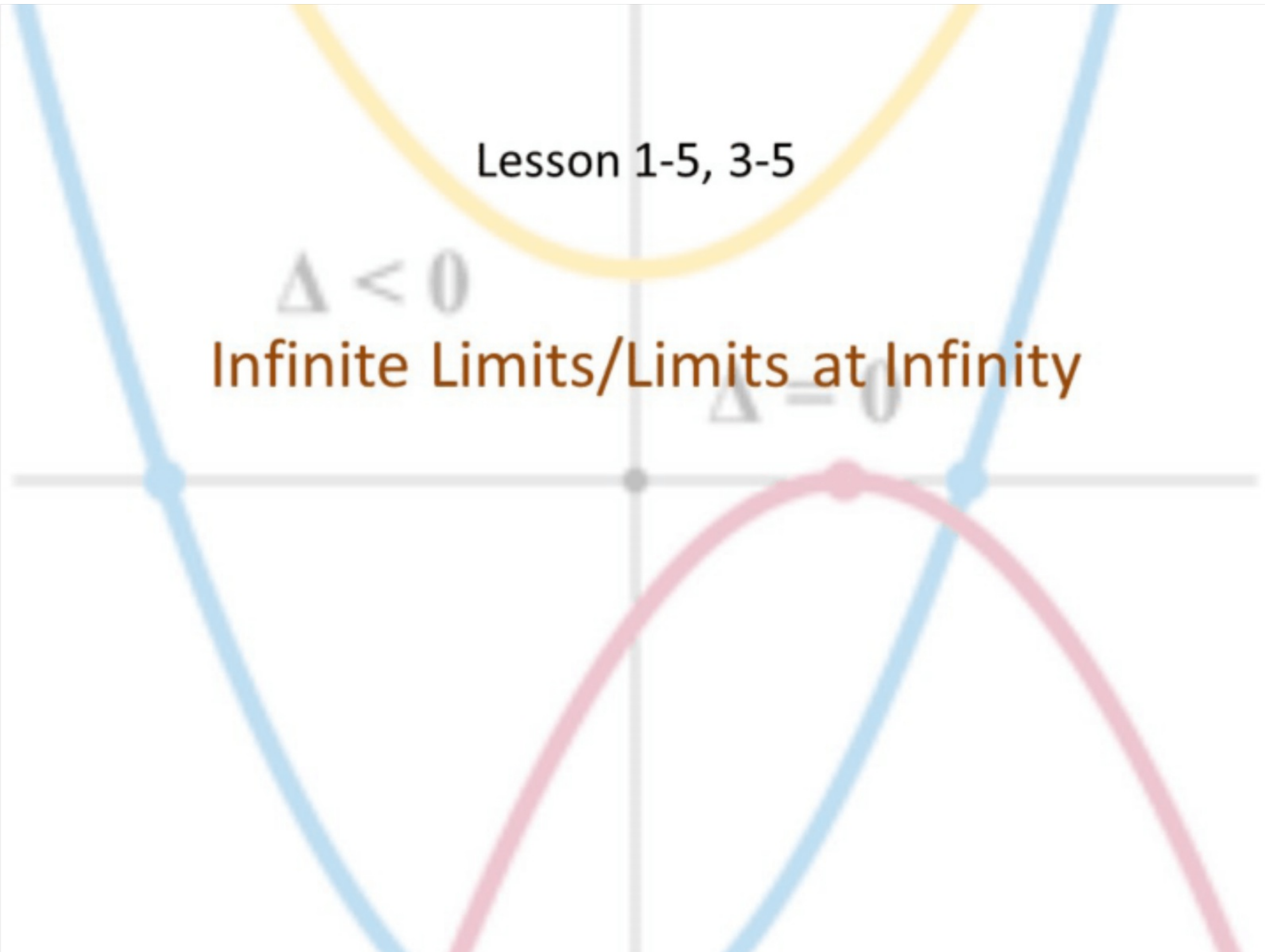
Lesson 1-5, 3-5

Infinite Limits/Limits at Infinity



Objective

Students will…

- Be able to define and determine infinite limits.
- Be able to determine (finite) limits at infinity.
- Be able to find limits of rational functions at infinity by finding its horizontal asymptotes.

Infinite Limits

$\lim_{x \to c} f(x) = \infty$, means…
"the limit of $f(x)$ as x approaches c is ∞."

Or, as x approaches c, y or $f(x)$ grows positively without bound.

On the other hand, $\lim_{x \to c} f(x) = -\infty$
"the limit of $f(x)$ as x approaches c is $-\infty$."

Or, as x approaches c, y or $f(x)$ grows negatively without bound.

Disclaimer: This actually shows that the limit **DOES NOT EXIST**. Infinity is **not** a number.

Vertical Asymptotes

We learned in our last study that vertical asymptotes are a type of a nonremovable discontinuity, i.e. the limit fails to exist. Better yet, the limit fails to exist because the limit is either ∞ or $-\infty$. Here is a quick way to find vertical asymptotes of a rational function.

Vertical asymptotes- For a rational function $h(x) = \frac{f(x)}{g(x)}$, and for some real number c, if $f(c) \neq 0$ and $g(c) = 0$, then $h(x)$ has a vertical asymptote at a $x = c$.

In other words, by default $h(x)$ has a nonremovable discontinuity at $x = c$.

Examples

a. $\lim_{x\to 1} \frac{x^2-3x}{x-1}$ = DNE.

$x-1 \neq 0$

$x \neq 1$

$1^2-3(1) = -2 \neq 0$

$\boxed{V.A @ x=1}$

b. $\lim_{x\to 0}\left(\cot x = \frac{\cos x}{\sin x}\right)$

$x \neq 0$

$\cos(0) = 1 \neq 0.$

V.A @ $x=0$

Limit DNE

Properties of Infinite Limits

THEOREM 1.15 PROPERTIES OF INFINITE LIMITS

Let c and L be real numbers and let f and g be functions such that

$$\lim_{x \to c} f(x) = \infty \qquad \text{and} \qquad \lim_{x \to c} g(x) = L.$$

$\infty \pm L$

$= \lim f(x) \pm \lim g(x)$

1. **Sum or difference:** $\lim_{x \to c} [f(x) \pm g(x)] = \infty$
2. **Product:** $\lim_{x \to c} [f(x)g(x)] = \infty, \quad L > 0$ $\quad \infty \cdot L = \infty$

   $\lim_{x \to c} [f(x)g(x)] = -\infty, \quad L < 0$ $\quad \infty \cdot -L = -\infty.$
3. **Quotient:** $\lim_{x \to c} \dfrac{g(x)}{f(x)} = 0$ $\quad \dfrac{L}{\infty} = 0$ $\qquad \lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{\infty}{L} = \infty$

Similar properties hold for one-sided limits and for functions for which the limit of $f(x)$ as x approaches c is $-\infty$.

Example

a. $\lim_{x\to 0}\left(1+\frac{1}{x^2}\right)$

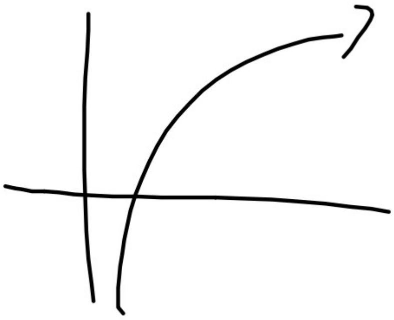
$$= \lim_{x\to 0} 1 + \lim_{x\to 0} \frac{1}{x^2}$$

$$1 + \infty = \infty$$

b. $\lim_{x\to 1^-} \frac{x^2+1}{\cot \pi x}$

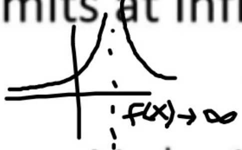
$$\frac{\lim_{x\to 1^-} x^2+1}{\lim_{x\to 1^-} \cot \pi x} = \frac{2}{\frac{\cos \pi x}{\sin \pi x}} = \frac{1}{\infty} = 0$$

c. $\lim_{x\to 0^+} 3\ln x$ $= -\infty$

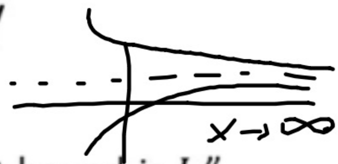


Limits at Infinity

$\lim_{x \to \infty} f(x) = L$, means…



vs.



"the limit of $f(x)$ as x grows positively without bound is L."

Or, as x approaches infinity, y or $f(x)$ approaches L.

On the other hand, $\lim_{x \to -\infty} f(x) = L$

"the limit of $f(x)$ as x grows negatively without bound is L."

Or, as x approaches negative infinity, y or $f(x)$ approaches L.

Horizontal Asymptote

The most useful way to evaluate limits at infinity is to find the **horizontal asymptote**. Recall from Pre-Calculus or Algebra 2…

1. If the degree of the numerator is less than the degree of the denominator, then the limit of the rational function is 0.
2. If the degree of the numerator is equal to the degree of the denominator, then the limit of the rational function is the ratio of the leading coefficients.
3. If the degree of the numerator is greater than the degree of the denominator, then the limit of the rational function does not exist, or the limit is $\pm\infty$.

Remember: The bigger the denominator gets, the closer your function gets to zero.

Example

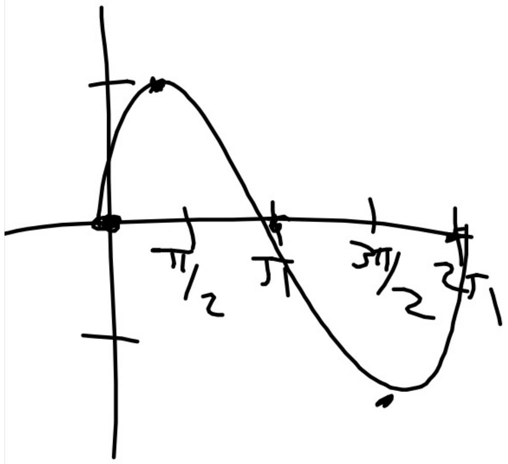
a. $\lim\limits_{x\to\infty} \frac{2x+5}{3x^2+1} = 0$

b. $\lim\limits_{x\to\infty} \frac{2x^2+5}{3x^2+1} = \boxed{\frac{2}{3}}$

c. $\lim\limits_{x\to\infty} \frac{2x^3+5}{3x^2+1} = \infty$

Limits at Infinity with Trig Functions

a. $\lim_{x\to\infty} \sin x$ = DNE.



b. $\lim_{x\to\infty} \frac{\sin x}{x} = \boxed{0}$

Example

a. $\lim\limits_{x\to\infty} \frac{2x^2-4x}{x+1} = \infty$

b. $\lim\limits_{x\to-\infty} \frac{2x^2-4x}{x+1} = -\infty$

c. $\lim\limits_{x\to\infty} \frac{2x^6-3}{x^2+1} = \infty$

Homework Due 8/30

TB 1.5 #1-4, 9-29 (e.o.o), 33-49 (e.o.o), 53, 54, 67-70, 73, 74

TB 3.5 #1-8, 17-45 (e.o.o)