1.

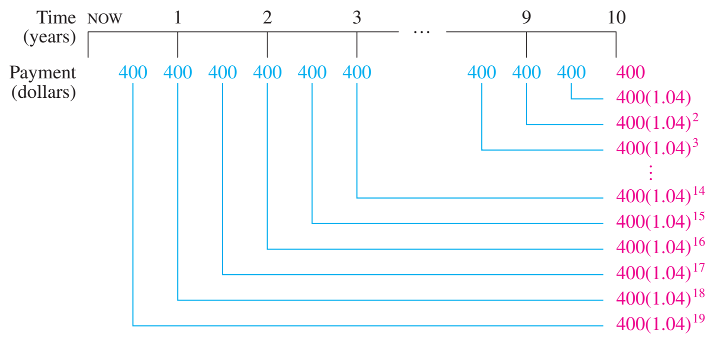


Figure 1

The amount A_f of the annuity is the sum of these 20 amounts. Thus

$$A_f = 400 + 400(1.04) + 400(1.04)^2 + \cdots + 400(1.04)^{19}$$

But this is a geometric series with $a = 400$, $r = 1.04$, and $n = 20$, so

$$A_f = 400\,\frac{1 - (1.04)^{20}}{1 - 1.04} \approx 11{,}911.23$$

Thus, the amount in the account after the last payment is \$11,911.23. ■

In general, the regular annuity payment is called the **periodic rent** and is denoted by R. We also let i denote the interest rate per time period and n the number of payments. *We always assume that the time period in which interest is compounded is equal to the time between payments*. By the same reasoning as in Example 1, we see that the amount A_f of an annuity is

$$A_f = R + R(1+i) + R(1+i)^2 + \cdots + R(1+i)^{n-1}$$

Since this is the nth partial sum of a geometric sequence with $a = R$ and $r = 1 + i$, the formula for the partial sum gives

$$A_f = R\,\frac{1-(1+i)^n}{1-(1+i)} = R\,\frac{1-(1+i)^n}{-i} = R\,\frac{(1+i)^n - 1}{i}$$

Amount of an Annuity

The amount A_f of an annuity consisting of n regular equal payments of size R with interest rate i per time period is given by

$$A_f = R\,\frac{(1+i)^n - 1}{i}$$

DRILL QUESTION

Every year, an investor deposits \$2000 in an IRA which earns an interest rate of 6% per year. How much is in the IRA after ten years?

Answer

$$2000\,\frac{(1+0.06)^{10} - 1}{0.06} = 26{,}361.59$$

IN-CLASS MATERIALS

Many states and companies hold lotteries and sweepstakes. The very large ones have options where the winner can chose a large lump sum award, or a larger award paid out over a period of years. Find the data for a local lottery or sweepstakes, and decide which option is the better option from a financial standpoint. For example, in the Midwestern United States Powerball, five white balls are drawn from a set of 53, and one red Powerball is drawn from a set of 42. The payouts are as follows:

Numbers Matched	Payout
All five numbers + Powerball	Jackpot (use \$10 million as an example)
All five numbers, without Powerball	\$100,000
Four numbers + Powerball	\$5,000
Four numbers, without Powerball	\$100
Three numbers + Powerball	\$100
Three numbers, without Powerball	\$7
Two numbers + Powerball	\$7
One number + Powerball	\$4
Powerball only	\$3

If the grand prize is \$10 million, it will be paid over 30 years, or the winner can choose to receive a lump sum payment of \$5.8 million.

ALTERNATE EXAMPLE 2

How much money should be invested every month at 6% per year, compounded monthly, in order to have \$2000 in 19 months? Please round the answer (expressed in dollars) to the nearest cent.

ANSWER

\$100.61

Mathematics in the Modern World

Mathematical Economics

The health of the global economy is determined by such interrelated factors as supply, demand, production, consumption, pricing, distribution, and thousands of other factors. These factors are in turn determined by economic decisions (for example, whether or not you buy a certain brand of toothpaste) made by billions of different individuals each day. How will today's creation and distribution of goods affect tomorrow's economy? Such questions are tackled by mathematicians who work on mathematical models of the economy. In the 1940s Wassily Leontief, a pioneer in this area, created a model consisting of thousands of equations that describe how different sectors of the economy, such as the oil industry, transportation, and communication, interact with each other. A different approach to economic models, one dealing with individuals in the economy as opposed to large sectors, was pioneered by John Nash in the 1950s. In his model, which uses *Game Theory*, the economy is a game where individual players make decisions that often lead to mutual gain. Leontief and Nash were awarded the Nobel Prize in Economics in 1973 and 1994, respectively. Economic theory continues to be a major area of mathematical research.

Example 2 Calculating the Amount of an Annuity

How much money should be invested every month at 12% per year, compounded monthly, in order to have \$4000 in 18 months?

Solution In this problem $i = 0.12/12 = 0.01$, $A_f = 4000$, and $n = 18$. We need to find the amount R of each payment. By the formula for the amount of an annuity,

$$4000 = R\,\frac{(1 + 0.01)^{18} - 1}{0.01}$$

Solving for R, we get

$$R = \frac{4000(0.01)}{(1 + 0.01)^{18} - 1} \approx 203.928$$

Thus, the monthly investment should be \$203.93. ■

The Present Value of an Annuity

If you were to receive \$10,000 five years from now, it would be worth much less than getting \$10,000 right now. This is because of the interest you could accumulate during the next five years if you invested the money now. What smaller amount would you be willing to accept *now* instead of receiving \$10,000 in five years? This is the amount of money that, together with interest, would be worth \$10,000 in five years. The amount we are looking for here is called the *discounted value* or *present value*. If the interest rate is 8% per year, compounded quarterly, then the interest per time period is $i = 0.08/4 = 0.02$, and there are $4 \times 5 = 20$ time periods. If we let PV denote the present value, then by the formula for compound interest (Section 4.1) we have

$$10{,}000 = PV(1 + i)^n = PV(1 + 0.02)^{20}$$

so

$$PV = 10{,}000(1 + 0.02)^{-20} \approx 6729.713$$

Thus, in this situation, the present value of \$10,000 is \$6729.71. This reasoning leads to a general formula for present value:

$$PV = A(1 + i)^{-n}$$

Similarly, the **present value of an annuity** is the amount A_p that must be invested now at the interest rate i per time period in order to provide n payments, each of amount R. Clearly, A_p is the sum of the present values of each individual payment (see Exercise 22). Another way of finding A_p is to note that A_p is the present value of A_f:

$$A_p = A_f(1 + i)^{-n} = R\,\frac{(1 + i)^n - 1}{i}(1 + i)^{-n} = R\,\frac{1 - (1 + i)^{-n}}{i}$$

The Present Value of an Annuity

The **present value** A_p of an annuity consisting of n regular equal payments of size R and interest rate i per time period is given by

$$A_p = R\,\frac{1 - (1 + i)^{-n}}{i}$$

Example 3 Calculating the Present Value of an Annuity

A person wins $10,000,000 in the California lottery, and the amount is paid in yearly installments of half a million dollars each for 20 years. What is the present value of his winnings? Assume that he can earn 10% interest, compounded annually.

Solution Since the amount won is paid as an annuity, we need to find its present value. Here $i = 0.1$, $R = \$500{,}000$, and $n = 20$. Thus

$$A_p = 500{,}000\,\frac{1 - (1 + 0.1)^{-20}}{0.1} \approx 4{,}256{,}781.859$$

This means that the winner really won only $4,256,781.86 if it were paid immediately. ■

Installment Buying

When you buy a house or a car by installment, the payments you make are an annuity whose present value is the amount of the loan.

Example 4 The Amount of a Loan

A student wishes to buy a car. He can afford to pay $200 per month but has no money for a down payment. If he can make these payments for four years and the interest rate is 12%, what purchase price can he afford?

Solution The payments the student makes constitute an annuity whose present value is the price of the car (which is also the amount of the loan, in this case). Here we have $i = 0.12/12 = 0.01$, $R = 200$, $n = 12 \times 4 = 48$, so

$$A_p = R\,\frac{1 - (1 + i)^{-n}}{i} = 200\,\frac{1 - (1 + 0.01)^{-48}}{0.01} \approx 7594.792$$

Thus, the student can buy a car priced at $7594.79. ■

When a bank makes a loan that is to be repaid with regular equal payments R, then the payments form an annuity whose present value A_p is the amount of the loan. So, to find the size of the payments, we solve for R in the formula for the amount of an annuity. This gives the following formula for R.

Installment Buying

If a loan A_p is to be repaid in n regular equal payments with interest rate i per time period, then the size R of each payment is given by

$$R = \frac{iA_p}{1 - (1 + i)^{-n}}$$

ALTERNATE EXAMPLE 3
A person wins $3,000,000 in the California lottery, and the amount is paid in yearly installments of half a million dollars each for 6 years. What is the present value of his winnings? Assume that he can earn 5% interest, compounded annually. Please round your answer (expressed in dollars) to the nearest cent.

ANSWER
$2,537,846.03

ALTERNATE EXAMPLE 4
A student wishes to buy a car. He can afford to pay $400 per month but has no money for a down payment. If he can make these payments for five years and the interest rate is 3%, what purchase price can he afford? Please round your answer (expressed in dollars) to the nearest cent.

ANSWER
$22,260.94

EXAMPLE
A person borrows $20,000 to buy a car, and wants to pay it off in 4 years. If the interest rate is 8% per year, compounded monthly, what is the amount of each monthly payment?

ANSWER
$488.26

IN-CLASS MATERIALS

A phone call to a cooperative automobile dealership will get a sample monthly payment on a 4-year car loan. Students can figure out the interest rate on cars in their community, and see if it varies based on the cost of the car.

ALTERNATE EXAMPLE 5
A couple borrows \$100,000 at 3% interest as a mortgage loan on a house. They expect to make monthly payments for 34 years to repay the loan. What is the size of each payment? Please round your answer (expressed in dollars) to the nearest cent.

ANSWER
\$391.27

ALTERNATE EXAMPLE 6
A car dealer sells a new car for \$19,000. He offers the buyer payments of \$367 per month for 5 years. What interest rate is this car dealer charging? Please round your answer to the nearest percent.

ANSWER
6%

Example 5 Calculating Monthly Mortgage Payments

A couple borrows \$100,000 at 9% interest as a mortage loan on a house. They expect to make monthly payments for 30 years to repay the loan. What is the size of each payment?

Solution The mortgage payments form an annuity whose present value is $A_p = \$100{,}000$. Also, $i = 0.09/12 = 0.0075$, and $n = 12 \times 30 = 360$. We are looking for the amount R of each payment. From the formula for installment buying, we get

$$R = \frac{iA_p}{1-(1+i)^{-n}}$$

$$= \frac{(0.0075)(100{,}000)}{1-(1+0.0075)^{-360}} \approx 804.623$$

Thus, the monthly payments are \$804.62. ■

We now illustrate the use of graphing devices in solving problems related to installment buying.

Example 6 Calculating the Interest Rate from the Size of Monthly Payments

A car dealer sells a new car for \$18,000. He offers the buyer payments of \$405 per month for 5 years. What interest rate is this car dealer charging?

Solution The payments form an annuity with present value $A_p = \$18{,}000$, $R = 405$, and $n = 12 \times 5 = 60$. To find the interest rate, we must solve for i in the equation

$$R = \frac{iA_p}{1-(1+i)^{-n}}$$

A little experimentation will convince you that it's not possible to solve this equation for i algebraically. So, to find i we use a graphing device to graph R as a function of the interest rate x, and we then use the graph to find the interest rate corresponding to the value of R we want (\$405 in this case). Since $i = x/12$, we graph the function

$$R(x) = \frac{\frac{x}{12}(18{,}000)}{1-\left(1+\frac{x}{12}\right)^{-60}}$$

in the viewing rectangle $[0.06, 0.16] \times [350, 450]$, as shown in Figure 2. We also graph the horizontal line $R(x) = 405$ in the same viewing rectangle. Then, by moving the cursor to the point of intersection of the two graphs, we find that the corresponding x-value is approximately 0.125. Thus, the interest rate is about $12\frac{1}{2}\%$. ■

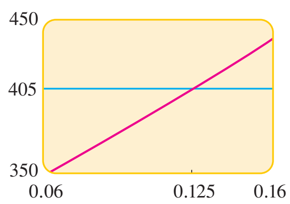


Figure 2

IN-CLASS MATERIALS

Students can be assigned to contact mortgage brokers to find the size of payments (and current interest rate) on an average 30-year mortgage on a house near their home or school. They then can explore the effects of making a larger or smaller down payment on the house.

11.4 Exercises

1. **Annuity** Find the amount of an annuity that consists of 10 annual payments of \$1000 each into an account that pays 6% interest per year.

2. **Annuity** Find the amount of an annuity that consists of 24 monthly payments of \$500 each into an account that pays 8% interest per year, compounded monthly.

3. **Annuity** Find the amount of an annuity that consists of 20 annual payments of \$5000 each into an account that pays interest of 12% per year.

4. **Annuity** Find the amount of an annuity that consists of 20 semiannual payments of \$500 each into an account that pays 6% interest per year, compounded semiannually.

5. **Annuity** Find the amount of an annuity that consists of 16 quarterly payments of \$300 each into an account that pays 8% interest per year, compounded quarterly.

6. **Saving** How much money should be invested every quarter at 10% per year, compounded quarterly, in order to have \$5000 in 2 years?

7. **Saving** How much money should be invested monthly at 6% per year, compounded monthly, in order to have \$2000 in 8 months?

8. **Annuity** What is the present value of an annuity that consists of 20 semiannual payments of \$1000 at the interest rate of 9% per year, compounded semiannually?

9. **Funding an Annuity** How much money must be invested now at 9% per year, compounded semiannually, to fund an annuity of 20 payments of \$200 each, paid every 6 months, the first payment being 6 months from now?

10. **Funding an Annuity** A 55-year-old man deposits \$50,000 to fund an annuity with an insurance company. The money will be invested at 8% per year, compounded semiannually. He is to draw semiannual payments until he reaches age 65. What is the amount of each payment?

11. **Financing a Car** A woman wants to borrow \$12,000 in order to buy a car. She wants to repay the loan by monthly installments for 4 years. If the interest rate on this loan is $10\frac{1}{2}$% per year, compounded monthly, what is the amount of each payment?

12. **Mortgage** What is the monthly payment on a 30-year mortgage of \$80,000 at 9% interest? What is the monthly payment on this same mortgage if it is to be repaid over a 15-year period?

13. **Mortgage** What is the monthly payment on a 30-year mortgage of \$100,000 at 8% interest per year, compounded monthly? What is the total amount paid on this loan over the 30-year period?

14. **Mortgage** A couple can afford to make a monthly mortgage payment of \$650. If the mortgage rate is 9% and the couple intends to secure a 30-year mortgage, how much can they borrow?

15. **Mortgage** A couple secures a 30-year loan of \$100,000 at $9\frac{3}{4}$% per year, compounded monthly, to buy a house.
   (a) What is the amount of their monthly payment?
   (b) What total amount will they pay over the 30-year period?
   (c) If, instead of taking the loan, the couple deposits the monthly payments in an account that pays $9\frac{3}{4}$% interest per year, compounded monthly, how much will be in the account at the end of the 30-year period?

16. **Financing a Car** Jane agrees to buy a car for a down payment of \$2000 and payments of \$220 per month for 3 years. If the interest rate is 8% per year, compounded monthly, what is the actual purchase price of her car?

17. **Financing a Ring** Mike buys a ring for his fiancee by paying \$30 a month for one year. If the interest rate is 10% per year, compounded monthly, what is the price of the ring?

18. **Interest Rate** Janet's payments on her \$12,500 car are \$420 a month for 3 years. Assuming that interest is compounded monthly, what interest rate is she paying on the car loan?

19. **Interest Rate** John buys a stereo system for \$640. He agrees to pay \$32 a month for 2 years. Assuming that interest is compounded monthly, what interest rate is he paying?

20. **Interest Rate** A man purchases a \$2000 diamond ring for a down payment of \$200 and monthly installments of \$88 for 2 years. Assuming that interest is compounded monthly, what interest rate is he paying?

21. **Interest Rate** An item at a department store is priced at \$189.99 and can be bought by making 20 payments of \$10.50. Find the interest rate, assuming that interest is compounded monthly.

Discovery • Discussion

22. **Present Value of an Annuity** **(a)** Draw a time line as in Example 1 to show that the present value of an annuity is the sum of the present values of each payment, that is,

$$A_p = \frac{R}{1+i} + \frac{R}{(1+i)^2} + \frac{R}{(1+i)^3} + \cdots + \frac{R}{(1+i)^n}$$

   (b) Use part (a) to derive the formula for A_p given in the text.

23. **An Annuity That Lasts Forever** An **annuity in perpetuity** is one that continues forever. Such annuities are useful in setting up scholarship funds to ensure that the award continues.

(a) Draw a time line (as in Example 1) to show that to set up an annuity in perpetuity of amount R per time period, the amount that must be invested now is

$$A_p = \frac{R}{1+i} + \frac{R}{(1+i)^2} + \frac{R}{(1+i)^3} + \cdots + \frac{R}{(1+i)^n} + \cdots$$

where i is the interest rate per time period.

(b) Find the sum of the infinite series in part (a) to show that

$$A_p = \frac{R}{i}$$

(c) How much money must be invested now at 10% per year, compounded annually, to provide an annuity in perpetuity of \$5000 per year? The first payment is due in one year.

(d) How much money must be invested now at 8% per year, compounded quarterly, to provide an annuity in perpetuity of \$3000 per year? The first payment is due in one year.

24. Amortizing a Mortgage When they bought their house, John and Mary took out a \$90,000 mortgage at 9% interest, repayable monthly over 30 years. Their payment is \$724.17 per month (check this using the formula in the text). The bank gave them an **amortization schedule**, which is a table showing how much of each payment is interest, how much goes toward the principal, and the remaining principal after each payment. The table below shows the first few entries in the amortization schedule.

Payment number	Total payment	Interest payment	Principal payment	Remaining principal
1	724.17	675.00	49.17	89,950.83
2	724.17	674.63	49.54	89,901.29
3	724.17	674.26	49.91	89,851.38
4	724.17	673.89	50.28	89,801.10

After 10 years they have made 120 payments and are wondering how much they still owe, but they have lost the amortization schedule.

(a) How much do John and Mary still owe on their mortgage? [*Hint*: The remaining balance is the present value of the 240 remaining payments.]

(b) How much of their next payment is interest and how much goes toward the principal? [*Hint*: Since $9\% \div 12 = 0.75\%$, they must pay 0.75% of the remaining principal in interest each month.]

SUGGESTED TIME AND EMPHASIS

$1\text{–}1\frac{1}{2}$ classes.
Optional material.

11.5 Mathematical Induction

There are two aspects to mathematics—discovery and proof—and both are of equal importance. We must discover something before we can attempt to prove it, and we can only be certain of its truth once it has been proved. In this section we examine the relationship between these two key components of mathematics more closely.

Conjecture and Proof

Let's try a simple experiment. We add more and more of the odd numbers as follows:

$$1 = 1$$
$$1 + 3 = 4$$
$$1 + 3 + 5 = 9$$
$$1 + 3 + 5 + 7 = 16$$
$$1 + 3 + 5 + 7 + 9 = 25$$

What do you notice about the numbers on the right side of these equations? They are in fact all perfect squares. These equations say the following:

The sum of the first 1 odd number is 1^2.

The sum of the first 2 odd numbers is 2^2.

The sum of the first 3 odd numbers is 3^2.

The sum of the first 4 odd numbers is 4^2.

The sum of the first 5 odd numbers is 5^2.

POINT TO STRESS

The concept and execution of a proof by mathematical induction.

Consider the polynomial

$$p(n) = n^2 - n + 41$$

Here are some values of $p(n)$:

$$p(1) = 41 \quad p(2) = 43$$
$$p(3) = 47 \quad p(4) = 53$$
$$p(5) = 61 \quad p(6) = 71$$
$$p(7) = 83 \quad p(8) = 97$$

All the values so far are prime numbers. In fact, if you keep going, you will find $p(n)$ is prime for all natural numbers up to $n = 40$. It may seem reasonable at this point to conjecture that $p(n)$ is prime for *every* natural number n. But out conjecture would be too hasty, because it is easily seen that $p(41)$ is *not* prime. This illustrates that we cannot be certain of the truth of a statement no matter how many special cases we check. We need a convincing argument—a *proof*—to determine the truth of a statement.

This leads naturally to the following question: Is it true that for every natural number n, the sum of the first n odd numbers is n^2? Could this remarkable property be true? We could try a few more numbers and find that the pattern persists for the first 6, 7, 8, 9, and 10 odd numbers. At this point, we feel quite sure that this is always true, so we make a *conjecture*:

The sum of the first n odd numbers is n^2.

Since we know that the nth odd number is $2n - 1$, we can write this statement more precisely as

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

It's important to realize that this is still a conjecture. We cannot conclude by checking a finite number of cases that a property is true for all numbers (there are infinitely many). To see this more clearly, suppose someone tells us he has added up the first trillion odd numbers and found that they do *not* add up to 1 trillion squared. What would you tell this person? It would be silly to say that you're sure it's true because you've already checked the first five cases. You could, however, take out paper and pencil and start checking it yourself, but this task would probably take the rest of your life. The tragedy would be that after completing this task you would still not be sure of the truth of the conjecture! Do you see why?

Herein lies the power of mathematical proof. A **proof** is a clear argument that demonstrates the truth of a statement beyond doubt.

Mathematical Induction

Let's consider a special kind of proof called **mathematical induction**. Here is how it works: Suppose we have a statement that says something about all natural numbers n. Let's call this statement P. For example, we could consider the statement

P: For every natural number n, the sum of the first n odd numbers is n^2.

Since this statement is about *all* natural numbers, it contains infinitely many statements; we will call them $P(1), P(2), \ldots$.

$P(1)$: The sum of the first 1 odd number is 1^2.

$P(2)$: The sum of the first 2 odd numbers is 2^2.

$P(3)$: The sum of the first 3 odd numbers is 3^2.

$\vdots \qquad \vdots$

How can we prove all of these statements at once? Mathematical induction is a clever way of doing just that.

The crux of the idea is this: Suppose we can prove that whenever one of these statements is true, then the one following it in the list is also true. In other words,

For every k, if $P(k)$ is true, then $P(k + 1)$ is true.

This is called the **induction step** because it leads us from the truth of one statement to the next. Now, suppose that we can also prove that

$P(1)$ is true.

The induction step now leads us through the following chain of statements:

$P(1)$ is true, so $P(2)$ is true.
$P(2)$ is true, so $P(3)$ is true.
$P(3)$ is true, so $P(4)$ is true.

$$\vdots \qquad\qquad \vdots$$

So we see that if both the induction step and $P(1)$ are proved, then statement P is proved for all n. Here is a summary of this important method of proof.

Principle of Mathematical Induction

For each natural number n, let $P(n)$ be a statement depending on n. Suppose that the following two conditions are satisfied.

1. $P(1)$ is true.
2. For every natural number k, if $P(k)$ is true then $P(k + 1)$ is true.

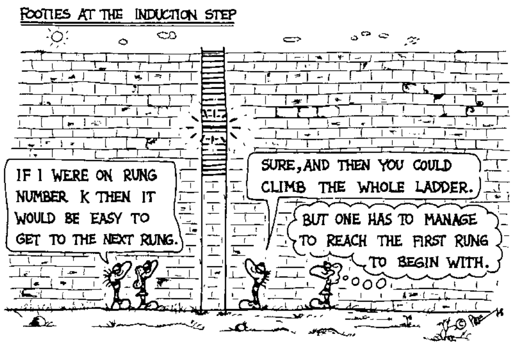
Then $P(n)$ is true for all natural numbers n.

To apply this principle, there are two steps:

Step 1 Prove that $P(1)$ is true.

Step 2 Assume that $P(k)$ is true and use this assumption to prove that $P(k + 1)$ is true.

Notice that in Step 2 we do not prove that $P(k)$ is true. We only show that *if* $P(k)$ is true, *then* $P(k + 1)$ is also true. The assumption that $P(k)$ is true is called the **induction hypothesis**.



We now use mathematical induction to prove that the conjecture we made at the beginning of this section is true.

IN-CLASS MATERIALS

The main thing for induction is to give the students plenty of examples to work from, and plenty of practice.

1. We can use induction to show that $\dfrac{(2n)!}{2^n n!}$ is an integer for all positive n. The base case is trivial. The key step in the inductive case:

$$\frac{(2n + 2)!}{2^{n+1}(n + 1)!} = \frac{(2n)!}{2^n n!} \cdot \frac{(2n + 1)(2n + 2)}{2(n + 1)}$$

and now cancellation occurs.

2. We can use induction to show that the sum of the cubes of three consecutive integers is divisible by 9. The base case is trivial. The key inductive step:

$$(n + 1)^3 + (n + 2)^3 + (n + 3)^3 = n^3 + (n + 1)^3 + (n + 2)^3 + 9n^2 + 27n + 27$$

3. We can use induction to show that

$$\frac{n^3 - n}{3} = (1 \cdot 2) + (2 \cdot 3) + \cdots + (n - 1)\,n$$

The base step should be $n = 2$, and the inductive step uses the fact that

$$\frac{(n + 1)^3 - (n + 1)}{3} = \frac{n^3 - n}{3} \frac{3n^2 + 3n}{3}$$

Example 1 A Proof by Mathematical Induction

Prove that for all natural numbers n,

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

Solution Let $P(n)$ denote the statement $1 + 3 + 5 + \cdots + (2n - 1) = n^2$.

Step 1 We need to show that $P(1)$ is true. But $P(1)$ is simply the statement that $1 = 1^2$, which is of course true.

Step 2 We assume that $P(k)$ is true. Thus, our induction hypothesis is

$$1 + 3 + 5 + \cdots + (2k - 1) = k^2$$

We want to use this to show that $P(k + 1)$ is true, that is,

$$1 + 3 + 5 + \cdots + (2k - 1) + [2(k + 1) - 1] = (k + 1)^2$$

[Note that we get $P(k + 1)$ by substituting $k + 1$ for each n in the statement $P(n)$.] We start with the left side and use the induction hypothesis to obtain the right side of the equation:

This equals k^2 by the induction hypothesis.

$$\begin{aligned}
&1 + 3 + 5 + \cdots + (2k - 1) + [2(k + 1) - 1] \\
&= [1 + 3 + 5 + \cdots + (2k - 1)] + [2(k + 1) - 1] && \text{Group the first } k \text{ terms} \\
&= k^2 + [2(k + 1) - 1] && \text{Induction hypothesis} \\
&= k^2 + [2k + 2 - 1] && \text{Distributive Property} \\
&= k^2 + 2k + 1 && \text{Simplify} \\
&= (k + 1)^2 && \text{Factor}
\end{aligned}$$

Thus, $P(k + 1)$ follows from $P(k)$ and this completes the induction step.

Having proved Steps 1 and 2, we conclude by the Principle of Mathematical Induction that $P(n)$ is true for all natural numbers n. ■

Example 2 A Proof by Mathematical Induction

Prove that for every natural number n,

$$1 + 2 + 3 + \cdots + n = \frac{n(n + 1)}{2}$$

Solution Let $P(n)$ be the statement $1 + 2 + 3 + \cdots + n = n(n + 1)/2$. We want to show that $P(n)$ is true for all natural numbers n.

Step 1 We need to show that $P(1)$ is true. But $P(1)$ says that

$$1 = \frac{1(1 + 1)}{2}$$

and this statement is clearly true.

ALTERNATE EXAMPLE 1
Use mathematical induction to determine whether it is true that for all natural numbers n, $2 + 6 + 10 + \cdots + (4n - 2) = 2n^2$.

ANSWER Yes

SAMPLE QUESTION
Text Question
There are two main steps to a proof by mathematical induction. What is the first one?

Answer
Any answer getting at the idea of a base case or a "proof for $n = 1$" should be accepted, even if the latter isn't technically true.

DRILL QUESTION
Prove, using mathematical induction, that $1 + 2 + \cdots + n = \frac{n(n + 1)}{2}$.

Answer
This is Example 2 from the text.

ALTERNATE EXAMPLE 2
Use mathematical induction to determine whether it is true that for all natural numbers n, $3 + 6 + 9 + \cdots + 3n = \frac{3n(n + 1)}{2}$.

ANSWER Yes

IN-CLASS MATERIALS
The students can do these, or you can write it on the blackboard.

1. Let $F(n)$ be the number of ways to climb n steps. We want to prove $F(n) = F(n - 1) + F(n - 2)$.
   Base Case: $n = 3$. It is true that $F(3) = F(2) + F(1)$.
   Inductive Step: Assume that this is true for 1 through n. Consider $F(n + 1)$. The first step is either a one-step or a two-step. If the first step is a one-step, there are n steps left to climb, and the number of ways to do that is $F(n)$. If the first step is a two-step then there are $n - 2$ steps left to climb, and the number of ways to do that is $F(n - 1)$. So we have $F(n) = F(n - 1) + F(n - 2)$.
2. Let $F(n)$ be the number of successor-free n-element sequences. We want to prove $F(n) = F(n - 1) + F(n - 2)$.
   Base Case: $n = 3$. It is true that $F(3) = F(2) + F(1)$.
   Inductive Step: Assume that this is true for 1 through n. Consider $F(n + 1)$. The first number in an $n + 1$ sequence will be either 0 or 1. If the first number is 0, we have n left to go. So the number of ways to finish the sequence is $F(n)$. If the first number is 1, the second must be 0, or we wouldn't be successor free. Then we have $n - 2$ numbers to go, so the number of ways to finish the sequence is $F(n - 1)$. So $F(n) = F(n - 1) + F(n - 2)$.
3. Let $F(n)$ be the number of ancestors of a female bee at stage n. In other words, $n = 1$ means the number of parents, $n = 2$ means the number of grandparents, etc. We want to prove $F(n) = F(n - 1) + F(n - 2)$.
   Base Case: $n = 3$. It is true that $F(3) = F(2) + F(1)$.
   Inductive Step: Assume that this is true for 1 through n. Consider $F(n + 1)$. The worker bee has a mommy, who has $F(n - 1)$ ancestors. She also has a daddy who has a mommy who has $F(n - 2)$ ancestors. So, again, $F(n) = F(n - 1) + F(n - 2)$.

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Blaise Pascal (1623–1662) is considered one of the most versatile minds in modern history. He was a writer and philosopher as well as a gifted mathematician and physicist. Among his contributions that appear in this book are Pascal's triangle and the Principle of Mathematical Induction.

Pascal's father, himself a mathematician, believed that his son should not study mathematics until he was 15 or 16. But at age 12, Blaise insisted on learning geometry, and proved most of its elementary theorems himself. At 19, he invented the first mechanical adding machine. In 1647, after writing a major treatise on the conic sections, he abruptly abandoned mathematics because he felt his intense studies were contributing to his ill health. He devoted himself instead to frivolous recreations such as gambling, but this only served to pique his interest in probability. In 1654 he miraculously survived a carriage accident in which his horses ran off a bridge. Taking this to be a sign from God, he entered a monastery, where he pursued theology and philosophy, writing his famous *Pensées*. He also continued his mathematical research. He valued faith and intuition more than reason as the source of truth, declaring that "the heart has its own reasons, which reason cannot know."

Step 2 Assume that $P(k)$ is true. Thus, our induction hypothesis is

$$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}$$

We want to use this to show that $P(k+1)$ is true, that is,

$$1 + 2 + 3 + \cdots + k + (k+1) = \frac{(k+1)[(k+1)+1]}{2}$$

So, we start with the left side and use the induction hypothesis to obtain the right side:

$$\begin{aligned}
&1 + 2 + 3 + \cdots + k + (k+1) \\
&\quad = [1 + 2 + 3 + \cdots + k] + (k+1) && \text{Group the first } k \text{ terms} \\
&\quad = \frac{k(k+1)}{2} + (k+1) && \text{Induction hypothesis} \\
&\quad = (k+1)\left(\frac{k}{2} + 1\right) && \text{Factor } k+1 \\
&\quad = (k+1)\left(\frac{k+2}{2}\right) && \text{Common denominator} \\
&\quad = \frac{(k+1)[(k+1)+1]}{2} && \text{Write } k+2 \text{ as } k+1+1
\end{aligned}$$

Thus, $P(k+1)$ follows from $P(k)$ and this completes the induction step.

Having proved Steps 1 and 2, we conclude by the Principle of Mathematical Induction that $P(n)$ is true for all natural numbers n. ■

Formulas for the sums of powers of the first n natural numbers are important in calculus. Formula 1 in the following box is proved in Example 2. The other formulas are also proved using mathematical induction (see Exercises 4 and 7).

Sums of Powers

0. $\displaystyle\sum_{k=1}^{n} 1 = n$

1. $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

2. $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

3. $\displaystyle\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$

It might happen that a statement $P(n)$ is false for the first few natural numbers, but true from some number on. For example, we may want to prove that $P(n)$ is true for $n \geq 5$. Notice that if we prove that $P(5)$ is true, then this fact, together with the induction step, would imply the truth of $P(5), P(6), P(7), \ldots$. The next example illustrates this point.

Example 3 **Proving an Inequality by Mathematical Induction**

Prove that $4n < 2^n$ for all $n \geq 5$.

ALTERNATE EXAMPLE 3
Use mathematical induction to answer the question:
Is $2n < 2^n$ for all $n \geq 3$?

ANSWER
Yes

Solution Let $P(n)$ denote the statement $4n < 2^n$.

Step 1 $P(5)$ is the statement that $4 \cdot 5 < 2^5$, or $20 < 32$, which is true.

Step 2 Assume that $P(k)$ is true. Thus, our induction hypothesis is

$$4k < 2^k$$

We want to use this to show that $P(k + 1)$ is true, that is,

$$4(k + 1) < 2^{k+1}$$

We get $P(k + 1)$ by replacing k by $k + 1$ in the statement $P(k)$.

So, we start with the left side of the inequality and use the induction hypothesis to show that it is less than the right side. For $k \geq 5$, we have

$$\begin{aligned} 4(k + 1) &= 4k + 4 \\ &< 2^k + 4 && \text{Induction hypothesis} \\ &< 2^k + 4k && \text{Because } 4 < 4k \\ &< 2^k + 2^k && \text{Induction hypothesis} \\ &= 2 \cdot 2^k \\ &= 2^{k+1} && \text{Property of exponents} \end{aligned}$$

Thus, $P(k + 1)$ follows from $P(k)$ and this completes the induction step.

Having proved Steps 1 and 2, we conclude by the Principle of Mathematical Induction that $P(n)$ is true for all natural numbers $n \geq 5$. ■

11.5 Exercises

1–12 ■ Use mathematical induction to prove that the formula is true for all natural numbers n.

1. $2 + 4 + 6 + \cdots + 2n = n(n + 1)$

2. $1 + 4 + 7 + \cdots + (3n - 2) = \dfrac{n(3n - 1)}{2}$

3. $5 + 8 + 11 + \cdots + (3n + 2) = \dfrac{n(3n + 7)}{2}$

4. $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n + 1)(2n + 1)}{6}$

5. $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n + 1) = \dfrac{n(n + 1)(n + 2)}{3}$

6. $1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + n(n + 2) = \dfrac{n(n + 1)(2n + 7)}{6}$

7. $1^3 + 2^3 + 3^3 + \cdots + n^3 = \dfrac{n^2(n + 1)^2}{4}$

8. $1^3 + 3^3 + 5^3 + \cdots + (2n - 1)^3 = n^2(2n^2 - 1)$

9. $2^3 + 4^3 + 6^3 + \cdots + (2n)^3 = 2n^2(n + 1)^2$

10. $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \dfrac{1}{3 \cdot 4} + \cdots + \dfrac{1}{n(n + 1)} = \dfrac{n}{(n + 1)}$

11. $1 \cdot 2 + 2 \cdot 2^2 + 3 \cdot 2^3 + 4 \cdot 2^4 + \cdots + n \cdot 2^n = 2[1 + (n - 1)2^n]$

12. $1 + 2 + 2^2 + \cdots + 2^{n-1} = 2^n - 1$

13. Show that $n^2 + n$ is divisible by 2 for all natural numbers n.

14. Show that $5^n - 1$ is divisible by 4 for all natural numbers n.

15. Show that $n^2 - n + 41$ is odd for all natural numbers n.

16. Show that $n^3 - n + 3$ is divisible by 3 for all natural numbers n.

17. Show that $8^n - 3^n$ is divisible by 5 for all natural numbers n.

18. Show that $3^{2n} - 1$ is divisible by 8 for all natural numbers n.

19. Prove that $n < 2^n$ for all natural numbers n.

20. Prove that $(n + 1)^2 < 2n^2$ for all natural numbers $n \geq 3$.

21. Prove that if $x > -1$, then $(1 + x)^n \geq 1 + nx$ for all natural numbers n.

22. Show that $100n \leq n^2$ for all $n \geq 100$.

23. Let $a_{n+1} = 3a_n$ and $a_1 = 5$. Show that $a_n = 5 \cdot 3^{n-1}$ for all natural numbers n.

24. A sequence is defined recursively by $a_{n+1} = 3a_n - 8$ and $a_1 = 4$. Find an explicit formula for a_n and then use mathematical induction to prove that the formula you found is true.

25. Show that $x - y$ is a factor of $x^n - y^n$ for all natural numbers n.

$[Hint: x^{k+1} - y^{k+1} = x^k(x - y) + (x^k - y^k)y]$

26. Show that $x + y$ is a factor of $x^{2n-1} + y^{2n-1}$ for all natural numbers n.

27–31 ■ F_n denotes the nth term of the Fibonacci sequence discussed in Section 11.1. Use mathematical induction to prove the statement.

27. F_{3n} is even for all natural numbers n.

28. $F_1 + F_2 + F_3 + \cdots + F_n = F_{n+2} - 1$

29. $F_1^2 + F_2^2 + F_3^2 + \cdots + F_n^2 = F_nF_{n+1}$

30. $F_1 + F_3 + \cdots + F_{2n-1} = F_{2n}$

31. For all $n \geq 2$,

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$$

32. Let a_n be the nth term of the sequence defined recursively by

$$a_{n+1} = \frac{1}{1 + a_n}$$

and $a_1 = 1$. Find a formula for a_n in terms of the Fibonacci numbers F_n. Prove that the formula you found is valid for all natural numbers n.

33. Let F_n be the nth term of the Fibonacci sequence. Find and prove an inequality relating n and F_n for natural numbers n.

34. Find and prove an inequality relating $100n$ and n^3.

Discovery • Discussion

35. True or False? Determine whether each statement is true or false. If you think the statement is true, prove it. If you think it is false, give an example where it fails.

(a) $p(n) = n^2 - n + 11$ is prime for all n.

(b) $n^2 > n$ for all $n \geq 2$.

(c) $2^{2n+1} + 1$ is divisible by 3 for all $n \geq 1$.

(d) $n^3 \geq (n + 1)^2$ for all $n \geq 2$.

(e) $n^3 - n$ is divisible by 3 for all $n \geq 2$.

(f) $n^3 - 6n^2 + 11n$ is divisible by 6 for all $n \geq 1$.

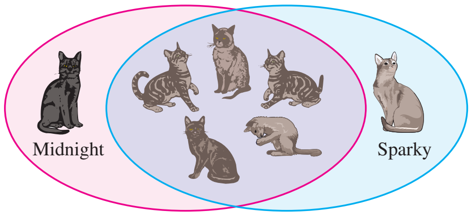
36. All Cats Are Black? What is wrong with the following "proof" by mathematical induction that all cats are black? Let $P(n)$ denote the statement: In any group of n cats, if one is black, then they are all black.

Step 1 The statement is clearly true for $n = 1$.

Step 2 Suppose that $P(k)$ is true. We show that $P(k + 1)$ is true.

Suppose we have a group of $k + 1$ cats, one of whom is black; call this cat "Midnight." Remove some other cat (call it "Sparky") from the group. We are left with k cats, one of whom (Midnight) is black, so by the induction hypothesis, all k of these are black. Now put Sparky back in the group and take out Midnight. We again have a group of k cats, all of whom—except possibly Sparky—are black. Then by the induction hypothesis, Sparky must be black, too. So all $k + 1$ cats in the original group are black.

Thus, by induction $P(n)$ is true for all n. Since everyone has seen at least one black cat, it follows that all cats are black.



11.6 The Binomial Theorem

SUGGESTED TIME AND EMPHASIS

$\frac{1}{2}$ class.
Optional material.

An expression of the form $a + b$ is called a **binomial**. Although in principle it's easy to raise $a + b$ to any power, raising it to a very high power would be tedious. In this section we find a formula that gives the expansion of $(a + b)^n$ for any natural number n and then prove it using mathematical induction.

POINTS TO STRESS

1. The expansion of $(a + b)^n$ where a and b are expressions, and n is a positive integer.

2. The computation of $\binom{n}{r}$ using factorials and using Pascal's triangle.

Expanding $(a + b)^n$

To find a pattern in the expansion of $(a + b)^n$, we first look at some special cases:

$$
\begin{aligned}
(a + b)^1 &= a + b \\
(a + b)^2 &= a^2 + 2ab + b^2 \\
(a + b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3 \\
(a + b)^4 &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 \\
(a + b)^5 &= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5 \\
&\ \ \vdots
\end{aligned}
$$

The following simple patterns emerge for the expansion of $(a + b)^n$:

1. There are $n + 1$ terms, the first being a^n and the last b^n.
2. The exponents of a decrease by 1 from term to term while the exponents of b increase by 1.
3. The sum of the exponents of a and b in each term is n.

For instance, notice how the exponents of a and b behave in the expansion of $(a + b)^5$.

The exponents of *a* decrease:

$$(a + b)^5 = a^{5} + 5a^{4}b^1 + 10a^{3}b^2 + 10a^{2}b^3 + 5a^{1}b^4 + b^5$$

The exponents of *b* increase:

$$(a + b)^5 = a^5 + 5a^4b^{1} + 10a^3b^{2} + 10a^2b^{3} + 5a^1b^{4} + b^{5}$$

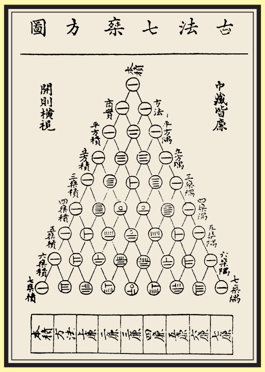
With these observations we can write the form of the expansion of $(a + b)^n$ for any natural number n. For example, writing a question mark for the missing coefficients, we have

$$(a + b)^8 = a^8 + \boxed{?}\,a^7b + \boxed{?}\,a^6b^2 + \boxed{?}\,a^5b^3 + \boxed{?}\,a^4b^4 + \boxed{?}\,a^3b^5 + \boxed{?}\,a^2b^6 + \boxed{?}\,ab^7 + b^8$$

To complete the expansion, we need to determine these coefficients. To find a pattern, let's write the coefficients in the expansion of $(a + b)^n$ for the first few values of n in a triangular array as shown in the following array, which is called **Pascal's triangle**.

$$
\begin{array}{lccccccccccc}
(a+b)^0 & & & & & & 1 & & & & & \\
(a+b)^1 & & & & & 1 & & 1 & & & & \\
(a+b)^2 & & & & 1 & & 2 & & 1 & & & \\
(a+b)^3 & & & 1 & & 3 & & 3 & & 1 & & \\
(a+b)^4 & & 1 & & 4 & & 6 & & 4 & & 1 & \\
(a+b)^5 & 1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

Pascal's triangle appears in this Chinese document by Chu Shikie, dated 1303. The title reads "The Old Method Chart of the Seven Multiplying Squares." The triangle was rediscovered by Pascal (see page 858).



The row corresponding to $(a + b)^0$ is called the zeroth row and is included to show the symmetry of the array. The key observation about Pascal's triangle is the following property.

Key Property of Pascal's Triangle

Every entry (other than a 1) is the sum of the two entries diagonally above it.

From this property it's easy to find any row of Pascal's triangle from the row above it. For instance, we find the sixth and seventh rows, starting with the fifth row:

$$\begin{array}{lccccccccccccccc}
(a+b)^5 & & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
(a+b)^6 & & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
(a+b)^7 & 1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1
\end{array}$$

To see why this property holds, let's consider the following expansions:

$$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

$$(a + b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$$

We arrive at the expansion of $(a + b)^6$ by multiplying $(a + b)^5$ by $(a + b)$. Notice, for instance, that the circled term in the expansion of $(a + b)^6$ is obtained via this multiplication from the two circled terms above it. We get this term when the two terms above it are multiplied by b and a, respectively. Thus, its coefficient is the sum of the coefficients of these two terms. We will use this observation at the end of this section when we prove the Binomial Theorem.

Having found these patterns, we can now easily obtain the expansion of any binomial, at least to relatively small powers.

ALTERNATE EXAMPLE 1
Find the expansion of $(a + b)^6$ using Pascal's triangle.

ANSWER
$a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$

Example 1 Expanding a Binomial Using Pascal's Triangle

Find the expansion of $(a + b)^7$ using Pascal's triangle.

Solution The first term in the expansion is a^7, and the last term is b^7. Using the fact that the exponent of a decreases by 1 from term to term and that of b increases by 1 from term to term, we have

$$(a + b)^7 = a^7 + ?\,a^6b + ?\,a^5b^2 + ?\,a^4b^3 + ?\,a^3b^4 + ?\,a^2b^5 + ?\,ab^6 + b^7$$

The appropriate coefficients appear in the seventh row of Pascal's triangle. Thus

$$(a + b)^7 = a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7 \quad ■$$

ALTERNATE EXAMPLE 2
Use Pascal's triangle to expand $(3 - x)^5$.

ANSWER
$243 - 405x + 270x^2 - 90x^3 + 15x^4 - x^5$

Example 2 Expanding a Binomial Using Pascal's Triangle

Use Pascal's triangle to expand $(2 - 3x)^5$.

Solution We find the expansion of $(a + b)^5$ and then substitute 2 for a and $-3x$ for b. Using Pascal's triangle for the coefficients, we get

$$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

IN-CLASS MATERIALS

Pascal's triangle is actually full of patterns. Have the students see how many they can find. They can look along the diagonals, look down "columns," find the sum of each row, etc. There are some instructors who spend an entire class on patterns visible in Pascal's triangle. As of this writing, the website http://ptri1.tripod.com has a good discussion of patterns to be found in Pascal's triangle.

Substituting $a = 2$ and $b = -3x$ gives

$$\begin{aligned}(2 - 3x)^5 &= (2)^5 + 5(2)^4(-3x) + 10(2)^3(-3x)^2 + 10(2)^2(-3x)^3 + 5(2)(-3x)^4 + (-3x)^5 \\ &= 32 - 240x + 720x^2 - 1080x^3 + 810x^4 - 243x^5\end{aligned}$$

■

The Binomial Coefficients

Although Pascal's triangle is useful in finding the binomial expansion for reasonably small values of n, it isn't practical for finding $(a + b)^n$ for large values of n. The reason is that the method we use for finding the successive rows of Pascal's triangle is recursive. Thus, to find the 100th row of this triangle, we must first find the preceding 99 rows.

We need to examine the pattern in the coefficients more carefully to develop a formula that allows us to calculate directly any coefficient in the binomial expansion. Such a formula exists, and the rest of this section is devoted to finding and proving it. However, to state this formula we need some notation.

The product of the first n natural numbers is denoted by $\boldsymbol{n!}$ and is called ***n* factorial**:

$4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$
$7! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 = 5040$
$10! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 = 3{,}628{,}800$

$$\boxed{n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot (n-1) \cdot n}$$

We also define 0! as follows:

$$\boxed{0! = 1}$$

This definition of 0! makes many formulas involving factorials shorter and easier to write.

The Binomial Coefficient

Let n and r be nonnegative integers with $r \le n$. The **binomial coefficient** is denoted by $\binom{n}{r}$ and is defined by

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Example 3 Calculating Binomial Coefficients

(a) $$\binom{9}{4} = \frac{9!}{4!(9-4)!} = \frac{9!}{4!5!} = \frac{\cancel{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \cdot 6 \cdot 7 \cdot 8 \cdot 9}{(1 \cdot 2 \cdot 3 \cdot 4)(\cancel{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5})} = \frac{6 \cdot 7 \cdot 8 \cdot 9}{1 \cdot 2 \cdot 3 \cdot 4} = 126$$

(b) $$\binom{100}{3} = \frac{100!}{3!(100-3)!} = \frac{\cancel{1 \cdot 2 \cdot 3 \cdot \cdots \cdot 97} \cdot 98 \cdot 99 \cdot 100}{(1 \cdot 2 \cdot 3)(\cancel{1 \cdot 2 \cdot 3 \cdot \cdots \cdot 97})} = \frac{98 \cdot 99 \cdot 100}{1 \cdot 2 \cdot 3} = 161{,}700$$

ALTERNATE EXAMPLE 3
Calculate the binomial coefficient $\binom{90}{4}$.

ANSWER
2,555,190

SAMPLE QUESTION
Text Question

Compute $\binom{5}{4}$.

Answer

5

(c) $\displaystyle \binom{100}{97} = \frac{100!}{97!(100-97)!} = \frac{\cancel{1\cdot 2\cdot 3\cdot \cdots \cdot 97}\cdot 98\cdot 99\cdot 100}{(\cancel{1\cdot 2\cdot 3\cdot \cdots \cdot 97})(1\cdot 2\cdot 3)}$

$$= \frac{98\cdot 99\cdot 100}{1\cdot 2\cdot 3} = 161{,}700$$

Although the binomial coefficient $\binom{n}{r}$ is defined in terms of a fraction, all the results of Example 3 are natural numbers. In fact, $\binom{n}{r}$ is always a natural number (see Exercise 50). Notice that the binomial coefficients in parts (b) and (c) of Example 3 are equal. This is a special case of the following relation, which you are asked to prove in Exercise 48.

$$\boxed{\binom{n}{r} = \binom{n}{n-r}}$$

To see the connection between the binomial coefficients and the binomial expansion of $(a+b)^n$, let's calculate the following binomial coefficients:

$$\binom{5}{2} = \frac{5!}{2!(5-2)!} = 10$$

$$\binom{5}{0} = 1 \quad \binom{5}{1} = 5 \quad \binom{5}{2} = 10 \quad \binom{5}{3} = 10 \quad \binom{5}{4} = 5 \quad \binom{5}{5} = 1$$

These are precisely the entries in the fifth row of Pascal's triangle. In fact, we can write Pascal's triangle as follows.

$$\begin{array}{ccccccccccc}
 & & & & & \binom{0}{0} & & & & & \\
 & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\
 & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
 & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
 & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\
\binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} \\
\cdot & & \cdot & & \cdot & & \cdot & & \cdot & & \cdot \\
\binom{n}{0} & & \binom{n}{1} & & \binom{n}{2} & \cdot\ \cdot\ \cdot & & \binom{n}{n-1} & & \binom{n}{n} &
\end{array}$$

To demonstrate that this pattern holds, we need to show that any entry in this version of Pascal's triangle is the sum of the two entries diagonally above it. In other words, we must show that each entry satisfies the key property of Pascal's triangle. We now state this property in terms of the binomial coefficients.

IN-CLASS MATERIALS

If every odd number in Pascal's triangle is colored black, and every even number colored white, a figure like Sierpinski's triangle is revealed. Students need only do a few rows before the recursive structure is revealed.

Key Property of the Binomial Coefficients

For any nonnegative integers r and k with $r \le k$,

$$\binom{k}{r-1} + \binom{k}{r} = \binom{k+1}{r}$$

Notice that the two terms on the left side of this equation are adjacent entries in the kth row of Pascal's triangle and the term on the right side is the entry diagonally below them, in the $(k + 1)$st row. Thus, this equation is a restatement of the key property of Pascal's triangle in terms of the binomial coefficients. A proof of this formula is outlined in Exercise 49.

The Binomial Theorem

We are now ready to state the Binomial Theorem.

The Binomial Theorem

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n$$

We prove this theorem at the end of this section. First, let's look at some of its applications.

Example 4 Expanding a Binomial Using the Binomial Theorem

Use the Binomial Theorem to expand $(x + y)^4$.

Solution By the Binomial Theorem,

$$(x+y)^4 = \binom{4}{0}x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}xy^3 + \binom{4}{4}y^4$$

Verify that

$$\binom{4}{0} = 1 \qquad \binom{4}{1} = 4 \qquad \binom{4}{2} = 6 \qquad \binom{4}{3} = 4 \qquad \binom{4}{4} = 1$$

It follows that

$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

■

EXAMPLE

$(4 - \sqrt{x})^5 = 1024 - 1280\sqrt{x} + 640x - 160(\sqrt{x})^3 + 20x^2 - (\sqrt{x})^5$

EXAMPLE

$\left(x + \dfrac{1}{x}\right)^8 = x^8 + 8x^6 + 28x^4 + 56x^2 + 70 + \dfrac{56}{x^2} + \dfrac{28}{x^4} + \dfrac{8}{x^6} + \dfrac{1}{x^8}$

ALTERNATE EXAMPLE 4

Use the Binomial Theorem to expand $(x + y)^5$.

ANSWER

$x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$

IN-CLASS MATERIALS

The binomial theorem can be extended to the real numbers, and then some very interesting things happen. First we generalize the definition of $\binom{n}{r}$:

$$\binom{n}{r} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}$$

Notice that this is equivalent to our former definition if n and r are positive integers, even if $r > n$ (in that case the numerator will be zero). But this new definition works even if n is an arbitrary real number. Now we can say

$$(a+b)^n = \sum_{r=0}^{\infty}\binom{n}{r}a^rb^{n-r}$$

Notice again that if n is a positive integer, then we are back to the standard binomial theorem, since $\binom{n}{r}$ is 0 for $r > n$. But now we can let $n = -1$ to obtain

$$\frac{1}{x+1} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$$

Similarly, we can let $n = \dfrac{1}{2}$ to obtain

$$x^5 + 5x^4 \cdot y + 10x\sqrt{x+1} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \cdots$$

with the general term being $(-1)^{n-1}\dfrac{1 \cdot 3 \cdot 5 \cdot 7 \cdot \cdots \cdot (2n-3)}{2^n n!}x^n$.

ALTERNATE EXAMPLE 5
Use the Binomial Theorem to expand $\left(\sqrt{x} - 2\right)^6$.

ANSWER
$x^3 - 12x^{5/2} + 60x^2 - 160x^{3/2} + 240x - 192x^{1/2} + 64$

DRILL QUESTION
Expand $(x + 2y)^6$.

Answer
$(x + 2y)^6 = x^6 + 12x^5y + 60x^4y^2 + 160x^3y^3 + 240x^2y^4 + 192xy^5 + 64y^6$

Example 5 Expanding a Binomial Using the Binomial Theorem

Use the Binomial Theorem to expand $\left(\sqrt{x} - 1\right)^8$.

Solution We first find the expansion of $(a + b)^8$ and then substitute $\sqrt{x}$ for a and -1 for b. Using the Binomial Theorem, we have

$$(a+b)^8 = \binom{8}{0}a^8 + \binom{8}{1}a^7b + \binom{8}{2}a^6b^2 + \binom{8}{3}a^5b^3 + \binom{8}{4}a^4b^4 + \binom{8}{5}a^3b^5 + \binom{8}{6}a^2b^6 + \binom{8}{7}ab^7 + \binom{8}{8}b^8$$

Verify that

$$\binom{8}{0} = 1 \quad \binom{8}{1} = 8 \quad \binom{8}{2} = 28 \quad \binom{8}{3} = 56 \quad \binom{8}{4} = 70$$
$$\binom{8}{5} = 56 \quad \binom{8}{6} = 28 \quad \binom{8}{7} = 8 \quad \binom{8}{8} = 1$$

So

$$(a+b)^8 = a^8 + 8a^7b + 28a^6b^2 + 56a^5b^3 + 70a^4b^4 + 56a^3b^5 + 28a^2b^6 + 8ab^7 + b^8$$

Performing the substitutions $a = x^{1/2}$ and $b = -1$ gives

$$\left(\sqrt{x} - 1\right)^8 = (x^{1/2})^8 + 8(x^{1/2})^7(-1) + 28(x^{1/2})^6(-1)^2 + 56(x^{1/2})^5(-1)^3 + 70(x^{1/2})^4(-1)^4 + 56(x^{1/2})^3(-1)^5 + 28(x^{1/2})^2(-1)^6 + 8(x^{1/2})(-1)^7 + (-1)^8$$

This simplifies to

$$\left(\sqrt{x} - 1\right)^8 = x^4 - 8x^{7/2} + 28x^3 - 56x^{5/2} + 70x^2 - 56x^{3/2} + 28x - 8x^{1/2} + 1 \quad ■$$

The Binomial Theorem can be used to find a particular term of a binomial expansion without having to find the entire expansion.

General Term of the Binomial Expansion

The term that contains a^r in the expansion of $(a + b)^n$ is

$$\binom{n}{n-r}a^rb^{n-r}$$

Example 6 Finding a Particular Term in a Binomial Expansion

Find the term that contains x^5 in the expansion of $(2x + y)^{20}$.

Solution The term that contains x^5 is given by the formula for the general term with $a = 2x$, $b = y$, $n = 20$, and $r = 5$. So, this term is

$$\binom{20}{15}a^5b^{15} = \frac{20!}{15!(20-15)!}(2x)^5y^{15} = \frac{20!}{15!5!}32x^5y^{15} = 496{,}128x^5y^{15}$$ ■

ALTERNATE EXAMPLE 6
Find the term that contains x^4 in the expansion of $(3x + y)^{15}$.

ANSWER
$110565x^4y^{11}$

Example 7 Finding a Particular Term in a Binomial Expansion

Find the coefficient of x^8 in the expansion of $\left(x^2 + \frac{1}{x}\right)^{10}$.

Solution Both x^2 and $1/x$ are powers of x, so the power of x in each term of the expansion is determined by both terms of the binomial. To find the required coefficient, we first find the general term in the expansion. By the formula we have $a = x^2$, $b = 1/x$, and $n = 10$, so the general term is

$$\binom{10}{10-r}(x^2)^r\left(\frac{1}{x}\right)^{10-r} = \binom{10}{10-r}x^{2r}(x^{-1})^{10-r} = \binom{10}{10-r}x^{3r-10}$$

Thus, the term that contains x^8 is the term in which

$$3r - 10 = 8$$
$$r = 6$$

So the required coefficient is

$$\binom{10}{10-6} = \binom{10}{4} = 210$$ ■

ALTERNATE EXAMPLE 7
Find the coefficient of x^6 in the expansion of $\left(x^2 + \frac{1}{x}\right)^9$.

ANSWER
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Proof of the Binomial Theorem

We now give a proof of the Binomial Theorem using mathematical induction.

■ **Proof** Let $P(n)$ denote the statement

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n$$

Step 1 We show that $P(1)$ is true. But $P(1)$ is just the statement

$$(a+b)^1 = \binom{1}{0}a^1 + \binom{1}{1}b^1 = 1a + 1b = a + b$$

which is certainly true.

Step 2 We assume that $P(k)$ is true. Thus, our induction hypothesis is

$$(a+b)^k = \binom{k}{0}a^k + \binom{k}{1}a^{k-1}b + \binom{k}{2}a^{k-2}b^2 + \cdots + \binom{k}{k-1}ab^{k-1} + \binom{k}{k}b^k$$

IN-CLASS MATERIALS

Point out that some of the formulas previously covered are just special cases of the Binomial Theorem, for example the formulas for $(x + y)^2$, $(x - y)^2$, $(x + y)^3$, and $(x - y)^3$.

We use this to show that $P(k+1)$ is true.

$$\begin{aligned}
(a+b)^{k+1} &= (a+b)[(a+b)^k] \\
&= (a+b)\left[\binom{k}{0}a^k + \binom{k}{1}a^{k-1}b + \binom{k}{2}a^{k-2}b^2 + \cdots + \binom{k}{k-1}ab^{k-1} + \binom{k}{k}b^k\right] && \text{Induction hypothesis} \\
&= a\left[\binom{k}{0}a^k + \binom{k}{1}a^{k-1}b + \binom{k}{2}a^{k-2}b^2 + \cdots + \binom{k}{k-1}ab^{k-1} + \binom{k}{k}b^k\right] \\
&\quad + b\left[\binom{k}{0}a^k + \binom{k}{1}a^{k-1}b + \binom{k}{2}a^{k-2}b^2 + \cdots + \binom{k}{k-1}ab^{k-1} + \binom{k}{k}b^k\right] && \text{Distributive Property} \\
&= \binom{k}{0}a^{k+1} + \binom{k}{1}a^k b + \binom{k}{2}a^{k-1}b^2 + \cdots + \binom{k}{k-1}a^2b^{k-1} + \binom{k}{k}ab^k \\
&\quad + \binom{k}{0}a^k b + \binom{k}{1}a^{k-1}b^2 + \binom{k}{2}a^{k-2}b^3 + \cdots + \binom{k}{k-1}ab^k + \binom{k}{k}b^{k+1} && \text{Distributive Property} \\
&= \binom{k}{0}a^{k+1} + \left[\binom{k}{0} + \binom{k}{1}\right]a^k b + \left[\binom{k}{1} + \binom{k}{2}\right]a^{k-1}b^2 \\
&\quad + \cdots + \left[\binom{k}{k-1} + \binom{k}{k}\right]ab^k + \binom{k}{k}b^{k+1} && \text{Group like terms}
\end{aligned}$$

Using the key property of the binomial coefficients, we can write each of the expressions in square brackets as a single binomial coefficient. Also, writing the first and last coefficients as $\binom{k+1}{0}$ and $\binom{k+1}{k+1}$ (these are equal to 1 by Exercise 46) gives

$$(a+b)^{k+1} = \binom{k+1}{0}a^{k+1} + \binom{k+1}{1}a^k b + \binom{k+1}{2}a^{k-1}b^2 + \cdots + \binom{k+1}{k}ab^k + \binom{k+1}{k+1}b^{k+1}$$

But this last equation is precisely $P(k+1)$, and this completes the induction step.

Having proved Steps 1 and 2, we conclude by the Principle of Mathematical Induction that the theorem is true for all natural numbers n. ■

11.6 Exercises

1–12 ■ Use Pascal's triangle to expand the expression.

1. $(x+y)^6$ **2.** $(2x+1)^4$ **3.** $\left(x+\frac{1}{x}\right)^4$

4. $(x-y)^5$ **5.** $(x-1)^5$ **6.** $(\sqrt{a}+\sqrt{b})^6$

7. $(x^2y-1)^5$ **8.** $(1+\sqrt{2})^6$ **9.** $(2x-3y)^3$

10. $(1+x^3)^3$ **11.** $\left(\frac{1}{x}-\sqrt{x}\right)^5$ **12.** $\left(2+\frac{x}{2}\right)^5$

13–20 ■ Evaluate the expression.

13. $\binom{6}{4}$ **14.** $\binom{8}{3}$ **15.** $\binom{100}{98}$

16. $\binom{10}{5}$ **17.** $\binom{3}{1}\binom{4}{2}$ **18.** $\binom{5}{2}\binom{5}{3}$

19. $\binom{5}{0}+\binom{5}{1}+\binom{5}{2}+\binom{5}{3}+\binom{5}{4}+\binom{5}{5}$

20. $\binom{5}{0} - \binom{5}{1} + \binom{5}{2} - \binom{5}{3} + \binom{5}{4} - \binom{5}{5}$

21–24 ■ Use the Binomial Theorem to expand the expression.

21. $(x + 2y)^4$ **22.** $(1 - x)^5$

23. $\left(1 + \frac{1}{x}\right)^6$ **24.** $(2A + B^2)^4$

25. Find the first three terms in the expansion of $(x + 2y)^{20}$.

26. Find the first four terms in the expansion of $(x^{1/2} + 1)^{30}$.

27. Find the last two terms in the expansion of $(a^{2/3} + a^{1/3})^{25}$.

28. Find the first three terms in the expansion of

$$\left(x + \frac{1}{x}\right)^{40}$$

29. Find the middle term in the expansion of $(x^2 + 1)^{18}$.

30. Find the fifth term in the expansion of $(ab - 1)^{20}$.

31. Find the 24th term in the expansion of $(a + b)^{25}$.

32. Find the 28th term in the expansion of $(A - B)^{30}$.

33. Find the 100th term in the expansion of $(1 + y)^{100}$.

34. Find the second term in the expansion of

$$\left(x^2 - \frac{1}{x}\right)^{25}$$

35. Find the term containing x^4 in the expansion of $(x + 2y)^{10}$.

36. Find the term containing y^3 in the expansion of $(\sqrt{2} + y)^{12}$.

37. Find the term containing b^8 in the expansion of $(a + b^2)^{12}$.

38. Find the term that does not contain x in the expansion of

$$\left(8x + \frac{1}{2x}\right)^8$$

39–42 ■ Factor using the Binomial Theorem.

39. $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

40. $(x - 1)^5 + 5(x - 1)^4 + 10(x - 1)^3 + 10(x - 1)^2 + 5(x - 1) + 1$

41. $8a^3 + 12a^2b + 6ab^2 + b^3$

42. $x^8 + 4x^6y + 6x^4y^2 + 4x^2y^3 + y^4$

43–44 ■ Simplify using the Binomial Theorem.

43. $\dfrac{(x + h)^3 - x^3}{h}$ **44.** $\dfrac{(x + h)^4 - x^4}{h}$

45. Show that $(1.01)^{100} > 2$.

[*Hint*: Note that $(1.01)^{100} = (1 + 0.01)^{100}$ and use the Binomial Theorem to show that the sum of the first two terms of the expansion is greater than 2.]

46. Show that $\binom{n}{0} = 1$ and $\binom{n}{n} = 1$.

47. Show that $\binom{n}{1} = \binom{n}{n-1} = n$.

48. Show that $\binom{n}{r} = \binom{n}{n-r}$ for $0 \le r \le n$.

49. In this exercise we prove the identity

$$\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$$

(a) Write the left side of this equation as the sum of two fractions.

(b) Show that a common denominator of the expression you found in part (a) is $r!(n - r + 1)!$.

(c) Add the two fractions using the common denominator in part (b), simplify the numerator, and note that the resulting expression is equal to the right side of the equation.

50. Prove that $\binom{n}{r}$ is an integer for all n and for $0 \le r \le n$. [*Suggestion:* Use induction to show that the statement is true for all n, and use Exercise 49 for the induction step.]

Discovery • Discussion

51. Powers of Factorials Which is larger, $(100!)^{101}$ or $(101!)^{100}$? [*Hint*: Try factoring the expressions. Do they have any common factors?]

52. Sums of Binomial Coefficients Add each of the first five rows of Pascal's triangle, as indicated. Do you see a pattern?

$$1 + 1 = \boxed{?}$$
$$1 + 2 + 1 = \boxed{?}$$
$$1 + 3 + 3 + 1 = \boxed{?}$$
$$1 + 4 + 6 + 4 + 1 = \boxed{?}$$
$$1 + 5 + 10 + 10 + 5 + 1 = \boxed{?}$$

Based on the pattern you have found, find the sum of the nth row:

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$$

Prove your result by expanding $(1 + 1)^n$ using the Binomial Theorem.

53. Alternating Sums of Binomial Coefficients Find the sum

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n\binom{n}{n}$$

by finding a pattern as in Exercise 52. Prove your result by expanding $(1 - 1)^n$ using the Binomial Theorem.

11 Review

Concept Check

1. **(a)** What is a sequence?
   (b) What is an arithmetic sequence? Write an expression for the nth term of an arithmetic sequence.
   (c) What is a geometric sequence? Write an expression for the nth term of a geometric sequence.
2. **(a)** What is a recursively defined sequence?
   (b) What is the Fibonacci sequence?
3. **(a)** What is meant by the partial sums of a sequence?
   (b) If an arithmetic sequence has first term a and common difference d, write an expression for the sum of its first n terms.
   (c) If a geometric sequence has first term a and common ratio r, write an expression for the sum of its first n terms.
   (d) Write an expression for the sum of an infinite geometric series with first term a and common ratio r. For what values of r is your formula valid?
4. **(a)** Write the sum $\sum_{k=1}^{n} a_k$ without using Σ-notation.
   (b) Write $b_1 + b_2 + b_3 + \cdots + b_n$ using Σ-notation.
5. Write an expression for the amount A_f of an annuity consisting of n regular equal payments of size R with interest rate i per time period.
6. State the Principle of Mathematical Induction.
7. Write the first five rows of Pascal's triangle. How are the entries related to each other?
8. **(a)** What does the symbol $n!$ mean?
   (b) Write an expression for the binomial coefficient $\binom{n}{r}$.
   (c) State the Binomial Theorem.
   (d) Write the term that contains a^r in the expansion of $(a + b)^n$.

Exercises

1–6 ■ Find the first four terms as well as the tenth term of the sequence with the given nth term.

1. $a_n = \dfrac{n^2}{n + 1}$ **2.** $a_n = (-1)^n\dfrac{2^n}{n}$

3. $a_n = \dfrac{(-1)^n + 1}{n^3}$ **4.** $a_n = \dfrac{n(n + 1)}{2}$

5. $a_n = \dfrac{(2n)!}{2^n n!}$ **6.** $a_n = \dbinom{n + 1}{2}$

7–10 ■ A sequence is defined recursively. Find the first seven terms of the sequence.

7. $a_n = a_{n-1} + 2n - 1, \quad a_1 = 1$

8. $a_n = \dfrac{a_{n-1}}{n}, \quad a_1 = 1$

9. $a_n = a_{n-1} + 2a_{n-2}, \quad a_1 = 1, a_2 = 3$

10. $a_n = \sqrt{3a_{n-1}}, \quad a_1 = \sqrt{3}$

11–14 ■ The nth term of a sequence is given.
(a) Find the first five terms of the sequence.
(b) Graph the terms you found in part (a).
(c) Determine if the series is arithmetic or geometric. Find the common difference or the common ratio.

11. $a_n = 2n + 5$ **12.** $a_n = \dfrac{5}{2^n}$

13. $a_n = \dfrac{3^n}{2^{n+1}}$ **14.** $a_n = 4 - \dfrac{n}{2}$

15–22 ■ The first four terms of a sequence are given. Determine whether they can be the terms of an arithmetic sequence, a geometric sequence, or neither. If the sequence is arithmetic or geometric, find the fifth term.

15. 5, 5.5, 6, 6.5, . . .

16. $1, -\frac{3}{2}, 2, -\frac{5}{2}, \ldots$

17. $\sqrt{2}, 2\sqrt{2}, 3\sqrt{2}, 4\sqrt{2}, \ldots$

18. $\sqrt{2}, 2, 2\sqrt{2}, 4, \ldots$

19. $t-3, t-2, t-1, t, \ldots$

20. $t^3, t^2, t, 1, \ldots$

21. $\frac{3}{4}, \frac{1}{2}, \frac{1}{3}, \frac{2}{9}, \ldots$

22. $a, 1, \frac{1}{a}, \frac{1}{a^2}, \ldots$

23. Show that $3, 6i, -12, -24i, \ldots$ is a geometric sequence, and find the common ratio. (Here $i = \sqrt{-1}$.)

24. Find the nth term of the geometric sequence $2, 2+2i, 4i, -4+4i, -8, \ldots$ (Here $i = \sqrt{-1}$.)

25. The sixth term of an arithmetic sequence is 17, and the fourth term is 11. Find the second term.

26. The 20th term of an arithmetic sequence is 96, and the common difference is 5. Find the nth term.

27. The third term of a geometric sequence is 9, and the common ratio is $\frac{3}{2}$. Find the fifth term.

28. The second term of a geometric sequence is 10, and the fifth term is $\frac{1250}{27}$. Find the nth term.

29. A teacher makes $32,000 in his first year at Lakeside School, and gets a 5% raise each year.

(a) Find a formula for his salary A_n in his nth year at this school.

(b) List his salaries for his first 8 years at this school.

30. A colleague of the teacher in Exercise 29, hired at the same time, makes $35,000 in her first year, and gets a $1200 raise each year.

(a) What is her salary A_n in her nth year at this school?

(b) Find her salary in her eighth year at this school, and compare it to the salary of the teacher in Exercise 29 in his eighth year.

31. A certain type of bacteria divides every 5 s. If three of these bacteria are put into a petri dish, how many bacteria are in the dish at the end of 1 min?

32. If $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ are arithmetic sequences, show that $a_1 + b_1, a_2 + b_2, a_3 + b_3, \ldots$ is also an arithmetic sequence.

33. If $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ are geometric sequences, show that $a_1b_1, a_2b_2, a_3b_3, \ldots$ is also a geometric sequence.

34. **(a)** If $a_1, a_2, a_3, \ldots$ is an arithmetic sequence, is the sequence $a_1 + 2, a_2 + 2, a_3 + 2, \ldots$ arithmetic?

(b) If $a_1, a_2, a_3, \ldots$ is a geometric sequence, is the sequence $5a_1, 5a_2, 5a_3, \ldots$ geometric?

35. Find the values of x for which the sequence $6, x, 12, \ldots$ is

(a) arithmetic **(b)** geometric

36. Find the values of x and y for which the sequence $2, x, y, 17, \ldots$ is

(a) arithmetic **(b)** geometric

37–40 ■ Find the sum.

37. $\displaystyle\sum_{k=3}^{6} (k+1)^2$

38. $\displaystyle\sum_{i=1}^{4} \frac{2i}{2i-1}$

39. $\displaystyle\sum_{k=1}^{6} (k+1)2^{k-1}$

40. $\displaystyle\sum_{m=1}^{5} 3^{m-2}$

41–44 ■ Write the sum without using sigma notation. Do not evaluate.

41. $\displaystyle\sum_{k=1}^{10} (k-1)^2$

42. $\displaystyle\sum_{j=2}^{100} \frac{1}{j-1}$

43. $\displaystyle\sum_{k=1}^{50} \frac{3^k}{2^{k+1}}$

44. $\displaystyle\sum_{n=1}^{10} n^2 2^n$

45–48 ■ Write the sum using sigma notation. Do not evaluate.

45. $3 + 6 + 9 + 12 + \cdots + 99$

46. $1^2 + 2^2 + 3^2 + \cdots + 100^2$

47. $1 \cdot 2^3 + 2 \cdot 2^4 + 3 \cdot 2^5 + 4 \cdot 2^6 + \cdots + 100 \cdot 2^{102}$

48. $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \dfrac{1}{3 \cdot 4} + \cdots + \dfrac{1}{999 \cdot 1000}$

49–54 ■ Determine whether the expression is a partial sum of an arithmetic or geometric sequence. Then find the sum.

49. $1 + 0.9 + (0.9)^2 + \cdots + (0.9)^5$

50. $3 + 3.7 + 4.4 + \cdots + 10$

51. $\sqrt{5} + 2\sqrt{5} + 3\sqrt{5} + \cdots + 100\sqrt{5}$

52. $\frac{1}{3} + \frac{2}{3} + 1 + \frac{4}{3} + \cdots + 33$

53. $\displaystyle\sum_{n=0}^{6} 3(-4)^n$

54. $\displaystyle\sum_{k=0}^{8} 7(5)^{k/2}$

55. The first term of an arithmetic sequence is $a = 7$, and the common difference is $d = 3$. How many terms of this sequence must be added to obtain 325?

56. The sum of the first three terms of a geometric series is 52, and the common ratio is $r = 3$. Find the first term.

57. A person has two parents, four grandparents, eight great-grandparents, and so on. What is the total number of a person's ancestors in 15 generations?

58. Find the amount of an annuity consisting of 16 annual payments of \$1000 each into an account that pays 8% interest per year, compounded annually.

59. How much money should be invested every quarter at 12% per year, compounded quarterly, in order to have \$10,000 in one year?

60. What are the monthly payments on a mortgage of \$60,000 at 9% interest if the loan is to be repaid in

(a) 30 years? **(b)** 15 years?

61–64 ■ Find the sum of the infinite geometric series.

61. $1 - \frac{2}{5} + \frac{4}{25} - \frac{8}{125} + \cdots$

62. $0.1 + 0.01 + 0.001 + 0.0001 + \cdots$

63. $1 + \dfrac{1}{3^{1/2}} + \dfrac{1}{3} + \dfrac{1}{3^{3/2}} + \cdots$

64. $a + ab^2 + ab^4 + ab^6 + \cdots$

65–67 ■ Use mathematical induction to prove that the formula is true for all natural numbers n.

65. $1 + 4 + 7 + \cdots + (3n - 2) = \dfrac{n(3n - 1)}{2}$

66. $\dfrac{1}{1 \cdot 3} + \dfrac{1}{3 \cdot 5} + \dfrac{1}{5 \cdot 7} + \cdots + \dfrac{1}{(2n - 1)(2n + 1)} = \dfrac{n}{2n + 1}$

67. $\left(1 + \dfrac{1}{1}\right)\left(1 + \dfrac{1}{2}\right)\left(1 + \dfrac{1}{3}\right)\cdots\left(1 + \dfrac{1}{n}\right) = n + 1$

68. Show that $7^n - 1$ is divisible by 6 for all natural numbers n.

69. Let $a_{n+1} = 3a_n + 4$ and $a_1 = 4$. Show that $a_n = 2 \cdot 3^n - 2$ for all natural numbers n.

70. Prove that the Fibonacci number F_{4n} is divisible by 3 for all natural numbers n.

71. Find and prove an inequality that relates 2^n and $n!$.

72–75 ■ Evaluate the expression.

72. $\binom{5}{2}\binom{5}{3}$

73. $\binom{10}{2} + \binom{10}{6}$

74. $\displaystyle\sum_{k=0}^{5} \binom{5}{k}$

75. $\displaystyle\sum_{k=0}^{8} \binom{8}{k}\binom{8}{8 - k}$

76–77 ■ Expand the expression.

76. $(1 - x^2)^6$

77. $(2x + y)^4$

78. Find the 20th term in the expansion of $(a + b)^{22}$.

79. Find the first three terms in the expansion of $(b^{-2/3} + b^{1/3})^{20}$.

80. Find the term containing A^6 in the expansion of $(A + 3B)^{10}$.

11 Test

1. Find the first four terms and the tenth term of the sequence whose nth term is $a_n = n^2 - 1$.

2. A sequence is defined recursively by $a_{n+2} = a_n^2 - a_{n+1}$, with $a_1 = 1$ and $a_2 = 1$. Find a_5.

3. An arithmetic sequence begins 2, 5, 8, 11, 14, ...
(a) Find the common difference d for this sequence.
(b) Find a formula for the nth term a_n of the sequence.
(c) Find the 35th term of the sequence.

4. A geometric sequence begins 12, 3, 3/4, 3/16, 3/64, ...
(a) Find the common ratio r for this sequence.
(b) Find a formula for the nth term a_n of the sequence.
(c) Find the tenth term of the sequence.

5. The first term of a geometric sequence is 25, and the fourth term is $\frac{1}{5}$.
(a) Find the common ratio r and the fifth term.
(b) Find the partial sum of the first eight terms.

6. The first term of an arithmetic sequence is 10 and the tenth term is 2.
(a) Find the common difference and the 100th term of the sequence.
(b) Find the partial sum of the first ten terms.

7. Let $a_1, a_2, a_3, \ldots$ be a geometric sequence with initial term a and common ratio r. Show that $a_1^2, a_2^2, a_3^2, \ldots$ is also a geometric sequence by finding its common ratio.

8. Write the expression without using sigma notation, and then find the sum.
(a) $\displaystyle\sum_{n=1}^{5}(1 - n^2)$ (b) $\displaystyle\sum_{n=3}^{6}(-1)^n 2^{n-2}$

9. Find the sum.
(a) $\dfrac{1}{3} + \dfrac{2}{3^2} + \dfrac{2^2}{3^3} + \dfrac{2^3}{3^4} + \cdots + \dfrac{2^9}{3^{10}}$
(b) $1 + \dfrac{1}{2^{1/2}} + \dfrac{1}{2} + \dfrac{1}{2^{3/2}} + \cdots$

10. Use mathematical induction to prove that, for all natural numbers n,

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

11. Expand $(2x + y^2)^5$.

12. Find the term containing x^3 in the binomial expansion of $(3x - 2)^{10}$.

13. A puppy weighs 0.85 lb at birth, and each week he gains 24% in weight. Let a_n be his weight in pounds at the end of his nth week of life.
(a) Find a formula for a_n.
(b) How much does the puppy weigh when he is six weeks old?
(c) Is the sequence $a_1, a_2, a_3, \ldots$ arithmetic, geometric, or neither?

Focus on Modeling
Modeling with Recursive Sequences

Many real-world processes occur in stages. Population growth can be viewed in stages—each new generation represents a new stage in population growth. Compound interest is paid in stages—each interest payment creates a new account balance. Many things that change continuously are more easily measured in discrete stages. For example, we can measure the temperature of a continuously cooling object in one-hour intervals. In this *Focus* we learn how recursive sequences are used to model such situations. In some cases, we can get an explicit formula for a sequence from the recursion relation that defines it by finding a pattern in the terms of the sequence.

Recursive Sequences as Models

Suppose you deposit some money in an account that pays 6% interest compounded monthly. The bank has a definite rule for paying interest: At the end of each month the bank adds to your account $\frac{1}{2}$% (or 0.005) of the amount in your account at that time. Let's express this rule as follows:

$$\boxed{\begin{array}{c}\text{amount at the end of}\\ \text{this month}\end{array}} = \boxed{\begin{array}{c}\text{amount at the end of}\\ \text{last month}\end{array}} + 0.005 \times \boxed{\begin{array}{c}\text{amount at the end of}\\ \text{last month}\end{array}}$$

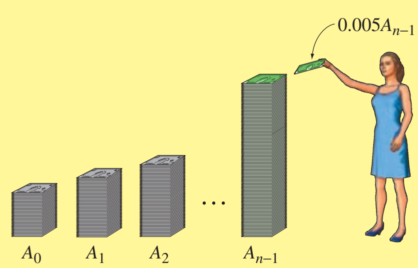
Using the Distributive Property, we can write this as

$$\boxed{\begin{array}{c}\text{amount at the end of}\\ \text{this month}\end{array}} = 1.005 \times \boxed{\begin{array}{c}\text{amount at the end of}\\ \text{last month}\end{array}}$$

To model this statement using algebra, let A_0 be the amount of the original deposit, A_1 the amount at the end of the first month, A_2 the amount at the end of the second month, and so on. So A_n is the amount at the end of the nth month. Thus

$$A_n = 1.005A_{n-1}$$

We recognize this as a recursively defined sequence—it gives us the amount at each stage in terms of the amount at the preceding stage.



To find a formula for A_n, let's find the first few terms of the sequence and look for a pattern.

$$\begin{aligned} A_1 &= 1.005A_0 \\ A_2 &= 1.005A_1 = (1.005)^2 A_0 \\ A_3 &= 1.005A_2 = (1.005)^3 A_0 \\ A_4 &= 1.005A_3 = (1.005)^4 A_0 \end{aligned}$$

We see that in general, $A_n = (1.005)^n A_0$.

Example 1 Population Growth

A certain animal population grows by 2% each year. The initial population is 5000.

(a) Find a recursive sequence that models the population P_n at the end of the nth year.

(b) Find the first five terms of the sequence P_n.

(c) Find a formula for P_n.

Solution

(a) We can model the population using the following rule:

population at the end of this year $= 1.02 \times$ population at the end of last year

Algebraically we can write this as the recursion relation

$$P_n = 1.02P_{n-1}$$

(b) Since the initial population is 5000, we have

$$\begin{aligned} P_0 &= 5000 \\ P_1 &= 1.02P_0 = (1.02)5000 \\ P_2 &= 1.02P_1 = (1.02)^2 5000 \\ P_3 &= 1.02P_2 = (1.02)^3 5000 \\ P_4 &= 1.02P_3 = (1.02)^4 5000 \end{aligned}$$

(c) We see from the pattern exhibited in part (b) that $P_n = (1.02)^n 5000$. (Note that P_n is a geometric sequence, with common ratio $r = 1.02$.) ■

Example 2 Daily Drug Dose

A patient is to take a 50-mg pill of a certain drug every morning. It is known that the body eliminates 40% of the drug every 24 hours.

(a) Find a recursive sequence that models the amount A_n of the drug in the patient's body after each pill is taken.

(b) Find the first four terms of the sequence A_n.

(c) Find a formula for A_n.

(d) How much of the drug remains in the patient's body after 5 days? How much will accumulate in his system after prolonged use?

Solution

(a) Each morning 60% of the drug remains in his system plus he takes an additional 50 mg (his daily dose).

$$\boxed{\text{amount of drug this morning}} = 0.6 \times \boxed{\text{amount of drug yesterday morning}} + 50 \text{ mg}$$

We can express this as a recursion relation

$$A_n = 0.6A_{n-1} + 50$$

(b) Since the initial dose is 50 mg, we have

$$\begin{aligned}
A_0 &= 50\\
A_1 &= 0.6A_0 + 50 = 0.6(50) + 50\\
A_2 &= 0.6A_1 + 50 = 0.6[0.6(50) + 50] + 50\\
&= 0.6^2(50) + 0.6(50) + 50\\
&= 50(0.6^2 + 0.6 + 1)\\
A_3 &= 0.6A_2 + 50 = 0.6[0.6^2(50) + 0.6(50) + 50] + 50\\
&= 0.6^3(50) + 0.6^2(50) + 0.6(50) + 50\\
&= 50(0.6^3 + 0.6^2 + 0.6 + 1)
\end{aligned}$$

(c) From the pattern in part (b), we see that

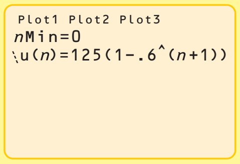
$$\begin{aligned}
A_n &= 50(1 + 0.6 + 0.6^2 + \cdots + 0.6^n)\\
&= 50\left(\frac{1 - 0.6^{n+1}}{1 - 0.6}\right) && \text{Sum of a geometric sequence: } S_n = a\left(\frac{1 - r^{n+1}}{1 - r}\right)\\
&= 125(1 - 0.6^{n+1}) && \text{Simplify}
\end{aligned}$$

(d) To find the amount remaining after 5 days, we substitute $n = 5$ and get $A_5 = 125(1 - 0.6^{5+1}) \approx 119$ mg.

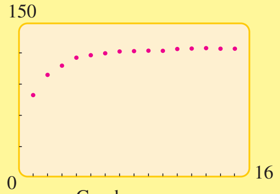
To find the amount remaining after prolonged use, we let n become large. As n gets large, 0.6^n approaches 0. That is, $0.6^n \to 0$ as $n \to \infty$ (see Section 4.1). So as $n \to \infty$,

$$A_n = 125(1 - 0.6^{n+1}) \to 125(1 - 0) = 125$$

Thus, after prolonged use the amount of drug in the patient's system approaches 125 mg (see Figure 1, where we have used a graphing calculator to graph the sequence). ■



Enter sequence



Graph sequence

Figure 1

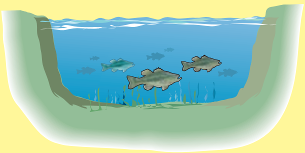
Problems

1. **Retirement Accounts** Many college professors keep retirement savings with TIAA, the largest annuity program in the world. Interest on these accounts is compounded and credited *daily*. Professor Brown has \$275,000 on deposit with TIAA at the start of 2006, and receives 3.65% interest per year on his account.
   (a) Find a recursive sequence that models the amount A_n in his account at the end of the nth day of 2006.
   (b) Find the first eight terms of the sequence A_n, rounded to the nearest cent.
   (c) Find a formula for A_n.

2. **Fitness Program** Sheila decides to embark on a swimming program as the best way to maintain cardiovascular health. She begins by swimming 5 min on the first day, then adds $1\frac{1}{2}$ min every day after that.
   (a) Find a recursive formula for the number of minutes T_n that she swims on the nth day of her program.
   (b) Find the first 6 terms of the sequence T_n.
   (c) Find a formula for T_n. What kind of sequence is this?
   (d) On what day does Sheila attain her goal of swimming at least 65 min a day?
   (e) What is the total amount of time she will have swum after 30 days?

3. **Monthly Savings Program** Alice opens a savings account paying 3% interest per year, compounded monthly. She begins by depositing \$100 at the start of the first month, and adds \$100 at the end of each month, when the interest is credited.
   (a) Find a recursive formula for the amount A_n in her account at the end of the nth month. (Include the interest credited for that month and her monthly deposit.)
   (b) Find the first 5 terms of the sequence A_n.
   (c) Use the pattern you observed in (b) to find a formula for A_n. [*Hint:* To find the pattern most easily, it's best *not* to simplify the terms *too* much.]
   (d) How much has she saved after 5 years?

4. **Stocking a Fish Pond** A pond is stocked with 4000 trout, and through reproduction the population increases by 20% per year. Find a recursive sequence that models the trout population P_n at the end of the nth year under each of the following circumstances. Find the trout population at the end of the fifth year in each case.
   (a) The trout population changes only because of reproduction.
   (b) Each year 600 trout are harvested.
   (c) Each year 250 additional trout are introduced into the pond.
   (d) Each year 10% of the trout are harvested and 300 additional trout are introduced into the pond.



5. **Pollution** A chemical plant discharges 2400 tons of pollutants every year into an adjacent lake. Through natural runoff, 70% of the pollutants contained in the lake at the beginning of the year are expelled by the end of the year.
   (a) Explain why the following sequence models the amount A_n of the pollutant in the lake at the end of the nth year that the plant is operating.

$$A_n = 0.30A_{n-1} + 2400$$

(b) Find the first five terms of the sequence A_n.

(c) Find a formula for A_n.

(d) How much of the pollutant remains in the lake after 6 years? How much will remain after the plant has been operating a long time?

(e) Verify your answer to part (d) by graphing A_n with a graphing calculator, for $n = 1$ to $n = 20$.

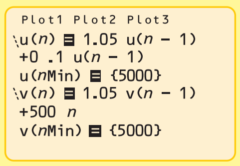
6. Annual Savings Program Ursula opens a one-year CD that yields 5% interest per year. She begins with a deposit of \$5000. At the end of each year when the CD matures, she reinvests at the same 5% interest rate, also adding 10% to the value of the CD from her other savings. (So for example, after the first year her CD has earned 5% of \$5000 in interest, for a value of \$5250 at maturity. She then adds 10%, or \$525, bringing the total value of her renewed CD to \$5775.)

(a) Find a recursive formula for the amount U_n in her CD when she reinvests at the end of the nth year.

(b) Find the first 5 terms of the sequence U_n. Does this appear to be a geometric sequence?

(c) Use the pattern you observed in (b) to find a formula for U_n.

(d) How much has she saved after 10 years?



Entering the sequences

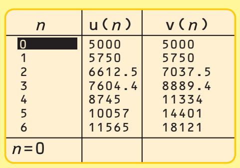


Table of values of the sequences

7. Annual Savings Program Victoria opens a one-year CD with a 5% annual interest yield at the same time as her friend Ursula in Problem 6. She also starts with an initial deposit of \$5000. However, Victoria decides to add \$500 to her CD when she reinvests at the end of the first year, \$1000 at the end of the second, \$1500 at the end of the third, and so on.

(a) Explain why the recursive formula displayed below gives the amount V_n in her CD when she reinvests at the end of the nth year.

$$V_n = 1.05V_{n-1} + 500n$$

(b) Using the `Seq` ("sequence") mode on your graphing calculator, enter the sequences U_n and V_n as shown in the figure to the left. Then use the TABLE command to compare the two sequences. For the first few years, Victoria seems to be accumulating more savings than Ursula. Scroll down in the table to verify that Ursula eventually pulls ahead of Victoria in the savings race. In what year does this occur?

8. Newton's Law of Cooling A tureen of soup at a temperature of 170 °F is placed on a table in a dining room in which the thermostat is set at 70 °F. The soup cools according to the following rule, a special case of Newton's Law of Cooling: Each minute, the temperature of the soup declines by 3% of the difference between the soup temperature and the room temperature.

(a) Find a recursive sequence that models the soup temperature T_n at the nth minute.

(b) Enter the sequence T_n in your graphing calculator, and use the TABLE command to find the temperature at 10-min increments from $n = 0$ to $n = 60$. (See Problem 7(b).)

(c) Graph the sequence T_n. What temperature will the soup be after a long time?

9. Logistic Population Growth Simple exponential models for population growth do not take into account the fact that when the population increases, survival becomes harder for each individual because of greater competition for food and other resources.

We can get a more accurate model by assuming that the birth rate is proportional to the size of the population, but the death rate is proportional to the square of the population. Using this idea, researchers find that the number of raccoons R_n on a certain island is modeled by the following recursive sequence:

$$\overbrace{R_n}^{\text{Population at end of year}} = \underbrace{R_{n-1}}_{\text{Population at beginning of year}} + \overbrace{0.08R_{n-1}}^{\text{Number of births}} - \underbrace{0.0004(R_{n-1})^2}_{\text{Number of deaths}}, \qquad R_0 = 100$$

Here n represents the number of years since observations began, R_0 is the initial population, 0.08 is the annual birth rate, and 0.0004 is a constant related to the death rate.

(a) Use the TABLE command on a graphing calculator to find the raccoon population for each year from $n = 1$ to $n = 7$.

(b) Graph the sequence R_n. What happens to the raccoon population as n becomes large?