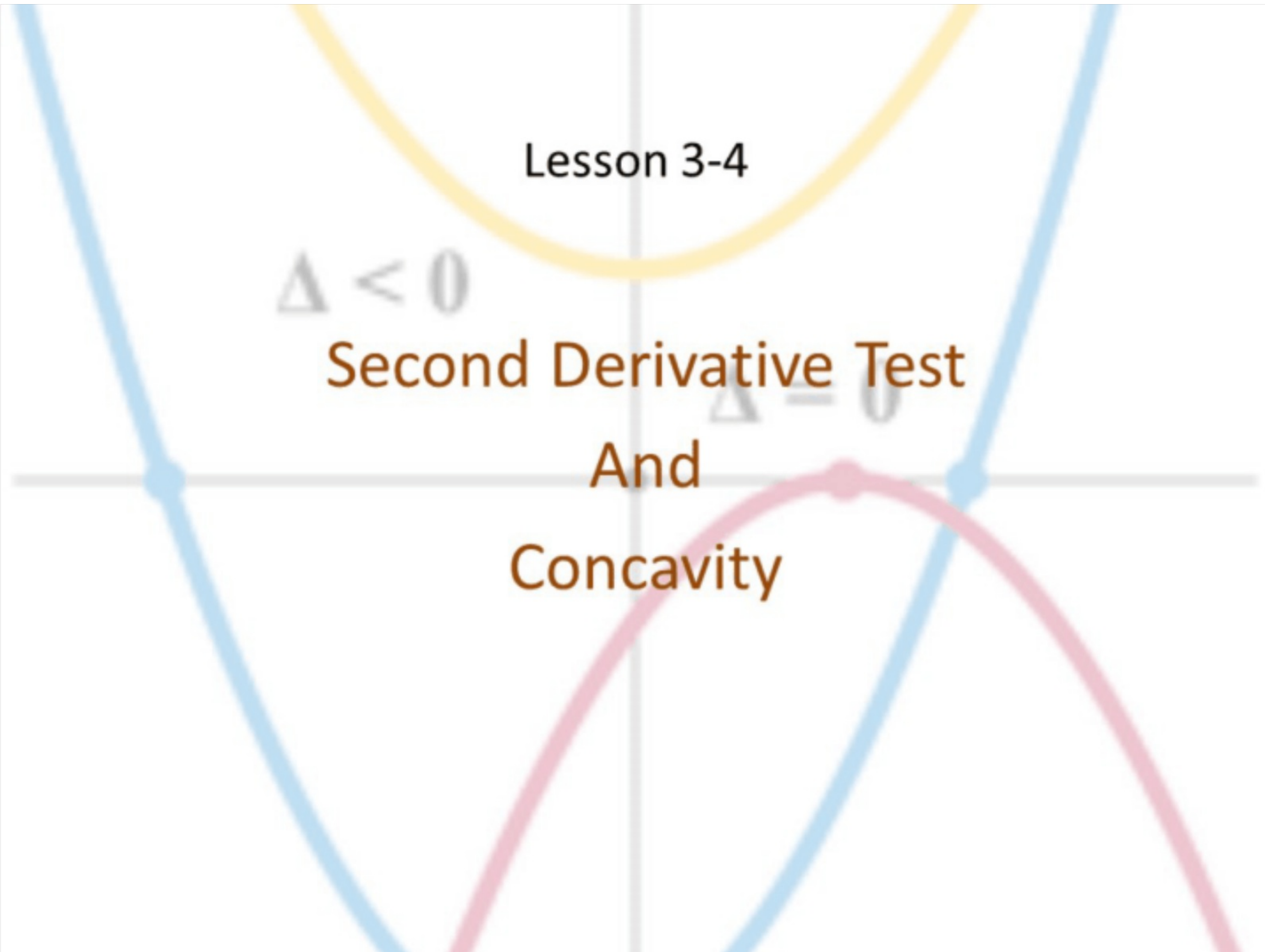
Lesson 3-4

Second Derivative Test
And
Concavity



Objective

Students will…

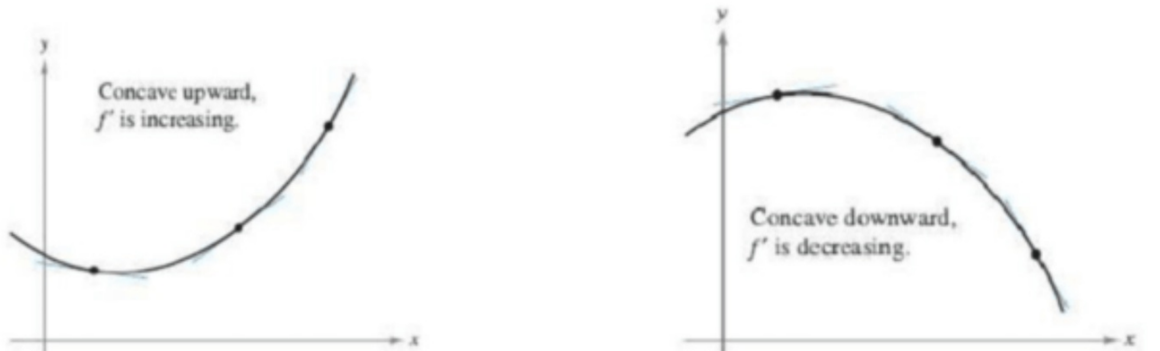
- Be able to define concavity.
- Be able to determine the different intervals of concavity.
- Be able to define and find points of inflection.
- Be able to know and apply the Second Derivative Test.

Concavity

Another useful information regarding graph is concavity.

<u>Concavity</u>- Let f be differentiable on an open interval I. The graph of f is **concave upward** on I if f' is increasing on the interval and **concave downward** on I if f' is decreasing on the interval.

Graphically speaking…



(a) The graph of f lies above its tangent lines. (b) The graph of f lies below its tangent lines.

Figure 3.24

Concavity and the Second Derivative

The being said, if we used the first derivative f' to determine the intervals in which f increases or decreases, we would naturally use the second derivative f'' to do the same for the graph of f'.

Test for Concavity- Let f be a function whose second derivative exists on an open interval I.

1. If $f''(x) > 0$ for all x in I, then f' is increasing on I. Therefore, f is concave upward in I.
2. If $f''(x) < 0$ for all x in I, then f' is decreasing on I. Therefore, f is concave downward in I.
3. If $f''(x) = 0$ for all x in I, then f' is constant on I. Therefore, f is neither concave upward nor downward in I.

Note: Concavity is **not** defined for a linear line.

Examples

Determine the open intervals on which the graph of $f(x) = \frac{6}{x^2+3}$ is concave upward or downward.

$= 6(x^2+3)^{-1}$

$f'(x) = -6(x^2+3)^{-2} \cdot 2x$

$= -12x(x^2+3)^{-2}$ (f, g)

$f''(x) = -12(x^2+3)^{-2} + -2(x^2+3)^{-3} \cdot 2x(-12x)$

$= -12(x^2+3)^{-2} + 48x^2(x^2+3)^{-3}$

CV: $0 = \frac{-12}{(x^2+3)^2} + \frac{48x^2}{(x^2+3)^3} = \frac{-12(x^2+3)+48x^2}{(x^2+3)^3}$ +

$0 = -12(x^2+3) + 48x^2$

$0 = -12(x^2+3-4x^2)$

$0 = -3x^2+3$

$1 = x^2$

$x = \pm 1$

-1		1
-2	0	2
+	−	+

f' inc: $(-\infty,-1) \cup (1,\infty)$

$\Rightarrow f$ is CU: $(-\infty,-1)$, $(1,\infty)$.

f' dec: $(-1,1)$

$\Rightarrow f$ is CD: $(-1,1)$.

Example

Determine the open intervals on which the graph of $f(x) = \frac{x^2+1}{x^2-4}$ is concave upward or downward.

$$f'(x) = \frac{2x(x^2-4) - 2x(x^2+1)}{(x^2-4)^2}$$

$$= \frac{2x^3 - 8x - 2x^3 - 2x}{(x^2-4)^2} = \frac{-10x}{(x^2-4)^2}$$

$$f''(x) = \frac{-10(x^2-4)^2 - 2(x^2-4)\cdot 2x(-10x)}{(x^2-4)^4}$$

$$= \frac{-10(x^2-4)^2 + 40x^2(x^2-4)}{(x^2-4)^4}$$

CV: $(x^2-4)(5x^2-4) = 0$

$x = \pm 2, \pm\sqrt{\frac{4}{5}} = \pm\frac{2\sqrt{5}}{5}$

	-2		$-\frac{2\sqrt{5}}{5}$		$\frac{2\sqrt{5}}{5}$		2	
-3		-1		0		1		3
$+$		$-$		$-$		$-$		$+$
CU		CD		CD		CD		CU

CU: $(-\infty, -2) \cup (2, \infty)$

CD: $\left(-2, -\frac{2\sqrt{5}}{5}\right) \cup \left(-\frac{2\sqrt{5}}{5}, \frac{2\sqrt{5}}{5}\right) \cup \left(\frac{2\sqrt{5}}{5}, 2\right)$

Points of Inflection

Recall from the First Derivative Test that **relative extrema** exist whenever f' switched signs ($+$ to $-$, or $-$ or $+$). With regards to the second derivative and concavity, such occurrence gives us the **points of inflection**.

Points of Inflection- Let f be a function that is continuous on an open interval and let c be a point in the interval. If the graph of f has a tangent line at this point $(c, f(c))$, then this point is a point of inflection of the graph of f if the concavity of f changes from upward to downward (or downward to upward) at the point.

Theorem 3.8- If $(c, f(c))$ is a point of inflection of the graph f, then either $f''(c) = 0$ or f'' does not exist at $x = c$. (i.e. critical values of f'')

Example

Determine the points of inflection and discuss the concavity of the graph of $f(x) = x^4 - 4x^3$

$f'(x) = 4x^3 - 12x^2$

$f''(x) = 12x^2 - 24x$

CV: $0 = 12x(x-2)$

$x = 0, 2$

	0		2	
−1		1		3
+		−		+
CU	PI	CD	PI	CU

CU: $(-\infty, 0) \cup (2, \infty)$

CD: $(0, 2)$

Pts of Inf: $(0, 0)$, $(2, -16)$.

Second Derivative Test

The second derivative can also be used to find the relative extrema of f.

Let f be a function such that $f'(c) = 0$ and the second derivative of f exists on an open interval containing c. (c is CV of f').

1. If $f''(c) > 0$, then f has a relative minimum at $(c, f(c))$.
2. If $f''(c) < 0$, then f has a relative maximum at $(c, f(c))$.
3. If $f''(c) = 0$, then the test is inconclusive. (Need to use the **First Derivative Test**).

Warning: Second Derivative test cannot be used for critical values that does not exist in f'.

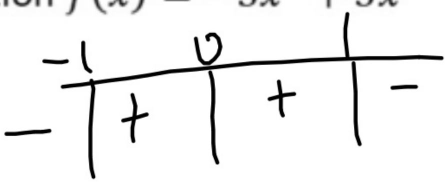
Example

Find the relative extrema of the function $f(x) = -3x^5 + 5x^3$

$f'(x) = -15x^4 + 15x^2$

CV: $0 = -15x^2(x^2 - 1)$

$x = 0, \pm 1$



$f''(x) = -60x^3 + 30x$

$f''(0) = 0$ inc.

$f''(-1) = 30 > 0$, Rel. min at $x = -1$ $= (-1, -2)$

$f''(1) = -30 < 0$, Rel max at $x = 1$ $= (1, 2)$

Example

Find the relative extrema of $f(x) = \sqrt{x^2 + 1} = (x^2+1)^{1/2}$

$f'(x) = \frac{1}{2}(x^2+1)^{-1/2} \cdot 2x$

CV

$0 = x(x^2+1)^{-1/2}$

$x = 0$

$f''(x)$

	0	
-1		1
−		+

Rel. min: (0, 1)

Example

Find the relative extrema of $f(x) = \frac{x}{x-1}$

Homework Due 10/9

TB 3.4 #1-6, 7-25 (odd), 27-39 (e.o.o)