
Lesson 3-9
Differentials
Δ < 0
Δ = 0

Objective

Students will…

- Be able to understand the concept of a tangent line approximation.
- Be able to compare the value of the differential, dy, with the actual change in y.
- Be able to estimate error using a differential.
- Be able to find the differential of a function.

$y - y_1 = m(x - x_1)$

Tangent Line Approximation

Another application of tangent line is **approximation**. Consider the following function: $f(x) = 1 + \sin x$. Find its tangent line at $(0, 1)$ and compare their outputs.

$c, f(c)$

$y - y(c) = f'(c)(x - c)$ ← tangent line

$f'(x) = \cos x$

$f'(0) = \cos(0) = 1$

$y - 1 = 1(x - 0) \Rightarrow y = x + 1$

Note: Tangent line approximation deals with the point of tangency. At a different point, the equation would be different!

Tangent Line Approximation

All in all, we arrive at the following:

Tangent Line Approximation- Consider a function f that is differentiable at c. The equation for the tangent line at the point $(c, f(c))$ is given by: $y = f(c) + f'(c)(x - c)$. This is called the tangent line (or linear) approximation.

Example

ⓐ $(16, 4)$

$y = \sqrt{x} = x^{1/2}$

Use the tangent line approximation to approximate $\sqrt{16.5}$. $\to$ $16.5 = 16 + 0.5$

$y' = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}} \Rightarrow \frac{1}{2\sqrt{16}} = \frac{1}{8}$

$y_t - 4 = \frac{1}{8}(x - 16)$

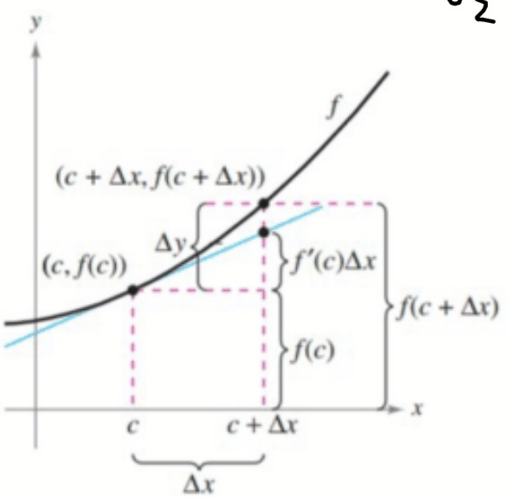
$y_t = \frac{1}{8}x + 2$

$y_{(16.5)} = \frac{1}{8}(16.5) + 2 \approx 4.0625$

Differentials

In the tangent line approximation equation: $y = f(c) + f'(c)(\overset{\Delta x}{x - c})$, the quantity $x - c$ is called the change in x (Δx). That being said, the change in y, or Δy, can be approximated by: (See figure below)

$$\Delta y = \underset{y_2}{f(c + \Delta x)} \underset{-y_1}{- f(c)} \approx \boxed{f'(c)\Delta x}$$



For such approximation, the quantity Δx is traditionally denoted by dx, and is called the **differential of x**. Moreover, Δy is called the **differential of y**. (dy)

$y' = \frac{dy}{dx}$

Example

Let $y = x^2$. Find dy when $x = 1$ and $dx = 0.01$. Compare this value with Δy for $x = 1$ and $\Delta x = 0.01$.

$y' = \boxed{\frac{dy}{\cancel{dx}} = 2x}\, dx$

$dy = 2x\, dx$

$dy = 2(1)(0.01)$

$= \boxed{0.020}$

$\approx$

$\Delta y = f(c + \Delta x) - f(c)$

$= f(1.01) - f(1)$

$= 0.0201$

Differential Form

Each of the differentiation rules that you have learned thus far can be written in **differential form**. This is where the **Leibniz derivative notation** comes in handy. Consider…

Function	Derivative	Differential
a. $y = x^2$	$y' = \frac{dy}{dx} = 2x$	$dy = 2x\,dx$
b. $y = x\cos x$	$\frac{dy}{dx} = \cos x - x\sin x$	$dy = \cos x\,dx - x\sin x\,dx$
c. $y = \frac{1}{x} = x^{-1}$	$\frac{dy}{dx} = -x^{-2} = -\frac{1}{x^2}$	$dy = -\frac{dx}{x^2}$

Error Approximation

Differentials can be also expressed and used in error measurement. Remember, every approximation has an error bound!

<u>Error Bound</u>: $|f(x) - [\underbrace{f(x) + f'(c)(x-c)}_{\text{tangent line app.}}]|$

$f(c)+f'(c)(x-c)$

Example

Let f be a function given by $f(x) = x^2 - 2x + 3$. The tangent line to the graph of f at $x = 2$ is used to approximate values of $f(x)$. What is the greatest value of x, to the nearest tenth, for which the error resulting from this tangent line approximation is less than 0.5?

$f'(x) = 2x-2$

$f'(2) = 2(2)-2$

$= 2$

$f(2) = 3$

$f(x)+f'(c)(x-c)$

$= x^2-2x+3+2(x-2)$

$= x^2 \cancel{-2x} + 3 \cancel{+2x} - 4$

x^2-1

$|3-(x^2-1)| < 0.5$

$|3-x^2+1| < 0.5$

$|-x^2+4| < 0.5 \Rightarrow$

$-x^2+4 < 0.5$

$x^2 < 3.5$

$\boxed{-\sqrt{3.5} < x < \sqrt{3.5}}$

$-x^2+4 > -0.5$

$+x^2 > 4.5$

$x > \sqrt{4.5}$

$x < -\sqrt{4.5}$

Homework Due 10/22

TB 3.9 #1-17 (e.o.o), 30, 43-46, 55-58