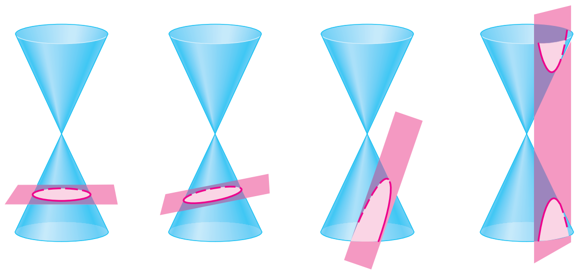
10 Analytic Geometry

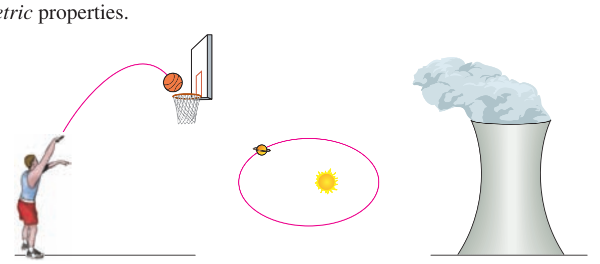
Chapter Overview

Conic sections are the curves we get when we make a straight cut in a cone, as shown in the figure. For example, if a cone is cut horizontally, the cross section is a circle. So a circle is a conic section. Other ways of cutting a cone produce parabolas, ellipses, and hyperbolas.



Circle Ellipse Parabola Hyperbola

Our goal in this chapter is to find equations whose graphs are the conic sections. We already know from Section 1.8 that the graph of the equation $x^2 + y^2 = r^2$ is a circle. We will find equations for each of the other conic sections by analyzing their *geometric* properties.



The trajectory of a basketball is a parabola. The orbit of a planet is an ellipse. The shape of a cooling tower is a hyperbola.

Conic sections are important because their shapes are hidden in the structure of many things. For example, the path of a planet moving around the sun is an ellipse.

The path of a projectile (such as a rocket, a basketball, or water spouting from a fountain) is a parabola—which makes the study of parabolas indispensable in rocket science. The conic sections also occur in many unexpected places. For example, the graph of crop yield as a function of amount of rainfall is a parabola (see page 321). We will examine some uses of the conics in medicine, engineering, navigation, and astronomy.

In Section 10.7 we study parametric equations, which we can use to describe the curve that a moving body traces out over time. In *Focus on Modeling*, page 816, we derive parametric equations for the path of a projectile.

SUGGESTED TIME AND EMPHASIS

$\frac{1}{2}$ class.

Recommended material.

POINTS TO STRESS

1. The definition and geometry of parabolas.
2. Using the equation of a parabola to find relevant constants.
3. Graphing a parabola given its equation.

10.1 Parabolas

We saw in Section 2.5 that the graph of the equation $y = ax^2 + bx + c$ is a U-shaped curve called a *parabola* that opens either upward or downward, depending on whether the sign of a is positive or negative.

In this section we study parabolas from a geometric rather than an algebraic point of view. We begin with the geometric definition of a parabola and show how this leads to the algebraic formula that we are already familiar with.

Geometric Definition of a Parabola

A **parabola** is the set of points in the plane equidistant from a fixed point F (called the **focus**) and a fixed line l (called the **directrix**).

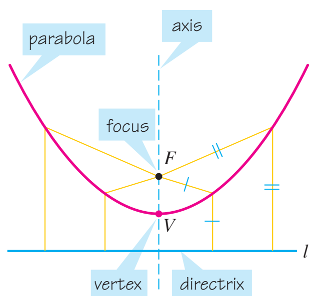


Figure 1

This definition is illustrated in Figure 1. The **vertex** V of the parabola lies halfway between the focus and the directrix, and the **axis of symmetry** is the line that runs through the focus perpendicular to the directrix.

In this section we restrict our attention to parabolas that are situated with the vertex at the origin and that have a vertical or horizontal axis of symmetry. (Parabolas in more general positions will be considered in Sections 10.4 and 10.5.) If the focus of such a parabola is the point $F(0, p)$, then the axis of symmetry must be vertical and the directrix has the equation $y = -p$. Figure 2 illustrates the case $p > 0$.

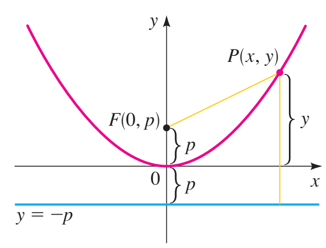


Figure 2

IN-CLASS MATERIALS

Have students sketch a parabola "from scratch." Hand out a sheet of paper with focus and directrix, and hand out rulers. Get the students to just plot points where the distance from the point to the directrix is equal to the distance from the point to the focus. Have them keep plotting points until a parabolic shape emerges.

If $P(x, y)$ is any point on the parabola, then the distance from P to the focus F (using the Distance Formula) is

$$\sqrt{x^2 + (y - p)^2}$$

The distance from P to the directrix is

$$|y - (-p)| = |y + p|$$

By the definition of a parabola, these two distances must be equal:

$$\begin{aligned}
\sqrt{x^2 + (y - p)^2} &= |y + p| \\
x^2 + (y - p)^2 &= |y + p|^2 = (y + p)^2 && \text{Square both sides} \\
x^2 + y^2 - 2py + p^2 &= y^2 + 2py + p^2 && \text{Expand} \\
x^2 - 2py &= 2py && \text{Simplify} \\
x^2 &= 4py
\end{aligned}$$

If $p > 0$, then the parabola opens upward, but if $p < 0$, it opens downward. When x is replaced by $-x$, the equation remains unchanged, so the graph is symmetric about the y-axis.

Equations and Graphs of Parabolas

The following box summarizes what we have just proved about the equation and features of a parabola with a vertical axis.

Parabola with Vertical Axis

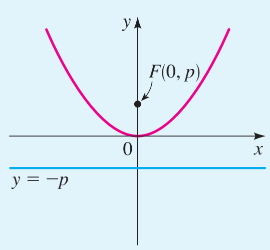
The graph of the equation

$$x^2 = 4py$$

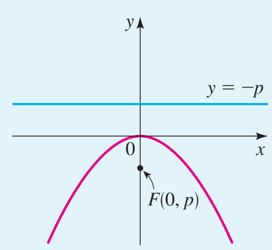
is a parabola with the following properties.

VERTEX	$V(0, 0)$
FOCUS	$F(0, p)$
DIRECTRIX	$y = -p$

The parabola opens upward if $p > 0$ or downward if $p < 0$.



$x^2 = 4py$ with $p > 0$



$x^2 = 4py$ with $p < 0$

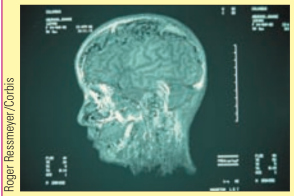
ALTERNATE EXAMPLE 1
Find the equation of the parabola with vertex $V(0, 0)$ and focus $F(0, -8)$.

ANSWER
$x^2 = -32y$

ALTERNATE EXAMPLE 2
Find the focus and directrix of the parabola $y = -5x^2$.

ANSWER
$\left(0, \frac{1}{20}\right), y = \frac{1}{20}$

Mathematics in the Modern World



Roger Ressmeyer/Corbis

Looking Inside Your Head

How would you like to look inside your head? The idea isn't particularly appealing to most of us, but doctors often need to do just that. If they can look without invasive surgery, all the better. An X-ray doesn't really give a look inside, it simply gives a "graph" of the density of tissue the X-rays must pass through. So an X-ray is a "flattened" view in one direction. Suppose you get an X-ray view from many different directions—can these "graphs" be used to reconstruct the three-dimensional inside view? This is a purely mathematical problem and was solved by mathematicians a long time ago. However, reconstructing the inside view requires thousands of tedious computations. Today, mathematics and high-speed computers make it possible to "look inside" by a process called Computer Aided Tomography (or CAT scan). Mathematicians continue to search for better ways of using mathematics to reconstruct images. One of the latest techniques, called magnetic resonance imaging (MRI), combines molecular biology and mathematics for a clear "look inside."

Example 1 Finding the Equation of a Parabola

Find the equation of the parabola with vertex $V(0, 0)$ and focus $F(0, 2)$, and sketch its graph.

Solution Since the focus is $F(0, 2)$, we conclude that $p = 2$ (and so the directrix is $y = -2$). Thus, the equation of the parabola is

$$x^2 = 4(2)y \qquad x^2 = 4py \text{ with } p = 2$$

$$x^2 = 8y$$

Since $p = 2 > 0$, the parabola opens upward. See Figure 3.

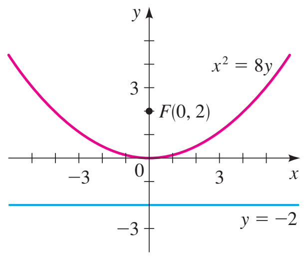


Figure 3 ■

Example 2 Finding the Focus and Directrix of a Parabola from Its Equation

Find the focus and directrix of the parabola $y = -x^2$, and sketch the graph.

Solution To find the focus and directrix, we put the given equation in the standard form $x^2 = -y$. Comparing this to the general equation $x^2 = 4py$, we see that $4p = -1$, so $p = -\frac{1}{4}$. Thus, the focus is $F\left(0, -\frac{1}{4}\right)$ and the directrix is $y = \frac{1}{4}$. The graph of the parabola, together with the focus and the directrix, is shown in Figure 4(a). We can also draw the graph using a graphing calculator as shown in Figure 4(b).

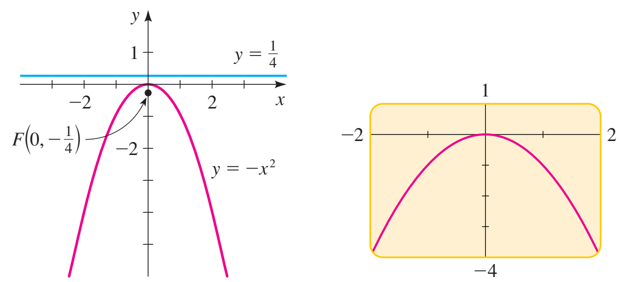


Figure 4 (a) (b) ■

IN-CLASS MATERIALS

As of this writing, it is possible to purchase a parabolic listening device for about $50 on eBay, or about $80 new. If this is feasible, many experiments and demonstrations can be done. For example, students can whisper from a long distance away and be heard using the device.

Reflecting the graph in Figure 2 about the diagonal line $y = x$ has the effect of interchanging the roles of x and y. This results in a parabola with horizontal axis. By the same method as before, we can prove the following properties.

Parabola with Horizontal Axis

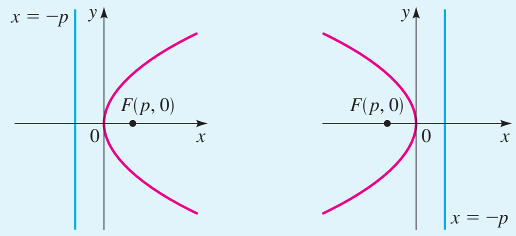
The graph of the equation

$$y^2 = 4px$$

is a parabola with the following properties.

VERTEX	$V(0, 0)$
FOCUS	$F(p, 0)$
DIRECTRIX	$x = -p$

The parabola opens to the right if $p > 0$ or to the left if $p < 0$.



$y^2 = 4px$ with $p > 0$ $y^2 = 4px$ with $p < 0$

SAMPLE QUESTION
Text Question

How can you tell if the axis of a parabola is vertical or horizontal?

Answer

If there is a y^2 term it is horizontal; if there is an x^2 term it is vertical.

Example 3 A Parabola with Horizontal Axis

A parabola has the equation $6x + y^2 = 0$.

(a) Find the focus and directrix of the parabola, and sketch the graph.

(b) Use a graphing calculator to draw the graph.

Solution

(a) To find the focus and directrix, we put the given equation in the standard form $y^2 = -6x$. Comparing this to the general equation $y^2 = 4px$, we see that $4p = -6$, so $p = -\frac{3}{2}$. Thus, the focus is $F\left(-\frac{3}{2}, 0\right)$ and the directrix is $x = \frac{3}{2}$. Since $p < 0$, the parabola opens to the left. The graph of the parabola, together with the focus and the directrix, is shown in Figure 5(a) on the next page.

(b) To draw the graph using a graphing calculator, we need to solve for y.

$$6x + y^2 = 0$$

$$y^2 = -6x \quad \text{Subtract } 6x$$

$$y = \pm\sqrt{-6x} \quad \text{Take square roots}$$

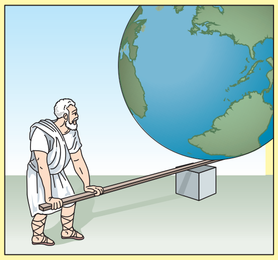
ALTERNATE EXAMPLE 3

Find the focus and directrix of the parabola.

$2x + y^2 = 0$

ANSWER

$\left(0, -\dfrac{1}{2}\right), x = \dfrac{1}{2}$



Archimedes (287–212 B.C.) was the greatest mathematician of the ancient world. He was born in Syracuse, a Greek colony on Sicily, a generation after Euclid (see page 532). One of his many discoveries is the Law of the Lever (see page 69). He famously said, "Give me a place to stand and a fulcrum for my lever, and I can lift the earth."

Renowned as a mechanical genius for his many engineering inventions, he designed pulleys for lifting heavy ships and the spiral screw for transporting water to higher levels. He is said to have used parabolic mirrors to concentrate the rays of the sun to set fire to Roman ships attacking Syracuse.

King Hieron II of Syracuse once suspected a goldsmith of keeping part of the gold intended for the king's crown and replacing it with an equal amount of silver. The king asked Archimedes for advice. While in deep thought at a public bath, Archimedes discovered the solution to the king's problem when he noticed that his body's volume was the same as the volume of water it displaced from the tub. As the story is told, he ran home naked, shouting "Eureka, eureka!" ("I have found it, I have found it!") This incident attests to his enormous powers of concentration.

In spite of his engineering prowess, Archimedes was most proud of his mathematical discov-

(continued)

To obtain the graph of the parabola, we graph both functions

$$y = \sqrt{-6x} \qquad \text{and} \qquad y = -\sqrt{-6x}$$

as shown in Figure 5(b).

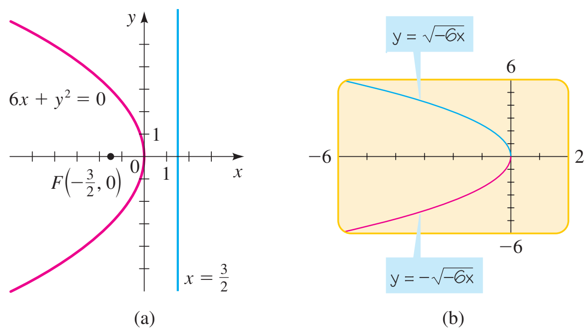


Figure 5

The equation $y^2 = 4px$ does not define y as a function of x (see page 164). So, to use a graphing calculator to graph a parabola with horizontal axis, we must first solve for y. This leads to two functions, $y = \sqrt{4px}$ and $y = -\sqrt{4px}$. We need to graph both functions to get the complete graph of the parabola. For example, in Figure 5(b) we had to graph both $y = \sqrt{-6x}$ and $y = -\sqrt{-6x}$ to graph the parabola $y^2 = -6x$.

We can use the coordinates of the focus to estimate the "width" of a parabola when sketching its graph. The line segment that runs through the focus perpendicular to the axis, with endpoints on the parabola, is called the **latus rectum**, and its length is the **focal diameter** of the parabola. From Figure 6 we can see that the distance from an endpoint Q of the latus rectum to the directrix is $|2p|$. Thus, the distance from Q to the focus must be $|2p|$ as well (by the definition of a parabola), and so the focal diameter is $|4p|$. In the next example we use the focal diameter to determine the "width" of a parabola when graphing it.

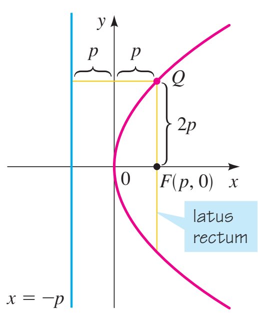


Figure 6

IN-CLASS MATERIALS

Make the connection between quadratic functions and parabolas explicit. Point out that any equation of the form $y = ax^2 + bx + c$ can be written as $y = n(x - h)^2 + k$ by completing the square, yielding a (possibly shifted) parabola. The constant n, of course, can be written as $4p$. Note that if $a = 0$ we have a line. So we can call a line a "degenerate parabola."

eries. These include the formulas for the volume of a sphere, $V = \frac{4}{3}\pi r^3$; the surface area of a sphere, $S = 4\pi r^2$; and a careful analysis of the properties of parabolas and other conics.

Example 4 The Focal Diameter of a Parabola

Find the focus, directrix, and focal diameter of the parabola $y = \frac{1}{2}x^2$, and sketch its graph.

Solution We first put the equation in the form $x^2 = 4py$.

$$y = \tfrac{1}{2}x^2$$

$$x^2 = 2y \qquad \text{Multiply each side by 2}$$

From this equation we see that $4p = 2$, so the focal diameter is 2. Solving for p gives $p = \frac{1}{2}$, so the focus is $\left(0, \frac{1}{2}\right)$ and the directrix is $y = -\frac{1}{2}$. Since the focal diameter is 2, the latus rectum extends 1 unit to the left and 1 unit to the right of the focus. The graph is sketched in Figure 7. ■

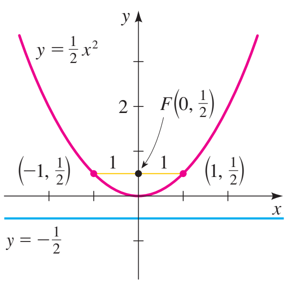


Figure 7

In the next example we graph a family of parabolas, to show how changing the distance between the focus and the vertex affects the "width" of a parabola.

Example 5 A Family of Parabolas

(a) Find equations for the parabolas with vertex at the origin and foci $F_1\left(0, \frac{1}{8}\right)$, $F_2\left(0, \frac{1}{2}\right)$, $F_3(0, 1)$, and $F_4(0, 4)$.

(b) Draw the graphs of the parabolas in part (a). What do you conclude?

Solution

(a) Since the foci are on the positive y-axis, the parabolas open upward and have equations of the form $x^2 = 4py$. This leads to the following equations.

Focus	p	Equation $x^2 = 4py$	Form of the equation for graphing calculator
$F_1\left(0, \frac{1}{8}\right)$	$p = \frac{1}{8}$	$x^2 = \frac{1}{2}y$	$y = 2x^2$
$F_2\left(0, \frac{1}{2}\right)$	$p = \frac{1}{2}$	$x^2 = 2y$	$y = 0.5x^2$
$F_3(0, 1)$	$p = 1$	$x^2 = 4y$	$y = 0.25x^2$
$F_4(0, 4)$	$p = 4$	$x^2 = 16y$	$y = 0.0625x^2$

(b) The graphs are drawn in Figure 8. We see that the closer the focus to the vertex, the narrower the parabola.

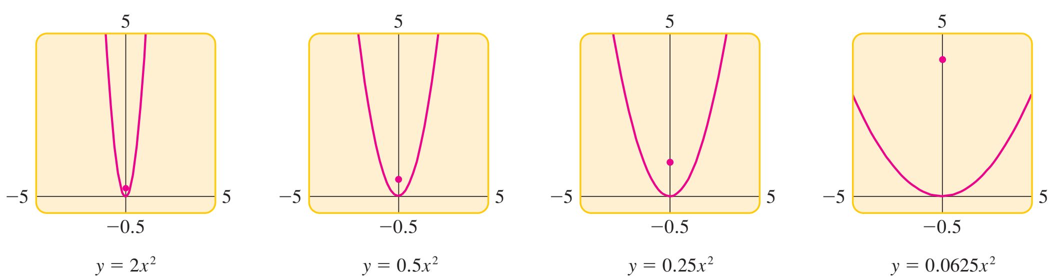


Figure 8
A family of parabolas ■

ALTERNATE EXAMPLE 4

Find the focus, directrix, and focal diameter of the parabola.

$$y = \frac{4}{9}x^2$$

ANSWER

$\left(0, \frac{9}{16}\right), y = -\frac{9}{16}, \frac{9}{4}$

ALTERNATE EXAMPLE 5a

Find equations for the parabolas with vertex at the origin and foci: $F_1\left(0, \frac{1}{8}\right), F_2\left(0, \frac{1}{4}\right), F_3(0, 1), F_4(0, 4)$

ANSWER

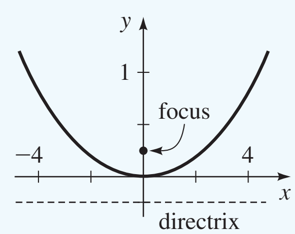
$y = 2x^2, y = x^2, y = 0.25x^2, y = 0.0625x^2$

DRILL QUESTION

Find the focus and directrix of $y = \frac{1}{16}x^2$ and sketch its graph.

Answer

Focus $(0, 4)$, directrix $y = -4$



IN-CLASS MATERIALS

Using dental floss and modeling compound (such as clay or Play-Doh®), it is easy for the students to make half a cone and slice it. Have the class attempt to do so to get a parabola. Most of them will wind up creating half of a hyperbola. Refer these students to the chapter overview. Make sure they note that a parabola is not as easy to create this way as hyperbolas or ellipses. If the cut is made at the wrong angle, even slightly, one of these two shapes will be formed instead of a parabola.

Applications

Parabolas have an important property that makes them useful as reflectors for lamps and telescopes. Light from a source placed at the focus of a surface with parabolic cross section will be reflected in such a way that it travels parallel to the axis of the parabola (see Figure 9). Thus, a parabolic mirror reflects the light into a beam of parallel rays. Conversely, light approaching the reflector in rays parallel to its axis of symmetry is concentrated to the focus. This *reflection property*, which can be proved using calculus, is used in the construction of reflecting telescopes.

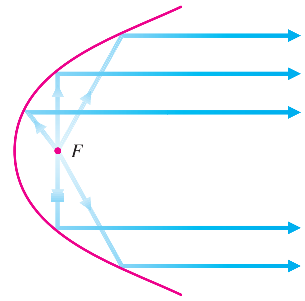


Figure 9
Parabolic reflector

Example 6 Finding the Focal Point of a Searchlight Reflector

A searchlight has a parabolic reflector that forms a "bowl," which is 12 in. wide from rim to rim and 8 in. deep, as shown in Figure 10. If the filament of the light bulb is located at the focus, how far from the vertex of the reflector is it?

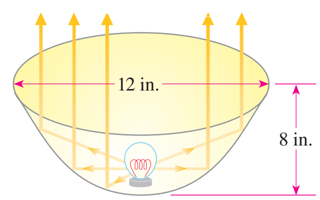


Figure 10
A parabolic reflector

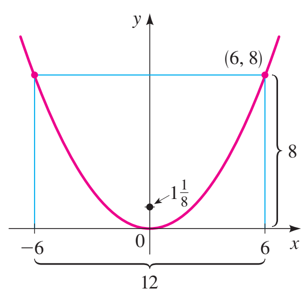


Figure 11

Solution We introduce a coordinate system and place a parabolic cross section of the reflector so that its vertex is at the origin and its axis is vertical (see Figure 11). Then the equation of this parabola has the form $x^2 = 4py$. From Figure 11 we see that the point $(6, 8)$ lies on the parabola. We use this to find p.

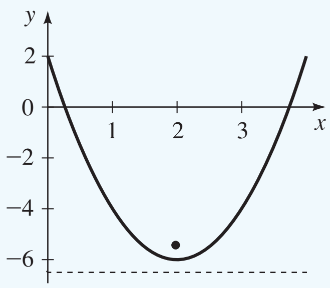
$$6^2 = 4p(8) \qquad \text{The point } (6, 8) \text{ satisfies the equation } x^2 = 4py$$
$$36 = 32p$$
$$p = \tfrac{9}{8}$$

The focus is $F\left(0, \frac{9}{8}\right)$, so the distance between the vertex and the focus is $\frac{9}{8} = 1\frac{1}{8}$ in. Because the filament is positioned at the focus, it is located $1\frac{1}{8}$ in. from the vertex of the reflector. ■

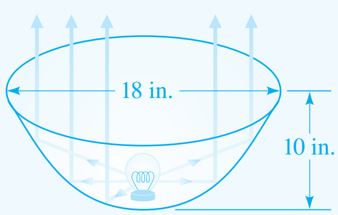
EXAMPLE
A shifted parabola:

$$\begin{aligned} y &= 2x^2 - 8x + 2 \\ &= 2(x^2 - 4x) + 2 \\ &= 2(x^2 - 4x + 4) - 6 \\ &= 2(x - 2)^2 - 6 \end{aligned}$$

$p = \frac{1}{2}$, focus $\left(0 + 2, \frac{1}{2} - 6\right) = \left(2, -\frac{11}{2}\right)$, directrix $y = -\frac{1}{2} - 6 = -\frac{13}{2}$.



ALTERNATE EXAMPLE 6
A searchlight has a parabolic reflector that forms a "bowl," which is 18 in. wide from rim to rim and 10 in. deep, as shown in the figure below. If the filament of the light bulb is located at the focus, how far from the vertex of the reflector is it?



ANSWER
$\frac{81}{40}$

IN-CLASS MATERIALS

Discuss the reflection properties of parabolas: A beam of light originating at the origin will emerge parallel to the parabola's axis of symmetry, and a beam of light that is parallel to the parabola's axis of symmetry will reflect off of the parabola in a direction that goes through its focus. One nice project is to construct a parabolic pool table or miniature golf hole. The students accurately graph a parabola, and glue erasers or wood along its border, placing the "hole" at the focus. A golf ball or pool ball that is rolled in a direction parallel to the axis will always bounce into the hole.

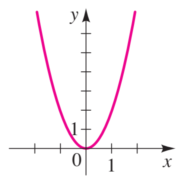
10.1 Exercises

1–6 ■ Match the equation with the graphs labeled I–VI. Give reasons for your answers.

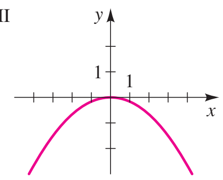
1. $y^2 = 2x$ **2.** $y^2 = -\frac{1}{4}x$ **3.** $x^2 = -6y$

4. $2x^2 = y$ **5.** $y^2 - 8x = 0$ **6.** $12y + x^2 = 0$

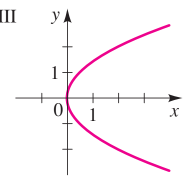
I



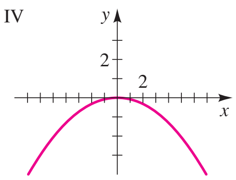
II



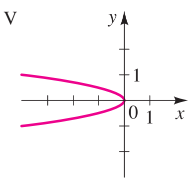
III



IV



V



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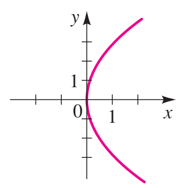


7–18 ■ Find the focus, directrix, and focal diameter of the parabola, and sketch its graph.

7. $y^2 = 4x$ **8.** $x^2 = y$

9. $x^2 = 9y$ **10.** $y^2 = 3x$

11. $y = 5x^2$ **12.** $y = -2x^2$

13. $x = -8y^2$ **14.** $x = \frac{1}{2}y^2$

15. $x^2 + 6y = 0$ **16.** $x - 7y^2 = 0$

17. $5x + 3y^2 = 0$ **18.** $8x^2 + 12y = 0$

19–24 ■ Use a graphing device to graph the parabola.

19. $x^2 = 16y$ **20.** $x^2 = -8y$

21. $y^2 = -\frac{1}{3}x$ **22.** $8y^2 = x$

23. $4x + y^2 = 0$ **24.** $x - 2y^2 = 0$

25–36 ■ Find an equation for the parabola that has its vertex at the origin and satisfies the given condition(s).

25. Focus $F(0, 2)$ **26.** Focus $F\left(0, -\frac{1}{2}\right)$

27. Focus $F(-8, 0)$ **28.** Focus $F(5, 0)$

29. Directrix $x = 2$ **30.** Directrix $y = 6$

31. Directrix $y = -10$ **32.** Directrix $x = -\frac{1}{8}$

33. Focus on the positive x-axis, 2 units away from the directrix

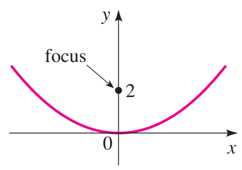
34. Directrix has y-intercept 6

35. Opens upward with focus 5 units from the vertex

36. Focal diameter 8 and focus on the negative y-axis

37–46 ■ Find an equation of the parabola whose graph is shown.

37.



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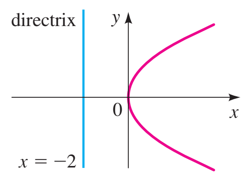

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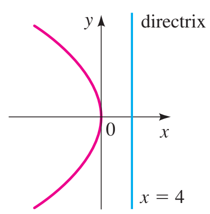

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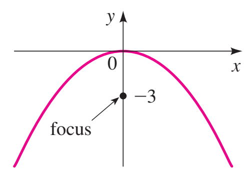

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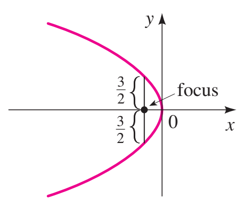

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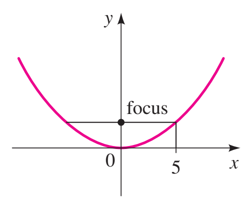

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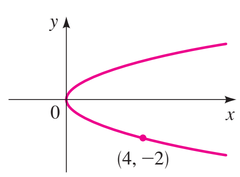

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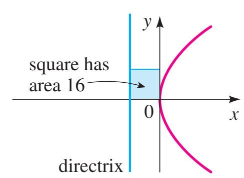

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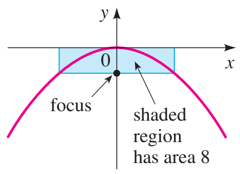

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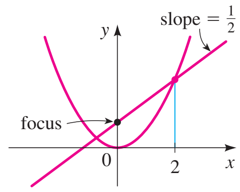


47. **(a)** Find equations for the family of parabolas with vertex at the origin and with directrixes $y = \frac{1}{2}$, $y = 1$, $y = 4$, and $y = 8$.

(b) Draw the graphs. What do you conclude?

48. **(a)** Find equations for the family of parabolas with vertex at the origin, focus on the positive y-axis, and with focal diameters 1, 2, 4, and 8.

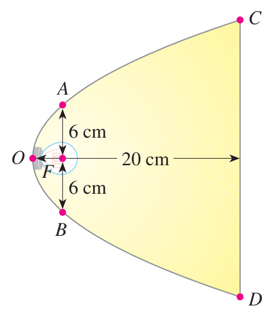
(b) Draw the graphs. What do you conclude?

Applications

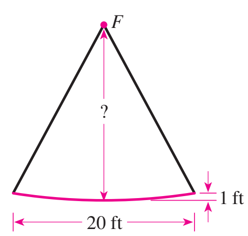
49. Parabolic Reflector A lamp with a parabolic reflector is shown in the figure. The bulb is placed at the focus and the focal diameter is 12 cm.

(a) Find an equation of the parabola.

(b) Find the diameter $d(C, D)$ of the opening, 20 cm from the vertex.

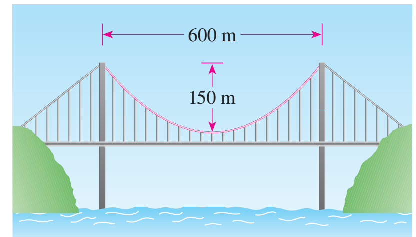


50. Satellite Dish A reflector for a satellite dish is parabolic in cross section, with the receiver at the focus F. The reflector is 1 ft deep and 20 ft wide from rim to rim (see the figure). How far is the receiver from the vertex of the parabolic reflector?

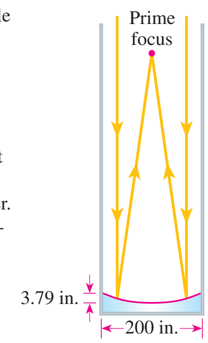


51. Suspension Bridge In a suspension bridge the shape of the suspension cables is parabolic. The bridge shown in the figure has towers that are 600 m apart, and the lowest point of the suspension cables is 150 m below the top of the towers. Find the equation of the parabolic part of the cables, placing the origin of the coordinate system at the vertex.

NOTE This equation is used to find the length of cable needed in the construction of the bridge.



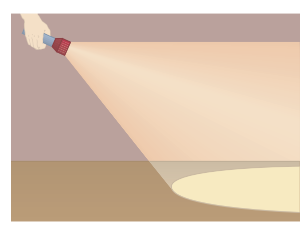
52. Reflecting Telescope The Hale telescope at the Mount Palomar Observatory has a 200-in. mirror, as shown. The mirror is constructed in a parabolic shape that collects light from the stars and focuses it at the **prime focus**, that is, the focus of the parabola. The mirror is 3.79 in. deep at its center. Find the **focal length** of this parabolic mirror, that is, the distance from the vertex to the focus.



Discovery • Discussion

53. Parabolas in the Real World Several examples of the uses of parabolas are given in the text. Find other situations in real life where parabolas occur. Consult a scientific encyclopedia in the reference section of your library, or search the Internet.

54. Light Cone from a Flashlight A flashlight is held to form a lighted area on the ground, as shown in the figure. Is it possible to angle the flashlight in such a way that the boundary of the lighted area is a parabola? Explain your answer.



10.2 Ellipses

An ellipse is an oval curve that looks like an elongated circle. More precisely, we have the following definition.

Geometric Definition of an Ellipse

An **ellipse** is the set of all points in the plane the sum of whose distances from two fixed points F_1 and F_2 is a constant. (See Figure 1.) These two fixed points are the **foci** (plural of **focus**) of the ellipse.

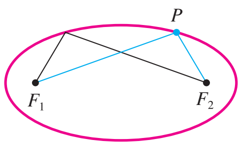


Figure 1

The geometric definition suggests a simple method for drawing an ellipse. Place a sheet of paper on a drawing board and insert thumbtacks at the two points that are to be the foci of the ellipse. Attach the ends of a string to the tacks, as shown in Figure 2(a). With the point of a pencil, hold the string taut. Then carefully move the pencil around the foci, keeping the string taut at all times. The pencil will trace out an ellipse, because the sum of the distances from the point of the pencil to the foci will always equal the length of the string, which is constant.

If the string is only slightly longer than the distance between the foci, then the ellipse traced out will be elongated in shape as in Figure 2(a), but if the foci are close together relative to the length of the string, the ellipse will be almost circular, as shown in Figure 2(b).

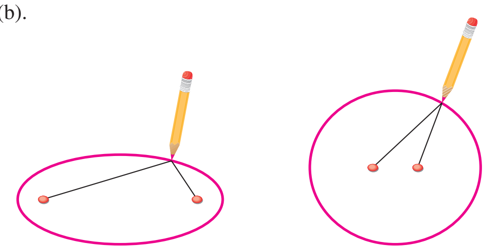


Figure 2 (a) (b)

To obtain the simplest equation for an ellipse, we place the foci on the x-axis at $F_1(-c, 0)$ and $F_2(c, 0)$, so that the origin is halfway between them (see Figure 3).

For later convenience we let the sum of the distances from a point on the ellipse to the foci be $2a$. Then if $P(x, y)$ is any point on the ellipse, we have

$$d(P, F_1) + d(P, F_2) = 2a$$

So, from the Distance Formula

$$\sqrt{(x + c)^2 + y^2} + \sqrt{(x - c)^2 + y^2} = 2a$$

or

$$\sqrt{(x - c)^2 + y^2} = 2a - \sqrt{(x + c)^2 + y^2}$$

Squaring each side and expanding, we get

$$x^2 - 2cx + c^2 + y^2 = 4a^2 - 4a\sqrt{(x + c)^2 + y^2} + (x^2 + 2cx + c^2 + y^2)$$

which simplifies to

$$4a\sqrt{(x + c)^2 + y^2} = 4a^2 + 4cx$$

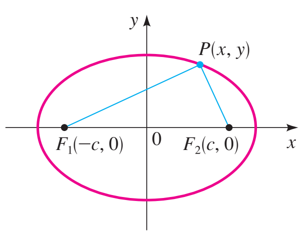


Figure 3

SUGGESTED TIME AND EMPHASIS

$\frac{1}{2}$–1 class.

Optional material.

POINTS TO STRESS

1. The definition and geometry of ellipses.
2. Using the equation of an ellipse to find relevant constants and to graph the ellipse.
3. Eccentricity.

IN-CLASS MATERIALS

Many representational artists never draw circles, noting that it is rare in nature to see a circle, since we are usually looking at an angle, thus seeing an ellipse. Perhaps have the students bring in photographs of manhole covers and other "circular" objects, noting that if the camera angle is not straight on, the resultant image is elliptical. (Because of the optical illusion of perspective, it makes things easier to draw the outline of the "circle" with a marker to see that it is an ellipse.)

Doug Shaw

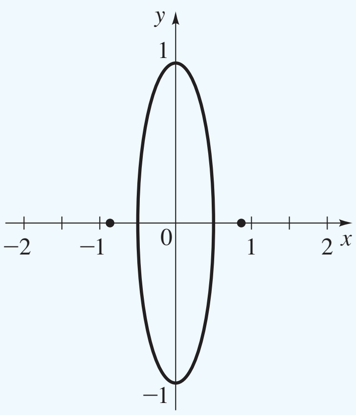
DRILL QUESTION

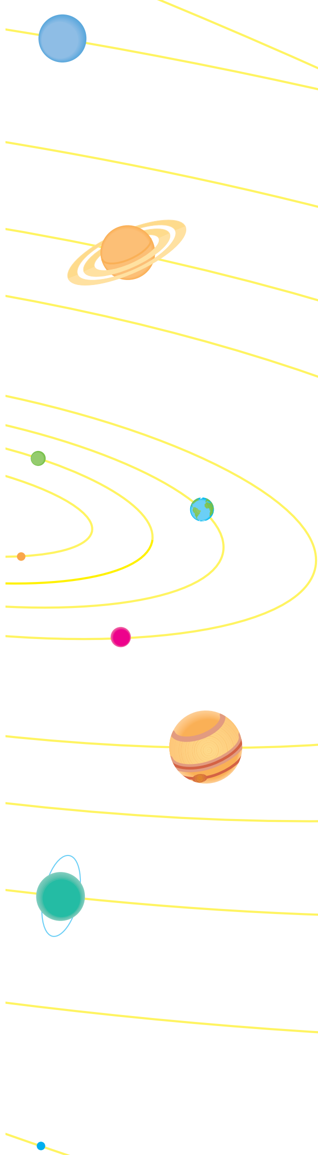
Find the vertices and foci of the ellipse $4x^2 + y^2 = 1$ and sketch its graph.

Answer

Vertices: $(0, 1)$, $(0, -1)$, $\left(\frac{1}{2}, 0\right)$, $\left(-\frac{1}{2}, 0\right)$

Foci: $\left(0, \frac{\sqrt{3}}{2}\right)\left(0, -\frac{\sqrt{3}}{2}\right)$





Dividing each side by 4 and squaring again, we get

$$a^2[(x + c)^2 + y^2] = (a^2 + cx)^2$$
$$a^2x^2 + 2a^2cx + a^2c^2 + a^2y^2 = a^4 + 2a^2cx + c^2x^2$$
$$(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2)$$

Since the sum of the distances from P to the foci must be larger than the distance between the foci, we have that $2a > 2c$, or $a > c$. Thus, $a^2 - c^2 > 0$, and we can divide each side of the preceding equation by $a^2(a^2 - c^2)$ to get

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1$$

For convenience let $b^2 = a^2 - c^2$ (with $b > 0$). Since $b^2 < a^2$, it follows that $b < a$. The preceding equation then becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{with } a > b$$

This is the equation of the ellipse. To graph it, we need to know the x- and y-intercepts. Setting $y = 0$, we get

$$\frac{x^2}{a^2} = 1$$

so $x^2 = a^2$, or $x = \pm a$. Thus, the ellipse crosses the x-axis at $(a, 0)$ and $(-a, 0)$, as in Figure 4. These points are called the **vertices** of the ellipse, and the segment that joins them is called the **major axis**. Its length is $2a$.

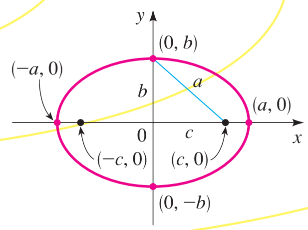


Figure 4
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with $a > b$

Similarly, if we set $x = 0$, we get $y = \pm b$, so the ellipse crosses the y-axis at $(0, b)$ and $(0, -b)$. The segment that joins these points is called the **minor axis**, and it has length $2b$. Note that $2a > 2b$, so the major axis is longer than the minor axis. The origin is the **center** of the ellipse.

If the foci of the ellipse are placed on the y-axis at $(0, \pm c)$ rather than on the x-axis, then the roles of x and y are reversed in the preceding discussion, and we get a vertical ellipse.

Equations and Graphs of Ellipses

The orbits of the planets are ellipses, with the sun at one focus.

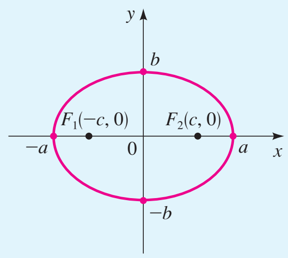
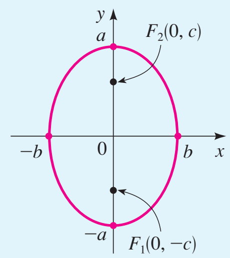
The following box summarizes what we have just proved about the equation and features of an ellipse centered at the origin.

IN-CLASS MATERIALS

Note that if $a = b$ then we have only one focus, at $(0, 0)$. In this case, the geometric definition breaks down, but it is clear from the equation that we have a circle. This is why a circle can be thought of as a "degenerate ellipse."

Ellipse with Center at the Origin

The graph of each of the following equations is an ellipse with center at the origin and having the given properties.

EQUATION	$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ $a>b>0$	$\dfrac{x^2}{b^2}+\dfrac{y^2}{a^2}=1$ $a>b>0$
VERTICES	$(\pm a, 0)$	$(0, \pm a)$
MAJOR AXIS	Horizontal, length $2a$	Vertical, length $2a$
MINOR AXIS	Vertical, length $2b$	Horizontal, length $2b$
FOCI	$(\pm c, 0)$, $c^2=a^2-b^2$	$(0, \pm c)$, $c^2=a^2-b^2$
GRAPH		

In the standard equation for an ellipse, a^2 is the *larger* denominator and b^2 is the *smaller*. To find c^2, we subtract: larger denominator minus smaller denominator.

Example 1 Sketching an Ellipse

An ellipse has the equation

$$\frac{x^2}{9}+\frac{y^2}{4}=1$$

(a) Find the foci, vertices, and the lengths of the major and minor axes, and sketch the graph.

(b) Draw the graph using a graphing calculator.

Solution

(a) Since the denominator of x^2 is larger, the ellipse has horizontal major axis. This gives $a^2 = 9$ and $b^2 = 4$, so $c^2 = a^2 - b^2 = 9 - 4 = 5$. Thus, $a = 3$, $b = 2$, and $c = \sqrt{5}$.

FOCI	$(\pm\sqrt{5}, 0)$
VERTICES	$(\pm 3, 0)$
LENGTH OF MAJOR AXIS	6
LENGTH OF MINOR AXIS	4

The graph is shown in Figure 5(a) on the next page.

SAMPLE QUESTION
Text Question

Does $\dfrac{x^2}{5^2}+\dfrac{y^2}{6^2}=1$ describe a horizontal or a vertical ellipse? How do you know?

Answer

It is vertical because the denominator of the y^2 term is larger than that of the x^2 term.

ALTERNATE EXAMPLE 1

Find the foci, vertices, and the lengths of the major and minor axes for the following ellipse.

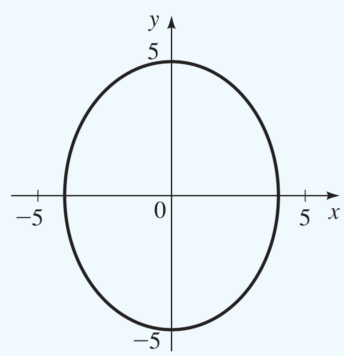
$$\frac{x^2}{36}+\frac{y^2}{25}=1$$

ANSWER

$(\sqrt{11}, 0)$, $(-\sqrt{11}, 0)$, $(6, 0)$, $(-6, 0)$, 12, 10

EXAMPLE

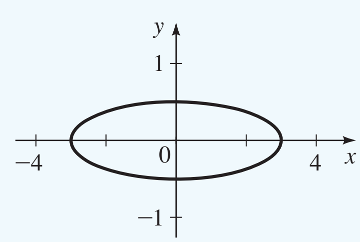
A vertical ellipse: $\dfrac{x^2}{16}+\dfrac{y^2}{25}=1$



EXAMPLE

A horizontal ellipse:

$\dfrac{x^2}{9}+4y^2=1$



IN-CLASS MATERIALS

Have students sketch an ellipse with thumbtacks and string, as suggested in the text. Have some use foci that are close together, and some that are farther apart. Make sure they see the connection between this activity and the idea that the summed distance from the foci is a constant.

(b) To draw the graph using a graphing calculator, we need to solve for y.

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

$$\frac{y^2}{4} = 1 - \frac{x^2}{9} \qquad \text{Subtract } x^2/9$$

$$y^2 = 4\left(1 - \frac{x^2}{9}\right) \qquad \text{Multiply by 4}$$

$$y = \pm 2\sqrt{1 - \frac{x^2}{9}} \qquad \text{Take square roots}$$

Note that the equation of an ellipse does not define y as a function of x (see page 164). That's why we need to graph two functions to graph an ellipse.

To obtain the graph of the ellipse, we graph both functions

$$y = 2\sqrt{1 - x^2/9} \qquad \text{and} \qquad y = -2\sqrt{1 - x^2/9}$$

as shown in Figure 5(b).

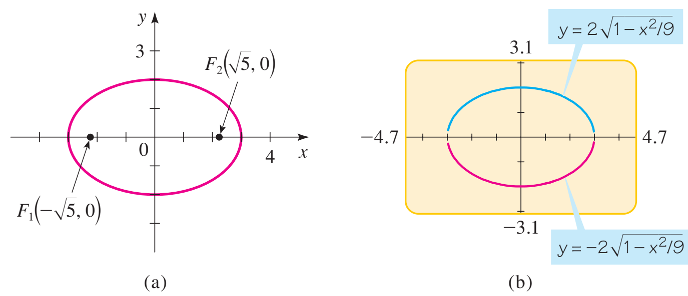


Figure 5
$\frac{x^2}{9} + \frac{y^2}{4} = 1$

■

ALTERNATE EXAMPLE 2
The vertices of an ellipse are $(\pm 6, 0)$ and the foci are $(\pm 4, 0)$. Find its equation.

ANSWER
$\frac{x^2}{36} + \frac{y^2}{20} = 1$

Example 2 Finding the Foci of an Ellipse

Find the foci of the ellipse $16x^2 + 9y^2 = 144$, and sketch its graph.

Solution First we put the equation in standard form. Dividing by 144, we get

$$\frac{x^2}{9} + \frac{y^2}{16} = 1$$

Since $16 > 9$, this is an ellipse with its foci on the y-axis, and with $a = 4$ and $b = 3$. We have

$$c^2 = a^2 - b^2 = 16 - 9 = 7$$
$$c = \sqrt{7}$$

Thus, the foci are $(0, \pm\sqrt{7})$. The graph is shown in Figure 6(a).

We can also draw the graph using a graphing calculator as shown in Figure 6(b).

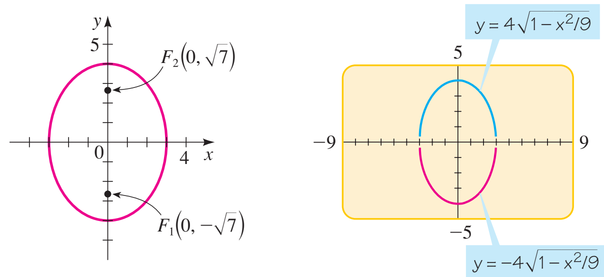


Figure 6
$16x^2 + 9y^2 = 144$

Example 3 Finding the Equation of an Ellipse

The vertices of an ellipse are $(\pm 4, 0)$ and the foci are $(\pm 2, 0)$. Find its equation and sketch the graph.

Solution Since the vertices are $(\pm 4, 0)$, we have $a = 4$. The foci are $(\pm 2, 0)$, so $c = 2$. To write the equation, we need to find b. Since $c^2 = a^2 - b^2$, we have

$$2^2 = 4^2 - b^2$$

$$b^2 = 16 - 4 = 12$$

Thus, the equation of the ellipse is

$$\frac{x^2}{16} + \frac{y^2}{12} = 1$$

The graph is shown in Figure 7.

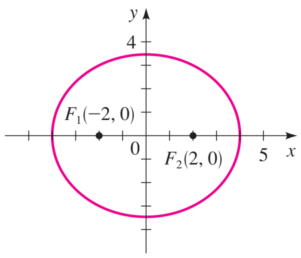


Figure 7
$\frac{x^2}{16} + \frac{y^2}{12} = 1$

Eccentricity of an Ellipse

We saw earlier in this section (Figure 2) that if $2a$ is only slightly greater than $2c$, the ellipse is long and thin, whereas if $2a$ is much greater than $2c$, the ellipse is almost circular. We measure the deviation of an ellipse from being circular by the ratio of a and c.

Definition of Eccentricity

For the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ or $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ (with $a > b > 0$), the **eccentricity *e*** is the number

$$e = \frac{c}{a}$$

where $c = \sqrt{a^2 - b^2}$. The eccentricity of every ellipse satisfies $0 < e < 1$.

ALTERNATE EXAMPLE 3
Find the foci of the ellipse $9x^2 + 4y^2 = 36$.

ANSWER
$(0, \sqrt{5}), (0, -\sqrt{5})$

Eccentricities of the Orbits of the Planets

The orbits of the planets are ellipses with the sun at one focus. For most planets these ellipses have very small eccentricity, so they are nearly circular. However, Mercury and Pluto, the innermost and outermost known planets, have visibly elliptical orbits.

Planet	Eccentricity
Mercury	0.206
Venus	0.007
Earth	0.017
Mars	0.093
Jupiter	0.048
Saturn	0.056
Uranus	0.046
Neptune	0.010
Pluto	0.248

Thus, if e is close to 1, then c is almost equal to a, and the ellipse is elongated in shape, but if e is close to 0, then the ellipse is close to a circle in shape. The eccentricity is a measure of how "stretched" the ellipse is.

In Figure 8 we show a number of ellipses to demonstrate the effect of varying the eccentricity e.

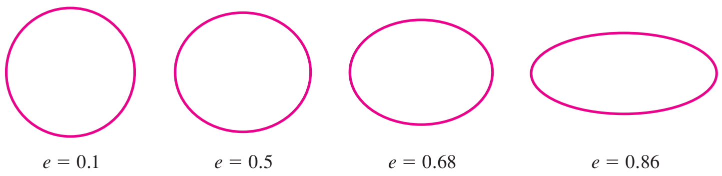


Figure 8
Ellipses with various eccentricities

ALTERNATE EXAMPLE 4
Find the equation of the ellipse with foci $(0, \pm 20)$ and eccentricity $e = \frac{4}{5}$.

ANSWER
$$\frac{x^2}{225} + \frac{y^2}{625} = 1$$

Example 4 Finding the Equation of an Ellipse from Its Eccentricity and Foci

Find the equation of the ellipse with foci $(0, \pm 8)$ and eccentricity $e = \frac{4}{5}$, and sketch its graph.

Solution We are given $e = \frac{4}{5}$ and $c = 8$. Thus

$$\frac{4}{5} = \frac{8}{a} \qquad \text{Eccentricity } e = \frac{c}{a}$$
$$4a = 40 \qquad \text{Cross multiply}$$
$$a = 10$$

To find b, we use the fact that $c^2 = a^2 - b^2$.

$$8^2 = 10^2 - b^2$$
$$b^2 = 10^2 - 8^2 = 36$$
$$b = 6$$

Thus, the equation of the ellipse is

$$\frac{x^2}{36} + \frac{y^2}{100} = 1$$

Because the foci are on the y-axis, the ellipse is oriented vertically. To sketch the ellipse, we find the intercepts: The x-intercepts are ± 6 and the y-intercepts are ± 10. The graph is sketched in Figure 9. ■

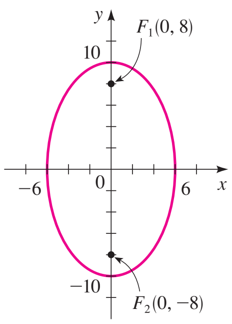


Figure 9
$\frac{x^2}{36} + \frac{y^2}{100} = 1$

Gravitational attraction causes the planets to move in elliptical orbits around the sun with the sun at one focus. This remarkable property was first observed by Johannes Kepler and was later deduced by Isaac Newton from his inverse square law of gravity, using calculus. The orbits of the planets have different eccentricities, but most are nearly circular (see the margin note above).

IN-CLASS MATERIALS

As noted in Section 10.1, one can use dental floss and a modeling compound (such as clay or Play-Doh®) to make half a cone and slice it. Have the class attempt to do so to get a circle. Note that if their angle is slightly off, they will get an ellipse. Make the analogy that just as a square is a particular kind of rectangle, a circle is a particular kind of ellipse.

Ellipses, like parabolas, have an interesting *reflection property* that leads to a number of practical applications. If a light source is placed at one focus of a reflecting surface with elliptical cross sections, then all the light will be reflected off the surface to the other focus, as shown in Figure 10. This principle, which works for sound waves as well as for light, is used in *lithotripsy*, a treatment for kidney stones. The patient is placed in a tub of water with elliptical cross sections in such a way that the kidney stone is accurately located at one focus. High-intensity sound waves generated at the other focus are reflected to the stone and destroy it with minimal damage to surrounding tissue. The patient is spared the trauma of surgery and recovers within days instead of weeks.

The reflection property of ellipses is also used in the construction of *whispering galleries*. Sound coming from one focus bounces off the walls and ceiling of an elliptical room and passes through the other focus. In these rooms even quiet whispers spoken at one focus can be heard clearly at the other. Famous whispering galleries include the National Statuary Hall of the U.S. Capitol in Washington, D.C. (see page 771), and the Mormon Tabernacle in Salt Lake City, Utah.

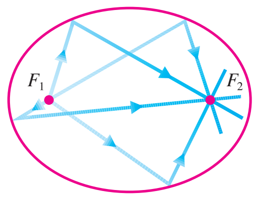


Figure 10

10.2 Exercises

1–4 ■ Match the equation with the graphs labeled I–IV. Give reasons for your answers.

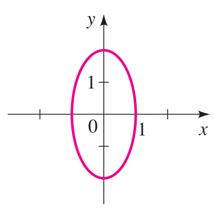
1. $\dfrac{x^2}{16} + \dfrac{y^2}{4} = 1$

2. $x^2 + \dfrac{y^2}{9} = 1$

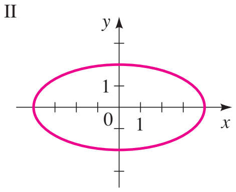
3. $4x^2 + y^2 = 4$

4. $16x^2 + 25y^2 = 400$

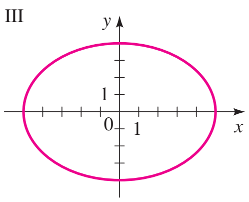
I



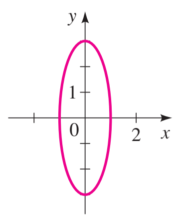
II



III



IV



5–18 ■ Find the vertices, foci, and eccentricity of the ellipse. Determine the lengths of the major and minor axes, and sketch the graph.

5. $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$

6. $\dfrac{x^2}{16} + \dfrac{y^2}{25} = 1$

7. $9x^2 + 4y^2 = 36$

8. $4x^2 + 25y^2 = 100$

9. $x^2 + 4y^2 = 16$

10. $4x^2 + y^2 = 16$

11. $2x^2 + y^2 = 3$

12. $5x^2 + 6y^2 = 30$

13. $x^2 + 4y^2 = 1$

14. $9x^2 + 4y^2 = 1$

15. $\frac{1}{2}x^2 + \frac{1}{8}y^2 = \frac{1}{4}$

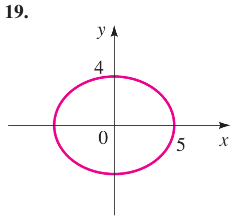
16. $x^2 = 4 - 2y^2$

17. $y^2 = 1 - 2x^2$

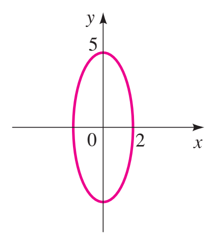
18. $20x^2 + 4y^2 = 5$

19–24 ■ Find an equation for the ellipse whose graph is shown.

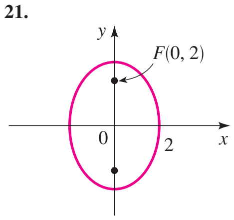
19.



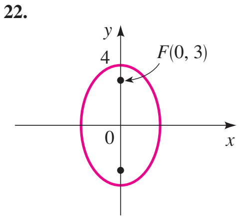
20.



21.



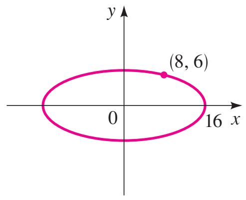
22.



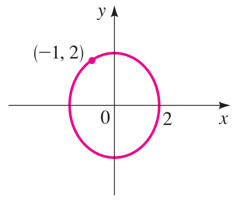
IN-CLASS MATERIALS

Discuss the reflection property of an ellipse: A beam of light originating at one focus will reflect off the ellipse and pass through the other focus. One nice project is to construct an elliptical pool table. The students accurately graph an ellipse and glue erasers or wood along its border, placing the "hole" at one focus and marking the second. A golf ball or pool ball that is placed on the mark and struck in any direction will ricochet into the hole.

23.



24.



25–28 ■ Use a graphing device to graph the ellipse.

25. $\frac{x^2}{25} + \frac{y^2}{20} = 1$

26. $x^2 + \frac{y^2}{12} = 1$

27. $6x^2 + y^2 = 36$

28. $x^2 + 2y^2 = 8$

29–40 ■ Find an equation for the ellipse that satisfies the given conditions.

29. Foci $(\pm 4, 0)$, vertices $(\pm 5, 0)$

30. Foci $(0, \pm 3)$, vertices $(0, \pm 5)$

31. Length of major axis 4, length of minor axis 2, foci on y-axis

32. Length of major axis 6, length of minor axis 4, foci on x-axis

33. Foci $(0, \pm 2)$, length of minor axis 6

34. Foci $(\pm 5, 0)$, length of major axis 12

35. Endpoints of major axis $(\pm 10, 0)$, distance between foci 6

36. Endpoints of minor axis $(0, \pm 3)$, distance between foci 8

37. Length of major axis 10, foci on x-axis, ellipse passes through the point $(\sqrt{5}, 2)$

38. Eccentricity $\frac{1}{9}$, foci $(0, \pm 2)$

39. Eccentricity 0.8, foci $(\pm 1.5, 0)$

40. Eccentricity $\sqrt{3}/2$, foci on y-axis, length of major axis 4

41–43 ■ Find the intersection points of the pair of ellipses. Sketch the graphs of each pair of equations on the same coordinate axes and label the points of intersection.

41. $\begin{cases} 4x^2 + y^2 = 4 \\ 4x^2 + 9y^2 = 36 \end{cases}$

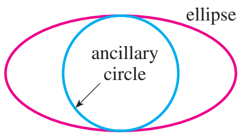
42. $\begin{cases} \dfrac{x^2}{16} + \dfrac{y^2}{9} = 1 \\ \dfrac{x^2}{9} + \dfrac{y^2}{16} = 1 \end{cases}$

43. $\begin{cases} 100x^2 + 25y^2 = 100 \\ x^2 + \dfrac{y^2}{9} = 1 \end{cases}$

44. The **ancillary circle** of an ellipse is the circle with radius equal to half the length of the minor axis and center the same as the ellipse (see the figure). The ancillary circle is thus the largest circle that can fit within an ellipse.

(a) Find an equation for the ancillary circle of the ellipse $x^2 + 4y^2 = 16$.

(b) For the ellipse and ancillary circle of part (a), show that if (s, t) is a point on the ancillary circle, then $(2s, t)$ is a point on the ellipse.



45. **(a)** Use a graphing device to sketch the top half (the portion in the first and second quadrants) of the family of ellipses $x^2 + ky^2 = 100$ for $k = 4$, 10, 25, and 50.

(b) What do the members of this family of ellipses have in common? How do they differ?

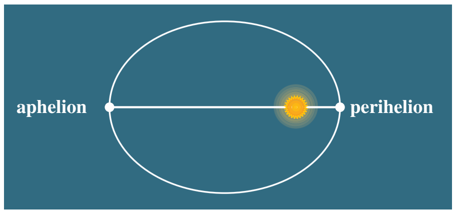
46. If $k > 0$, the following equation represents an ellipse:

$$\frac{x^2}{k} + \frac{y^2}{4 + k} = 1$$

Show that all the ellipses represented by this equation have the same foci, no matter what the value of k.

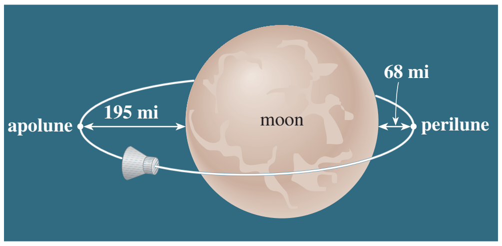
Applications

47. Perihelion and Aphelion The planets move around the sun in elliptical orbits with the sun at one focus. The point in the orbit at which the planet is closest to the sun is called **perihelion**, and the point at which it is farthest is called **aphelion**. These points are the vertices of the orbit. The earth's distance from the sun is 147,000,000 km at perihelion and 153,000,000 km at aphelion. Find an equation for the earth's orbit. (Place the origin at the center of the orbit with the sun on the x-axis.)

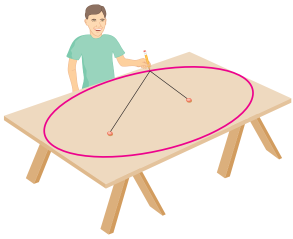


48. The Orbit of Pluto With an eccentricity of 0.25, Pluto's orbit is the most eccentric in the solar system. The length of the minor axis of its orbit is approximately 10,000,000,000 km. Find the distance between Pluto and the sun at perihelion and at aphelion. (See Exercise 47.)

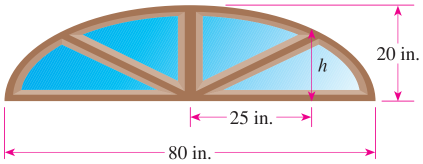
49. Lunar Orbit For an object in an elliptical orbit around the moon, the points in the orbit that are closest to and farthest from the center of the moon are called **perilune** and **apolune**, respectively. These are the vertices of the orbit. The center of the moon is at one focus of the orbit. The *Apollo 11* spacecraft was placed in a lunar orbit with perilune at 68 mi and apolune at 195 mi above the surface of the moon. Assuming the moon is a sphere of radius 1075 mi, find an equation for the orbit of *Apollo 11*. (Place the coordinate axes so that the origin is at the center of the orbit and the foci are located on the x-axis.)



50. Plywood Ellipse A carpenter wishes to construct an elliptical table top from a sheet of plywood, 4 ft by 8 ft. He will trace out the ellipse using the "thumbtack and string" method illustrated in Figures 2 and 3. What length of string should he use, and how far apart should the tacks be located, if the ellipse is to be the largest possible that can be cut out of the plywood sheet?



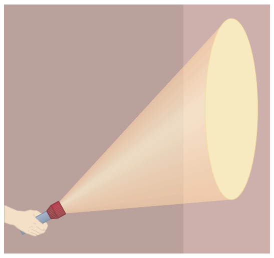
51. Sunburst Window A "sunburst" window above a doorway is constructed in the shape of the top half of an ellipse, as shown in the figure. The window is 20 in. tall at its highest point and 80 in. wide at the bottom. Find the height of the window 25 in. from the center of the base.



Discovery • Discussion

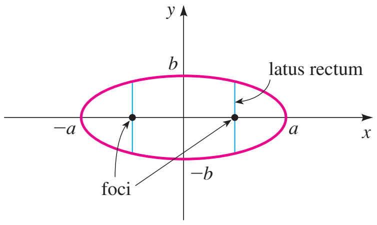
52. Drawing an Ellipse on a Blackboard Try drawing an ellipse as accurately as possible on a blackboard. How would a piece of string and two friends help this process?

53. Light Cone from a Flashlight A flashlight shines on a wall, as shown in the figure. What is the shape of the boundary of the lighted area? Explain your answer.

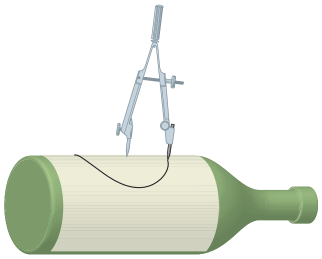


54. How Wide Is an Ellipse at Its Foci? A *latus rectum* for an ellipse is a line segment perpendicular to the major axis at a focus, with endpoints on the ellipse, as shown. Show that the length of a latus rectum is $2b^2/a$ for the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{with } a > b$$



55. Is It an Ellipse? A piece of paper is wrapped around a cylindrical bottle, and then a compass is used to draw a circle on the paper, as shown in the figure. When the paper is laid flat, is the shape drawn on the paper an ellipse? (You don't need to prove your answer, but you may want to do the experiment and see what you get.)



SUGGESTED TIME AND EMPHASIS

$\frac{1}{2}$ class.

Optional material.

10.3 Hyperbolas

Although ellipses and hyperbolas have completely different shapes, their definitions and equations are similar. Instead of using the *sum* of distances from two fixed foci, as in the case of an ellipse, we use the *difference* to define a hyperbola.

Geometric Definition of a Hyperbola

A **hyperbola** is the set of all points in the plane, the difference of whose distances from two fixed points F_1 and F_2 is a constant. (See Figure 1.) These two fixed points are the **foci** of the hyperbola.

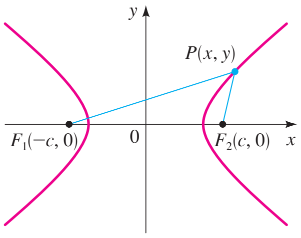


Figure 1

P is on the hyperbola if $|d(P, F_1) - d(P, F_2)| = 2a$.

As in the case of the ellipse, we get the simplest equation for the hyperbola by placing the foci on the x-axis at $(\pm c, 0)$, as shown in Figure 1. By definition, if $P(x, y)$ lies on the hyperbola, then either $d(P, F_1) - d(P, F_2)$ or $d(P, F_2) - d(P, F_1)$ must equal some positive constant, which we call $2a$. Thus, we have

$$d(P, F_1) - d(P, F_2) = \pm 2a$$

or

$$\sqrt{(x + c)^2 + y^2} - \sqrt{(x - c)^2 + y^2} = \pm 2a$$

Proceeding as we did in the case of the ellipse (Section 10.2), we simplify this to

$$(c^2 - a^2)x^2 - a^2y^2 = a^2(c^2 - a^2)$$

From triangle PF_1F_2 in Figure 1 we see that $|d(P, F_1) - d(P, F_2)| < 2c$. It follows that $2a < 2c$, or $a < c$. Thus, $c^2 - a^2 > 0$, so we can set $b^2 = c^2 - a^2$. We then simplify the last displayed equation to get

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

This is the *equation of the hyperbola*. If we replace x by $-x$ or y by $-y$ in this equation, it remains unchanged, so the hyperbola is symmetric about both the x- and y-axes and about the origin. The x-intercepts are $\pm a$, and the points $(a, 0)$ and $(-a, 0)$ are the **vertices** of the hyperbola. There is no y-intercept, because setting $x = 0$ in the equation of the hyperbola leads to $-y^2 = b^2$, which has no real solution. Furthermore, the equation of the hyperbola implies that

$$\frac{x^2}{a^2} = \frac{y^2}{b^2} + 1 \geq 1$$

so $x^2/a^2 \geq 1$; thus, $x^2 \geq a^2$, and hence $x \geq a$ or $x \leq -a$. This means that the hyperbola consists of two parts, called its **branches**. The segment joining the two vertices on the separate branches is the **transverse axis** of the hyperbola, and the origin is called its **center**.

If we place the foci of the hyperbola on the y-axis rather than on the x-axis, then this has the effect of reversing the roles of x and y in the derivation of the equation of the hyperbola. This leads to a hyperbola with a vertical transverse axis.

POINTS TO STRESS

1. The definition and geometry of hyperbolas.
2. Using the equation of a hyperbola to find relevant constants.
3. Graphing a hyperbola given its equation.

Equations and Graphs of Hyperbolas

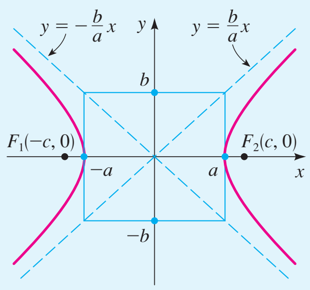
The main properties of hyperbolas are listed in the following box.

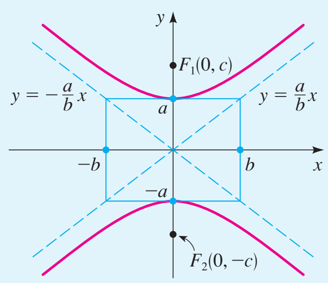
Hyperbola with Center at the Origin

The graph of each of the following equations is a hyperbola with center at the origin and having the given properties.

EQUATION	$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1 \quad (a > 0, b > 0)$	$\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1 \quad (a > 0, b > 0)$
VERTICES	$(\pm a, 0)$	$(0, \pm a)$
TRANSVERSE AXIS	Horizontal, length $2a$	Vertical, length $2a$
ASYMPTOTES	$y = \pm\dfrac{b}{a}x$	$y = \pm\dfrac{a}{b}x$
FOCI	$(\pm c, 0), \quad c^2 = a^2 + b^2$	$(0, \pm c), \quad c^2 = a^2 + b^2$

GRAPH





Asymptotes of rational functions are discussed in Section 3.6.

The *asymptotes* mentioned in this box are lines that the hyperbola approaches for large values of x and y. To find the asymptotes in the first case in the box, we solve the equation for y to get

$$\begin{aligned} y &= \pm\frac{b}{a}\sqrt{x^2 - a^2} \\ &= \pm\frac{b}{a}x\sqrt{1 - \frac{a^2}{x^2}} \end{aligned}$$

As x gets large, a^2/x^2 gets closer to zero. In other words, as $x \to \infty$ we have $a^2/x^2 \to 0$. So, for large x the value of y can be approximated as $y = \pm(b/a)x$. This shows that these lines are asymptotes of the hyperbola.

Asymptotes are an essential aid for graphing a hyperbola; they help us determine its shape. A convenient way to find the asymptotes, for a hyperbola with horizontal transverse axis, is to first plot the points $(a, 0)$, $(-a, 0)$, $(0, b)$, and $(0, -b)$. Then sketch horizontal and vertical segments through these points to construct a rectangle, as shown in Figure 2(a) on the next page. We call this rectangle the **central box** of the hyperbola. The slopes of the diagonals of the central box are $\pm b/a$, so by extending them we obtain the asymptotes $y = \pm(b/a)x$, as sketched in part (b) of the figure. Finally, we plot the vertices and use the asymptotes as a guide in sketching the

SAMPLE QUESTION
Text Question

Is $\frac{x^2}{5^2} - \frac{y^2}{6^2} = 1$ the equation of a horizontal or a vertical hyperbola? How do you know?

Answer

It is horizontal because the x^2 term is positive.

ALTERNATE EXAMPLE 1

Find the vertices, foci, and asymptotes of the hyperbola $25x^2 - 144y^2 = 3600$.

ANSWER

$(\pm 12, 0)$, $(\pm 13, 0)$, $y = \pm \frac{5}{12}x$

hyperbola shown in part (c). (A similar procedure applies to graphing a hyperbola that has a vertical transverse axis.)

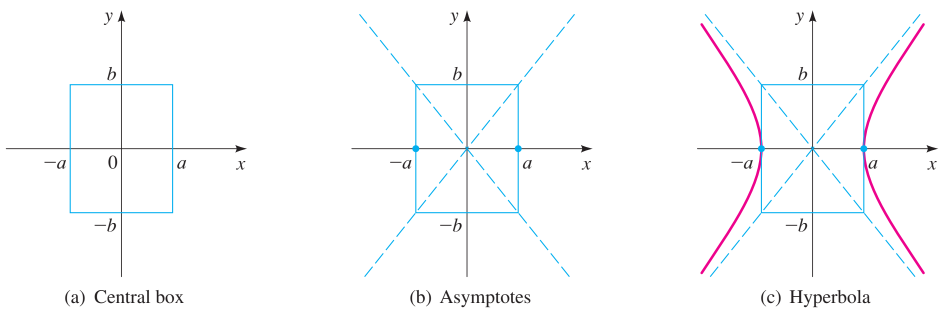


Figure 2

Steps in graphing the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

How to Sketch a Hyperbola

1. **Sketch the Central Box.** This is the rectangle centered at the origin, with sides parallel to the axes, that crosses one axis at $\pm a$, the other at $\pm b$.
2. **Sketch the Asymptotes.** These are the lines obtained by extending the diagonals of the central box.
3. **Plot the Vertices.** These are the two x-intercepts or the two y-intercepts.
4. **Sketch the Hyperbola.** Start at a vertex and sketch a branch of the hyperbola, approaching the asymptotes. Sketch the other branch in the same way.

Example 1 A Hyperbola with Horizontal Transverse Axis

A hyperbola has the equation

$$9x^2 - 16y^2 = 144$$

(a) Find the vertices, foci, and asymptotes, and sketch the graph.
(b) Draw the graph using a graphing calculator.

Solution

(a) First we divide both sides of the equation by 144 to put it into standard form:

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

IN-CLASS MATERIALS

Have students sketch a hyperbola "from scratch." Hand out a sheet of paper with two foci, and hand out rulers. Get the students to plot points where the difference of the distances between the points and the foci is 1 inch. Have them keep plotting points until a hyperbolic shape emerges.

Because the x^2-term is positive, the hyperbola has a horizontal transverse axis; its vertices and foci are on the x-axis. Since $a^2 = 16$ and $b^2 = 9$, we get $a = 4$, $b = 3$, and $c = \sqrt{16 + 9} = 5$. Thus, we have

VERTICES	$(\pm 4, 0)$
FOCI	$(\pm 5, 0)$
ASYMPTOTES	$y = \pm \frac{3}{4}x$

After sketching the central box and asymptotes, we complete the sketch of the hyperbola as in Figure 3(a).

(b) To draw the graph using a graphing calculator, we need to solve for y.

Note that the equation of a hyperbola does not define y as a function of x (see page 164). That's why we need to graph two functions to graph a hyperbola.

$$9x^2 - 16y^2 = 144$$

$$-16y^2 = -9x^2 + 144 \quad \text{Subtract } 9x^2$$

$$y^2 = 9\left(\frac{x^2}{16} - 1\right) \quad \text{Divide by } -16 \text{ and factor } 9$$

$$y = \pm 3\sqrt{\frac{x^2}{16} - 1} \quad \text{Take square roots}$$

To obtain the graph of the hyperbola, we graph the functions

$$y = 3\sqrt{(x^2/16) - 1} \quad \text{and} \quad y = -3\sqrt{(x^2/16) - 1}$$

as shown in Figure 3(b).

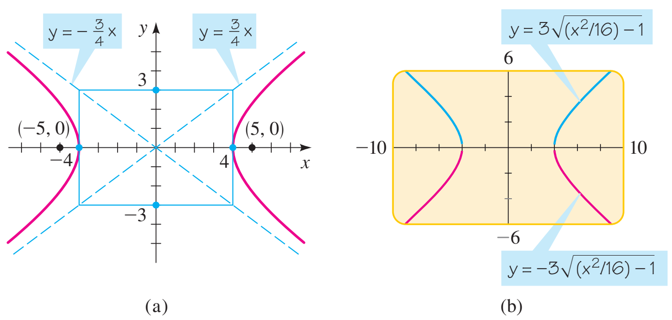


Figure 3
$9x^2 - 16y^2 = 144$

Example 2 A Hyperbola with Vertical Transverse Axis

Find the vertices, foci, and asymptotes of the hyperbola, and sketch its graph.

$$x^2 - 9y^2 + 9 = 0$$

Solution We begin by writing the equation in the standard form for a hyperbola.

$$x^2 - 9y^2 = -9$$

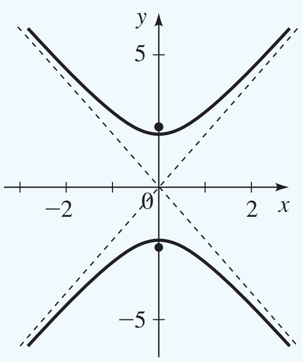
$$y^2 - \frac{x^2}{9} = 1 \quad \text{Divide by } -9$$

DRILL QUESTION

Find the vertices, foci, and asymptotes of the hyperbola given by $\frac{y^2}{4} - x^2 = 1$. Then graph it.

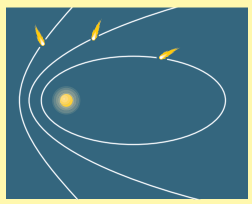
Answer

Vertices $(0, \pm 2)$, foci $(0, \pm\sqrt{5})$ asymptotes $y = \pm 2x$



Paths of Comets

The path of a comet is an ellipse, a parabola, or a hyperbola with the sun at a focus. This fact can be proved using calculus and Newton's laws of motion.* If the path is a parabola or a hyperbola, the comet will never return. If the path is an ellipse, it can be determined precisely when and where the comet can be seen again. Halley's comet has an elliptical path and returns every 75 years; it was last seen in 1987. The brightest comet of the 20th century was comet Hale-Bopp, seen in 1997. Its orbit is a very eccentric ellipse; it is expected to return to the inner solar system around the year 4377.



*James Stewart, *Calculus*, 5th ed. (Pacific Grove, CA: Brooks/Cole, 2003), pp. 912–914.

Because the y^2-term is positive, the hyperbola has a vertical transverse axis; its foci and vertices are on the y-axis. Since $a^2 = 1$ and $b^2 = 9$, we get $a = 1$, $b = 3$, and $c = \sqrt{1 + 9} = \sqrt{10}$. Thus, we have

VERTICES	$(0, \pm 1)$
FOCI	$(0, \pm\sqrt{10})$
ASYMPTOTES	$y = \pm\frac{1}{3}x$

We sketch the central box and asymptotes, then complete the graph, as shown in Figure 4(a).

We can also draw the graph using a graphing calculator, as shown in Figure 4(b).

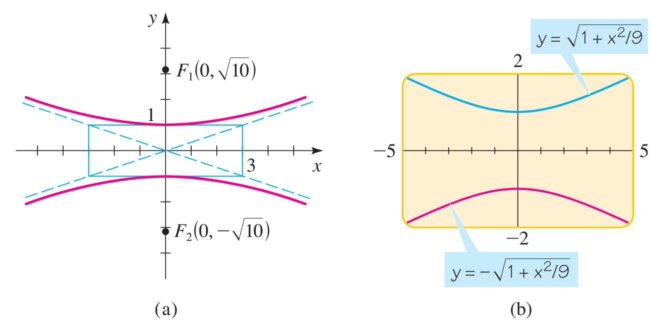


Figure 4
$x^2 - 9y^2 + 9 = 0$

■

ALTERNATE EXAMPLE 3

Find the equation of the hyperbola in standard form with vertices $(\pm 2, 0)$ and foci $(\pm 3, 0)$.

ANSWER

$\frac{x^2}{4} - \frac{y^2}{5} = 1$

Example 3 Finding the Equation of a Hyperbola from Its Vertices and Foci

Find the equation of the hyperbola with vertices $(\pm 3, 0)$ and foci $(\pm 4, 0)$. Sketch the graph.

Solution Since the vertices are on the x-axis, the hyperbola has a horizontal transverse axis. Its equation is of the form

$$\frac{x^2}{3^2} - \frac{y^2}{b^2} = 1$$

We have $a = 3$ and $c = 4$. To find b, we use the relation $a^2 + b^2 = c^2$:

$$3^2 + b^2 = 4^2$$
$$b^2 = 4^2 - 3^2 = 7$$
$$b = \sqrt{7}$$

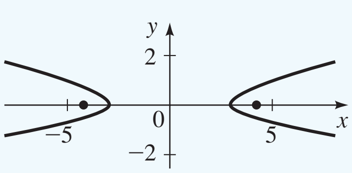
Thus, the equation of the hyperbola is

$$\frac{x^2}{9} - \frac{y^2}{7} = 1$$

EXAMPLE

A horizontal hyperbola: $\frac{x^2}{9} - 2y^2 = 1$

Vertices $(\pm 3, 0)$, asymptotes $y = \pm\frac{1}{3\sqrt{2}}x$, foci $\left(\pm\sqrt{\frac{37}{2}}, 0\right)$



IN-CLASS MATERIALS

As noted in Section 10.1, one can use dental floss and a modeling compound (such as clay or Play-Doh®) to make a half-cone and slice it. Have the students attempt to do so to get a hyperbola. Notice that the slice does not have to be straight up and down, as shown in the prologue to the chapter. As long as the slice would cut the other half of the cone, the resultant curve is a hyperbola.

The graph is shown in Figure 5.

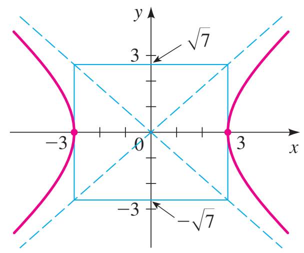


Figure 5
$\frac{x^2}{9} - \frac{y^2}{7} = 1$

■

Example 4 Finding the Equation of a Hyperbola from Its Vertices and Asymptotes

Find the equation and the foci of the hyperbola with vertices $(0, \pm 2)$ and asymptotes $y = \pm 2x$. Sketch the graph.

Solution Since the vertices are on the y-axis, the hyperbola has a vertical transverse axis with $a = 2$. From the asymptote equation we see that $a/b = 2$. Since $a = 2$, we get $2/b = 2$, and so $b = 1$. Thus, the equation of the hyperbola is

$$\frac{y^2}{4} - x^2 = 1$$

To find the foci, we calculate $c^2 = a^2 + b^2 = 2^2 + 1^2 = 5$, so $c = \sqrt{5}$. Thus, the foci are $(0, \pm\sqrt{5})$. The graph is shown in Figure 6. ■

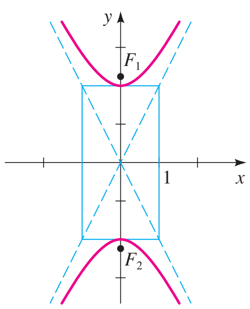


Figure 6
$\frac{y^2}{4} - x^2 = 1$

Like parabolas and ellipses, hyperbolas have an interesting *reflection property*. Light aimed at one focus of a hyperbolic mirror is reflected toward the other focus, as shown in Figure 7. This property is used in the construction of Cassegrain-type telescopes. A hyperbolic mirror is placed in the telescope tube so that light reflected from the primary parabolic reflector is aimed at one focus of the hyperbolic mirror. The light is then refocused at a more accessible point below the primary reflector (Figure 8).

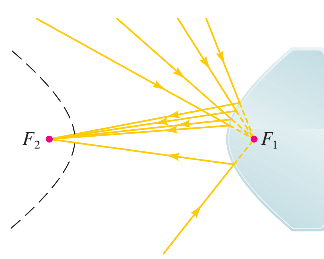


Figure 7
Reflection property of hyperbolas

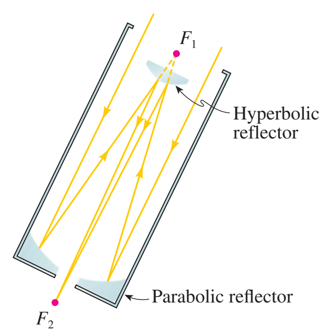
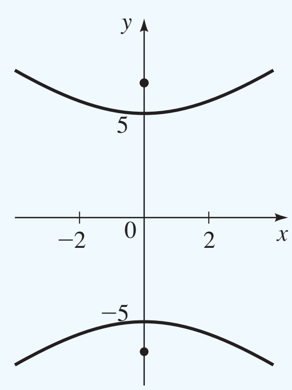


Figure 8
Cassegrain-type telescope

ALTERNATE EXAMPLE 4
Find the equation and the foci of the hyperbola with vertices $(0, \pm 4)$ and asymptotes $y = 4x$.

ANSWER
$\frac{y^2}{16} - x^2 = 1$, $(0, \sqrt{17})$, $(0, -\sqrt{17})$

EXAMPLE
A vertical hyperbola:
$-\frac{x^2}{16} + \frac{y^2}{25} = 1$
Vertices $(0, \pm 5)$, asymptotes $y = \pm\frac{5}{4}x$, foci $(0, \pm\sqrt{41})$



IN-CLASS MATERIALS

Discuss the reflection property of hyperbolas: Take a point between the branches, and aim a beam of light at one of the foci. It will reflect off the hyperbola, and go in a path aimed directly at the other focus. This reflection property is harder to model physically than those of the ellipse and the parabola.

The LORAN (LOng RAnge Navigation) system was used until the early 1990s; it has now been superseded by the GPS system (see page 656). In the LORAN system, hyperbolas are used onboard a ship to determine its location. In Figure 9 radio stations at A and B transmit signals simultaneously for reception by the ship at P. The onboard computer converts the time difference in reception of these signals into a distance difference $d(P, A) - d(P, B)$. From the definition of a hyperbola this locates the ship on one branch of a hyperbola with foci at A and B (sketched in black in the figure). The same procedure is carried out with two other radio stations at C and D, and this locates the ship on a second hyperbola (shown in red in the figure). (In practice, only three stations are needed because one station can be used as a focus for both hyperbolas.) The coordinates of the intersection point of these two hyperbolas, which can be calculated precisely by the computer, give the location of P.

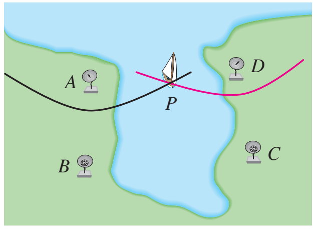


Figure 9
LORAN system for finding the location of a ship

10.3 Exercises

1–4 ■ Match the equation with the graphs labeled I–IV. Give reasons for your answers.

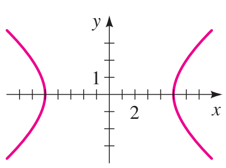
1. $\dfrac{x^2}{4} - y^2 = 1$

2. $y^2 - \dfrac{x^2}{9} = 1$

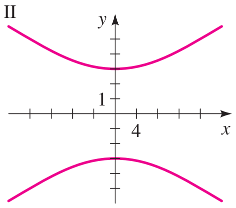
3. $16y^2 - x^2 = 144$

4. $9x^2 - 25y^2 = 225$

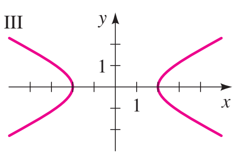
I



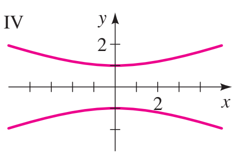
II



III



IV



5–16 ■ Find the vertices, foci, and asymptotes of the hyperbola, and sketch its graph.

5. $\dfrac{x^2}{4} - \dfrac{y^2}{16} = 1$

6. $\dfrac{y^2}{9} - \dfrac{x^2}{16} = 1$

7. $y^2 - \dfrac{x^2}{25} = 1$

8. $\dfrac{x^2}{2} - y^2 = 1$

9. $x^2 - y^2 = 1$

10. $9x^2 - 4y^2 = 36$

11. $25y^2 - 9x^2 = 225$

12. $x^2 - y^2 + 4 = 0$

13. $x^2 - 4y^2 - 8 = 0$

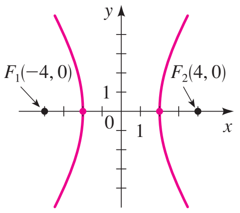
14. $x^2 - 2y^2 = 3$

15. $4y^2 - x^2 = 1$

16. $9x^2 - 16y^2 = 1$

17–22 ■ Find the equation for the hyperbola whose graph is shown.

17.



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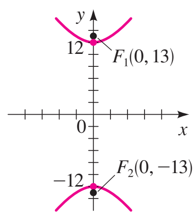

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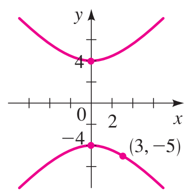

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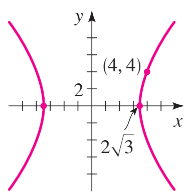

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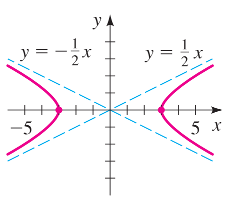
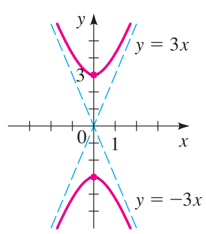


22.



23–26 ■ Use a graphing device to graph the hyperbola.

23. $x^2 - 2y^2 = 8$

24. $3y^2 - 4x^2 = 24$

25. $\dfrac{y^2}{2} - \dfrac{x^2}{6} = 1$

26. $\dfrac{x^2}{100} - \dfrac{y^2}{64} = 1$

27–38 ■ Find an equation for the hyperbola that satisfies the given conditions.

27. Foci $(\pm 5, 0)$, vertices $(\pm 3, 0)$

28. Foci $(0, \pm 10)$, vertices $(0, \pm 8)$

29. Foci $(0, \pm 2)$, vertices $(0, \pm 1)$

30. Foci $(\pm 6, 0)$, vertices $(\pm 2, 0)$

31. Vertices $(\pm 1, 0)$, asymptotes $y = \pm 5x$

32. Vertices $(0, \pm 6)$, asymptotes $y = \pm\frac{1}{3}x$

33. Foci $(0, \pm 8)$, asymptotes $y = \pm\frac{1}{2}x$

34. Vertices $(0, \pm 6)$, hyperbola passes through $(-5, 9)$

35. Asymptotes $y = \pm x$, hyperbola passes through $(5, 3)$

36. Foci $(\pm 3, 0)$, hyperbola passes through $(4, 1)$

37. Foci $(\pm 5, 0)$, length of transverse axis 6

38. Foci $(0, \pm 1)$, length of transverse axis 1

39. **(a)** Show that the asymptotes of the hyperbola $x^2 - y^2 = 5$ are perpendicular to each other.

(b) Find an equation for the hyperbola with foci $(\pm c, 0)$ and with asymptotes perpendicular to each other.

40. The hyperbolas

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{and} \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$$

are said to be **conjugate** to each other.

(a) Show that the hyperbolas

$$x^2 - 4y^2 + 16 = 0 \quad \text{and} \quad 4y^2 - x^2 + 16 = 0$$

are conjugate to each other, and sketch their graphs on the same coordinate axes.

(b) What do the hyperbolas of part (a) have in common?

(c) Show that any pair of conjugate hyperbolas have the relationship you discovered in part (b).

41. In the derivation of the equation of the hyperbola at the beginning of this section, we said that the equation

$$\sqrt{(x + c)^2 + y^2} - \sqrt{(x - c)^2 + y^2} = \pm 2a$$

simplifies to

$$(c^2 - a^2)x^2 - a^2y^2 = a^2(c^2 - a^2)$$

Supply the steps needed to show this.

42. **(a)** For the hyperbola

$$\frac{x^2}{9} - \frac{y^2}{16} = 1$$

determine the values of a, b, and c, and find the coordinates of the foci F_1 and F_2.

(b) Show that the point $P(5, \frac{16}{3})$ lies on this hyperbola.

(c) Find $d(P, F_1)$ and $d(P, F_2)$.

(d) Verify that the difference between $d(P, F_1)$ and $d(P, F_2)$ is $2a$.

43. Hyperbolas are called **confocal** if they have the same foci.

(a) Show that the hyperbolas

$$\frac{y^2}{k} - \frac{x^2}{16 - k} = 1 \qquad \text{with } 0 < k < 16$$

are confocal.

(b) Use a graphing device to draw the top branches of the family of hyperbolas in part (a) for $k = 1, 4, 8$, and 12. How does the shape of the graph change as k increases?

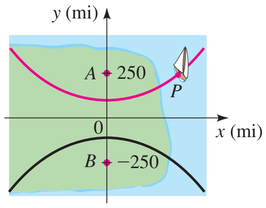
Applications

44. Navigation In the figure, the LORAN stations at A and B are 500 mi apart, and the ship at P receives station A's signal 2640 microseconds (μs) before it receives the signal from B.

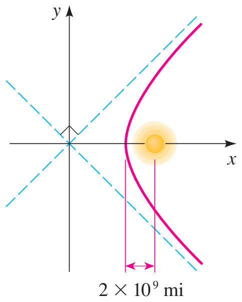
(a) Assuming that radio signals travel at 980 ft/μs, find $d(P, A) - d(P, B)$

(b) Find an equation for the branch of the hyperbola indicated in red in the figure. (Use miles as the unit of distance.)

(c) If A is due north of B, and if P is due east of A, how far is P from A?



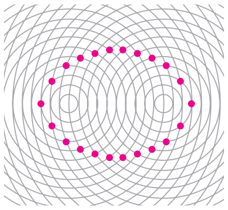
45. Comet Trajectories Some comets, such as Halley's comet, are a permanent part of the solar system, traveling in elliptical orbits around the sun. Others pass through the solar system only once, following a hyperbolic path with the sun at a focus. The figure shows the path of such a comet. Find an equation for the path, assuming that the closest the comet comes to the sun is 2×10^9 mi and that the path the comet was taking before it neared the solar system is at a right angle to the path it continues on after leaving the solar system.

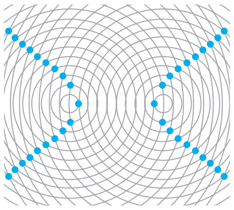


46. Ripples in Pool Two stones are dropped simultaneously in a calm pool of water. The crests of the resulting waves form equally spaced concentric circles, as shown in the figures. The waves interact with each other to create certain interference patterns.

(a) Explain why the red dots lie on an ellipse.

(b) Explain why the blue dots lie on a hyperbola.

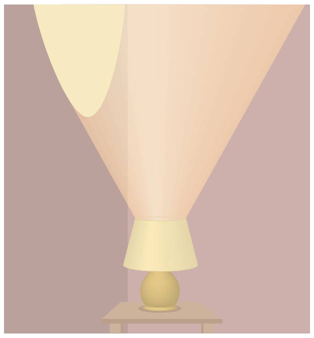




Discovery • Discussion

47. Hyperbolas in the Real World Several examples of the uses of hyperbolas are given in the text. Find other situations in real life where hyperbolas occur. Consult a scientific encyclopedia in the reference section of your library, or search the Internet.

48. Light from a Lamp The light from a lamp forms a lighted area on a wall, as shown in the figure. Why is the boundary of this lighted area a hyperbola? How can one hold a flashlight so that its beam forms a hyperbola on the ground?



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DISCOVERY PROJECT

Conics in Architecture

In ancient times architecture was part of mathematics, so architects had to be mathematicians. Many of the structures they built—pyramids, temples, amphitheaters, and irrigation projects—still stand. In modern times architects employ even more sophisticated mathematical principles. The photographs below show some structures that employ conic sections in their design.

Roman Amphitheater in Alexandria, Egypt (circle)

Nik Wheeler/Corbis

Ceiling of Statuary Hall in the U.S. Capitol (ellipse)

Architect of the Capitol

Roof of the Skydome in Toronto, Canada (parabola)

Walter Schmid/Stone/Getty Images

Roof of Washington Dulles Airport (hyperbola and parabola)

Richard T. Nowitz/Corbis

McDonnell Planetarium, St. Louis, MO (hyperbola)

Courtesy of Chamber of Commerce, St. Louis, MO

Attic in La Pedrera, Barcelona, Spain (parabola)

O. Alamany and Vincens/Corbis

Architects have different reasons for using conics in their designs. For example, the Spanish architect Antoni Gaudi used parabolas in the attic of La Pedrera (see photo above). He reasoned that since a rope suspended between two points with an equally distributed load (like in a suspension bridge) has the shape of a parabola, an inverted parabola would provide the best support for a flat roof.

Constructing Conics

The equations of the conics are helpful in manufacturing small objects, because a computer-controlled cutting tool can accurately trace a curve given by an equation. But in a building project, how can we construct a portion of a

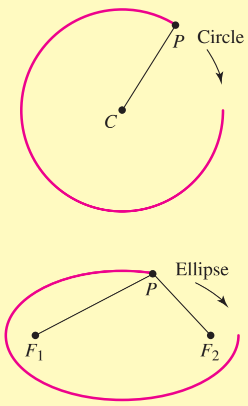


Figure 1
Constructing a circle and an ellipse

parabola, ellipse, or hyperbola that spans the ceiling or walls of a building? The geometric properties of the conics provide practical ways of constructing them. For example, if you were building a circular tower, you would choose a center point, then make sure that the walls of the tower are a fixed distance from that point. Elliptical walls can be constructed using a string anchored at two points, as shown in Figure 1.

To construct a parabola, we can use the apparatus shown in Figure 2. A piece of string of length a is anchored at F and A. The T-square, also of length a, slides along the straight bar L. A pencil at P holds the string taut against the T-square. As the T-square slides to the right the pencil traces out a curve.

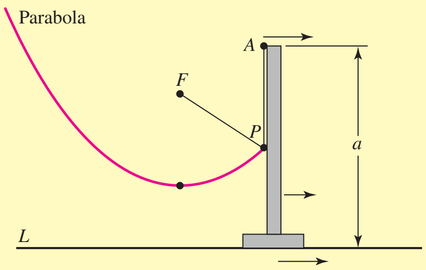


Figure 2
Constructing a parabola

From the figure we see that

$$d(F, P) + d(P, A) = a \qquad \text{The string is of length } a$$

$$d(L, P) + d(P, A) = a \qquad \text{The T-square is of length } a$$

It follows that $d(F, P) + d(P, A) = d(L, P) + d(P, A)$. Subtracting $d(P, A)$ from each side, we get

$$d(F, P) = d(L, P)$$

The last equation says that the distance from F to P is equal to the distance from P to the line L. Thus, the curve is a parabola with focus F and directrix L.

In building projects it's easier to construct a straight line than a curve. So in some buildings, such as in the Kobe Tower (see problem 4), a curved surface is produced by using many straight lines. We can also produce a curve using straight lines, such as the parabola shown in Figure 3.

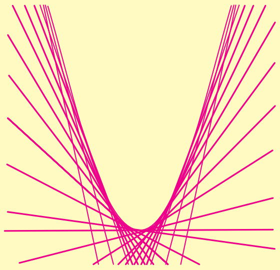


Figure 3
Tangent lines to a parabola

Each line is **tangent** to the parabola; that is, the line meets the parabola at exactly one point and does not cross the parabola. The line tangent to the parabola $y = x^2$ at the point (a, a^2) is

$$y = 2ax - a^2$$

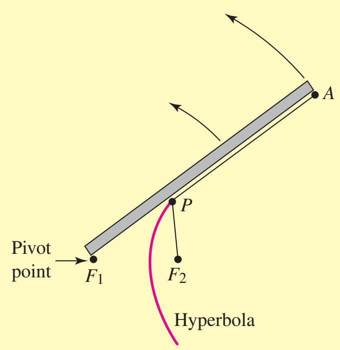
You are asked to show this in problem 6. The parabola is called the **envelope** of all such lines.

1. The photographs on page 771 show six examples of buildings that contain conic sections. Search the Internet to find other examples of structures that employ parabolas, ellipses, or hyperbolas in their design. Find at least one example for each type of conic.

2. In this problem we construct a hyperbola. The wooden bar in the figure can pivot at F_1. A string shorter than the bar is anchored at F_2 and at A, the other end of the bar. A pencil at P holds the string taut against the bar as it moves counterclockwise around F_1.

(a) Show that the curve traced out by the pencil is one branch of a hyperbola with foci at F_1 and F_2.

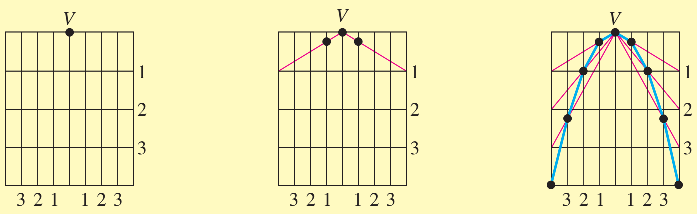
(b) How should the apparatus be reconfigured to draw the other branch of the hyperbola?



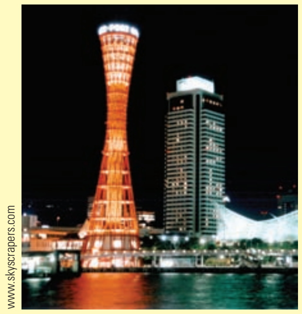
3. The following method can be used to construct a parabola that fits in a given rectangle. The parabola will be approximated by many short line segments.

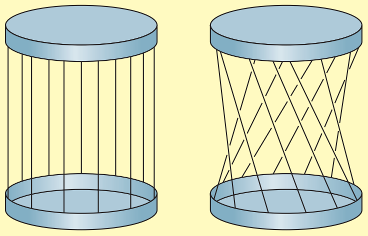
First, draw a rectangle. Divide the rectangle in half by a vertical line segment and label the top endpoint V. Next, divide the length and width of each half rectangle into an equal number of parts to form grid lines, as shown in the figure on the next page. Draw lines from V to the endpoints of horizontal grid line 1, and mark the points where these lines cross the vertical grid lines labeled 1. Next, draw lines from V to the endpoints of horizontal grid line 2, and mark the points where these lines cross the vertical grid lines labeled 2. Continue in this way until you have used all the horizontal grid lines.

Now, use line segments to connect the points you have marked to obtain an approximation to the desired parabola. Apply this procedure to draw a parabola that fits into a 6 ft by 10 ft rectangle on a lawn.



4. In this problem we construct hyperbolic shapes using straight lines. Punch equally spaced holes into the edges of two large plastic lids. Connect corresponding holes with strings of equal lengths as shown in the figure. Holding the strings taut, twist one lid against the other. An imaginary surface passing through the strings has hyperbolic cross sections. (An architectural example of this is the Kobe Tower in Japan shown in the photograph.) What happens to the vertices of the hyperbolic cross sections as the lids are twisted more?





5. In this problem we show that the line tangent to the parabola $y = x^2$ at the point (a, a^2) has the equation $y = 2ax - a^2$.

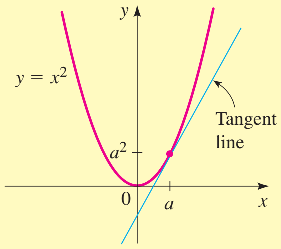
(a) Let m be the slope of the tangent line at (a, a^2). Show that the equation of the tangent line is $y - a^2 = m(x - a)$.

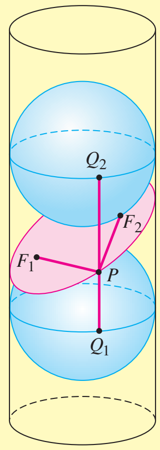
(b) Use the fact that the tangent line intersects the parabola at only one point to show that (a, a^2) is the only solution of the system.

$$\begin{cases} y - a^2 = m(x - a) \\ y = x^2 \end{cases}$$

(c) Eliminate y from the system in part (b) to get a quadratic equation in x. Show that the discriminant of this quadratic is $(m - 2a)^2$. Since the system in (b) has exactly one solution, the discriminant must equal 0. Find m.

(d) Substitute the value for m you found in part (c) into the equation in part (a) and simpify to get the equation of the tangent line.





6. In this problem we prove that when a cylinder is cut by a plane an ellipse is formed. An architectural example of this is the Tycho Brahe Planetarium in Copenhagen (see the photograph). In the figure a cylinder is cut by a plane resulting in the red curve. Two spheres with the same radius as the cylinder slide inside the cylinder so that they just touch the plane at F_1 and F_2. Choose an arbitrary point P on the curve and let Q_1 and Q_2 be the two points on the cylinder where a vertical line through P touches the "equator" of each sphere.

(a) Show that $PF_1 = PQ_1$ and $PF_2 = PQ_2$. [*Hint:* Use the fact that all tangents to a sphere from a given point outside the sphere are of the same length.]

(b) Explain why $PQ_1 + PQ_2$ is the same for all points P on the curve.

(c) Show that $PF_1 + PF_2$ is the same for all points P on the curve.

(d) Conclude that the curve is an ellipse with foci F_1 and F_2.

Bob Krist/Corbis

10.4 Shifted Conics

In the preceding sections we studied parabolas with vertices at the origin and ellipses and hyperbolas with centers at the origin. We restricted ourselves to these cases because these equations have the simplest form. In this section we consider conics whose vertices and centers are not necessarily at the origin, and we determine how this affects their equations.

In Section 2.4 we studied transformations of functions that have the effect of shifting their graphs. In general, for any equation in x and y, if we replace x by $x - h$ or by $x + h$, the graph of the new equation is simply the old graph shifted horizontally; if y is replaced by $y - k$ or by $y + k$, the graph is shifted vertically. The following box gives the details.

SUGGESTED TIME AND EMPHASIS

1 class.
Optional material. May be covered in conjunction with Sections 10.1–10.3.

POINTS TO STRESS

1. Completing the square in a general equation of a conic in order to apply the techniques of Section 2.5 to graph the conic.
2. Identifying conic sections by the constants in the general equation.
3. Understanding degenerate conic sections.

Shifting Graphs of Equations

If h and k are positive real numbers, then replacing x by $x - h$ or by $x + h$ and replacing y by $y - k$ or by $y + k$ has the following effect(s) on the graph of any equation in x and y.

Replacement	How the graph is shifted
1. x replaced by $x - h$	Right h units
2. x replaced by $x + h$	Left h units
3. y replaced by $y - k$	Upward k units
4. y replaced by $y + k$	Downward k units

Shifted Ellipses

Let's apply horizontal and vertical shifting to the ellipse with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

whose graph is shown in Figure 1. If we shift it so that its center is at the point (h, k) instead of at the origin, then its equation becomes

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

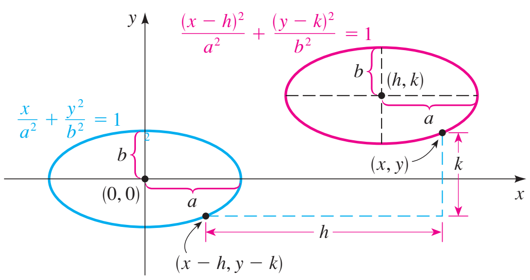


Figure 1
Shifted ellipse

Example 1 Sketching the Graph of a Shifted Ellipse

Sketch the graph of the ellipse

$$\frac{(x+1)^2}{4} + \frac{(y-2)^2}{9} = 1$$

and determine the coordinates of the foci.

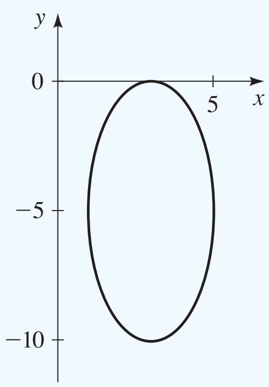
Solution The ellipse

$$\frac{(x+1)^2}{4} + \frac{(y-2)^2}{9} = 1 \qquad \text{Shifted ellipse}$$

SAMPLE QUESTION
Text Question
Identify the conic section with equation $9x^2 - 36x + 4y^2 = 0$.

Answer
Ellipse

EXAMPLE
A non-degenerate ellipse:
$25x^2 + 4y^2 - 150x + 40y + 324 = 0 \Leftrightarrow$
$$\frac{(x-3)^2}{4} + \frac{(y+5)^2}{25} = 1$$



ALTERNATE EXAMPLE 1
For the graph of the ellipse
$$\frac{(x-5)^2}{16} + \frac{(y-3)^2}{36} = 1$$
determine the coordinates of the foci.

ANSWER
$(-5, 3 + 2\sqrt{5}), (-5, 3 - 2\sqrt{5})$

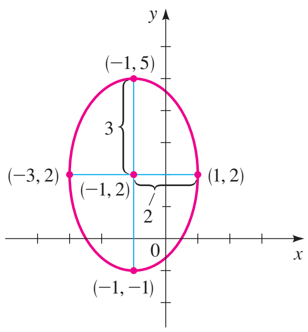


Figure 2
$\dfrac{(x+1)^2}{4} + \dfrac{(y-2)^2}{9} = 1$

is shifted so that its center is at $(-1, 2)$. It is obtained from the ellipse

$$\frac{x^2}{4} + \frac{y^2}{9} = 1 \qquad \text{Ellipse with center at origin}$$

by shifting it left 1 unit and upward 2 units. The endpoints of the minor and major axes of the unshifted ellipse are $(2, 0)$, $(-2, 0)$, $(0, 3)$, $(0, -3)$. We apply the required shifts to these points to obtain the corresponding points on the shifted ellipse:

$$\begin{aligned}
(2, 0) &\rightarrow (2 - 1, 0 + 2) = (1, 2)\\
(-2, 0) &\rightarrow (-2 - 1, 0 + 2) = (-3, 2)\\
(0, 3) &\rightarrow (0 - 1, 3 + 2) = (-1, 5)\\
(0, -3) &\rightarrow (0 - 1, -3 + 2) = (-1, -1)
\end{aligned}$$

This helps us sketch the graph in Figure 2.

To find the foci of the shifted ellipse, we first find the foci of the ellipse with center at the origin. Since $a^2 = 9$ and $b^2 = 4$, we have $c^2 = 9 - 4 = 5$, so $c = \sqrt{5}$. So the foci are $(0, \pm\sqrt{5})$. Shifting left 1 unit and upward 2 units, we get

$$\begin{aligned}
(0, \sqrt{5}) &\rightarrow (0 - 1, \sqrt{5} + 2) = (-1, 2 + \sqrt{5})\\
(0, -\sqrt{5}) &\rightarrow (0 - 1, -\sqrt{5} + 2) = (-1, 2 - \sqrt{5})
\end{aligned}$$

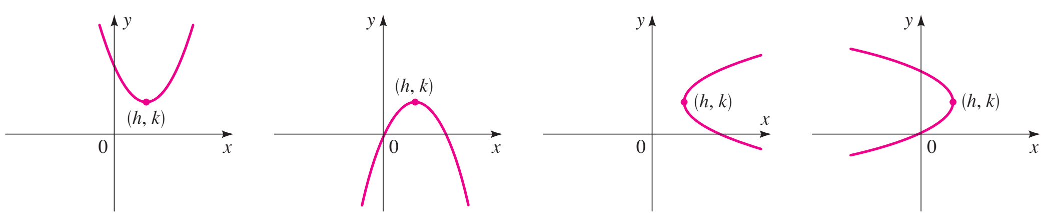
Thus, the foci of the shifted ellipse are

$$(-1, 2 + \sqrt{5}) \qquad \text{and} \qquad (-1, 2 - \sqrt{5})$$

■

Shifted Parabolas

Applying shifts to parabolas leads to the equations and graphs shown in Figure 3.



(a) $(x - h)^2 = 4p(y - k)$, $p > 0$ (b) $(x - h)^2 = 4p(y - k)$, $p < 0$ (c) $(y - k)^2 = 4p(x - h)$, $p > 0$ (d) $(y - k)^2 = 4p(x - h)$, $p < 0$

Figure 3
Shifted parabolas

Example 2 Graphing a Shifted Parabola

Determine the vertex, focus, and directrix and sketch the graph of the parabola.

$$x^2 - 4x = 8y - 28$$

DRILL QUESTION

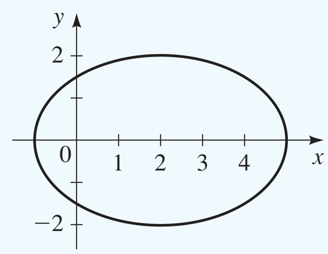
Graph the conic section with equation
$4x^2 - 16x + 9y^2 - 20 = 0$.

Answer

$4x^2 - 16x + 9y^2 - 20 = 0$

$\Leftrightarrow 4(x - 2)^2 + 9y^2 = 36$

$\Leftrightarrow \dfrac{(x-2)^2}{9} + \dfrac{y^2}{4} = 1$



ALTERNATE EXAMPLE 2

Determine the vertex, focus, and directrix of the parabola.
$x^2 - 4x = 36y - 112$

ANSWER

$(2, 3)$, $(2, 12)$, $y = -6$

IN-CLASS MATERIALS

Class time can be saved if this section is covered along with Sections 10.1–10.3. For example, after discussing parabolas, immediately do an example of a shifted parabola. Teach ellipses and hyperbolas similarly.

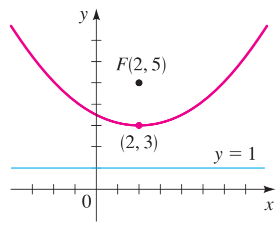


Figure 4
$x^2 - 4x = 8y - 28$

Solution We complete the square in x to put this equation into one of the forms in Figure 3.

$$x^2 - 4x + 4 = 8y - 28 + 4 \qquad \text{Add 4 to complete the square}$$
$$(x-2)^2 = 8y - 24$$
$$(x-2)^2 = 8(y-3) \qquad \text{Shifted parabola}$$

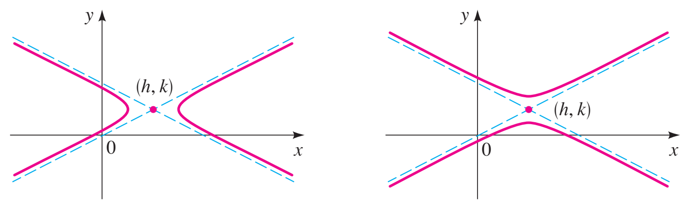
This parabola opens upward with vertex at $(2, 3)$. It is obtained from the parabola

$$x^2 = 8y \qquad \text{Parabola with vertex at origin}$$

by shifting right 2 units and upward 3 units. Since $4p = 8$, we have $p = 2$, so the focus is 2 units above the vertex and the directrix is 2 units below the vertex. Thus, the focus is $(2, 5)$ and the directrix is $y = 1$. The graph is shown in Figure 4. ■

Shifted Hyperbolas

Applying shifts to hyperbolas leads to the equations and graphs shown in Figure 5.



(a) $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$ (b) $-\dfrac{(x-h)^2}{b^2} + \dfrac{(y-k)^2}{a^2} = 1$

Figure 5
Shifted hyperbolas

ALTERNATE EXAMPLE 3
Find the center, vertices, foci, and asymptotes of the given hyperbola.
$4x^2 - 32x - 16y^2 - 32y = 16$

ANSWER
$(4, -1), (4 - 2\sqrt{5}, -1),$
$(4 + 2\sqrt{5}, -1)(0, -1),$
$(8, -1), y = \frac{1}{2}x - 3,$
$y = -\frac{1}{2}x + 1$

Example 3 Graphing a Shifted Hyperbola

A shifted conic has the equation

$$9x^2 - 72x - 16y^2 - 32y = 16$$

(a) Complete the square in x and y to show that the equation represents a hyperbola.

(b) Find the center, vertices, foci, and asymptotes of the hyperbola and sketch its graph.

(c) Draw the graph using a graphing calculator.

Solution

(a) We complete the squares in both x and y:

$$9(x^2 - 8x \qquad) - 16(y^2 + 2y \qquad) = 16$$
$$9(x^2 - 8x + 16) - 16(y^2 + 2y + 1) = 16 + 9 \cdot 16 - 16 \cdot 1 \qquad \text{Complete the squares}$$
$$9(x-4)^2 - 16(y+1)^2 = 144 \qquad \text{Divide this by 144}$$
$$\frac{(x-4)^2}{16} - \frac{(y+1)^2}{9} = 1 \qquad \text{Shifted hyperbola}$$

Comparing this to Figure 5(a), we see that this is the equation of a shifted hyperbola.

IN-CLASS MATERIALS

Help the class recognize the difference between the formulas of conic sections by pointing out that the equation of a parabola has degree 2 in only one variable (which variable it is determines which type of parabola it is). Equations of ellipses and hyperbolas have degree 2 in both variables; for ellipses the second-degree terms have the same sign, and for hyperbolas they have opposite signs.

(b) The shifted hyperbola has center $(4, -1)$ and a horizontal transverse axis.

CENTER $(4, -1)$

Its graph will have the same shape as the unshifted hyperbola

$$\frac{x^2}{16} - \frac{y^2}{9} = 1 \qquad \text{Hyperbola with center at origin}$$

Since $a^2 = 16$ and $b^2 = 9$, we have $a = 4$, $b = 3$, and $c = \sqrt{a^2 + b^2} = \sqrt{16 + 9} = 5$. Thus, the foci lie 5 units to the left and to the right of the center, and the vertices lie 4 units to either side of the center.

FOCI $(-1, -1)$ and $(9, -1)$

VERTICES $(0, -1)$ and $(8, -1)$

The asymptotes of the unshifted hyperbola are $y = \pm\frac{3}{4}x$, so the asymptotes of the shifted hyperbola are found as follows.

$$\textbf{ASYMPTOTES} \quad y + 1 = \pm\tfrac{3}{4}(x - 4)$$
$$y + 1 = \pm\tfrac{3}{4}x \mp 3$$
$$y = \tfrac{3}{4}x - 4 \qquad \text{and} \qquad y = -\tfrac{3}{4}x + 2$$

To help us sketch the hyperbola, we draw the central box; it extends 4 units left and right from the center and 3 units upward and downward from the center. We then draw the asymptotes and complete the graph of the shifted hyperbola as shown in Figure 6(a).

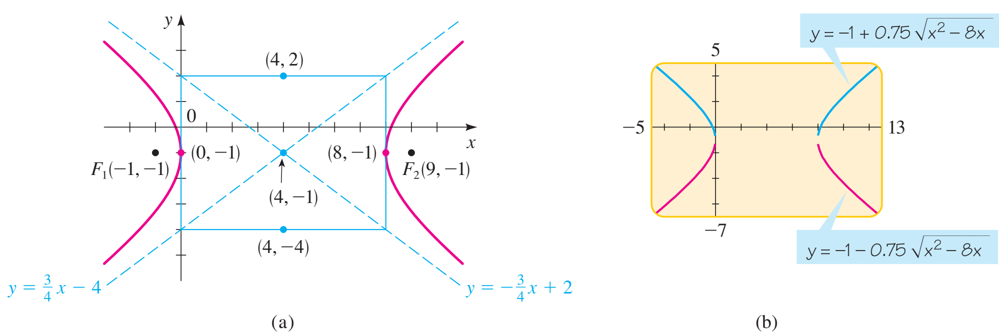
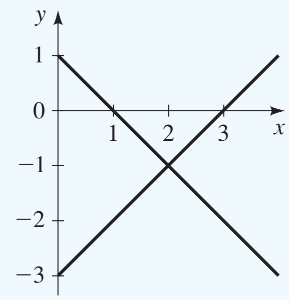


Figure 6
$9x^2 - 72x - 16y^2 - 32y = 16$

(c) To draw the graph using a graphing calculator, we need to solve for y. The given equation is a quadratic equation in y, so we use the quadratic formula to solve for y. Writing the equation in the form

$$16y^2 + 32y - 9x^2 + 72x + 16 = 0$$

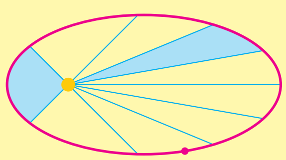
EXAMPLE
A degenerate hyperbola:
$x^2 - y^2 - 4x - 2y + 3 = 0$
$\Leftrightarrow (x - 2)^2 - (y + 1)^2 = 0$



Johannes Kepler (1571–1630) was the first to give a correct description of the motion of the planets. The cosmology of his time postulated complicated systems of circles moving on circles to describe these motions. Kepler sought a simpler and more harmonious description. As the official astronomer at the imperial court in Prague, he studied the astronomical observations of the Danish astronomer Tycho Brahe, whose data were the most accurate available at the time. After numerous attempts to find a theory, Kepler made the momentous discovery that the orbits of the planets are elliptical. His three great laws of planetary motion are

1. The orbit of each planet is an ellipse with the sun at one focus.
2. The line segment that joins the sun to a planet sweeps out equal areas in equal time (see the figure).
3. The square of the period of revolution of a planet is proportional to the cube of the length of the major axis of its orbit.

His formulation of these laws is perhaps the most impressive deduction from empirical data in the history of science.



we get

$$y = \frac{-32 \pm \sqrt{32^2 - 4(16)(-9x^2 + 72x + 16)}}{2(16)} \qquad \text{Quadratic formula}$$

$$= \frac{-32 \pm \sqrt{576x^2 - 4608x}}{32} \qquad \text{Expand}$$

$$= \frac{-32 \pm 24\sqrt{x^2 - 8x}}{32} \qquad \text{Factor 576 from under the radical}$$

$$= -1 \pm \tfrac{3}{4}\sqrt{x^2 - 8x} \qquad \text{Simplify}$$

To obtain the graph of the hyperbola, we graph the functions

$$y = -1 + 0.75\sqrt{x^2 - 8x} \quad \text{and} \quad y = -1 - 0.75\sqrt{x^2 - 8x}$$

as shown in Figure 6(b). ■

The General Equation of a Shifted Conic

If we expand and simplify the equations of any of the shifted conics illustrated in Figures 1, 3, and 5, then we will always obtain an equation of the form

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

where A and C are not both 0. Conversely, if we begin with an equation of this form, then we can complete the square in x and y to see which type of conic section the equation represents. In some cases, the graph of the equation turns out to be just a pair of lines, a single point, or there may be no graph at all. These cases are called **degenerate conics**. If the equation is not degenerate, then we can tell whether it represents a parabola, an ellipse, or a hyperbola simply by examining the signs of A and C, as described in the following box.

General Equation of a Shifted Conic

The graph of the equation

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

where A and C are not both 0, is a conic or a degenerate conic. In the nondegenerate cases, the graph is

1. a parabola if A or C is 0

2. an ellipse if A and C have the same sign (or a circle if $A = C$)

3. a hyperbola if A and C have opposite signs

Example 4 An Equation That Leads to a Degenerate Conic

Sketch the graph of the equation

$$9x^2 - y^2 + 18x + 6y = 0$$

Solution Because the coefficients of x^2 and y^2 are of opposite sign, this equation looks as if it should represent a hyperbola (like the equation of Example 3). To see

ALTERNATE EXAMPLE 4
The graph of the equation $49x^2 - y^2 + 196x + 28y = 0$ is equivalent to a pair of intersecting lines. Find the equations of these lines.

ANSWER
$y = 7x + 28,\ y = -7x$

IN-CLASS MATERIALS

The text's guide to the general equation of a shifted conic is important, but students should not memorize it blindly. Point out that if they've learned the previous section, these rules of thumb should make perfect sense. Have them look at the three general formulas they have learned, generalizing them to allow for a shift:

$$y = 4p\,(x - h)^2$$

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

whether this is in fact the case, we complete the squares:

$$9(x^2 + 2x \quad\;) - (y^2 - 6y \quad\;) = 0 \qquad \text{Group terms and factor 9}$$
$$9(x^2 + 2x + 1) - (y^2 - 6y + 9) = 0 + 9 \cdot 1 - 9 \qquad \text{Complete the square}$$
$$9(x + 1)^2 - (y - 3)^2 = 0 \qquad \text{Factor}$$
$$(x + 1)^2 - \frac{(y - 3)^2}{9} = 0 \qquad \text{Divide by 9}$$

For this to fit the form of the equation of a hyperbola, we would need a nonzero constant to the right of the equal sign. In fact, further analysis shows that this is the equation of a pair of intersecting lines:

$$(y - 3)^2 = 9(x + 1)^2$$
$$y - 3 = \pm 3(x + 1) \qquad \text{Take square roots}$$
$$y = 3(x + 1) + 3 \quad \text{or} \quad y = -3(x + 1) + 3$$
$$y = 3x + 6 \qquad\qquad y = -3x$$

These lines are graphed in Figure 7. ■

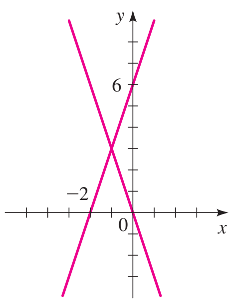


Figure 7
$9x^2 - y^2 + 18x + 6y = 0$

Because the equation in Example 4 looked at first glance like the equation of a hyperbola but, in fact, turned out to represent simply a pair of lines, we refer to its graph as a **degenerate hyperbola**. Degenerate ellipses and parabolas can also arise when we complete the square(s) in an equation that seems to represent a conic. For example, the equation

$$4x^2 + y^2 - 8x + 2y + 6 = 0$$

looks as if it should represent an ellipse, because the coefficients of x^2 and y^2 have the same sign. But completing the squares leads to

$$(x - 1)^2 + \frac{(y + 1)^2}{4} = -\frac{1}{4}$$

which has no solution at all (since the sum of two squares cannot be negative). This equation is therefore degenerate.

10.4 Exercises

1–4 ■ Find the center, foci, and vertices of the ellipse, and determine the lengths of the major and minor axes. Then sketch the graph.

1. $\dfrac{(x - 2)^2}{9} + \dfrac{(y - 1)^2}{4} = 1$ **2.** $\dfrac{(x - 3)^2}{16} + (y + 3)^2 = 1$

3. $\dfrac{x^2}{9} + \dfrac{(y + 5)^2}{25} = 1$ **4.** $\dfrac{(x + 2)^2}{4} + y^2 = 1$

5–8 ■ Find the vertex, focus, and directrix of the parabola, and sketch the graph.

5. $(x - 3)^2 = 8(y + 1)$ **6.** $(y + 5)^2 = -6x + 12$

7. $-4\left(x + \frac{1}{2}\right)^2 = y$ **8.** $y^2 = 16x - 8$

9–12 ■ Find the center, foci, vertices, and asymptotes of the hyperbola. Then sketch the graph.

9. $\dfrac{(x + 1)^2}{9} - \dfrac{(y - 3)^2}{16} = 1$

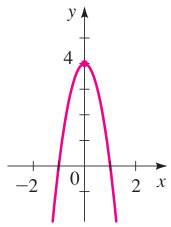
10. $(x - 8)^2 - (y + 6)^2 = 1$

11. $y^2 - \dfrac{(x + 1)^2}{4} = 1$

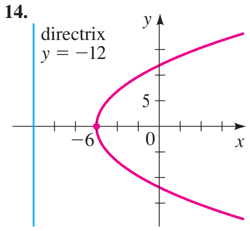
12. $\dfrac{(y - 1)^2}{25} - (x + 3)^2 = 1$

13–18 ■ Find an equation for the conic whose graph is shown.

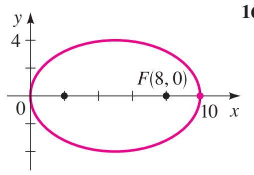
13.



14.



15.



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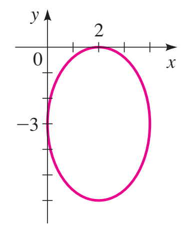

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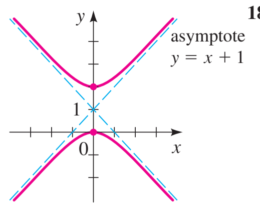

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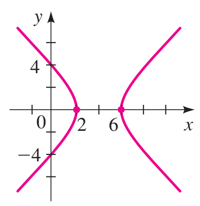


19–30 ■ Complete the square to determine whether the equation represents an ellipse, a parabola, a hyperbola, or a degenerate conic. If the graph is an ellipse, find the center, foci, vertices, and lengths of the major and minor axes. If it is a parabola, find the vertex, focus, and directrix. If it is a hyperbola, find the center, foci, vertices, and asymptotes. Then sketch the graph of the equation. If the equation has no graph, explain why.

19. $9x^2 - 36x + 4y^2 = 0$

20. $y^2 = 4(x + 2y)$

21. $x^2 - 4y^2 - 2x + 16y = 20$

22. $x^2 + 6x + 12y + 9 = 0$

23. $4x^2 + 25y^2 - 24x + 250y + 561 = 0$

24. $2x^2 + y^2 = 2y + 1$

25. $16x^2 - 9y^2 - 96x + 288 = 0$

26. $4x^2 - 4x - 8y + 9 = 0$

27. $x^2 + 16 = 4(y^2 + 2x)$

28. $x^2 - y^2 = 10(x - y) + 1$

29. $3x^2 + 4y^2 - 6x - 24y + 39 = 0$

30. $x^2 + 4y^2 + 20x - 40y + 300 = 0$

31–34 ■ Use a graphing device to graph the conic.

31. $2x^2 - 4x + y + 5 = 0$

32. $4x^2 + 9y^2 - 36y = 0$

33. $9x^2 + 36 = y^2 + 36x + 6y$

34. $x^2 - 4y^2 + 4x + 8y = 0$

35. Determine what the value of F must be if the graph of the equation

$$4x^2 + y^2 + 4(x - 2y) + F = 0$$

is **(a)** an ellipse, **(b)** a single point, or **(c)** the empty set.

36. Find an equation for the ellipse that shares a vertex and a focus with the parabola $x^2 + y = 100$ and has its other focus at the origin.

37. This exercise deals with **confocal parabolas**, that is, families of parabolas that have the same focus.

(a) Draw graphs of the family of parabolas

$$x^2 = 4p(y + p)$$

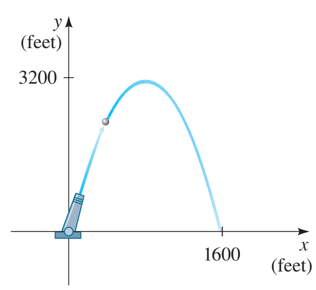
for $p = -2, -\frac{3}{2}, -1, -\frac{1}{2}, \frac{1}{2}, 1, \frac{3}{2}, 2$.

(b) Show that each parabola in this family has its focus at the origin.

(c) Describe the effect on the graph of moving the vertex closer to the origin.

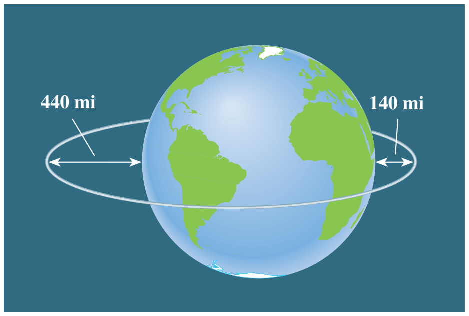
Applications

38. Path of a Cannonball A cannon fires a cannonball as shown in the figure. The path of the cannonball is a parabola with vertex at the highest point of the path. If the cannonball lands 1600 ft from the cannon and the highest point it reaches is 3200 ft above the ground, find an equation for the path of the cannonball. Place the origin at the location of the cannon.



39. Orbit of a Satellite A satellite is in an elliptical orbit around the earth with the center of the earth at one focus. The height of the satellite above the earth varies between 140 mi and 440 mi. Assume the earth is a sphere with radius

3960 mi. Find an equation for the path of the satellite with the origin at the center of the earth.



Discovery • Discussion

40. A Family of Confocal Conics Conics that share a focus are called **confocal**. Consider the family of conics that have a focus at $(0, 1)$ and a vertex at the origin (see the figure).

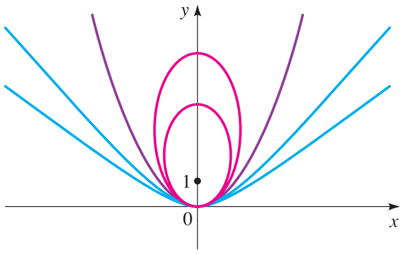
(a) Find equations of two different ellipses that have these properties.

(b) Find equations of two different hyperbolas that have these properties.

(c) Explain why only one parabola satisfies these properties. Find its equation.

(d) Sketch the conics you found in parts (a), (b), and (c) on the same coordinate axes (for the hyperbolas, sketch the top branches only).

(e) How are the ellipses and hyperbolas related to the parabola?



10.5 Rotation of Axes

In Section 10.4 we studied conics with equations of the form

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

We saw that the graph is always an ellipse, parabola, or hyperbola with horizontal or vertical axes (except in the degenerate cases). In this section we study the most general second-degree equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

We will see that the graph of an equation of this form is also a conic. In fact, by rotating the coordinate axes through an appropriate angle, we can eliminate the term Bxy and then use our knowledge of conic sections to analyze the graph.

Rotation of Axes

In Figure 1 the x- and y-axes have been rotated through an acute angle ϕ about the origin to produce a new pair of axes, which we call the X- and Y-axes. A point P that has coordinates (x, y) in the old system has coordinates (X, Y) in the new system. If we let r denote the distance of P from the origin and let θ be the angle that the segment OP makes with the new X-axis, then we can see from Figure 2 on the next page (by considering the two right triangles in the figure) that

$$X = r\cos\theta \qquad Y = r\sin\theta$$

$$x = r\cos(\theta + \phi) \qquad y = r\sin(\theta + \phi)$$

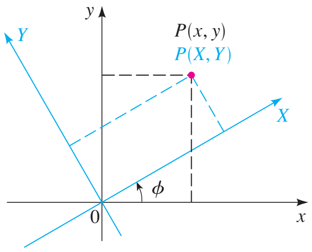


Figure 1

SUGGESTED TIME AND EMPHASIS

$\frac{1}{2}$–1 class.
Optional material.

POINTS TO STRESS

1. Rotation of axes formulas.
2. Graphing a general quadratic equation, including middle terms.

IN-CLASS MATERIALS

Point out the progression of the past few sections. Write out $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$. When the course started we studied lines, which was just the case where $A = B = C = 0$. When we discussed circles, that was the case where $A = C$, $B = D = E = 0$. Shifting the center of a circle allowed D and E to take on nonzero values. Our study of conic sections removed the restriction that $A = C$, so this section really is the last step in a logical progression.

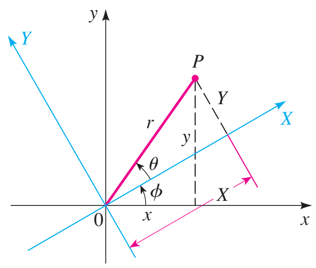


Figure 2

Using the addition formula for cosine, we see that

$$\begin{aligned} x &= r\cos(\theta + \phi) \\ &= r(\cos\theta\cos\phi - \sin\theta\sin\phi) \\ &= (r\cos\theta)\cos\phi - (r\sin\theta)\sin\phi \\ &= X\cos\phi - Y\sin\phi \end{aligned}$$

Similarly, we can apply the addition formula for sine to the expression for y to obtain $y = X\sin\phi + Y\cos\phi$. By treating these equations for x and y as a system of linear equations in the variables X and Y (see Exercise 33), we obtain expressions for X and Y in terms of x and y, as detailed in the following box.

Rotation of Axes Formulas

Suppose the x- and y-axes in a coordinate plane are rotated through the acute angle ϕ to produce the X- and Y-axes, as shown in Figure 1. Then the coordinates (x, y) and (X, Y) of a point in the xy- and the XY-planes are related as follows:

$$x = X\cos\phi - Y\sin\phi \qquad X = x\cos\phi + y\sin\phi$$

$$y = X\sin\phi + Y\cos\phi \qquad Y = -x\sin\phi + y\cos\phi$$

ALTERNATE EXAMPLE 1

If the coordinate axes are rotated through an angle of $\frac{\pi}{3}$, find the XY coordinates of the point with xy coordinates (1, 4).

ANSWER

$\left(\dfrac{1 + 4\sqrt{3}}{2}, \dfrac{4 - \sqrt{3}}{2}\right)$

Example 1 Rotation of Axes

If the coordinate axes are rotated through 30°, find the XY-coordinates of the point with xy-coordinates $(2, -4)$.

Solution Using the Rotation of Axes Formulas with $x = 2$, $y = -4$, and $\phi = 30°$, we get

$$X = 2\cos 30° + (-4)\sin 30° = 2\left(\frac{\sqrt{3}}{2}\right) - 4\left(\frac{1}{2}\right) = \sqrt{3} - 2$$

$$Y = -2\sin 30° + (-4)\cos 30° = -2\left(\frac{1}{2}\right) - 4\left(\frac{\sqrt{3}}{2}\right) = -1 - 2\sqrt{3}$$

The XY-coordinates are $(-2 + \sqrt{3}, -1 - 2\sqrt{3})$. ■

ALTERNATE EXAMPLE 2

Rotate the coordinate axes through 45° to show that $xy = 4$ is a hyperbola.

ANSWER

Rotation as in the text gives $\dfrac{X^2}{8} - \dfrac{Y^2}{8} = 1$.

Example 2 Rotating a Hyperbola

Rotate the coordinate axes through 45° to show that the graph of the equation $xy = 2$ is a hyperbola.

Solution We use the Rotation of Axes Formulas with $\phi = 45°$ to obtain

$$x = X\cos 45° - Y\sin 45° = \frac{X}{\sqrt{2}} - \frac{Y}{\sqrt{2}}$$

$$y = X\sin 45° + Y\cos 45° = \frac{X}{\sqrt{2}} + \frac{Y}{\sqrt{2}}$$

Substituting these expressions into the original equation gives

$$\left(\frac{X}{\sqrt{2}} - \frac{Y}{\sqrt{2}}\right)\left(\frac{X}{\sqrt{2}} + \frac{Y}{\sqrt{2}}\right) = 2$$

$$\frac{X^2}{2} - \frac{Y^2}{2} = 2$$

$$\frac{X^2}{4} - \frac{Y^2}{4} = 1$$

We recognize this as a hyperbola with vertices $(\pm 2, 0)$ in the XY-coordinate system. Its asymptotes are $Y = \pm X$, which correspond to the coordinate axes in the xy-system (see Figure 3).

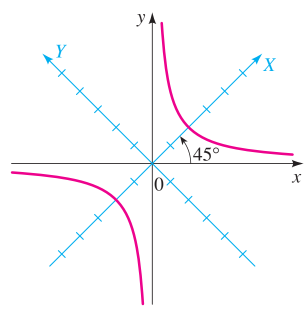


Figure 3
$xy = 2$

■

General Equation of a Conic

The method of Example 2 can be used to transform any equation of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

into an equation in X and Y that doesn't contain an XY-term by choosing an appropriate angle of rotation. To find the angle that works, we rotate the axes through an angle ϕ and substitute for x and y using the Rotation of Axes Formulas:

$$\begin{aligned} A(X\cos\phi - Y\sin\phi)^2 &+ B(X\cos\phi - Y\sin\phi)(X\sin\phi + Y\cos\phi) \\ &+ C(X\sin\phi + Y\cos\phi)^2 + D(X\cos\phi - Y\sin\phi) \\ &+ E(X\sin\phi + Y\cos\phi) + F = 0 \end{aligned}$$

If we expand this and collect like terms, we obtain an equation of the form

$$A'X^2 + B'XY + C'Y^2 + D'X + E'Y + F' = 0$$

where

$$\begin{aligned} A' &= A\cos^2\phi + B\sin\phi\cos\phi + C\sin^2\phi \\ B' &= 2(C - A)\sin\phi\cos\phi + B(\cos^2\phi - \sin^2\phi) \\ C' &= A\sin^2\phi - B\sin\phi\cos\phi + C\cos^2\phi \end{aligned}$$

$$D' = D\cos\phi + E\sin\phi$$
$$E' = -D\sin\phi + E\cos\phi$$
$$F' = F$$

To eliminate the XY-term, we would like to choose ϕ so that $B' = 0$, that is,

Double-angle formulas
$\sin 2\phi = 2\sin\phi\cos\phi$
$\cos 2\phi = \cos^2\phi - \sin^2\phi$

$$2(C - A)\sin\phi\cos\phi + B(\cos^2\phi - \sin^2\phi) = 0 \qquad \text{Double-angle formulas for sine and cosine}$$
$$(C - A)\sin 2\phi + B\cos 2\phi = 0$$
$$B\cos 2\phi = (A - C)\sin 2\phi$$
$$\cot 2\phi = \frac{A - C}{B} \qquad \text{Divide by } B\sin 2\phi$$

The preceding calculation proves the following theorem.

Simplifying the General Conic Equation

To eliminate the xy-term in the general conic equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

rotate the axes through the acute angle ϕ that satisfies

$$\cot 2\phi = \frac{A - C}{B}$$

ALTERNATE EXAMPLE 3
Use rotation of axes to eliminate the xy-term in the equation $7x^2 - 6\sqrt{3}xy + 13y^2 = 16$.

ANSWER
Rotation angle:

$$\cot 2\phi = \frac{1}{\sqrt{3}}$$
$$\phi = \frac{\pi}{6}$$

Transformation equations:

$$x = \frac{\sqrt{3}}{2}X - \frac{1}{2}Y$$
$$y = \frac{1}{2}X + \frac{\sqrt{3}}{2}Y$$

New equation:
$4X^2 + 16Y^2 = 16$

Example 3 Eliminating the xy-Term

Use a rotation of axes to eliminate the xy-term in the equation

$$6\sqrt{3}x^2 + 6xy + 4\sqrt{3}y^2 = 21\sqrt{3}$$

Identify and sketch the curve.

Solution To eliminate the xy-term, we rotate the axes through an angle ϕ that satisfies

$$\cot 2\phi = \frac{A - C}{B} = \frac{6\sqrt{3} - 4\sqrt{3}}{6} = \frac{\sqrt{3}}{3}$$

Thus, $2\phi = 60°$ and hence $\phi = 30°$. With this value of ϕ, we get

$$x = X\left(\frac{\sqrt{3}}{2}\right) - Y\left(\frac{1}{2}\right) \qquad \text{Rotation of Axes Formulas}$$
$$y = X\left(\frac{1}{2}\right) + Y\left(\frac{\sqrt{3}}{2}\right) \qquad \cos\phi = \frac{\sqrt{3}}{2},\ \sin\phi = \frac{1}{2}$$

Substituting these values for x and y into the given equation leads to

$$6\sqrt{3}\left(\frac{X\sqrt{3}}{2} - \frac{Y}{2}\right)^2 + 6\left(\frac{X\sqrt{3}}{2} - \frac{Y}{2}\right)\left(\frac{X}{2} + \frac{Y\sqrt{3}}{2}\right) + 4\sqrt{3}\left(\frac{X}{2} + \frac{Y\sqrt{3}}{2}\right)^2 = 21\sqrt{3}$$

IN-CLASS MATERIALS

This is a very good section to teach with the aid of a CAS, or even a good place to introduce such a device. A great deal of the work in graphing an arbitrary conic section is multiplying out long algebraic expressions, which a CAS can do easily.

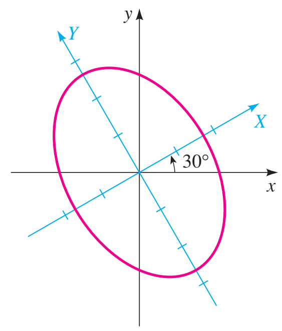


Figure 4
$6\sqrt{3}x^2 + 6xy + 4\sqrt{3}y^2 = 21\sqrt{3}$

Expanding and collecting like terms, we get

$$7\sqrt{3}X^2 + 3\sqrt{3}Y^2 = 21\sqrt{3}$$

$$\frac{X^2}{3} + \frac{Y^2}{7} = 1 \qquad \text{Divide by } 21\sqrt{3}$$

This is the equation of an ellipse in the XY-coordinate system. The foci lie on the Y-axis. Because $a^2 = 7$ and $b^2 = 3$, the length of the major axis is $2\sqrt{7}$, and the length of the minor axis is $2\sqrt{3}$. The ellipse is sketched in Figure 4. ■

In the preceding example we were able to determine ϕ without difficulty, since we remembered that $\cot 60° = \sqrt{3}/3$. In general, finding ϕ is not quite so easy. The next example illustrates how the following half-angle formulas, which are valid for $0 < \phi < \pi/2$, are useful in determining ϕ (see Section 7.3):

$$\cos\phi = \sqrt{\frac{1 + \cos 2\phi}{2}} \qquad \sin\phi = \sqrt{\frac{1 - \cos 2\phi}{2}}$$

Example 4 Graphing a Rotated Conic

A conic has the equation

$$64x^2 + 96xy + 36y^2 - 15x + 20y - 25 = 0$$

(a) Use a rotation of axes to eliminate the xy-term.

(b) Identify and sketch the graph.

(c) Draw the graph using a graphing calculator.

Solution

(a) To eliminate the xy-term, we rotate the axes through an angle ϕ that satisfies

$$\cot 2\phi = \frac{A - C}{B} = \frac{64 - 36}{96} = \frac{7}{24}$$

In Figure 5 we sketch a triangle with $\cot 2\phi = \frac{7}{24}$. We see that

$$\cos 2\phi = \tfrac{7}{25}$$

so, using the half-angle formulas, we get

$$\cos\phi = \sqrt{\frac{1 + \frac{7}{25}}{2}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$$

$$\sin\phi = \sqrt{\frac{1 - \frac{7}{25}}{2}} = \sqrt{\frac{9}{25}} = \frac{3}{5}$$

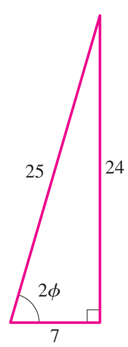


Figure 5

The Rotation of Axes Formulas then give

$$x = \tfrac{4}{5}X - \tfrac{3}{5}Y \qquad \text{and} \qquad y = \tfrac{3}{5}X + \tfrac{4}{5}Y$$

Substituting into the given equation, we have

$$64\left(\tfrac{4}{5}X - \tfrac{3}{5}Y\right)^2 + 96\left(\tfrac{4}{5}X - \tfrac{3}{5}Y\right)\left(\tfrac{3}{5}X + \tfrac{4}{5}Y\right)$$
$$+ 36\left(\tfrac{3}{5}X + \tfrac{4}{5}Y\right)^2 - 15\left(\tfrac{4}{5}X - \tfrac{3}{5}Y\right) + 20\left(\tfrac{3}{5}X + \tfrac{4}{5}Y\right) - 25 = 0$$

ALTERNATE EXAMPLE 4

A conic has the equation

$7x^2 - 6\sqrt{3}xy + 13y^2 = 16.$

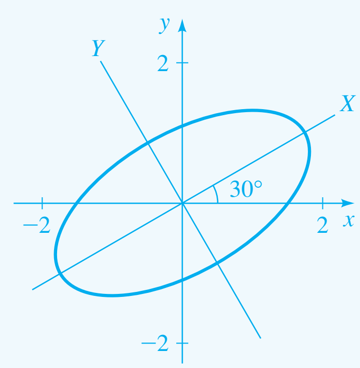
(a) Identify and sketch the graph.

(b) Draw the graph using a graphing device.

ANSWERS

(a) As shown in Alternate Example 3, this is a conic rotated by 30º with equation $4X^2 + 16Y^2 = 16$. The sketch will be the ellipse $\frac{x^2}{4} + y^2 = 1$ with axes rotated by 30º.

(b)



Expanding and collecting like terms, we get

$$100X^2 + 25Y - 25 = 0$$

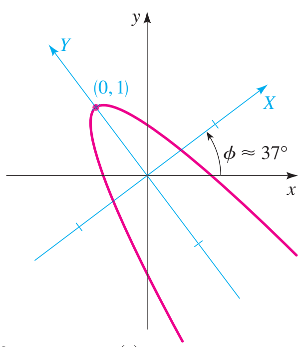
$$-4X^2 = Y - 1 \qquad \text{Simplify}$$

$$X^2 = -\tfrac{1}{4}(Y - 1) \qquad \text{Divide by 4}$$

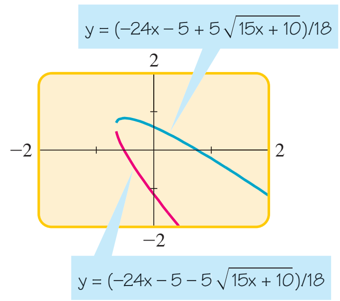
(b) We recognize this as the equation of a parabola that opens along the negative Y-axis and has vertex $(0, 1)$ in XY-coordinates. Since $4p = -\frac{1}{4}$, we have $p = -\frac{1}{16}$, so the focus is $\left(0, \frac{15}{16}\right)$ and the directrix is $Y = \frac{17}{16}$. Using

$$\phi = \cos^{-1}\tfrac{4}{5} \approx 37°$$

we sketch the graph in Figure 6(a).



(a)



(b)

Figure 6

$64x^2 + 96xy + 36y^2 - 15x + 20y - 25 = 0$

(c) To draw the graph using a graphing calculator, we need to solve for y. The given equation is a quadratic equation in y, so we can use the quadratic formula to solve for y. Writing the equation in the form

$$36y^2 + (96x + 20)y + (64x^2 - 15x - 25) = 0$$

we get

$$y = \frac{-(96x + 20) \pm \sqrt{(96x + 20)^2 - 4(36)(64x^2 - 15x - 25)}}{2(36)} \qquad \text{Quadratic formula}$$

$$= \frac{-(96x + 20) \pm \sqrt{6000x + 4000}}{72} \qquad \text{Expand}$$

$$= \frac{-96x - 20 \pm 20\sqrt{15x + 10}}{72} \qquad \text{Simplify}$$

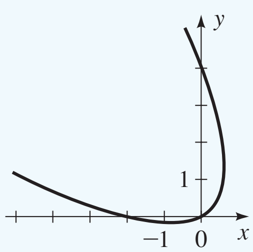
$$= \frac{-24x - 5 \pm 5\sqrt{15x + 10}}{18} \qquad \text{Simplify}$$

EXAMPLE

The rotated parabola $x^2 + 2xy + y^2 + 2x - 4y = 0$: $\cot(2\phi) = 0 \Rightarrow 2\phi = \dfrac{\pi}{2} \Rightarrow \phi = \pi$. Now we have $x = \sqrt{2}X - \sqrt{2}Y,\ y = \sqrt{2}X + \sqrt{2}Y$.

After substituting, and a lot of algebra, we get $8x^2 - 2\sqrt{2}x - 6\sqrt{2}y = 0$. Completing the square,

$$y = \frac{2\sqrt{2}}{3}\left(x - \frac{\sqrt{2}}{8}\right)^2 - \frac{\sqrt{2}}{48} \approx 0.9(x - 0.2)^2 \text{ (for sketching purposes).}$$



To obtain the graph of the parabola, we graph the functions

$$y = \left(-24x - 5 + 5\sqrt{15x + 10}\right)/18 \qquad \text{and} \qquad y = \left(-24x - 5 - 5\sqrt{15x + 10}\right)/18$$

as shown in Figure 6(b). ■

The Discriminant

In Examples 3 and 4 we were able to identify the type of conic by rotating the axes. The next theorem gives rules for identifying the type of conic directly from the equation, without rotating axes.

Identifying Conics by the Discriminant

The graph of the equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

is either a conic or a degenerate conic. In the nondegenerate cases, the graph is

1. a parabola if $B^2 - 4AC = 0$
2. an ellipse if $B^2 - 4AC < 0$
3. a hyperbola if $B^2 - 4AC > 0$

The quantity $B^2 - 4AC$ is called the **discriminant** of the equation.

■ **Proof** If we rotate the axes through an angle ϕ, we get an equation of the form

$$A'X^2 + B'XY + C'Y^2 + D'X + E'Y + F' = 0$$

where $A', B', C', \ldots$ are given by the formulas on pages 785–786. A straightforward calculation shows that

$$(B')^2 - 4A'C' = B^2 - 4AC$$

Thus, the expression $B^2 - 4AC$ remains unchanged for any rotation. In particular, if we choose a rotation that eliminates the xy-term $(B' = 0)$, we get

$$A'X^2 + C'Y^2 + D'X + E'Y + F' = 0$$

In this case, $B^2 - 4AC = -4A'C'$. So $B^2 - 4AC = 0$ if either A' or C' is zero; $B^2 - 4AC < 0$ if A' and C' have the same sign; and $B^2 - 4AC > 0$ if A' and C' have opposite signs. According to the box on page 780, these cases correspond to the graph of the last displayed equation being a parabola, an ellipse, or a hyperbola, respectively. ■

In the proof we indicated that the discriminant is unchanged by any rotation; for this reason, the discriminant is said to be **invariant** under rotation.

Example 5 Identifying a Conic by the Discriminant

A conic has the equation

$$3x^2 + 5xy - 2y^2 + x - y + 4 = 0$$

(a) Use the discriminant to identify the conic.

(b) Confirm your answer to part (a) by graphing the conic with a graphing calculator.

Solution

(a) Since $A = 3$, $B = 5$, and $C = -2$, the discriminant is

$$B^2 - 4AC = 5^2 - 4(3)(-2) = 49 > 0$$

So the conic is a hyperbola.

(b) Using the quadratic formula, we solve for y to get

$$y = \frac{5x - 1 \pm \sqrt{49x^2 - 2x + 33}}{4}$$

We graph these functions in Figure 7. The graph confirms that this is a hyperbola. ■

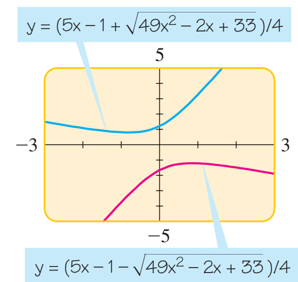


Figure 7

10.5 Exercises

1–6 ■ Determine the XY-coordinates of the given point if the coordinate axes are rotated through the indicated angle.

1. $(1, 1)$, $\phi = 45°$
2. $(-2, 1)$, $\phi = 30°$
3. $(3, -\sqrt{3})$, $\phi = 60°$
4. $(2, 0)$, $\phi = 15°$
5. $(0, 2)$, $\phi = 55°$
6. $(\sqrt{2}, 4\sqrt{2})$, $\phi = 45°$

7–12 ■ Determine the equation of the given conic in XY-coordinates when the coordinate axes are rotated through the indicated angle.

7. $x^2 - 3y^2 = 4$, $\phi = 60°$
8. $y = (x - 1)^2$, $\phi = 45°$
9. $x^2 - y^2 = 2y$, $\phi = \cos^{-1}\frac{3}{5}$
10. $x^2 + 2y^2 = 16$, $\phi = \sin^{-1}\frac{3}{5}$
11. $x^2 + 2\sqrt{3}xy - y^2 = 4$, $\phi = 30°$
12. $xy = x + y$, $\phi = \pi/4$

ALTERNATE EXAMPLE 5

A conic has the equation $17x^2 - 48xy + 31y^2 + 49 = 0$.

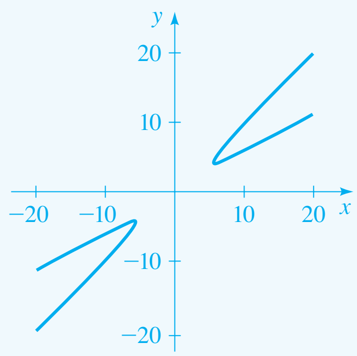
(a) Use the discriminant to identify the conic.

(b) Confirm your answer to part (a) by graphing the conic with a graphing calculator.

ANSWERS

(a) $B^2 - 4AC = 196 > 0$, so the conic is a hyperbola.

(b)



SAMPLE QUESTION

Text Question

If you are presented with the equation of a conic such as $3x^2 + 5xy - 2y^2 + x - y + 4 = 0$, how would you go about finding if it is an ellipse, parabola, or hyperbola?

Answer

Evaluate the discriminant.

DRILL QUESTION

Consider the equation $3x^2 + 5xy - 2y^2 + x - y + 4 = 0$. What shape is its graph?

Answer

It is a hyperbola.

13–26 ■ **(a)** Use the discriminant to determine whether the graph of the equation is a parabola, an ellipse, or a hyperbola. **(b)** Use a rotation of axes to eliminate the xy-term. **(c)** Sketch the graph.

13. $xy = 8$

14. $xy + 4 = 0$

15. $x^2 + 2xy + y^2 + x - y = 0$

16. $13x^2 + 6\sqrt{3}xy + 7y^2 = 16$

17. $x^2 + 2\sqrt{3}xy - y^2 + 2 = 0$

18. $21x^2 + 10\sqrt{3}xy + 31y^2 = 144$

19. $11x^2 - 24xy + 4y^2 + 20 = 0$

20. $25x^2 - 120xy + 144y^2 - 156x - 65y = 0$

21. $\sqrt{3}x^2 + 3xy = 3$

22. $153x^2 + 192xy + 97y^2 = 225$

23. $2\sqrt{3}x^2 - 6xy + \sqrt{3}x + 3y = 0$

24. $9x^2 - 24xy + 16y^2 = 100(x - y - 1)$

25. $52x^2 + 72xy + 73y^2 = 40x - 30y + 75$

26. $(7x + 24y)^2 = 600x - 175y + 25$

27–30 ■ **(a)** Use the discriminant to identify the conic. **(b)** Confirm your answer by graphing the conic using a graphing device.

27. $2x^2 - 4xy + 2y^2 - 5x - 5 = 0$

28. $x^2 - 2xy + 3y^2 = 8$

29. $6x^2 + 10xy + 3y^2 - 6y = 36$

30. $9x^2 - 6xy + y^2 + 6x - 2y = 0$

31. **(a)** Use rotation of axes to show that the following equation represents a hyperbola:

$$7x^2 + 48xy - 7y^2 - 200x - 150y + 600 = 0$$

(b) Find the XY- and xy-coordinates of the center, vertices, and foci.

(c) Find the equations of the asymptotes in XY- and xy-coordinates.

32. **(a)** Use rotation of axes to show that the following equation represents a parabola:

$$2\sqrt{2}(x + y)^2 = 7x + 9y$$

(b) Find the XY- and xy-coordinates of the vertex and focus.

(c) Find the equation of the directrix in XY- and xy-coordinates.

33. Solve the equations:

$$x = X\cos\phi - Y\sin\phi$$
$$y = X\sin\phi + Y\cos\phi$$

for X and Y in terms of x and y. [*Hint:* To begin, multiply the first equation by $\cos\phi$ and the second by $\sin\phi$, and then add the two equations to solve for X.]

34. Show that the graph of the equation

$$\sqrt{x} + \sqrt{y} = 1$$

is part of a parabola by rotating the axes through an angle of 45°. [*Hint:* First convert the equation to one that does not involve radicals.]

Discovery • Discussion

35. Matrix Form of Rotation of Axes Formulas
Let Z, Z', and R be the matrices

$$Z = \begin{bmatrix} x \\ y \end{bmatrix} \qquad Z' = \begin{bmatrix} X \\ Y \end{bmatrix}$$

$$R = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

Show that the Rotation of Axes Formulas can be written as

$$Z = RZ' \qquad \text{and} \qquad Z' = R^{-1}Z$$

36. Algebraic Invariants A quantity is invariant under rotation if it does not change when the axes are rotated. It was stated in the text that for the general equation of a conic, the quantity $B^2 - 4AC$ is invariant under rotation.

(a) Use the formulas for A', B', and C' on page 785 to prove that the quantity $B^2 - 4AC$ is invariant under rotation; that is, show that

$$B^2 - 4AC = B'^2 - 4A'C'$$

(b) Prove that $A + C$ is invariant under rotation.

(c) Is the quantity F invariant under rotation?

37. Geometric Invariants Do you expect that the distance between two points is invariant under rotation? Prove your answer by comparing the distance $d(P, Q)$ and $d(P', Q')$ where P' and Q' are the images of P and Q under a rotation of axes.

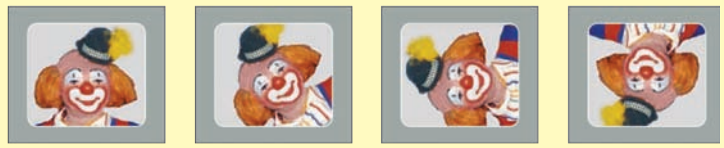
IN-CLASS MATERIALS

If the students are going to be moving on to linear algebra in the future, Exercise 35 is an excellent one to go over in class. It addresses the idea of linear transformations, which will be a major topic in that course.

DISCOVERY PROJECT

Computer Graphics II

In the *Discovery Project* on page 700 we saw how matrix multiplication is used in computer graphics. We found matrices that reflect, expand, or shear an image. We now consider matrices that rotate an image, as in the graphics shown here.



Rotating Points in the Plane

Recall that a point (x, y) in the plane is represented by the 2×1 matrix $\begin{bmatrix} x \\ y \end{bmatrix}$. The matrix that rotates this point about the origin through an angle ϕ is

$$R = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \qquad \text{Rotation matrix}$$

Compare this matrix with the rotation of axes matrix in Exercise 35, Section 10.5. Note that rotating a point counterclockwise corresponds to rotating the axes clockwise.

When the point $P = \begin{bmatrix} x \\ y \end{bmatrix}$ is rotated clockwise about the origin through an angle ϕ, it moves to a new location $P' = \begin{bmatrix} x' \\ y' \end{bmatrix}$ given by the matrix product $P' = RP$, as shown in Figure 1.

$$P' = RP = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos\phi - y\sin\phi \\ x\sin\phi + y\cos\phi \end{bmatrix}$$

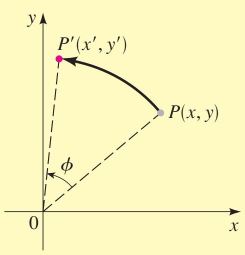


Figure 1

For example, if $\phi = 90°$, the rotation matrix is

$$R = \begin{bmatrix} \cos 90° & -\sin 90° \\ \sin 90° & \cos 90° \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \text{Rotation matrix } (\phi = 90°)$$

Applying a 90° rotation to the point $P = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ moves it to the point

$$P' = RP = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

See Figure 2.

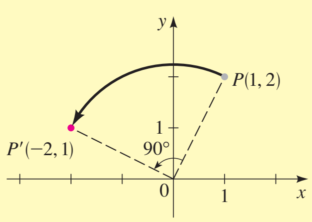


Figure 2

Rotating Images in the Plane

If the rotation matrix is applied to every point in an image, then the entire image is rotated. To rotate the house in Figure 3(a) through a 30° angle about the origin, we multiply its data matrix (described on page 701) by the rotation matrix that has $\phi = 30°$.

$$RD = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 & 2 & 4 & 4 & 3 & 3 & 2 & 2 & 3 \\ 0 & 0 & 3 & 5 & 3 & 0 & 0 & 2 & 2 & 0 & 0 \end{bmatrix}$$

$$\approx \begin{bmatrix} 1.73 & 0 & -1.50 & -0.77 & 1.96 & 3.46 & 2.60 & 1.60 & 0.73 & 1.73 & 2.60 \\ 1 & 0 & 2.60 & 5.33 & 4.60 & 2 & 1.50 & 3.23 & 2.73 & 1 & 1.50 \end{bmatrix}$$

The new data matrix RD represents the rotated house in Figure 3(b).

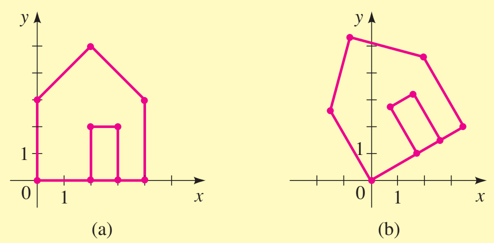


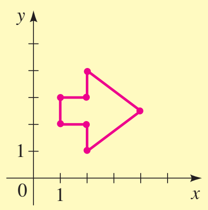
Figure 3

The *Discovery Project* on page 702 describes a TI-83 program that draws the image corresponding to a given data matrix. You may find it convenient to use this program in some of the following activities.

1. Use a rotation matrix to find the new coordinates of the given point when it is rotated through the given angle.

(a) $(1, 4)$, $\phi = 90°$ (b) $(-2, 1)$, $\phi = 60°$

(c) $(-2, -2)$, $\phi = 135°$ (d) $(7, 3)$, $\phi = -60°$



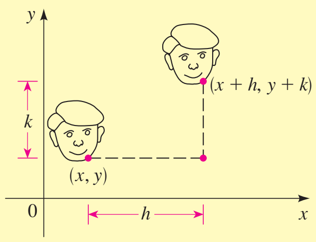
2. Find a data matrix for the line drawing in the figure shown in the margin. Multiply the data matrix by a suitable rotation matrix to rotate the image about the origin by $\phi = 120°$. Sketch the rotated image given by the new data matrix.

3. Sketch the image represented by the data matrix D.

$$D = \begin{bmatrix} 2 & 3 & 3 & 4 & 4 & 1 & 1 & 2 & 2 \\ 1 & 1 & 3 & 3 & 4 & 4 & 3 & 3 & 1 \end{bmatrix}$$

Find the rotation matrix R that corresponds to a 45° rotation, and the transformation matrix T that corresponds to an expansion by a factor of 2 in the x-direction (see page 701). How does multiplying the data matrix by RT change the image? How about multiplying by TR? Calculate the products RTD and TRD, and sketch the corresponding images to confirm your answers.

4. Let R be the rotation matrix for the angle ϕ. Show that R^{-1} is the rotation matrix for the angle $-\phi$.



5. To **translate** an image by (h, k), we add h to each x-coordinate and k to each y-coordinate of each point in the image (see the figure in the margin). This can be done by adding an appropriate matrix M to D, but the dimension of M would change depending on the dimension of D. In practice, translation is accomplished by matrix multiplication. To see how this is done, we introduce **homogeneous coordinates**; that is, we represent the point (x, y) by a 3×1 matrix:

$$(x, y) \quad \leftrightarrow \quad \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

(a) Let T be the matrix

$$T = \begin{bmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix}$$

Show that T translates the point (x, y) to the point $(x + h, y + h)$ by verifying the following matrix multiplication.

$$\begin{bmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + h \\ y + k \\ 1 \end{bmatrix}$$

(b) Find T^{-1} and describe how T^{-1} translates points.

(c) Verify that multiplying by the following matrices has the indicated effects on a point (x, y) represented by its homogeneous coordinates $\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} c & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & c & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Reflection in x-axis	Expansion (or contraction) in x-direction	Shear in x-direction	Rotation about the origin by the angle ϕ

(d) Sketch the image represented (in homogeneous coordinates) by this data matrix:

$$D = \begin{bmatrix} 3 & 5 & 5 & 7 & 7 & 9 & 9 & 7 & 7 & 5 & 5 & 3 & 3 \\ 7 & 7 & 5 & 5 & 7 & 7 & 9 & 9 & 11 & 11 & 9 & 9 & 7 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Find a matrix T that translates the image by $(-6, -8)$ and a matrix R that rotates the image by 45°. Sketch the images represented by the data matrices TD, RTD, and $T^{-1}RTD$. Describe how an image is changed when its data matrix is multiplied by T, by RT, and by $T^{-1}RT$.

10.6 Polar Equations of Conics

Earlier in this chapter we defined a parabola in terms of a focus and directrix, but we defined the ellipse and hyperbola in terms of two foci. In this section we give a more unified treatment of all three types of conics in terms of a focus and directrix. If we place the focus at the origin, then a conic section has a simple polar equation. Moreover, in polar form, rotation of conics becomes a simple matter. Polar equations of ellipses are crucial in the derivation of Kepler's Laws (see page 780).

Equivalent Description of Conics

Let F be a fixed point (the **focus**), ℓ a fixed line (the **directrix**), and e a fixed positive number (the **eccentricity**). The set of all points P such that the ratio of the distance from P to F to the distance from P to ℓ is the constant e is a conic. That is, the set of all points P such that

$$\frac{d(P, F)}{d(P, \ell)} = e$$

is a conic. The conic is a parabola if $e = 1$, an ellipse if $e < 1$, or a hyperbola if $e > 1$.

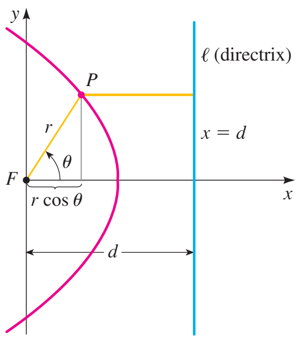


Figure 1

Proof If $e = 1$, then $d(P, F) = d(P, \ell)$, and so the given condition becomes the definition of a parabola as given in Section 10.1.

Now, suppose $e \neq 1$. Let's place the focus F at the origin and the directrix parallel to the y-axis and d units to the right. In this case the directrix has equation $x = d$ and is perpendicular to the polar axis. If the point P has polar coordinates (r, θ), we see from Figure 1 that $d(P, F) = r$ and $d(P, \ell) = d - r\cos\theta$. Thus, the condition $d(P, F)/d(P, \ell) = e$, or $d(P, F) = e \cdot d(P, \ell)$, becomes

$$r = e(d - r\cos\theta)$$

If we square both sides of this polar equation and convert to rectangular coordinates, we get

$$x^2 + y^2 = e^2(d - x)^2$$

$$(1 - e^2)x^2 + 2de^2x + y^2 = e^2d^2 \qquad \text{Expand and simplify}$$

$$\left(x + \frac{e^2d}{1 - e^2}\right)^2 + \frac{y^2}{1 - e^2} = \frac{e^2d^2}{(1 - e^2)^2} \qquad \text{Divide by } 1 - e^2 \text{ and complete the square}$$

If $e < 1$, then dividing both sides of this equation by $e^2d^2/(1 - e^2)^2$ gives an equation of the form

$$\frac{(x - h)^2}{a^2} + \frac{y^2}{b^2} = 1$$

where

$$h = \frac{-e^2d}{1 - e^2} \qquad a^2 = \frac{e^2d^2}{(1 - e^2)^2} \qquad b^2 = \frac{e^2d^2}{1 - e^2}$$

SUGGESTED TIME AND EMPHASIS

1 class.
Optional material.

POINTS TO STRESS

1. Expressing conic sections in polar form.
2. Rotating conic sections in parametric form.
3. The eccentricity of a conic.

SAMPLE QUESTION

Text Question

State a reason that we would want to express a conic in polar form.

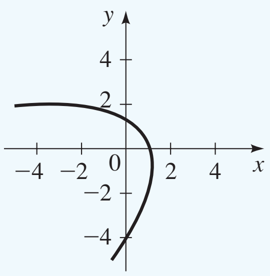
Answer

Answers will vary. One reason is that they are easier to rotate when expressed in polar form.

EXAMPLE
Identify and graph conic section.

$$r = \frac{6}{3 + 3\cos\left(\theta - \frac{\pi}{6}\right)} = \frac{2}{1 + \cos\left(\theta - \frac{\pi}{6}\right)}$$

is a parabola.



This is the equation of an ellipse with center $(h, 0)$. In Section 10.2 we found that the foci of an ellipse are a distance c from the center, where $c^2 = a^2 - b^2$. In our case

$$c^2 = a^2 - b^2 = \frac{e^4 d^2}{(1 - e^2)^2}$$

Thus, $c = e^2 d/(1 - e^2) = -h$, which confirms that the focus defined in the theorem is the same as the focus defined in Section 10.2. It also follows that

$$\boxed{e = \frac{c}{a}}$$

If $e > 1$, a similar proof shows that the conic is a hyperbola with $e = c/a$, where $c^2 = a^2 + b^2$. ■

In the proof we saw that the polar equation of the conic in Figure 1 is $r = e(d - r\cos\theta)$. Solving for r, we get

$$r = \frac{ed}{1 + e\cos\theta}$$

If the directrix is chosen to be to the *left* of the focus $(x = -d)$, then we get the equation $r = ed/(1 - e\cos\theta)$. If the directrix is *parallel* to the polar axis $(y = d$ or $y = -d)$, then we get $\sin\theta$ instead of $\cos\theta$ in the equation. These observations are summarized in the following box and in Figure 2.

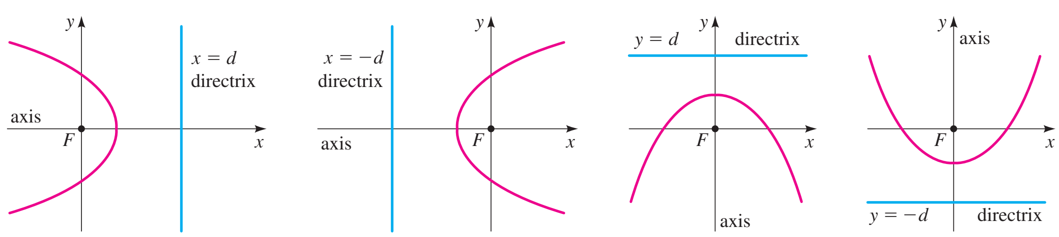
Polar Equations of Conics

A polar equation of the form

$$r = \frac{ed}{1 \pm e\cos\theta} \quad \text{or} \quad r = \frac{ed}{1 \pm e\sin\theta}$$

represents a conic with one focus at the origin and with eccentricity e. The conic is

1. a parabola if $e = 1$
2. an ellipse if $0 < e < 1$
3. a hyperbola if $e > 1$



(a) $r = \dfrac{ed}{1 + e\cos\theta}$ (b) $r = \dfrac{ed}{1 - e\cos\theta}$ (c) $r = \dfrac{ed}{1 + e\sin\theta}$ (d) $r = \dfrac{ed}{1 - e\sin\theta}$

Figure 2
The form of the polar equation of a conic indicates the location of the directrix.

IN-CLASS MATERIALS

Notice the advantages of working with conics in polar form. They are easier to rotate, for example. It is also easier to identify a conic as an ellipse, parabola, or hyperbola by inspection. After some practice, we can even look at such an equation and have a good idea of what the graph will look like, again by inspection.

To graph the polar equation of a conic, we first determine the location of the directrix from the form of the equation. The four cases that arise are shown in Figure 2. (The figure shows only the parts of the graphs that are close to the focus at the origin. The shape of the rest of the graph depends on whether the equation represents a parabola, an ellipse, or a hyperbola.) The axis of a conic is perpendicular to the directrix—specifically we have the following:

1. For a parabola, the axis of symmetry is perpendicular to the directrix.
2. For an ellipse, the major axis is perpendicular to the directrix.
3. For a hyperbola, the transverse axis is perpendicular to the directrix.

Example 1 Finding a Polar Equation for a Conic

Find a polar equation for the parabola that has its focus at the origin and whose directrix is the line $y = -6$.

Solution Using $e = 1$ and $d = 6$, and using part (d) of Figure 2, we see that the polar equation of the parabola is

$$r = \frac{6}{1 - \sin\theta}$$

■

To graph a polar conic, it is helpful to plot the points for which $\theta = 0, \pi/2, \pi$, and $3\pi/2$. Using these points and a knowledge of the type of conic (which we obtain from the eccentricity), we can easily get a rough idea of the shape and location of the graph.

Example 2 Identifying and Sketching a Conic

A conic is given by the polar equation

$$r = \frac{10}{3 - 2\cos\theta}$$

(a) Show that the conic is an ellipse and sketch the graph.

(b) Find the center of the ellipse, and the lengths of the major and minor axes.

Solution

(a) Dividing the numerator and denominator by 3, we have

$$r = \frac{\frac{10}{3}}{1 - \frac{2}{3}\cos\theta}$$

Since $e = \frac{2}{3} < 1$, the equation represents an ellipse. For a rough graph we plot the points for which $\theta = 0, \pi/2, \pi, 3\pi/2$ (see Figure 3 on the next page).

(b) Comparing the equation to those in Figure 2, we see that the major axis is horizontal. Thus, the endpoints of the major axis are $V_1(10, 0)$ and $V_2(2, \pi)$.

DRILL QUESTION

Identify the conic:

$$r = \frac{6}{1 \pm 3\cos\theta}$$

Answer

It is a hyperbola.

ALTERNATE EXAMPLE 1

Find a polar equation for the parabola that has its focus at the origin and whose directrix is the line $y = -5$.

ANSWER

$$r = \frac{5}{1 - \sin(\theta)}$$

ALTERNATE EXAMPLE 2

A conic is given by the polar equation:

$$r = \frac{25}{3 - 2\cos\theta}$$

Identify the conic. If the conic is an ellipse or a hyperbola, find the distances a and b. If the conic is a parabola, find the focus and the equation of the directrix.

ANSWER

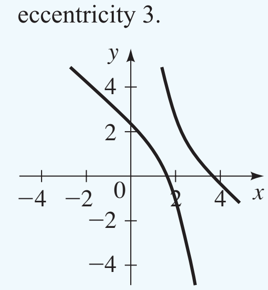
Ellipse, 15, 11.18

EXAMPLE

Identify and graph conic section:

$$r = \frac{6}{1 + 3\cos\left(\theta - \frac{\pi}{6}\right)}$$

is a hyperbola, rotated 30º from standard position, with eccentricity 3.

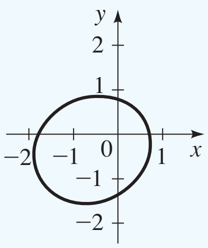


EXAMPLE

Identify and graph conic section.

$$r = \frac{6}{6 + 3\cos\left(\theta - \frac{\pi}{6}\right)} = \frac{1}{1 + \frac{1}{2}\cos\left(\theta - \frac{\pi}{6}\right)}$$

is an ellipse.



ALTERNATE EXAMPLE 3

A conic is given by the polar equation:

$$r = \frac{36}{4 + 8\sin\theta}$$

Identify the conic. If the conic is an ellipse or a hyperbola, find the distances a and b. If the conic is a parabola, find the focus and the equation of the directrix.

ANSWER

Hyperbola, 3, 5.20

So the center of the ellipse is at $C(4, 0)$, the midpoint of V_1V_2.

θ	r
0	10
$\pi/2$	$\frac{10}{3}$
π	2
$3\pi/2$	$\frac{10}{3}$

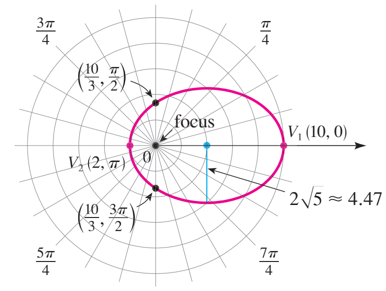


Figure 3

$$r = \frac{10}{3 - 2\cos\theta}$$

The distance between the vertices V_1 and V_2 is 12; thus, the length of the major axis is $2a = 12$, and so $a = 6$. To determine the length of the minor axis, we need to find b. From page 796 we have $c = ae = 6\left(\frac{2}{3}\right) = 4$, so

$$b^2 = a^2 - c^2 = 6^2 - 4^2 = 20$$

Thus, $b = \sqrt{20} = 2\sqrt{5} \approx 4.47$, and the length of the minor axis is $2b = 4\sqrt{5} \approx 8.94$. ■

Example 3 Identifying and Sketching a Conic

A conic is given by the polar equation

$$r = \frac{12}{2 + 4\sin\theta}$$

(a) Show that the conic is a hyperbola and sketch the graph.

(b) Find the center of the hyperbola and sketch the asymptotes.

θ	r
0	6
$\pi/2$	2
π	6
$3\pi/2$	-6

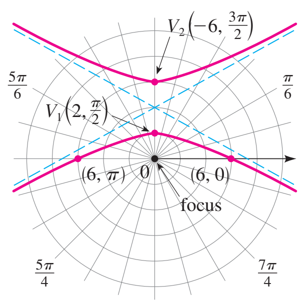


Figure 4

$$r = \frac{12}{2 + 4\sin\theta}$$

Solution

(a) Dividing the numerator and denominator by 2, we have

$$r = \frac{6}{1 + 2\sin\theta}$$

Since $e = 2 > 1$, the equation represents a hyperbola. For a rough graph we plot the points for which $\theta = 0, \pi/2, \pi, 3\pi/2$ (see Figure 4).

(b) Comparing the equation to those in Figure 2, we see that the transverse axis is vertical. Thus, the endpoints of the transverse axis (the vertices of the hyperbola) are $V_1(2, \pi/2)$ and $V_2(-6, 3\pi/2) = V_2(6, \pi/2)$. So the center of the hyperbola is $C(4, \pi/2)$, the midpoint of V_1V_2.

To sketch the asymptotes, we need to find a and b. The distance between V_1 and V_2 is 4; thus, the length of the transverse axis is $2a = 4$, and so $a = 2$. To find b, we first find c. From page 796 we have $c = ae = 2 \cdot 2 = 4$, so

$$b^2 = c^2 - a^2 = 4^2 - 2^2 = 12$$

Thus, $b = \sqrt{12} = 2\sqrt{3} \approx 3.46$. Knowing a and b allows us to sketch the central box, from which we obtain the asymptotes shown in Figure 4. ■

When we rotate conic sections, it is much more convenient to use polar equations than Cartesian equations. We use the fact that the graph of $r = f(\theta - \alpha)$ is the graph of $r = f(\theta)$ rotated counterclockwise about the origin through an angle α (see Exercise 55 in Section 8.2).

Example 4 Rotating an Ellipse

Suppose the ellipse of Example 2 is rotated through an angle $\pi/4$ about the origin. Find a polar equation for the resulting ellipse, and draw its graph.

Solution We get the equation of the rotated ellipse by replacing θ with $\theta - \pi/4$ in the equation given in Example 2. So the new equation is

$$r = \frac{10}{3 - 2\cos(\theta - \pi/4)}$$

We use this equation to graph the rotated ellipse in Figure 5. Notice that the ellipse has been rotated about the focus at the origin. ■

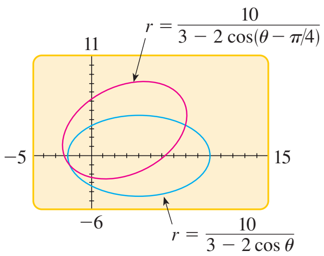


Figure 5

In Figure 6 we use a computer to sketch a number of conics to demonstrate the effect of varying the eccentricity e. Notice that when e is close to 0, the ellipse is nearly circular and becomes more elongated as e increases. When $e = 1$, of course, the conic is a parabola. As e increases beyond 1, the conic is an ever steeper hyperbola.

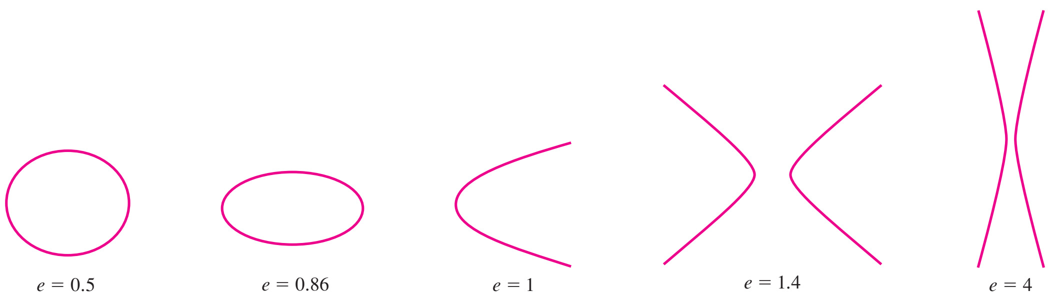


Figure 6

10.6 Exercises

1–8 ■ Write a polar equation of a conic that has its focus at the origin and satisfies the given conditions.

1. Ellipse, eccentricity $\frac{2}{3}$, directrix $x = 3$

2. Hyperbola, eccentricity $\frac{4}{3}$, directrix $x = -3$

3. Parabola, directrix $y = 2$

4. Ellipse, eccentricity $\frac{1}{2}$, directrix $y = -4$

5. Hyperbola, eccentricity 4, directrix $r = 5 \sec\theta$

6. Ellipse, eccentricity 0.6, directrix $r = 2 \csc\theta$

7. Parabola, vertex at $(5, \pi/2)$

8. Ellipse, eccentricity 0.4, vertex at $(2, 0)$

ALTERNATE EXAMPLE 4

Suppose the ellipse $r = \dfrac{8}{4 - 3\cos\theta}$ is rotated through the angle $\dfrac{\pi}{4}$ about the origin. Find a polar equation for the resulting ellipse.

ANSWER

$$r = \frac{8}{4 - 3\cos\left(\theta - \dfrac{\pi}{4}\right)}$$

IN-CLASS MATERIALS

It is good to go through an example like Example 4 with the class, having them use their calculators to vary the eccentricity e of a given conic section, and see how it affects the graph. Most graphic calculators have a mode for graphing polar functions as well.

IN-CLASS MATERIALS

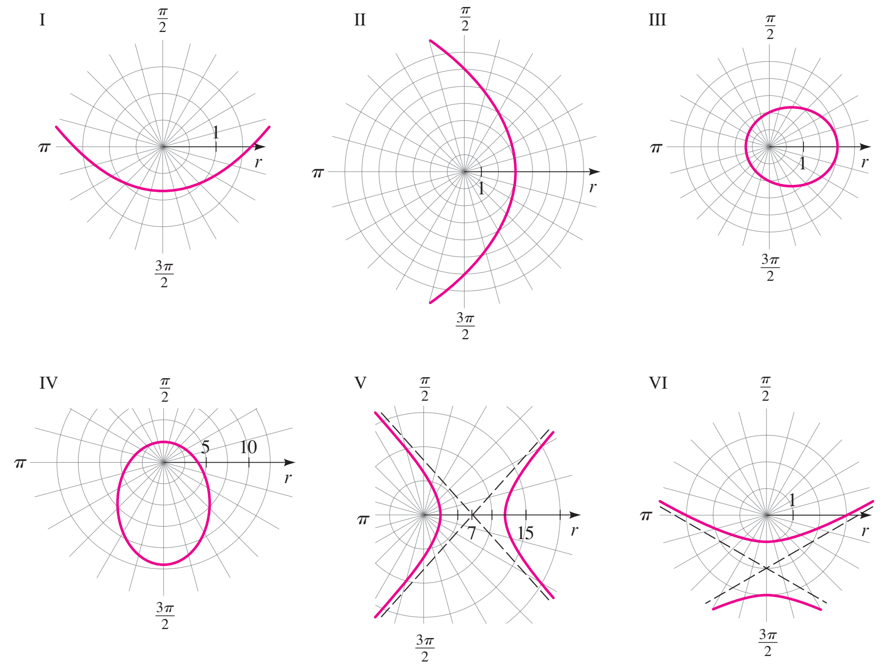
We have called a circle a "degenerate ellipse." The text notes that when e is close to zero, the resultant ellipse is close to circular. Examine the equation of a conic when $e = 0$ to see how it is undefined for that value of e.

9–14 ■ Match the polar equations with the graphs labeled I–VI. Give reasons for your answer.

9. $r = \dfrac{6}{1 + \cos\theta}$ **10.** $r = \dfrac{2}{2 - \cos\theta}$

11. $r = \dfrac{3}{1 - 2\sin\theta}$ **12.** $r = \dfrac{5}{3 - 3\sin\theta}$

13. $r = \dfrac{12}{3 + 2\sin\theta}$ **14.** $r = \dfrac{12}{2 + 3\cos\theta}$



15–22 ■ **(a)** Find the eccentricity and identify the conic. **(b)** Sketch the conic and label the vertices.

15. $r = \dfrac{4}{1 + 3\cos\theta}$ **16.** $r = \dfrac{8}{3 + 3\cos\theta}$

17. $r = \dfrac{2}{1 - \cos\theta}$ **18.** $r = \dfrac{10}{3 - 2\sin\theta}$

19. $r = \dfrac{6}{2 + \sin\theta}$ **20.** $r = \dfrac{5}{2 - 3\sin\theta}$

21. $r = \dfrac{7}{2 - 5\sin\theta}$ **22.** $r = \dfrac{8}{3 + \cos\theta}$

23. **(a)** Find the eccentricity and directrix of the conic $r = 1/(4 - 3\cos\theta)$ and graph the conic and its directrix.

(b) If this conic is rotated about the origin through an angle $\pi/3$, write the resulting equation and draw its graph.

24. Graph the parabola $r = 5/(2 + 2\sin\theta)$ and its directrix. Also graph the curve obtained by rotating this parabola about its focus through an angle $\pi/6$.

25. Graph the conics $r = e/(1 - e\cos\theta)$ with $e = 0.4, 0.6, 0.8,$ and 1.0 on a common screen. How does the value of e affect the shape of the curve?

26. **(a)** Graph the conics

$$r = \frac{ed}{(1 + e\sin\theta)}$$

for $e = 1$ and various values of d. How does the value of d affect the shape of the conic?

(b) Graph these conics for $d = 1$ and various values of e. How does the value of e affect the shape of the conic?

Applications

27. Orbit of the Earth The polar equation of an ellipse can be expressed in terms of its eccentricity e and the length a of its major axis.

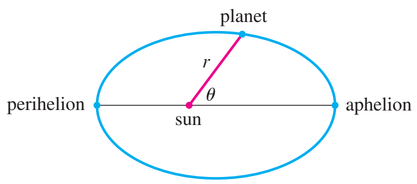
(a) Show that the polar equation of an ellipse with directrix $x = -d$ can be written in the form

$$r = \frac{a(1 - e^2)}{1 - e\cos\theta}$$

[*Hint*: Use the relation $a^2 = e^2d^2/(1 - e^2)^2$ given in the proof on page 795.]

(b) Find an approximate polar equation for the elliptical orbit of the earth around the sun (at one focus) given that the eccentricity is about 0.017 and the length of the major axis is about 2.99×10^8 km.

28. Perihelion and Aphelion The planets move around the sun in elliptical orbits with the sun at one focus. The positions of a planet that are closest to, and farthest from, the sun are called its **perihelion** and **aphelion**, respectively.



(a) Use Exercise 27(a) to show that the perihelion distance from a planet to the sun is $a(1 - e)$ and the aphelion distance is $a(1 + e)$.

(b) Use the data of Exercise 27(b) to find the distances from the earth to the sun at perihelion and at aphelion.

29. Orbit of Pluto The distance from the planet Pluto to the sun is 4.43×10^9 km at perihelion and 7.37×10^9 km at aphelion. Use Exercise 28 to find the eccentricity of Pluto's orbit.

Discovery • Discussion

30. Distance to a Focus When we found polar equations for the conics, we placed one focus at the pole. It's easy to find the distance from that focus to any point on the conic. Explain how the polar equation gives us this distance.

31. Polar Equations of Orbits When a satellite orbits the earth, its path is an ellipse with one focus at the center of the earth. Why do scientists use polar (rather than rectangular) coordinates to track the position of satellites? [*Hint:* Your answer to Exercise 30 is relevant here.]

10.7 Plane Curves and Parametric Equations

So far we've described a curve by giving an equation (in rectangular or polar coordinates) that the coordinates of all the points on the curve must satisfy. But not all curves in the plane can be described in this way. In this section we study parametric equations, which are a general method for describing any curve.

Plane Curves

We can think of a curve as the path of a point moving in the plane; the x- and y-coordinates of the point are then functions of time. This idea leads to the following definition.

Plane Curves and Parametric Equations

If f and g are functions defined on an interval I, then the set of points $(f(t), g(t))$ is a **plane curve**. The equations

$$x = f(t) \qquad y = g(t)$$

where $t \in I$, are **parametric equations** for the curve, with **parameter** t.

SUGGESTED TIME AND EMPHASIS

1 class.
Recommended material.

POINTS TO STRESS

1. Definition of parametric equations.
2. Sketching parametric curves.
3. Eliminating the parameter.
4. Polar equations in parametric form.

SAMPLE QUESTION
Text Question

What is the difference between a function and a parametric curve?

Answer

Many answers are possible. The graph of a function can be made into a parametric curve, but not necessarily the other way around. A function has to pass the vertical line test and a parametric curve does not.

ALTERNATE EXAMPLE 1
For the curve defined by the parametric equations $x = t^2 - 7t$ and $y = t - 3$ eliminate the parameter t and obtain a single equation for the curve in variables x and y.

ANSWER
$x = y^2 - y - 12t$

Bettmann/Corbis

Maria Gaetana Agnesi (1718–1799) is famous for having written *Instituzioni Analitiche*, considered to be the first calculus textbook.

Maria was born into a wealthy family in Milan, Italy, the oldest of 21 children. She was a child prodigy, mastering many languages at an early age, including Latin, Greek, and Hebrew. At the age of 20 she published a series of essays on philosophy and natural science. After Maria's mother died, she took on the task of educating her brothers. In 1748 Agnesi published her famous textbook, which she originally wrote as a text for tutoring her brothers. The book compiled and explained the mathematical knowledge of the day. It contains many carefully chosen examples, one of which is the curve now known as the "witch of Agnesi" (see page 809). One review calls her book an "exposition by examples rather than by theory." The book gained Agnesi immediate recognition. Pope Benedict XIV appointed her to a position at the University of Bologna, writing "we have had the idea that you should be awarded the well-known chair of mathematics, by which it comes of itself that you should not thank us but we you." This appointment was an extremely high honor for a woman, since very few women then were even allowed

(continued)

Example 1 Sketching a Plane Curve

Sketch the curve defined by the parametric equations

$$x = t^2 - 3t \qquad y = t - 1$$

Solution For every value of t, we get a point on the curve. For example, if $t = 0$, then $x = 0$ and $y = -1$, so the corresponding point is $(0, -1)$. In Figure 1 we plot the points (x, y) determined by the values of t shown in the following table.

t	x	y
−2	10	−3
−1	4	−2
0	0	−1
1	−2	0
2	−2	1
3	0	2
4	4	3
5	10	4

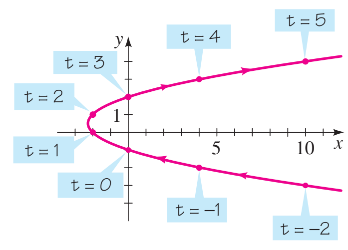


Figure 1

As t increases, a particle whose position is given by the parametric equations moves along the curve in the direction of the arrows. ■

If we replace t by $-t$ in Example 1, we obtain the parametric equations

$$x = t^2 + 3t \qquad y = -t - 1$$

The graph of these parametric equations (see Figure 2) is the same as the curve in Figure 1, but traced out in the opposite direction. On the other hand, if we replace t by $2t$ in Example 1, we obtain the parametric equations

$$x = 4t^2 - 6t \qquad y = 2t - 1$$

The graph of these parametric equations (see Figure 3) is again the same, but is traced out "twice as fast." *Thus, a parametrization contains more information than just the shape of the curve; it also indicates* how *the curve is being traced out.*

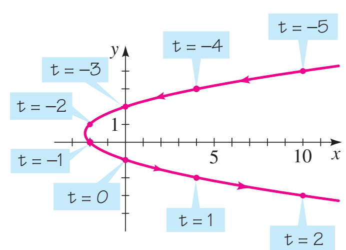


Figure 2
$x = t^2 + 3t$, $y = -t - 1$

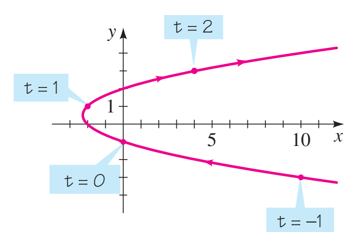


Figure 3
$x = 4t^2 - 6t$, $y = 2t - 1$

to attend university. Just two years later, Agnesi's father died and she left mathematics completely. She became a nun and devoted the rest of her life and her wealth to caring for sick and dying women, herself dying in poverty at a poorhouse of which she had once been director.

Eliminating the Parameter

Often a curve given by parametric equations can also be represented by a single rectangular equation in x and y. The process of finding this equation is called *eliminating the parameter*. One way to do this is to solve for t in one equation, then substitute into the other.

Example 2 Eliminating the Parameter

Eliminate the parameter in the parametric equations of Example 1.

Solution First we solve for t in the simpler equation, then we substitute into the other equation. From the equation $y = t - 1$, we get $t = y + 1$. Substituting into the equation for x, we get

$$x = t^2 - 3t = (y+1)^2 - 3(y+1) = y^2 - y - 2$$

Thus, the curve in Example 1 has the rectangular equation $x = y^2 - y - 2$, so it is a parabola. ■

Eliminating the parameter often helps us identify the shape of a curve, as we see in the next two examples.

Example 3 Eliminating the Parameter

Describe and graph the curve represented by the parametric equations

$$x = \cos t \qquad y = \sin t \qquad 0 \le t \le 2\pi$$

Solution To identify the curve, we eliminate the parameter. Since $\cos^2 t + \sin^2 t = 1$ and since $x = \cos t$ and $y = \sin t$ for every point (x, y) on the curve, we have

$$x^2 + y^2 = (\cos t)^2 + (\sin t)^2 = 1$$

This means that all points on the curve satisfy the equation $x^2 + y^2 = 1$, so the graph is a circle of radius 1 centered at the origin. As t increases from 0 to 2π, the point given by the parametric equations starts at $(1, 0)$ and moves counterclockwise once around the circle, as shown in Figure 4. Notice that the parameter t can be interpreted as the angle shown in the figure. ■

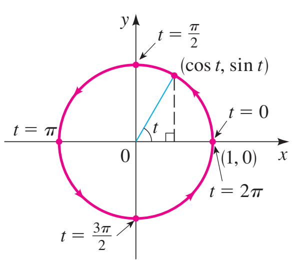


Figure 4

Example 4 Sketching a Parametric Curve

Eliminate the parameter and sketch the graph of the parametric equations

$$x = \sin t \qquad y = 2 - \cos^2 t$$

Solution To eliminate the parameter, we first use the trigonometric identity $\cos^2 t = 1 - \sin^2 t$ to change the second equation:

$$y = 2 - \cos^2 t = 2 - (1 - \sin^2 t) = 1 + \sin^2 t$$

Now we can substitute $\sin t = x$ from the first equation to get

$$y = 1 + x^2$$

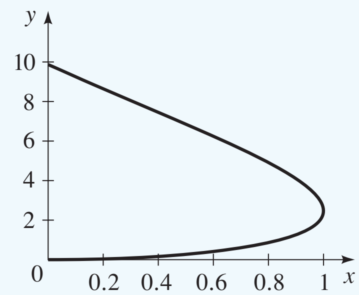
DRILL QUESTION

Sketch the parametric curve $x(t) = \sin t\ y(t) = t^2,\ 0 \le t \le \pi$.

Is the point $\left(1, \frac{\pi}{4}\right)$ on this curve?

Answer

$\left(1, \frac{\pi}{4}\right)$ is not on the curve.



ALTERNATE EXAMPLE 3

Eliminate the parameter t to obtain a single equation for the curve $x = 5 \cos t$, $y = 5 \sin t$ in the variables x and y.

ANSWER

$x^2 + y^2 = 25$

ALTERNATE EXAMPLE 4

For the graph of a curve defined by the parametric equations $x = \sin t$ and $y = 7 - \cos 2t$ find a single equation and its domain.

ANSWER

$y = 6 + x^2, x \in [-1, 1]$

IN-CLASS MATERIALS

Give an example of a curve such as $x(t) = \cos(et)$, $y(t) = \sin(\sqrt{3}t)$. This curve essentially fills the square $-1 \le x \le 1, -1 \le y \le 1$ in that the curve gets arbitrarily close to any point in the square. (It is not what mathematicians call a "space filling curve" because it does not actually hit every point in the square.) It can be simulated using a graphing calculator with the approximations $e \approx 2.7183$ and $\sqrt{3} \approx 1.7321$. The range $0 \le t \le 200$ should be sufficient to convey this property to the students. Next, describe the family of functions $x(t) = a\cos(et)$, $y(t) = b\sin(\sqrt{3}t)$. If the students are following well, perhaps consider the family $x(t) = \cos(at)$, $y(t) = \sin(\sqrt{3}t)$. The students might be tempted to conclude that all these curves behave the same way, but they do not for some values of a, such as $a = 2\sqrt{3}$.

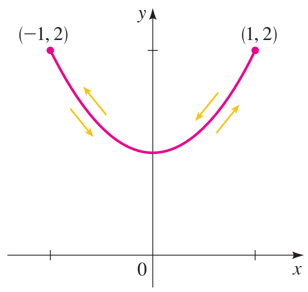


Figure 5

and so the point (x, y) moves along the parabola $y = 1 + x^2$. However, since $-1 \le \sin t \le 1$, we have $-1 \le x \le 1$, so the parametric equations represent only the part of the parabola between $x = -1$ and $x = 1$. Since $\sin t$ is periodic, the point $(x, y) = (\sin t, 2 - \cos^2 t)$ moves back and forth infinitely often along the parabola between the points $(-1, 2)$ and $(1, 2)$ as shown in Figure 5. ■

Finding Parametric Equations for a Curve

It is often possible to find parametric equations for a curve by using some geometric properties that define the curve, as in the next two examples.

ALTERNATE EXAMPLE 5
Find a single equation for the curve in the variables x and y for the line of slope 9 that passes through the point (4, 36).

ANSWER
$y = 9x$

Example 5 Finding Parametric Equations for a Graph

Find parametric equations for the line of slope 3 that passes through the point $(2, 6)$.

Solution Let's start at the point $(2, 6)$ and move up and to the right along this line. Because the line has slope 3, for every 1 unit we move to the right, we must move up 3 units. In other words, if we increase the x-coordinate by t units, we must correspondingly increase the y-coordinate by $3t$ units. This leads to the parametric equations

$$x = 2 + t \qquad y = 6 + 3t$$

To confirm that these equations give the desired line, we eliminate the parameter. We solve for t in the first equation and substitute into the second to get

$$y = 6 + 3(x - 2) = 3x$$

Thus, the slope-intercept form of the equation of this line is $y = 3x$, which is a line of slope 3 that does pass through $(2, 6)$ as required. The graph is shown in Figure 6. ■

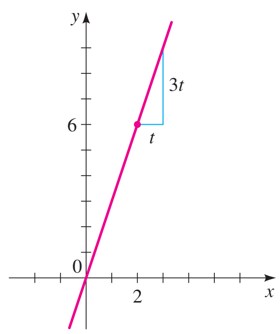


Figure 6

Example 6 Parametric Equations for the Cycloid

As a circle rolls along a straight line, the curve traced out by a fixed point P on the circumference of the circle is called a **cycloid** (see Figure 7). If the circle has radius a and rolls along the x-axis, with one position of the point P being at the origin, find parametric equations for the cycloid.

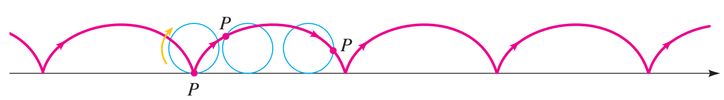
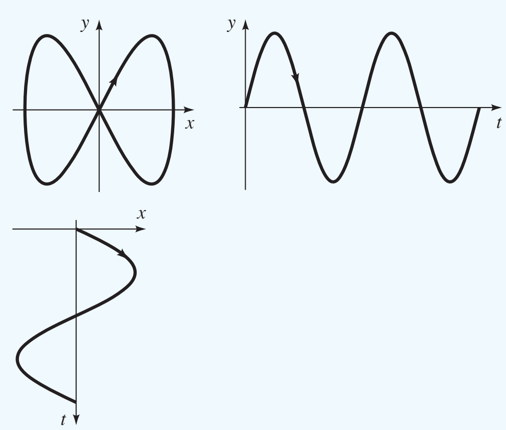


Figure 7

Solution Figure 8 shows the circle and the point P after the circle has rolled through an angle θ (in radians). The distance $d(O, T)$ that the circle has rolled must be the same as the length of the arc PT, which, by the arc length formula, is $a\theta$ (see Section 6.1). This means that the center of the circle is $C(a\theta, a)$.

IN-CLASS MATERIALS

Discuss the process of going from a parametric curve to a relation between x and y. The figure at right is meant to help students see the way that parametrized curves are sketched out over time. First sketch $(x(t), y(t))$, starting at the initial point (the origin), and moving up and to the right. (Try to keep your speed constant.) Stop when the cycle is about to repeat. Then, to the right of the figure, graph the motion in the y-direction only. Then, below the figure, graph the motion in the x-direction. That graph is sideways because the x-axis is horizontal.



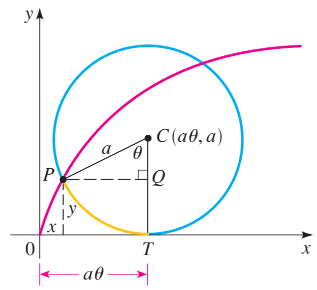


Figure 8

Let the coordinates of P be (x, y). Then from Figure 8 (which illustrates the case $0 < \theta < \pi/2$), we see that

$$x = d(O, T) - d(P, Q) = a\theta - a \sin\theta = a(\theta - \sin\theta)$$
$$y = d(T, C) - d(Q, C) = a - a\cos\theta = a(1 - \cos\theta)$$

so parametric equations for the cycloid are

$$x = a(\theta - \sin\theta) \qquad y = a(1 - \cos\theta)$$ ■

The cycloid has a number of interesting physical properties. It is the "curve of quickest descent" in the following sense. Let's choose two points P and Q that are not directly above each other, and join them with a wire. Suppose we allow a bead to slide down the wire under the influence of gravity (ignoring friction). Of all possible shapes that the wire can be bent into, the bead will slide from P to Q the fastest when the shape is half of an arch of an inverted cycloid (see Figure 9). The cycloid is also the "curve of equal descent" in the sense that no matter where we place a bead B on a cycloid-shaped wire, it takes the same time to slide to the bottom (see Figure 10). These rather surprising properties of the cycloid were proved (using calculus) in the 17th century by several mathematicians and physicists, including Johann Bernoulli, Blaise Pascal, and Christiaan Huygens.

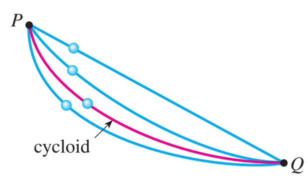


Figure 9

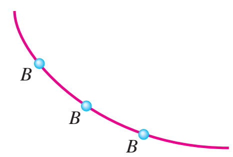


Figure 10

Using Graphing Devices to Graph Parametric Curves

Most graphing calculators and computer graphing programs can be used to graph parametric equations. Such devices are particularly useful when sketching complicated curves like the one shown in Figure 11.

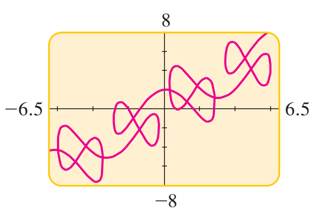


Figure 11
$x = t + 2\sin 2t,\ y = t + 2\cos 5t$

Example 7 Graphing Parametric Curves

Use a graphing device to draw the following parametric curves. Discuss their similarities and differences.

(a) $x = \sin 2t$
$\quad y = 2\cos t$

(b) $x = \sin 3t$
$\quad y = 2\cos t$

Solution In both parts (a) and (b), the graph will lie inside the rectangle given by $-1 \le x \le 1$, $-2 \le y \le 2$, since both the sine and the cosine of any number will be between -1 and 1. Thus, we may use the viewing rectangle $[-1.5, 1.5]$ by $[-2.5, 2.5]$.

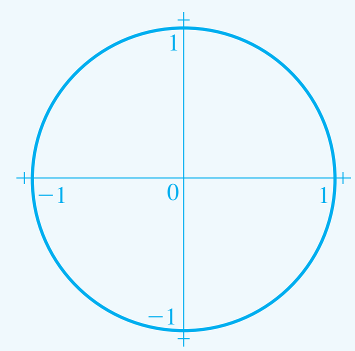
EXAMPLE
Construction of an ellipse with center $(-1, -2)$ and axis lengths 2 and 3: We start with the circle $x = \cos\theta$, $y = \sin\theta$. We stretch it into shape: $x = 2\cos\theta$, $y = 3\sin\theta$. Then we move it into position: $x = 2\cos\theta - 1$, $y = 3\sin\theta - 2$.

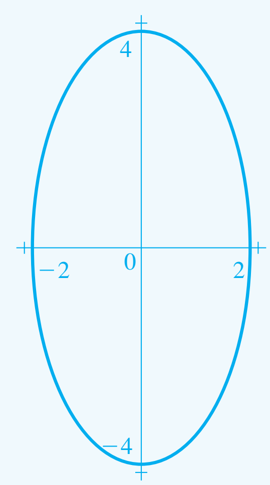
ALTERNATE EXAMPLE 7
Use a graphing device to draw the following parametric curves. Compare the two curves.

(a) $x = \cos t$
$\quad y = \sin t$

(b) $x = 2\cos t$
$\quad y = 4\sin t$

ANSWERS

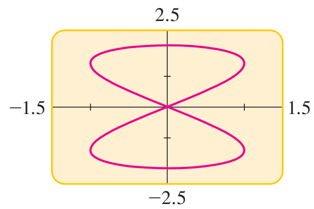




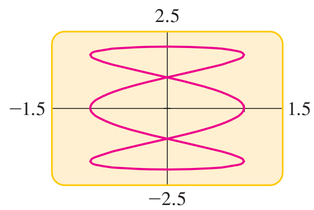
The first curve is a circle. The second one stretches the x- and y-coordinates by different factors, resulting in an ellipse.

IN-CLASS MATERIALS

Show how reversing the functions $x(t)$ and $y(t)$ yields the inverse of a given relation. For example, $\begin{cases} x(t) = t \\ y(t) = \sin t \end{cases}$ is the sine function, so $\begin{cases} x(t) = \sin t \\ y(t) = t \end{cases}$ is the general arcsine function. Display an inverse for $f(x) = x^3 + x + 2$ graphically using parametric equations. Explain the difficulties with the algebraic approach.



(a) $x = \sin 2t,\ y = 2\cos t$



(b) $x = \sin 3t,\ y = 2\cos t$

Figure 12

(a) Since $2\cos t$ is periodic with period 2π (see Section 5.3), and since $\sin 2t$ has period π, letting t vary over the interval $0 \le t \le 2\pi$ gives us the complete graph, which is shown in Figure 12(a).

(b) Again, letting t take on values between 0 and 2π gives the complete graph shown in Figure 12(b).

Both graphs are *closed curves*, which means they form loops with the same starting and ending point; also, both graphs cross over themselves. However, the graph in Figure 12(a) has two loops, like a figure eight, whereas the graph in Figure 12(b) has three loops. ■

The curves graphed in Example 7 are called Lissajous figures. A **Lissajous figure** is the graph of a pair of parametric equations of the form

$$x = A\sin\omega_1 t \qquad y = B\cos\omega_2 t$$

where A, B, ω_1, and ω_2 are real constants. Since $\sin\omega_1 t$ and $\cos\omega_2 t$ are both between -1 and 1, a Lissajous figure will lie inside the rectangle determined by $-A \le x \le A$, $-B \le y \le B$. This fact can be used to choose a viewing rectangle when graphing a Lissajous figure, as in Example 7.

Recall from Section 8.1 that rectangular coordinates (x, y) and polar coordinates (r, θ) are related by the equations $x = r\cos\theta$, $y = r\sin\theta$. Thus, we can graph the polar equation $r = f(\theta)$ by changing it to parametric form as follows:

$$x = r\cos\theta = f(\theta)\cos\theta \qquad \text{Since } r = f(\theta)$$
$$y = r\sin\theta = f(\theta)\sin\theta$$

Replacing θ by the standard parametric variable t, we have the following result.

Polar Equations in Parametric Form

The graph of the polar equation $r = f(\theta)$ is the same as the graph of the parametric equations

$$x = f(t)\cos t \qquad y = f(t)\sin t$$

Example 8 Parametric Form of a Polar Equation

Consider the polar equation $r = \theta$, $1 \le \theta \le 10\pi$.

(a) Express the equation in parametric form.

(b) Draw a graph of the parametric equations from part (a).

Solution

(a) The given polar equation is equivalent to the parametric equations

$$x = t\cos t \qquad y = t\sin t$$

(b) Since $10\pi \approx 31.42$, we use the viewing rectangle $[-32, 32]$ by $[-32, 32]$, and we let t vary from 1 to 10π. The resulting graph shown in Figure 13 is a *spiral*. ■

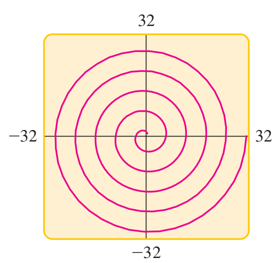


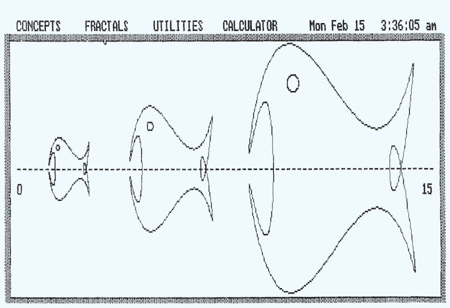
Figure 13
$x = t\cos t,\ y = t\sin t$

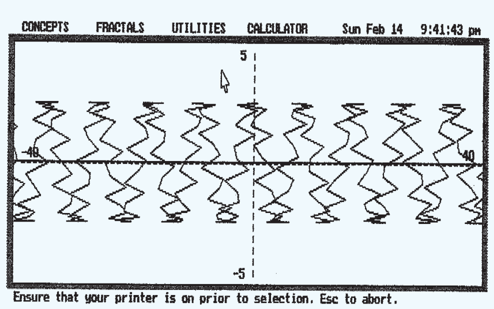
ALTERNATE EXAMPLE 8a
Express the equation in parametric form:
$r = 3\ln\theta$

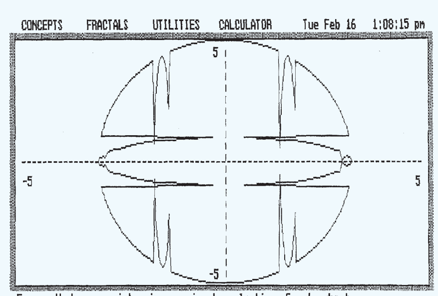
ANSWER
$x = 3\ln(t)\cdot\cos(t)$,
$y = 3\ln(t)\cdot\sin(t)$

IN-CLASS MATERIALS

Have the students get into groups, and have each group try to come up with the most interesting looking parametric curve. After displaying their best ones, perhaps show them the following examples:







10.7 Exercises

1–22 ■ A pair of parametric equations is given.

(a) Sketch the curve represented by the parametric equations.

(b) Find a rectangular-coordinate equation for the curve by eliminating the parameter.

1. $x = 2t, \quad y = t + 6$

2. $x = 6t - 4, \quad y = 3t, \quad t \geq 0$

3. $x = t^2, \quad y = t - 2, \quad 2 \leq t \leq 4$

4. $x = 2t + 1, \quad y = \left(t + \frac{1}{2}\right)^2$

5. $x = \sqrt{t}, \quad y = 1 - t$

6. $x = t^2, \quad y = t^4 + 1$

7. $x = \dfrac{1}{t}, \quad y = t + 1$

8. $x = t + 1, \quad y = \dfrac{t}{t + 1}$

9. $x = 4t^2, \quad y = 8t^3$

10. $x = |t|, \quad y = |1 - |t||$

11. $x = 2 \sin t, \quad y = 2 \cos t, \quad 0 \leq t \leq \pi$

12. $x = 2 \cos t, \quad y = 3 \sin t, \quad 0 \leq t \leq 2\pi$

13. $x = \sin^2 t, \quad y = \sin^4 t$

14. $x = \sin^2 t, \quad y = \cos t$

15. $x = \cos t, \quad y = \cos 2t$

16. $x = \cos 2t, \quad y = \sin 2t$

17 $x = \sec t, \quad y = \tan t, \quad 0 \leq t < \pi/2$

18 $x = \cot t, \quad y = \csc t, \quad 0 < t < \pi$

19 $x = \tan t, \quad y = \cot t, \quad 0 < t < \pi/2$

20. $x = \sec t, \quad y = \tan^2 t, \quad 0 \leq t < \pi/2$

21. $x = \cos^2 t, \quad y = \sin^2 t$

22. $x = \cos^3 t, \quad y = \sin^3 t, \quad 0 \leq t \leq 2\pi$

23–26 ■ Find parametric equations for the line with the given properties.

23. Slope $\frac{1}{2}$, passing through $(4, -1)$

24. Slope -2, passing through $(-10, -20)$

25. Passing through $(6, 7)$ and $(7, 8)$

26. Passing through $(12, 7)$ and the origin

27. Find parametric equations for the circle $x^2 + y^2 = a^2$.

28. Find parametric equations for the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

29. Show by eliminating the parameter θ that the following parametric equations represent a hyperbola:

$$x = a \tan \theta \qquad y = b \sec \theta$$

30. Show that the following parametric equations represent a part of the hyperbola of Exercise 29:

$$x = a\sqrt{t} \qquad y = b\sqrt{t + 1}$$

31–34 ■ Sketch the curve given by the parametric equations.

31. $x = t \cos t, \quad y = t \sin t, \quad t \geq 0$

32. $x = \sin t, \quad y = \sin 2t$

33. $x = \dfrac{3t}{1 + t^3}, \quad y = \dfrac{3t^2}{1 + t^3}$

34. $x = \cot t, \quad y = 2 \sin^2 t, \quad 0 < t < \pi$

35. If a projectile is fired with an initial speed of v_0 ft/s at an angle α above the horizontal, then its position after t seconds is given by the parametric equations

$$x = (v_0 \cos \alpha)t \qquad y = (v_0 \sin \alpha)t - 16t^2$$

(where x and y are measured in feet). Show that the path of the projectile is a parabola by eliminating the parameter t.

36. Referring to Exercise 35, suppose a gun fires a bullet into the air with an initial speed of 2048 ft/s at an angle of 30° to the horizontal.

(a) After how many seconds will the bullet hit the ground?

(b) How far from the gun will the bullet hit the ground?

(c) What is the maximum height attained by the bullet?

37–42 ■ Use a graphing device to draw the curve represented by the parametric equations.

37. $x = \sin t, \quad y = 2 \cos 3t$

38. $x = 2 \sin t, \quad y = \cos 4t$

39. $x = 3 \sin 5t, \quad y = 5 \cos 3t$

40. $x = \sin 4t, \quad y = \cos 3t$

41. $x = \sin(\cos t), \quad y = \cos(t^{3/2}), \quad 0 \leq t \leq 2\pi$

42. $x = 2 \cos t + \cos 2t, \quad y = 2 \sin t - \sin 2t$

43–46 ■ A polar equation is given.

(a) Express the polar equation in parametric form.

(b) Use a graphing device to graph the parametric equations you found in part (a).

43. $r = 2^{\theta/12}, \quad 0 \leq \theta \leq 4\pi$

44. $r = \sin \theta + 2 \cos \theta$

45. $r = \dfrac{4}{2 - \cos \theta}$

46. $r = 2^{\sin \theta}$

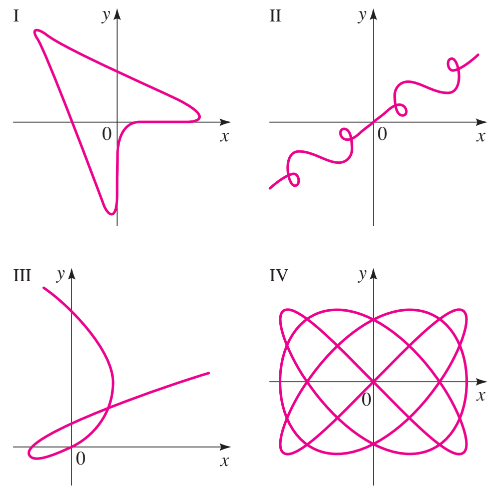
47–50 ■ Match the parametric equations with the graphs labeled I–IV. Give reasons for your answers.

47. $x = t^3 - 2t, \quad y = t^2 - t$

48. $x = \sin 3t, \quad y = \sin 4t$

49. $x = t + \sin 2t, \quad y = t + \sin 3t$

50. $x = \sin(t + \sin t), \quad y = \cos(t + \cos t)$



51. **(a)** In Example 6 suppose the point P that traces out the curve lies not on the edge of the circle, but rather at a fixed point inside the rim, at a distance b from the center (with $b < a$). The curve traced out by P is called a **curtate cycloid** (or **trochoid**). Show that parametric equations for the curtate cycloid are

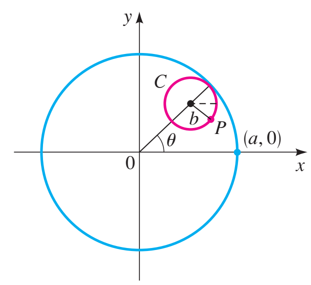
$$x = a\theta - b\sin\theta \qquad y = a - b\cos\theta$$

(b) Sketch the graph using $a = 3$ and $b = 2$.

52. **(a)** In Exercise 51 if the point P lies *outside* the circle at a distance b from the center (with $b > a$), then the curve traced out by P is called a **prolate cycloid**. Show that parametric equations for the prolate cycloid are the same as the equations for the curtate cycloid.

(b) Sketch the graph for the case where $a = 1$ and $b = 2$.

53. A circle C of radius b rolls on the inside of a larger circle of radius a centered at the origin. Let P be a fixed point on the smaller circle, with initial position at the point $(a, 0)$ as shown in the figure. The curve traced out by P is called a **hypocycloid**.



(a) Show that parametric equations for the hypocycloid are

$$x = (a - b)\cos\theta + b\cos\left(\frac{a - b}{b}\theta\right)$$

$$y = (a - b)\sin\theta - b\sin\left(\frac{a - b}{b}\theta\right)$$

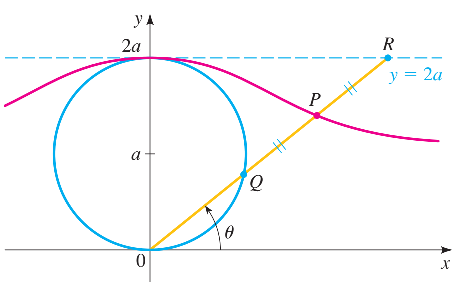
(b) If $a = 4b$, the hypocycloid is called an **astroid**. Show that in this case the parametric equations can be reduced to

$$x = a\cos^3\theta \qquad y = a\sin^3\theta$$

Sketch the curve. Eliminate the parameter to obtain an equation for the astroid in rectangular coordinates.

54. If the circle C of Exercise 53 rolls on the *outside* of the larger circle, the curve traced out by P is called an **epicycloid**. Find parametric equations for the epicycloid.

55. In the figure, the circle of radius a is stationary and, for every θ, the point P is the midpoint of the segment QR. The curve traced out by P for $0 < \theta < \pi$ is called the **longbow curve**. Find parametric equations for this curve.

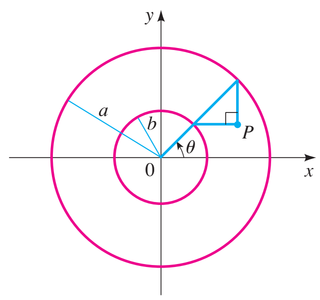


56. Two circles of radius a and b are centered at the origin, as shown in the figure. As the angle θ increases, the point P traces out a curve that lies between the circles.

(a) Find parametric equations for the curve, using θ as the parameter.

(b) Graph the curve using a graphing device, with $a = 3$ and $b = 2$.

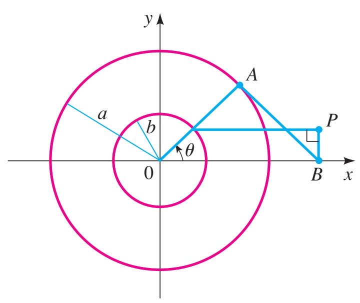
(c) Eliminate the parameter and identify the curve.



57. Two circles of radius a and b are centered at the origin, as shown in the figure.

(a) Find parametric equations for the curve traced out by the point P, using the angle θ as the parameter. (Note that the line segment AB is always tangent to the larger circle.)

(b) Graph the curve using a graphing device, with $a = 3$ and $b = 2$.

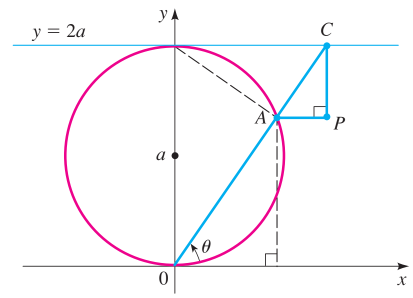


58. A curve, called a **witch of Maria Agnesi**, consists of all points P determined as shown in the figure.

(a) Show that parametric equations for this curve can be written as

$$x = 2a \cot \theta \qquad y = 2a \sin^2\theta$$

(b) Graph the curve using a graphing device, with $a = 3$.



59. Eliminate the parameter θ in the parametric equations for the cycloid (Example 6) to obtain a rectangular coordinate equation for the section of the curve given by $0 \le \theta \le \pi$.

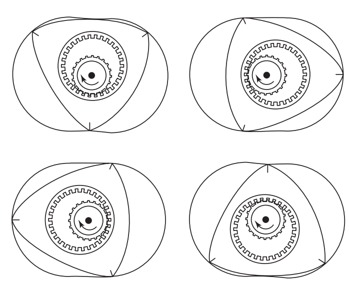
Applications

60. The Rotary Engine The Mazda RX-8 uses an unconventional engine (invented by Felix Wankel in 1954) in which the pistons are replaced by a triangular rotor that turns in a special housing as shown in the figure. The vertices of the rotor maintain contact with the housing at all times, while the center of the triangle traces out a circle of radius r, turning the drive shaft. The shape of the housing is given by the parametric equations below (where R is the distance between the vertices and center of the rotor).

$$x = r \cos 3\theta + R \cos \theta \qquad y = r \sin 3\theta + R \sin \theta$$

(a) Suppose that the drive shaft has radius $r = 1$. Graph the curve given by the parametric equations for the following values of R: 0.5, 1, 3, 5.

(b) Which of the four values of R given in part (a) seems to best model the engine housing illustrated in the figure?



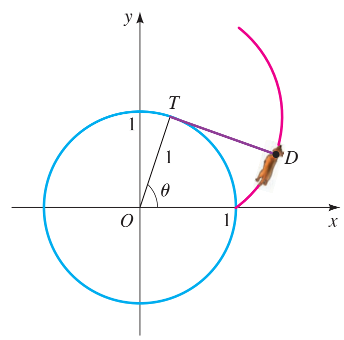
61. Spiral Path of a Dog A dog is tied to a circular tree trunk of radius 1 ft by a long leash. He has managed to wrap the entire leash around the tree while playing in the yard, and finds himself at the point $(1, 0)$ in the figure. Seeing a squirrel, he runs around the tree counterclockwise, keeping the leash taut while chasing the intruder.

(a) Show that parametric equations for the dog's path (called an **involute of a circle**) are

$$x = \cos\theta + \theta\sin\theta \qquad y = \sin\theta - \theta\cos\theta$$

[*Hint:* Note that the leash is always tangent to the tree, so OT is perpendicular to TD.]

(b) Graph the path of the dog for $0 \le \theta \le 4\pi$.



Discovery • Discussion

62. More Information in Parametric Equations In this section we stated that parametric equations contain more information than just the shape of a curve. Write a short paragraph explaining this statement. Use the following example and your answers to parts (a) and (b) below in your explanation.

The position of a particle is given by the parametric equations

$$x = \sin t \qquad y = \cos t$$

where t represents time. We know that the shape of the path of the particle is a circle.

(a) How long does it take the particle to go once around the circle? Find parametric equations if the particle moves twice as fast around the circle.

(b) Does the particle travel clockwise or counterclockwise around the circle? Find parametric equations if the particle moves in the opposite direction around the circle.

63. Different Ways of Tracing Out a Curve The curves C, D, E, and F are defined parametrically as follows, where the parameter t takes on all real values unless otherwise stated:

$$C\colon \quad x = t, \quad y = t^2$$

$$D\colon \quad x = \sqrt{t}, \quad y = t, \quad t \ge 0$$

$$E\colon \quad x = \sin t, \quad y = \sin^2 t$$

$$F\colon \quad x = 3^t, \quad y = 3^{2t}$$

(a) Show that the points on all four of these curves satisfy the same rectangular coordinate equation.

(b) Draw the graph of each curve and explain how the curves differ from one another.

10 Review

Concept Check

1. (a) Give the geometric definition of a parabola. What are the focus and directrix of the parabola?

(b) Sketch the parabola $x^2 = 4py$ for the case $p > 0$. Identify on your diagram the vertex, focus, and directrix. What happens if $p < 0$?

(c) Sketch the parabola $y^2 = 4px$, together with its vertex, focus, and directrix, for the case $p > 0$. What happens if $p < 0$?

2. (a) Give the geometric definition of an ellipse. What are the foci of the ellipse?

(b) For the ellipse with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where $a > b > 0$, what are the coordinates of the vertices and the foci? What are the major and minor axes? Illustrate with a graph.

(c) Give an expression for the eccentricity of the ellipse in part (b).

(d) State the equation of an ellipse with foci on the y-axis.

3. (a) Give the geometric definition of a hyperbola. What are the foci of the hyperbola?
(b) For the hyperbola with equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

what are the coordinates of the vertices and foci? What are the equations of the asymptotes? What is the transverse axis? Illustrate with a graph.
(c) State the equation of a hyperbola with foci on the y-axis.
(d) What steps would you take to sketch a hyperbola with a given equation?

4. Suppose h and k are positive numbers. What is the effect on the graph of an equation in x and y if
(a) x is replaced by $x - h$? By $x + h$?
(b) y is replaced by $y - k$? By $y + k$?

5. How can you tell whether the following nondegenerate conic is a parabola, an ellipse, or a hyperbola?

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

6. Suppose the x- and y-axes are rotated through an acute angle ϕ to produce the X- and Y-axes. Write equations that relate the coordinates (x, y) and (X, Y) of a point in the xy-plane and XY-plane, respectively.

7. (a) How do you eliminate the xy-term in this equation?

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

(b) What is the discriminant of the conic in part (a)? How can you use the discriminant to determine whether the conic is a parabola, an ellipse, or a hyperbola?

8. (a) Write polar equations that represent a conic with eccentricity e.
(b) For what values of e is the conic an ellipse? A hyperbola? A parabola?

9. A curve is given by the parametric equations $x = f(t)$, $y = g(t)$.
(a) How do you sketch the curve?
(b) How do you eliminate the parameter?

Exercises

1–8 ■ Find the vertex, focus, and directrix of the parabola, and sketch the graph.

1. $y^2 = 4x$
2. $x = \frac{1}{12}y^2$
3. $x^2 + 8y = 0$
4. $2x - y^2 = 0$
5. $x - y^2 + 4y - 2 = 0$
6. $2x^2 + 6x + 5y + 10 = 0$
7. $\frac{1}{2}x^2 + 2x = 2y + 4$
8. $x^2 = 3(x + y)$

9–16 ■ Find the center, vertices, foci, and the lengths of the major and minor axes of the ellipse, and sketch the graph.

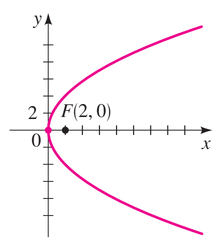
9. $\dfrac{x^2}{9} + \dfrac{y^2}{25} = 1$
10. $\dfrac{x^2}{49} + \dfrac{y^2}{9} = 1$
11. $x^2 + 4y^2 = 16$
12. $9x^2 + 4y^2 = 1$
13. $\dfrac{(x-3)^2}{9} + \dfrac{y^2}{16} = 1$
14. $\dfrac{(x-2)^2}{25} + \dfrac{(y+3)^2}{16} = 1$
15. $4x^2 + 9y^2 = 36y$
16. $2x^2 + y^2 = 2 + 4(x - y)$

17–24 ■ Find the center, vertices, foci, and asymptotes of the hyperbola, and sketch the graph.

17. $-\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$
18. $\dfrac{x^2}{49} - \dfrac{y^2}{32} = 1$
19. $x^2 - 2y^2 = 16$
20. $x^2 - 4y^2 + 16 = 0$
21. $\dfrac{(x+4)^2}{16} - \dfrac{y^2}{16} = 1$
22. $\dfrac{(x-2)^2}{8} - \dfrac{(y+2)^2}{8} = 1$
23. $9y^2 + 18y = x^2 + 6x + 18$
24. $y^2 = x^2 + 6y$

25–30 ■ Find an equation for the conic whose graph is shown.

25.



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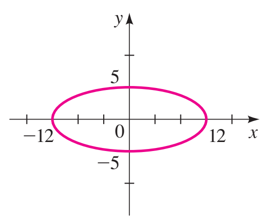

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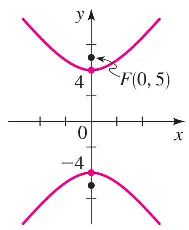

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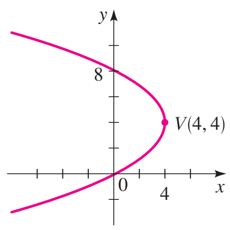

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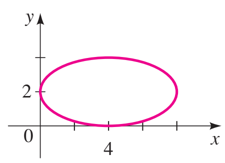

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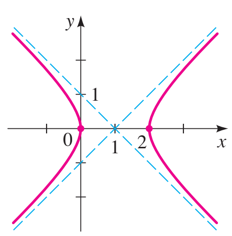


31–42 ■ Determine the type of curve represented by the equation. Find the foci and vertices (if any), and sketch the graph.

31. $\dfrac{x^2}{12} + y = 1$

32. $\dfrac{x^2}{12} + \dfrac{y^2}{144} = \dfrac{y}{12}$

33. $x^2 - y^2 + 144 = 0$

34. $x^2 + 6x = 9y^2$

35. $4x^2 + y^2 = 8(x + y)$

36. $3x^2 - 6(x + y) = 10$

37. $x = y^2 - 16y$

38. $2x^2 + 4 = 4x + y^2$

39. $2x^2 - 12x + y^2 + 6y + 26 = 0$

40. $36x^2 - 4y^2 - 36x - 8y = 31$

41. $9x^2 + 8y^2 - 15x + 8y + 27 = 0$

42. $x^2 + 4y^2 = 4x + 8$

43–50 ■ Find an equation for the conic section with the given properties.

43. The parabola with focus $F(0, 1)$ and directrix $y = -1$

44. The ellipse with center $C(0, 4)$, foci $F_1(0, 0)$ and $F_2(0, 8)$, and major axis of length 10

45. The hyperbola with vertices $V(0, \pm 2)$ and asymptotes $y = \pm\frac{1}{2}x$

46. The hyperbola with center $C(2, 4)$, foci $F_1(2, 1)$ and $F_2(2, 7)$, and vertices $V_1(2, 6)$ and $V_2(2, 2)$

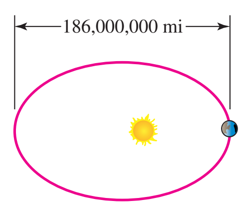
47. The ellipse with foci $F_1(1, 1)$ and $F_2(1, 3)$, and with one vertex on the x-axis

48. The parabola with vertex $V(5, 5)$ and directrix the y-axis

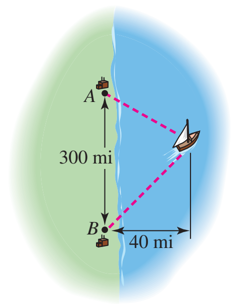
49. The ellipse with vertices $V_1(7, 12)$ and $V_2(7, -8)$, and passing through the point $P(1, 8)$

50. The parabola with vertex $V(-1, 0)$ and horizontal axis of symmetry, and crossing the y-axis at $y = 2$

51. The path of the earth around the sun is an ellipse with the sun at one focus. The ellipse has major axis 186,000,000 mi and eccentricity 0.017. Find the distance between the earth and the sun when the earth is **(a)** closest to the sun and **(b)** farthest from the sun.



52. A ship is located 40 mi from a straight shoreline. LORAN stations A and B are located on the shoreline, 300 mi apart. From the LORAN signals, the captain determines that his ship is 80 mi closer to A than to B. Find the location of the ship. (Place A and B on the y-axis with the x-axis halfway between them. Find the x- and y-coordinates of the ship.)



53. **(a)** Draw graphs of the following family of ellipses for $k = 1, 2, 4$, and 8.

$$\frac{x^2}{16 + k^2} + \frac{y^2}{k^2} = 1$$

(b) Prove that all the ellipses in part (a) have the same foci.

54. **(a)** Draw graphs of the following family of parabolas for $k = \frac{1}{2}, 1, 2$, and 4.

$$y = kx^2$$

(b) Find the foci of the parabolas in part (a).

(c) How does the location of the focus change as k increases?

55–58 ■ An equation of a conic is given.

(a) Use the discriminant to determine whether the graph of the equation is a parabola, an ellipse, or a hyperbola.

(b) Use a rotation of axes to eliminate the xy-term.

(c) Sketch the graph.

55. $x^2 + 4xy + y^2 = 1$

56. $5x^2 - 6xy + 5y^2 - 8x + 8y - 8 = 0$

57. $7x^2 - 6\sqrt{3}xy + 13y^2 - 4\sqrt{3}x - 4y = 0$

58. $9x^2 + 24xy + 16y^2 = 25$

59–62 ■ Use a graphing device to graph the conic. Identify the type of conic from the graph.

59. $5x^2 + 3y^2 = 60$

60. $9x^2 - 12y^2 + 36 = 0$

61. $6x + y^2 - 12y = 30$

62. $52x^2 - 72xy + 73y^2 = 100$

63–66 ■ A polar equation of a conic is given.

(a) Find the eccentricity and identify the conic.

(b) Sketch the conic and label the vertices.

63. $r = \dfrac{1}{1 - \cos\theta}$

64. $r = \dfrac{2}{3 + 2\sin\theta}$

65. $r = \dfrac{4}{1 + 2\sin\theta}$

66. $r = \dfrac{12}{1 - 4\cos\theta}$

67–70 ■ A pair of parametric equations is given.

(a) Sketch the curve represented by the parametric equations.

(b) Find a rectangular-coordinate equation for the curve by eliminating the parameter.

67. $x = 1 - t^2, \quad y = 1 + t$

68. $x = t^2 - 1, \quad y = t^2 + 1$

69. $x = 1 + \cos t, \quad y = 1 - \sin t, \quad 0 \le t \le \pi/2$

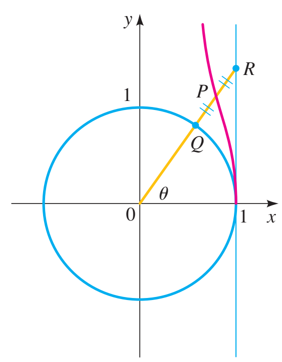
70. $x = \dfrac{1}{t} + 2, \quad y = \dfrac{2}{t^2}, \quad 0 < t \le 2$

71–72 ■ Use a graphing device to draw the parametric curve.

71. $x = \cos 2t, \quad y = \sin 3t$

72. $x = \sin(t + \cos 2t), \quad y = \cos(t + \sin 3t)$

73. In the figure the point P is the midpoint of the segment QR and $0 \le \theta < \pi/2$. Using θ as the parameter, find a parametric representation for the curve traced out by P.



10 Test

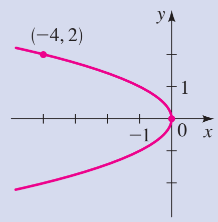
1. Find the focus and directrix of the parabola $x^2 = -12y$, and sketch its graph.

2. Find the vertices, foci, and the lengths of the major and minor axes for the ellipse $\frac{x^2}{16} + \frac{y^2}{4} = 1$. Then sketch its graph.

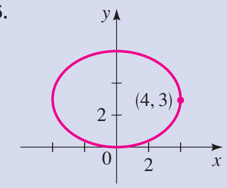
3. Find the vertices, foci, and asymptotes of the hyperbola $\frac{y^2}{9} - \frac{x^2}{16} = 1$. Then sketch its graph.

4–6 ■ Find an equation for the conic whose graph is shown.

4.



5.



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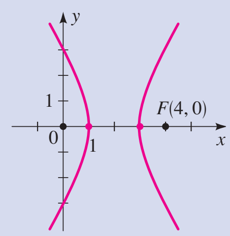


7–9 ■ Sketch the graph of the equation.

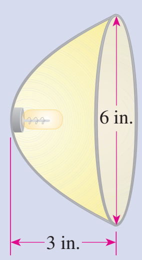
7. $16x^2 + 36y^2 - 96x + 36y + 9 = 0$

8. $9x^2 - 8y^2 + 36x + 64y = 164$

9. $2x + y^2 + 8y + 8 = 0$

10. Find an equation for the hyperbola with foci $(0, \pm 5)$ and with asymptotes $y = \pm\frac{3}{4}x$.

11. Find an equation for the parabola with focus $(2, 4)$ and directrix the x-axis.



12. A parabolic reflector for a car headlight forms a bowl shape that is 6 in. wide at its opening and 3 in. deep, as shown in the figure at the left. How far from the vertex should the filament of the bulb be placed if it is to be located at the focus?

13. (a) Use the discriminant to determine whether the graph of this equation is a parabola, an ellipse, or a hyperbola:

$$5x^2 + 4xy + 2y^2 = 18$$

(b) Use rotation of axes to eliminate the xy-term in the equation.

(c) Sketch the graph of the equation.

(d) Find the coordinates of the vertices of this conic (in the xy-coordinate system).

14. (a) Find the polar equation of the conic that has a focus at the origin, eccentricity $e = \frac{1}{2}$, and directrix $x = 2$. Sketch the graph.

(b) What type of conic is represented by the following equation? Sketch its graph.

$$r = \frac{3}{2 - \sin\theta}$$

15. (a) Sketch the graph of the parametric curve

$$x = 3\sin\theta + 3 \qquad y = 2\cos\theta \qquad 0 \le \theta \le \pi$$

(b) Eliminate the parameter θ in part (a) to obtain an equation for this curve in rectangular coordinates.

Focus on Modeling
The Path of a Projectile

Modeling motion is one of the most important ideas in both classical and modern physics. Much of Isaac Newton's work dealt with creating a mathematical model for how objects move and interact—this was the main reason for his invention of calculus. Albert Einstein developed his Special Theory of Relativity in the early 1900s to refine Newton's laws of motion.

In this section we use coordinate geometry to model the motion of a projectile, such as a ball thrown upward into the air, a bullet fired from a gun, or any other sort of missile. A similar model was created by Galileo, but we have the advantage of using our modern mathematical notation to make describing the model much easier than it was for Galileo!

Parametric Equations for the Path of a Projectile

Suppose that we fire a projectile into the air from ground level, with an initial speed v_0 and at an angle θ upward from the ground. The initial *velocity* of the projectile is a vector (see Section 8.4) with horizontal component $v_0 \cos\theta$ and vertical component $v_0 \sin\theta$, as shown in Figure 1.

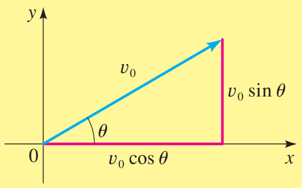


Figure 1

If there were no gravity (and no air resistance), the projectile would just keep moving indefinitely at the same speed and in the same direction. Since distance = speed × time, the projectile's position at time t would therefore be given by the following parametric equations (assuming the origin of our coordinate system is placed at the initial location of the projectile):

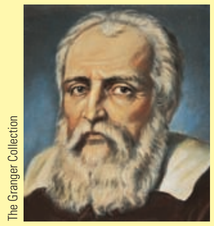
$$x = (v_0 \cos\theta)t \qquad y = (v_0 \sin\theta)t \qquad \text{No gravity}$$

But, of course, we know that gravity will pull the projectile back to ground level. Using calculus, it can be shown that the effect of gravity can be accounted for by subtracting $\frac{1}{2}gt^2$ from the vertical position of the projectile. In this expression, g is the gravitational acceleration: $g \approx 32$ ft/s^2 ≈ 9.8 m/s^2. Thus, we have the following parametric equations for the path of the projectile:

$$x = (v_0 \cos\theta)t \qquad y = (v_0 \sin\theta)t - \tfrac{1}{2}gt^2 \qquad \text{With gravity}$$

Example **The Path of a Cannonball**

Find parametric equations that model the path of a cannonball fired into the air with an initial speed of 150.0 m/s at a 30° angle of elevation. Sketch the path of the cannonball.



Galileo Galilei (1564–1642) was born in Pisa, Italy. He studied medicine, but later abandoned this in favor of science and mathematics. At the age of 25 he demonstrated that light objects fall at the same rate as heavier ones, by dropping cannonballs of various sizes from the Leaning Tower of Pisa. This contradicted the then-accepted view of Aristotle that heavier objects fall more quickly. He also showed that the distance an object falls is proportional to the square of the time it has been falling, and from this was able to prove that the path of a projectile is a parabola.

Galileo constructed the first telescope, and using it, discovered the moons of Jupiter. His advocacy of the Copernican view that the earth revolves around the sun (rather than being stationary) led to his being called before the Inquisition. By then an old man, he was forced to recant his views, but he is said to have muttered under his breath "the earth nevertheless does move." Galileo revolutionized science by expressing scientific principles in the language of mathematics. He said, "The great book of nature is written in mathematical symbols."

Solution Substituting the given initial speed and angle into the general parametric equations of the path of a projectile, we get

$$x = (150.0 \cos 30°)t \qquad y = (150.0 \sin 30°)t - \tfrac{1}{2}(9.8)t^2 \qquad \text{Substitute } v_0 = 150.0,\ \theta = 30°$$

$$x = 129.9t \qquad y = 75.0t - 4.9t^2 \qquad \text{Simplify}$$

This path is graphed in Figure 2.

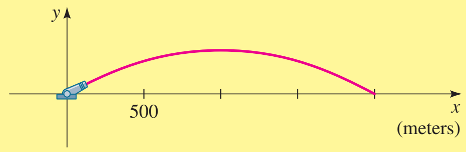


Figure 2
Path of a cannonball

Range of a Projectile

How can we tell where and when the cannonball of the above example hits the ground? Since ground level corresponds to $y = 0$, we substitute this value for y and solve for t.

$$0 = 75.0t - 4.9t^2 \qquad \text{Set } y = 0$$

$$0 = t(75.0 - 4.9t) \qquad \text{Factor}$$

$$t = 0 \qquad \text{or} \qquad t = \frac{75.0}{4.9} \approx 15.3 \qquad \text{Solve for } t$$

The first solution, $t = 0$, is the time when the cannon was fired; the second solution means that the cannonball hits the ground after 15.3 s of flight. To see *where* this happens, we substitute this value into the equation for x, the horizontal location of the cannonball.

$$x = 129.9(15.3) \approx 1987.5 \text{ m}$$

The cannonball travels almost 2 km before hitting the ground.

Figure 3 shows the paths of several projectiles, all fired with the same initial speed but at different angles. From the graphs we see that if the firing angle is too high or too low, the projectile doesn't travel very far.

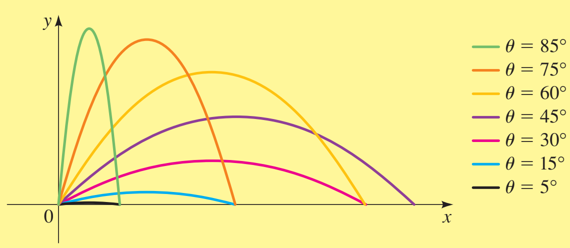


Figure 3
Paths of projectiles

Let's try to find the optimal firing angle—the angle that shoots the projectile as far as possible. We'll go through the same steps as we did in the preceding example, but we'll use the general parametric equations instead. First, we solve for the time when the projectile hits the ground by substituting $y = 0$.

$$0 = (v_0 \sin\theta)t - \tfrac{1}{2}gt^2 \qquad \text{Substitute } y = 0$$

$$0 = t(v_0 \sin\theta - \tfrac{1}{2}gt) \qquad \text{Factor}$$

$$0 = v_0 \sin\theta - \tfrac{1}{2}gt \qquad \text{Set second factor equal to 0}$$

$$t = \frac{2v_0 \sin\theta}{g} \qquad \text{Solve for } t$$

Now we substitute this into the equation for x to see how far the projectile has traveled horizontally when it hits the ground.

$$x = (v_0 \cos\theta)t \qquad \text{Parametric equation for } x$$

$$= (v_0 \cos\theta)\left(\frac{2v_0 \sin\theta}{g}\right) \qquad \text{Substitute } t = (2v_0 \sin\theta)/g$$

$$= \frac{2v_0^2 \sin\theta \cos\theta}{g} \qquad \text{Simplify}$$

$$= \frac{v_0^2 \sin 2\theta}{g} \qquad \text{Use identity } \sin 2\theta = 2\sin\theta\cos\theta$$

We want to choose θ so that x is as large as possible. The largest value that the sine of any angle can have is 1, the sine of 90°. Thus, we want $2\theta = 90°$, or $\theta = 45°$. So to send the projectile as far as possible, it should be shot up at an angle of 45°. From the last equation in the preceding display, we can see that it will then travel a distance $x = v_0^2/g$.

Problems

1. Trajectories are Parabolas From the graphs in Figure 3 the paths of projectiles appear to be parabolas that open downward. Eliminate the parameter t from the general parametric equations to verify that these are indeed parabolas.

2. Path of a Baseball Suppose a baseball is thrown at 30 ft/s at a 60° angle to the horizontal, from a height of 4 ft above the ground.

(a) Find parametric equations for the path of the baseball, and sketch its graph.

(b) How far does the baseball travel, and when does it hit the ground?

3. Path of a Rocket Suppose that a rocket is fired at an angle of 5° from the vertical, with an initial speed of 1000 ft/s.

(a) Find the length of time the rocket is in the air.

(b) Find the greatest height it reaches.

(c) Find the horizontal distance it has traveled when it hits the ground.

(d) Graph the rocket's path.

4. Firing a Missile The initial speed of a missile is 330 m/s.

(a) At what angle should the missile be fired so that it hits a target 10 km away? (You should find that there are two possible angles.) Graph the missile paths for both angles.

(b) For which angle is the target hit sooner?

5. Maximum Height Show that the maximum height reached by a projectile as a function of its initial speed v_0 and its firing angle θ is

$$y = \frac{v_0^2 \sin^2\theta}{2g}$$

6. Shooting into the Wind Suppose that a projectile is fired into a headwind that pushes it back so as to reduce its horizontal speed by a constant amount w. Find parametric equations for the path of the projectile.

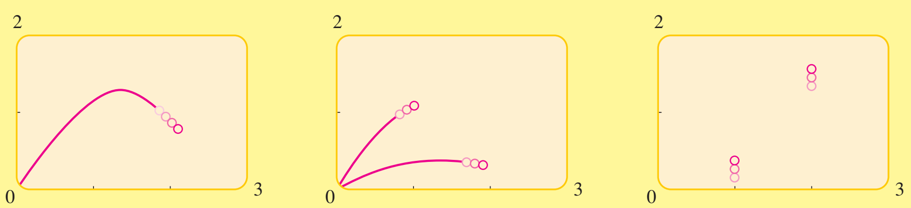
7. Shooting into the Wind Using the parametric equations you derived in Problem 6, draw graphs of the path of a projectile with initial speed $v_0 = 32$ ft/s, fired into a headwind of $w = 24$ ft/s, for the angles $\theta = 5°, 15°, 30°, 40°, 45°, 55°, 60°$, and $75°$. Is it still true that the greatest range is attained when firing at $45°$? Draw some more graphs for different angles, and use these graphs to estimate the optimal firing angle.

8. Simulating the Path of a Projectile The path of a projectile can be simulated on a graphing calculator. On the TI-83 use the "Path" graph style to graph the general parametric equations for the path of a projectile and watch as the circular cursor moves, simulating the motion of the projectile. Selecting the size of the `Tstep` determines the speed of the "projectile."

(a) Simulate the path of a projectile. Experiment with various values of θ. Use $v_0 = 10$ ft/s and `Tstep` $= 0.02$. Part (a) of the figure below shows one such path.

(b) Simulate the path of two projectiles, fired simultaneously, one at $\theta = 30°$ and the other at $\theta = 60°$. This can be done on the TI-83 using `Simul` mode ("simultaneous" mode). Use $v_0 = 10$ ft/s and `Tstep` $= 0.02$. See part (b) of the figure. Where do the projectiles land? Which lands first?

(c) Simulate the path of a ball thrown straight up ($\theta = 90°$). Experiment with values of v_0 between 5 and 20 ft/s. Use the "Animate" graph style and `Tstep` $= 0.02$. Simulate the path of two balls thrown simultaneously at different speeds. To better distinguish the two balls, place them at different x-coordinates (for example, $x = 1$ and $x = 2$). See part (c) of the figure. How does doubling v_0 change the maximum height the ball reaches?



(a) (b) (c)