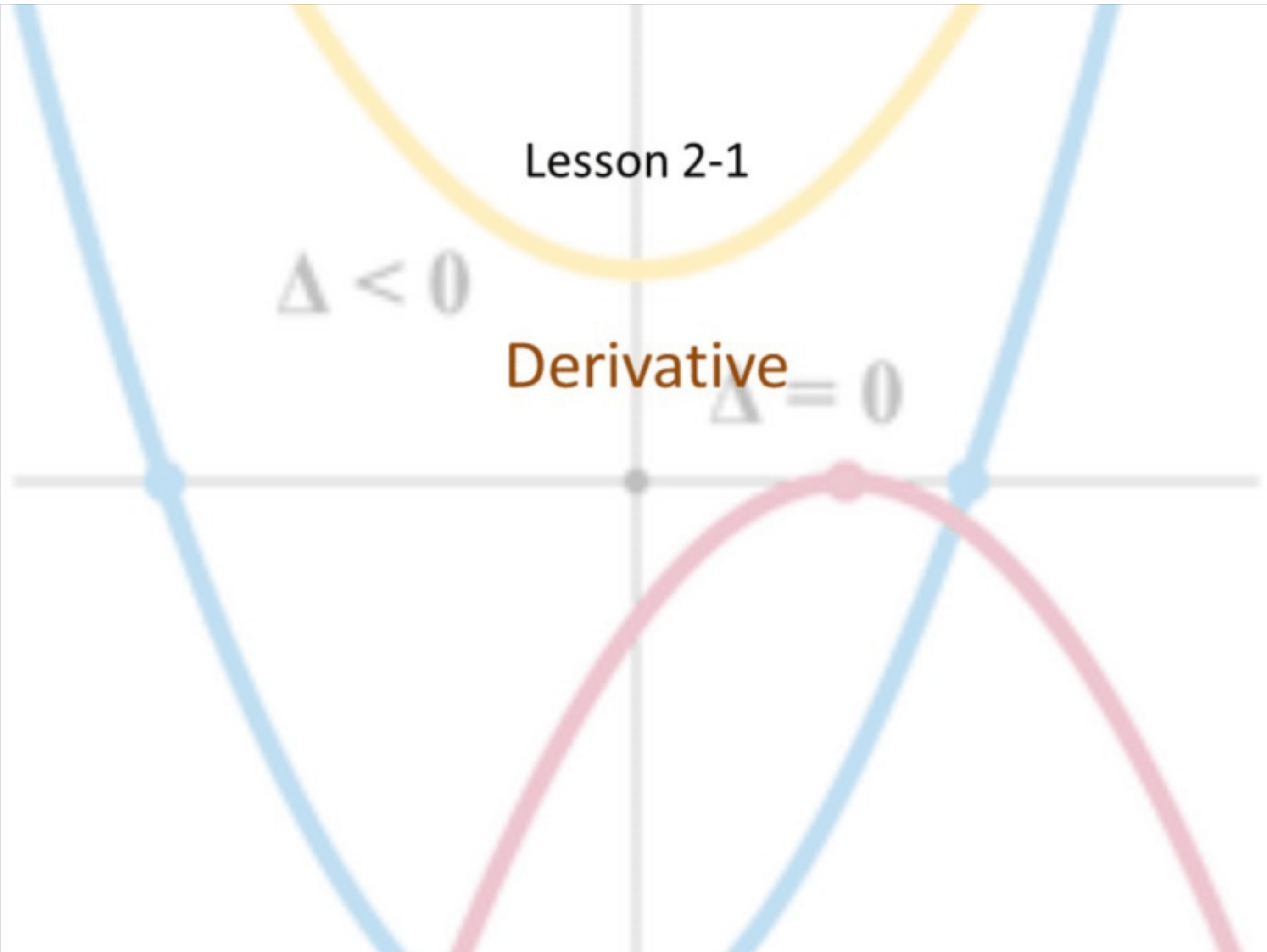
Lesson 2-1

Derivative



Objective

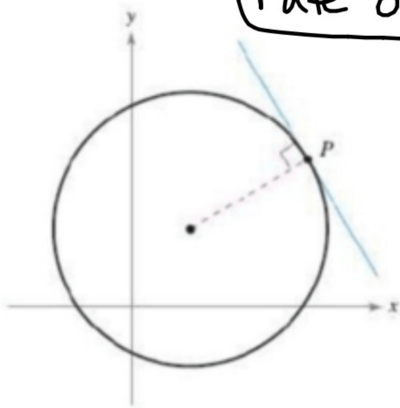
Students will…

- Be able to define what a tangent line is.
- Be able to make connections between tangent lines to the rate of change (slope).
- Be able to define derivative and find it.
- Be able to understand the relationship between differentiability and continuity.

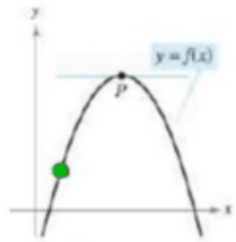
The Tangent Line Problem

Calculus is said to have grown out of 4 major problems. First of these problems involve the **tangent line**. Recall that a tangent line is **a line that represents the slope at a certain point**. See examples below:

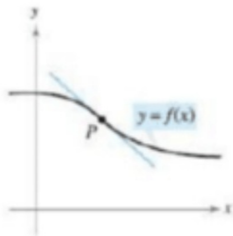
rate of change.

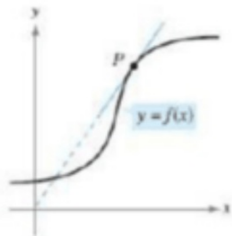


Tangent line to a circle
Figure 2.1

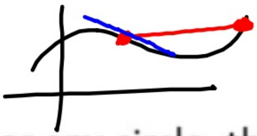


Tangent line to a curve at a point
Figure 2.2





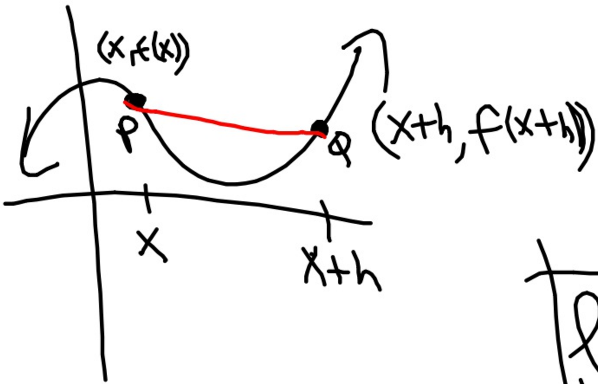
ex.



From Secant to Tangent

For any circle, the tangent line is always **perpendicular to the radius**. However, for a curve this isn't an easy thing to find. In order to find the tangent line, we need to use the **secant line**, which is a line created by connecting two points on the curve. (Think **average** rate of change!)

* secant line = avg. rate of change $\left(m = \frac{y_2 - y_1}{x_2 - x_1}\right)$.



$$m = \frac{f(x+h) - f(x)}{\cancel{x} + h - \cancel{x}}$$

$$= \frac{f(x+h) - f(x)}{h}$$

$$\boxed{\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

Derivative

To find the slope of any point on a function is known as finding its derivative at that point. It is also known as differentiating a function at a certain point. So now, we can define what a derivative is at x:

$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$, provided the limit exists.

$f'(x)$ is read as "f prime of x."

Notice the different notation for derivatives.

$$f'(x), \quad \frac{dy}{dx}, \quad y', \quad \frac{d}{dx}[f(x)], \quad D_x[y].$$

Again, derivative is simply finding slope, or ~~average~~ instantaneous. rate of change.

Examples

$$f(x+h) = 2(x+h) - 3$$
$$2x + 2h - \cancel{3}$$

Find the derivative of $f(x) = 2x^{1} - 3$

$$\lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} = \frac{\cancel{2x} + 2h \cancel{- 3} - (\cancel{2x} \cancel{- 3})}{h} = \frac{2\cancel{h}}{\cancel{h}} = 2 \right).$$

$$= \lim_{h \to 0} 2 = \boxed{2x^0}$$

$f(x+h) = (x+h)^3 + 2(x+h)$

$= x^3 + 3x^2h + 3xh^2 + h^3 + 2x + 2h$

Example

1 1

1 2 1

1 3 3 1

$(x+a)^2$

$(x+a)^3$

Find the derivative of $f(x) = x^3 + 2x$

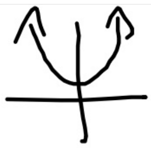
$$\lim_{h\to 0}\left(\frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 \cancel{+2x} + 2h - (\cancel{x^3} \cancel{+2x})}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{\cancel{h}(3x^2 \cancel{+3xh} \cancel{+h^2} + 2)}{\cancel{h}}\right) = \boxed{3x^2+2}$$

$(x+h)^2+1$

$x^2+2xh+h^2+1$

Example



Find the derivative of $f(x) = x^2 + 1$

$$\lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h} = \frac{\cancel{x^2}+2xh+h^2\cancel{+1}-(\cancel{x^2}\cancel{+1})}{h} = \frac{2xh+h^2}{h} = \frac{\cancel{h}(2x+h)}{\cancel{h}}\right)$$

$$\lim_{h\to 0} 2x+h = \boxed{2x}$$

$f(t+h) = \frac{2}{t+h}$

Example

$\frac{2\cancel{h}}{\cancel{h}\, t(t+h)}$

Find the derivative of $f(t) = \frac{2}{t}$. Then, find the tangent to the graph at point $\boxed{(1, 2)}$

$$\lim_{h \to 0} \left(\frac{\frac{2 \cdot t}{t+h} - \frac{2(t+h)}{t}}{h} = \frac{\frac{\cancel{2t} - (\cancel{2t} + 2h)}{t(t+h)}}{\cancel{h}} = \frac{-2}{t(t+h)} \right)$$

$$= \frac{-2}{t^2}$$

$m = \frac{-2}{1^2} = -2$

tangent line @ $(1, 2)$

$$\boxed{y - 2 = -2(x - 1)}$$

Differentiability and Continuity

Recall that limit only exists if the right side and the left side limits match. It turns out, this is also true for differentiability (derivatives).

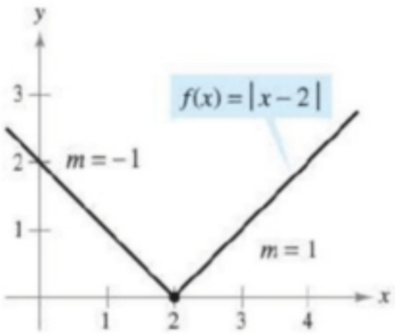
A function, say f, is differentiable if and only if,

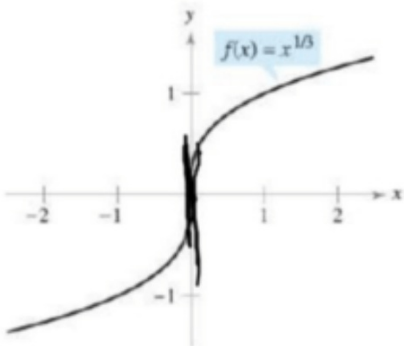
$$\lim_{h\to 0^-} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0^+} \frac{f(x+h)-f(x)}{h}$$

That being said, a function is not differentiable at these instances:

1. Cusp (sharp turn, or corners) 
2. Holes
3. Vertical asymptotes
4. Vertical line
5. Jump discontinuities.

Example





Differentiability and Continuity

Somethings to keep in mind regarding derivatives and continuity…

1. When a function is **<u>not</u>** continuous at $x = c$, it is also **<u>not</u>** **<u>differentiable</u>** at $x = c$.
2. If the function is **<u>differentiable</u>** at $x = c$, it is also **<u>continuous</u>** there.

However, the converse is NOT necessarily true!!

1. If function is **<u>not</u>** differentiable at $x = c$, it is also **<u>not</u>** continuous there.
2. If the function is continuous at $x = c$, it is also differentiable there .

Homework Due 9/9

TB 2.1- #5, 7, 9, 37-40, 59, 81-86, 99-103